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# The class of *n*-entire operators

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**Abstract**
We introduce a classification of simple, regular, closed symmetric operators with deficiency indices $(1, 1)$ according to a geometric criterion that extends the classical notions of entire operators and entire operators in the generalized sense due to M G Krein. We show that these classes of operators have several distinctive properties, some of them related to the spectra of their canonical self-adjoint extensions. In particular, we provide necessary and sufficient conditions on the spectra of two canonical self-adjoint extensions of an operator for it to belong to one of our classes. Our discussion is based on some recent results in the theory of de Branges spaces.

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## 1. Introduction

Let $\mathcal{S}(\mathcal{H})$ be the class of regular, closed symmetric operators on a separable Hilbert space $\mathcal{H}$, whose deficiency indices are $(1, 1)$ (see details in section 2). It is well known that operators of this class share a number of distinctive properties. For instance, all the canonical self-adjoint extensions of a given operator $A \in \mathcal{S}(\mathcal{H})$ have simple discrete spectra, pairwise interlaced, whose union is the real line. Also, associated with each $A \in \mathcal{S}(\mathcal{H})$, there exists a unitary transformation that maps $\mathcal{H}$ onto a de Branges space (a special kind of Hilbert space of entire functions [6]), on which $A$ is unitarily transformed into the multiplication operator by the independent variable [30]. These facts, among others, have been exploited in more or less explicit form in the study of diverse questions of interest in mathematical physics, like boundary-value and inverse problems of canonical systems [12, 38], the spectral analysis of Krein strings [14], inverse spectral problems of one-dimensional Schrödinger operators [27] (see also [8] for recent developments in the case of strongly singular potentials), analysis of minimum uncertainty for quantum observables [25] and some related problems in quantum gravity [15–17], to mention a few of them. Besides applications in mathematical physics, operators in $\mathcal{S}(\mathcal{H})$ have been used in some aspects of signal processing and analytical sampling theory (see for instance [29]).

In this paper, we introduce a classification of operators within $\mathcal{S}(\mathcal{H})$. Namely, for every given $n \in \mathbb{Z}^+ = \{0, 1, \ldots\}$, we consider those operators $A \in \mathcal{E}_n(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ for which one can find $n + 1$ vectors $\mu_0, \ldots, \mu_n \in \mathcal{H}$ such that

$$\mathcal{H} = \operatorname{ran}(A - zI) \dot{+} \operatorname{span}\{\mu_0 + z\mu_1 + \cdots + z^n\mu_n\}, \qquad \text{for all } z \in \mathbb{C}. \tag{1.1}$$

The aim of this paper is to discuss a number of properties that are common to all operators within each class $\mathcal{E}_n(\mathcal{H})$, some of them related to the spectra of their canonical self-adjoint extensions, and some others connected to their associated de Branges spaces. It will be shown that our classification carries out a refinement in the characterization of some (but not all) operators in $\mathcal{S}(\mathcal{H})$.

Among the operators that obey (1.1) are the entire operators as well as the entire operators in the generalized sense. These classes of operators, which include operators frequently appearing in mathematical physics, were originally concocted by M G Krein as a tool for treating in a unified way several classical problems in mathematical analysis [18–20, 22]. A detailed review of entire operators and their many remarkable properties is given in [10]. Because of this connection with entire operators, the class $\mathcal{E}_n(\mathcal{H})$ will henceforth be referred to as the class of *n*-entire operators.

Let us describe briefly the relation between Krein's definitions and ours here, referring the details to section 2. In what follows, let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathcal{H}$, assumed antilinear in its first argument. We recall that a simple, regular, closed symmetric operator $A$, *densely defined* on $\mathcal{H}$, with deficiency indices $(1, 1)$, is entire (according to Krein) if there exists $\mu \in \mathcal{H}$, such that $\mathcal{H} = \operatorname{ran}(A - zI) \dot{+} \operatorname{span}\{\mu\}$ for all $z \in \mathcal{H}$. Equivalently, $A$ is entire if $\langle \xi(\bar{z}), \mu \rangle$ is a zero-free entire function, where $\xi(z)$ is a certain vector-valued zero-free entire function such that $\xi(z) \in \ker(A^* - zI)$ (for details see [10, 30]). The operator $A$ is entire in the generalized sense if there exists $\mu \in \mathcal{H}_-$ such that $[\xi(\bar{z}), \mu]$ is a zero-free entire function, where $\mathcal{H}_-$ is the dual of $\mathcal{H}_+ := \operatorname{dom}(A^*)$ equipped with the graph norm and $[\cdot, \cdot]$ denotes the associated duality bracket. Clearly, an operator entire according to Krein's definition is 0-entire. It is a bit less apparent that an operator entire in the generalized sense is indeed 1-entire. To see this, we observe that, as a direct consequence of [30, proposition 5.1], given $\mu \in \mathcal{H}_{-1} \setminus \mathcal{H}$, one can find $\mu_0, \mu_1 \in \mathcal{H}$ such that

$$[\xi(\bar{z}), \mu] = \langle \xi(\bar{z}), \mu_0 \rangle + z\langle \xi(\bar{z}), \mu_1 \rangle$$

for all $z \in \mathbb{C}$, hence reducing Krein's to our definition with $n = 1$. It is worth remarking here that the class $\mathcal{S}(\mathcal{H})$ includes operators with a non-dense domain. That is, our classes of 0-entire and 1-entire operators are themselves larger than the corresponding Krein's classes.

As first discussed in [30], it is possible to determine whether an operator is either entire or entire in the generalized sense under conditions that rely exclusively on the distribution of the spectra of self-adjoint extensions of the operator. This spectral characterization was obtained on the basis of recent results in the theory of de Branges spaces [39, 40]. One of the main results of this paper is a generalization of this spectral characterization to *n*-entire operators.

The remainder of this paper is organized as follows. In section 2, we introduce the main concepts relevant to this work. We also present here the mathematical background (including some new results) needed later. In section 3, we discuss several characterizations for the classes of operators discussed in this paper. Section 4 is devoted to the construction of a Gelfand triplet associated with *n*-entire operators, in an attempt to recover the original way that Krein used to introduce the notion of operator entire in the generalized sense. Finally, we draw some conclusions and point out some ideas for further investigation in section 5.

## 2. Symmetric operators and de Branges spaces

In this section, we lay out the notation and introduce some of the main objects to be considered in this work. The first part of the section deals with symmetric operators. The operator classes which will be discussed in this work are defined here. The second part is devoted to the theory of de Branges spaces. Finally, the last part of this section deals with the construction of the functional model for the operators considered in the first part. The functional model serves as a bridge that relates every operator in $\mathcal{S}(\mathcal{H})$ to a certain de Branges space.

### *2.1. On symmetric operators with not necessarily a dense domain*

Let $\mathcal{H}$ be a Hilbert space whose inner product $\langle\cdot,\cdot\rangle$ is assumed antilinear in its first argument. In this space, we consider a closed, symmetric linear operator $A$ with deficiency indices $(1,1)$. It is not presumed that its domain is dense in $\mathcal{H}$; therefore, one should deal with the case when the adjoint of $A$ is a closed linear relation. Recall that a closed linear relation in $\mathcal{H}$ is a subspace of $\mathcal{H}\oplus\mathcal{H}$, and therefore, closed operators are closed linear relations when they are identified with their graphs. Thus, in general,

$$A^* := \{(\eta,\omega)\in\mathcal{H}\oplus\mathcal{H} : \langle\eta, A\varphi\rangle = \langle\omega,\varphi\rangle \text{ for all } \varphi\in\operatorname{dom}(A)\}. \tag{2.1}$$

Whenever the orthogonal complement of $\operatorname{dom}(A)$ is trivial, the set

$$A^*(0) := \{\omega\in\mathcal{H} : (0,\omega)\in A^*\}$$

is also trivial, i.e. $A^*(0) = \{0\}$, so $A^*$ is an operator; otherwise, $A^*$ is a multivalued closed linear relation.

For $z\in\mathbb{C}$, one has

$$A^* - zI := \{(\eta,\omega - z\eta)\in\mathcal{H}\oplus\mathcal{H} : (\eta,\omega)\in A^*\}, \tag{2.2}$$

so accordingly

$$\ker(A^* - zI) := \{\eta\in\mathcal{H} : (\eta,0)\in A^* - zI\}. \tag{2.3}$$

Therefore, on the basis of the decomposition,

$$\mathcal{H} = \operatorname{ran}(A - zI)\oplus\ker(A^* - \overline{z}I), \tag{2.4}$$

which holds independently of the fact that $A$ is or not densely defined [3, proposition 3.31], our assumption on the deficiency indices implies $\dim\ker(A^* - zI) = 1$ for all $z\in\mathbb{C}\setminus\mathbb{R}$. Moreover, since

$$A^*(0) = \{\omega\in\mathcal{H} : \langle\omega,\psi\rangle = 0 \text{ for all } \psi\in\operatorname{dom}(A)\},$$

it is obvious that $A^*(0) = \operatorname{dom}(A)^\perp$.

In this work, we deal not only with symmetric operators but also with their canonical self-adjoint extensions. A canonical self-adjoint extension of a given symmetric operator is a self-adjoint extension within the original space $\mathcal{H}$, i.e. a self-adjoint extension of $A$ being a restriction of $A^*$. If $A$ turns out not to be densely defined, then a canonical self-adjoint extension $A_\gamma$ of $A$ is a subspace of $A^*$ that extends the graph of $A$ and that satisfies $A_\gamma^* = A_\gamma$ (as subsets of $\mathcal{H}\oplus\mathcal{H}$).

The following proposition concerns non-densely defined symmetric operators. It shows that the condition for the deficiency indices to be $(1,1)$ implies that these operators are not quite dissimilar to the densely defined ones. A proof of this proposition follows from [11, section 1, lemma 2.2 and theorem 2.4] (see [11, proposition 5.4] and the comment below it).

**Proposition 2.1.** *Let $A$ be a closed, non-densely defined, symmetric operator in a Hilbert space. If $A$ has deficiency indices $(1,1)$, then* *(i) the codimension of* $\operatorname{dom}(A)$ *equals 1,*

*(ii) all except one of the canonical self-adjoint extensions of A are operators.*

Let us now bring up a simple result which does not depend on whether the operator is densely defined or not. The proof of it can be found in [11, section 1] for the nondensely defined case and in [10, section 2.1] for the densely defined one. Before stating it, we remind the reader that the spectrum of a closed linear relation $B$ in $\mathcal{H}$ is the complement of the set of all $z \in \mathbb{C}$ such that $(B - zI)^{-1}$ is a bounded operator defined on all $\mathcal{H}$. Moreover, $\operatorname{spec}(B) \subset \mathbb{R}$ when $B$ is a self-adjoint linear relation [7].

**Proposition 2.2.** *Let A be a closed, symmetric operator in a Hilbert space. If* $A_\gamma$ *is a canonical self-adjoint extension of A, then the operator*

$$I + (z - w)(A_\gamma - zI)^{-1}, \quad z \in \mathbb{C} \setminus \operatorname{spec}(A_\gamma), \quad w \in \mathbb{C},$$

*maps* $\ker(A^* - wI)$ *injectively onto* $\ker(A^* - zI)$.

The operator given in this proposition is the generalized Cayley transform and we use it to define a function taking values in $\ker(A^* - zI)$ as follows:

$$\psi(z) := [I + (z - w_0)(A_\gamma - zI)^{-1}]\psi_{w_0}, \tag{2.5}$$

for given $\psi_{w_0} \in \ker(A^* - w_0 I)$ and $w_0 \in \mathbb{C} \setminus \mathbb{R}$. Clearly, $\psi(\cdot)$ is an analytic function in the upper and lower half-planes because of the analytic properties of the resolvent. Obviously, $\psi(w_0) = \psi_{w_0}$. Moreover, a computation involving the resolvent identity yields

$$\psi(z) = [I + (z - v)(A_\gamma - zI)^{-1}]\psi(v), \tag{2.6}$$

for any pair $z, v \in \mathbb{C} \setminus \mathbb{R}$. This identity will be used later on.

For the sake of completeness, and also for future reference, we recall the notion of simplicity of a closed symmetric nonself-adjoint operator. A closed symmetric nonself-adjoint operator is said to be simple (or completely nonself-adjoint) if it is not a nontrivial orthogonal sum of a symmetric and a self-adjoint operator. Since an invariant subspace of a symmetric operator is a subspace reducing that operator [5, theorem 4.6.1], a symmetric operator $A$ is simple when there is not a nontrivial invariant subspace of $A$ on which $A$ is self-adjoint.

By [23, proposition 1.1] (see [10, theorem 1.2.1] for the densely defined case), a necessary and sufficient condition for the symmetric nonself-adjoint operator $A$ to be simple is

$$\bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \operatorname{ran}(A - zI) = \{0\}. \tag{2.7}$$

Simplicity plays an important role in our further considerations. Here, we briefly discuss some of the distinctive features that a closed symmetric operator with deficiency indices $(1, 1)$ has when it is simple. Consider the function $\psi(\cdot)$ given by (2.5) and take a sequence $\{z_k\}_{k=1}^\infty$ with elements in $\mathbb{C} \setminus \mathbb{R}$ having accumulation points in the upper and lower half-planes. Suppose that there is $\eta \in \mathcal{H}$ such that $\langle \eta, \psi(z_k) \rangle = 0$ for all $k \in \mathbb{N}$. This implies that $\langle \eta, \psi(z) \rangle = 0$ for $z \in \mathbb{C} \setminus \mathbb{R}$ because of the analyticity of the function $\langle \eta, \psi(\cdot) \rangle$. Therefore, by (2.7), $\eta = 0$. We have thus arrived at the conclusion that simple, closed symmetric operators with deficiency indices $(1, 1)$ can exist only in *separable* Hilbert spaces. From now on, the reader should assume that $\mathcal{H}$ is separable.

Another property of simple, closed symmetric operators with deficiency indices $(1, 1)$ concerns their commutativity with involutions and it is the content of the next proposition. We say that an involution $J$ commutes with a self-adjoint relation $B$ if

$$J(B - zI)^{-1}\varphi = (B - \bar{z}I)^{-1}J\varphi,$$

for every $\varphi \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. If $B$ is moreover an operator, this is equivalent to the usual notion of commutativity:

$$J \operatorname{dom}(B) \subseteq \operatorname{dom}(B), \qquad JB\varphi = BJ\varphi,$$

for every $\varphi \in \operatorname{dom}(B)$.

**Proposition 2.3.** *Let $A$ be a simple, closed symmetric operator with deficiency indices* $(1, 1)$. *Then, there exists an involution $J$ that commutes with all its canonical self-adjoint extensions.*

**Proof.** Choose a self-adjoint extension $A_\gamma$ and consider $\psi(z)$ as defined by (2.5). Recalling (2.6) along with the unitary character of the generalized Cayley transform, and applying the resolvent identity, one can verify that

$$\langle \psi(\overline{z}), \psi(\overline{v}) \rangle = \langle \psi(v), \psi(z) \rangle \tag{2.8}$$

for every pair $z, v \in \mathbb{C} \setminus \mathbb{R}$.

Now define the action of $J$ on $\psi(z)$ $(z \in \mathbb{C} \setminus \mathbb{R})$ by the rule

$$J\psi(z) = \psi(\overline{z}),$$

and on the set $\mathcal{D}$ of finite linear combinations of such elements by

$$J\left(\sum_n c_n \psi(z_n)\right) := \sum_n \overline{c_n} \psi(\overline{z_n}),$$

where the sequence $\{z_k\}_{k=1}^\infty$ is defined as in the paragraph following (2.7). Then, on one hand, (2.8) implies that $J$ is an involution on $\mathcal{D}$ which can be extended to all $\mathcal{H}$ because of the simplicity of $A$. On the other hand, since by the resolvent identity

$$(A_\gamma - wI)^{-1}\psi(z) = \frac{\psi(z) - \psi(w)}{z - w},$$

one obtains the identity

$$J(A_\gamma - wI)^{-1}\psi(z) = (A_\gamma - \overline{w}I)^{-1} J\psi(z),$$

which by linearity holds on $\mathcal{D}$ and in turn it extends to all $\mathcal{H}$.

So far we know that $J$ commutes with $A_\gamma$. By resorting to the well-known resolvent formula due to Krein (see [11, theorem 3.2] for a generalized formulation), one immediately obtains the commutativity of $J$ with all the self-adjoint extensions of $A$ within $\mathcal{H}$. □

We now remind the reader of the notion of regularity of a closed operator. A closed operator $A$ in $\mathcal{H}$ is regular if for every $z \in \mathbb{C}$ there exists $d_z > 0$ such that

$$\|(A - zI)\psi\| \geqslant d_z \|\psi\|, \tag{2.9}$$

for all $\psi \in \operatorname{dom}(A)$. In other words, $A$ is regular if every point of the complex plane is a point of regular type.

It is easy to see that a regular, closed symmetric operator is necessarily simple, this is so because the regularity implies the lack of spectral kernel. However, the converse statement is not true.

Let us define the operator class $\mathcal{S}(\mathcal{H})$ as the set of all regular, closed symmetric operators with deficiency indices $(1, 1)$. By what has just been said in the paragraph above, all the operators in $\mathcal{S}(\mathcal{H})$ are simple. But regularity adds also further properties to the class $\mathcal{S}(\mathcal{H})$. Indeed, the combination of regularity and the fact that the deficiency indices are $(1, 1)$ leads to the following proposition which extends to the whole class $\mathcal{S}(\mathcal{H})$ the well-known facts for densely defined operators in $\mathcal{S}(\mathcal{H})$.

**Proposition 2.4.** *For $A \in \mathcal{S}(\mathcal{H})$, the following assertions hold true.*

*(i) The spectrum of every canonical self-adjoint extension of A consists solely of isolated eigenvalues of multiplicity 1.*
*(ii) Every real number is part of the spectrum of one, and only one, canonical self-adjoint extension of A.*
*(iii) The spectra of the canonical self-adjoint extensions of A are pairwise interlaced.*

**Proof.** We will prove (i) using similar ideas as in the proofs of propositions 3.1 and 3.2 of [10], but taking into account that the operator is not necessarily densely defined.

For $A \in \mathcal{S}(\mathcal{H})$ and any $r \in \mathbb{R}$, consider the constant $d_r$ of (2.9). Thus, the symmetric operator $(A-rI)^{-1}$, defined on the subspace $\operatorname{ran}(A-rI)$, is such that $\|(A-rI)^{-1}\| \leqslant d_r^{-1}$. By [21, theorem 2], there is a self-adjoint extension $B$ of $(A-rI)^{-1}$ defined on the whole space and such that $\|B\| \leqslant d_r^{-1}$. Now, $B^{-1}$ is a self-adjoint extension of $A-rI$ and $\|B^{-1}f\| \geqslant d_r\|f\|$ for any $f \in \operatorname{dom}(B^{-1})$, which implies that $(-d_r, d_r) \cap \operatorname{spec}(B^{-1}) = \emptyset$. By appropriately shifting $B^{-1}$, one obtains a self-adjoint extension of $A$ with no spectrum in the spectral lacuna $(r-d_r, r+d_r)$. Now, according to perturbation theory, any self-adjoint extension of $A$ which is an operator has no points of the spectrum in this spectral lacuna other than one eigenvalue of multiplicity 1. To prove (i) for operator extensions, consider any closed interval of $\mathbb{R}$, cover it with spectral lacunae and take a finite subcover. Actually (i) also holds for the only self-adjoint multivalued relation in the case $\overline{\operatorname{dom}(A)} \neq \mathcal{H}$. This follows from the simplicity of the operator self-adjoint extensions and [11, equation 3.10].

Once (i) has been proven, the assertions (ii) and (iii) follow again from [11, equation 3.10] and the properties of Herglotz meromorphic functions. □

We now turn to the discussion of the notion of entire operators and their generalizations. A vector $\mu \in \mathcal{H}$ is said to be a gauge for (a given operator) $A \in \mathcal{S}(\mathcal{H})$ if and only if

$$\mathcal{H} = \operatorname{ran}(A - z_0 I) \dot{+} \operatorname{span}\{\mu\}, \tag{2.10}$$

for some $z_0 \in \mathbb{C}$, where $\dot{+}$ denotes the direct sum. Once a gauge has been chosen, we look for the set of complex numbers for which (2.10) fails to hold:

$$\{z \in \mathbb{C} : \mu \perp \ker(A^* - \bar{z}I)\}. \tag{2.11}$$

The set (2.11) is at most an infinite countable set with no finite accumulation points (see [30, section 2]). Moreover, depending on the choice of the gauge $\mu$, the set (2.11) could be entirely contained in $\mathbb{R}$ [30, lemma 2.1] or placed completely outside $\mathbb{R}$ [30, theorem 2.2].

If the gauge $\mu$ can be chosen so that the set (2.11) is empty, then the gauge is said to be entire. In other words, $\mu \in \mathcal{H}$ is an entire gauge if and only if

$$\mathcal{H} = \operatorname{ran}(A - zI) \dot{+} \operatorname{span}\{\mu\}, \tag{2.12}$$

for all $z \in \mathbb{C}$.

Within $\mathcal{S}(\mathcal{H})$, we single out the class $\mathcal{E}_0(\mathcal{H})$ of operators for which there exists an entire gauge. The operators in $\mathcal{E}_0(\mathcal{H})$ are called the entire operators. The densely defined operators in $\mathcal{E}_0(\mathcal{H})$ were originally introduced by Krein in the 1940s for the purpose of treating in a unified way several classical problems in mathematical analysis [18–20, 22]. It is worth remarking that the extension of the concept of entire operators from the densely defined ones to the not necessarily densely defined operators is completely natural in the light of the investigations carried out by de Branges on certain Hilbert spaces of entire functions in the 1960s. This will become clear in subsection 2.3.

Krein's theory of entire operators is constructed on the basis of a particular functional model for densely defined operators in the class $\mathcal{S}(\mathcal{H})$. This functional model was generalized in an abstract way in [34, 35] to include operator classes broader than $\mathcal{S}(\mathcal{H})$. Subsection 2.3 provides a realization of the abstract construction of [34, 35] based on the function given in (2.5). Basically, the idea behind our functional model is to construct a function which associates with any complex number $z$ a vector $\xi(z) \in \ker(A^* - zI)$. By means of this function, one says that $A$ is in $\mathcal{E}_0(\mathcal{H})$ if and only if there exists $\mu \in \mathcal{H}$ such that for all $z \in \mathbb{C}$

$$\langle \xi(\overline{z}), \mu \rangle \neq 0. \tag{2.13}$$

Besides entire operators, Krein considered the so-called entire operators in the generalized sense. These operators were studied by Šmuljan (see for instance [33]) and their definition is as follows. Take a densely defined operator $A \in \mathcal{S}(\mathcal{H})$ and consider the Hilbert space $\mathcal{H}_+$ to be the linear set $\operatorname{dom}(A^*)$ equipped with the graph norm. Let $\mathcal{H}_-$ be the dual of $\mathcal{H}_+$, i.e. the collection of $\mathcal{H}_+$-continuous antilinear functionals. Clearly, $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ (for the details see section 4). Then, $A$ is entire in the generalized sense when there is $\mu \in \mathcal{H}_- \setminus \mathcal{H}$ such that for all $z \in \mathbb{C}$, one has, instead of (2.13),

$$[\xi(\overline{z}), \mu] \neq 0,$$

where $[\cdot, \cdot]$ denotes the duality bracket between $\mathcal{H}_+$ and $\mathcal{H}_-$.

It will be proven below (see proposition 4.8) that a densely defined operator $A$ is entire in the generalized sense if and only if there are vectors $\mu_0, \mu_1 \in \mathcal{H}$ such that

$$\mathcal{H} = \operatorname{ran}(A - zI) \dot{+} \operatorname{span}\{\mu_0 + z\mu_1\}. \tag{2.14}$$

Clearly, this definition makes sense whether or not the operator is densely defined. This motivated us to single out the class $\mathcal{E}_1(\mathcal{H})$ of operators entire in the generalized sense as the collection of operators in $\mathcal{S}(\mathcal{H})$ that satisfies (2.14).

At this point, it is clear that our definition of the classes $\mathcal{E}_n(\mathcal{H})$, of operators in $\mathcal{S}(\mathcal{H})$ that fulfil (1.1) for a given $n \in \mathbb{Z}^+$, is the natural generalization of the classes $\mathcal{E}_0(\mathcal{H})$ and $\mathcal{E}_1(\mathcal{H})$. These classes are ordered in the following sense:

$$\mathcal{E}_0(\mathcal{H}) \subset \mathcal{E}_1(\mathcal{H}) \subset \mathcal{E}_2(\mathcal{H}) \subset \cdots \subset \mathcal{S}(\mathcal{H}).$$

However,

$$\bigcup_{n \in \mathbb{Z}^+} \mathcal{E}_n(\mathcal{H}) \subsetneq \mathcal{S}(\mathcal{H}),$$

as will become clear in section 3 and illustrated by example 3.13.

**Example 2.5.** Here, we construct densely and nondensely defined 0-entire operators using Jacobi matrices. These matrices appear often in the mathematical physics literature not only because of the theoretical significance the corresponding operators have for being the discrete analogue of Sturm–Liouville operators, but also because they are used for modeling physical processes as in solid state physics within the so-called tight binding approximation [9, chapter 9], quantum optics [37] and mechanics [36, section 1.5 and part 2].

Consider the semi-infinite Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \tag{2.15}$$

where $b_k > 0$ and $q_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Fix an orthonormal basis $\{\delta_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$. Let $B$ be the operator in $\mathcal{H}$ whose matrix representation with respect to $\{\delta_k\}_{k\in\mathbb{N}}$ is (2.15) (cf [2, section 47]). We assume that $B \neq B^*$, which in this case is equivalent to assuming that $B$ has deficiency indices $(1, 1)$ [1, chapter 4, section 1.2]. A classical result tells us that the orthogonal polynomials of the first kind $P_k(z)$ associated with (2.15) are such that

$$\sum_{k=0}^{\infty} |P_k(z)|^2 < \infty$$

uniformly in any compact domain of the complex plane [1, theorem 1.3.2]. Therefore, for any $z \in \mathbb{C}$, $\pi(z) := \sum_{k=1}^{\infty} P_{k-1}(z)\delta_k$ is in $\mathcal{H}$, and more specifically in $\ker(B^* - zI)$ [1, chapter 4, section 1.2]. By construction of the polynomials of the first kind,

$$\langle \pi(\overline{z}), \delta_1 \rangle = P_0(\overline{z}) \equiv 1,$$

so $B$ is a densely defined 0-entire and $\delta_1$ is an entire gauge.

Now we outline how one may construct a 0-entire operator which is not densely defined. Let $B_0$ be the restriction of $B$ to the set $\{\phi \in \operatorname{dom}(B) : \langle \phi, \delta_1 \rangle = 0\}$. It follows from (2.1), (2.2) and (2.3) that $\eta \in \ker(B_0^* - zI)$ if and only if it satisfies the equation

$$\langle B\phi, \eta \rangle = \langle \phi, z\eta \rangle \qquad \forall \phi \in \operatorname{dom}(B_0).$$

Thus, $\ker(B_0^* - zI)$ is the set of $\eta$'s in $\mathcal{H}$ that satisfy

$$b_{k-1} \langle \delta_{k-1}, \eta \rangle + q_k \langle \delta_k, \eta \rangle + b_k \langle \delta_{k+1}, \eta \rangle = z \langle \delta_k, \eta \rangle \quad \forall k > 1 \tag{2.16}$$

Hence, $\dim \ker(B_0^* - zI) \leqslant 2$. Now, let $\theta(z) := \sum_{k=1}^{\infty} Q_{k-1}(z)\delta_k$, where $Q_k(z)$ is the $k$th polynomial of second kind associated with (2.15). By the definition of the polynomials $P_k(z)$ and $Q_k(z)$ [1, chapter 1, section 2.1], $\pi(z)$ and $\theta(z)$ are linearly independent solutions of (2.16) for every fix $z \in \mathbb{C}$. Moreover, since $B \neq B^*$, $\pi(z)$ and $\theta(z)$ are in $\mathcal{H}$ for all $z \in \mathbb{C}$ [1, theorems 1.3.1 and 1.3.2], [32, theorem 3]. So one arrives at the conclusion that, for every fix $z \in \mathbb{C}$,

$$\ker(B_0^* - zI) = \operatorname{span}\{\pi(z), \theta(z)\}.$$

Any symmetric nonself-adjoint extension of $B_0$ has deficiency indices (1,1). Furthermore, if $\kappa(z)$ is a ($z$-dependent) linear combination of $\pi(z)$ and $\theta(z)$ such that $\langle \kappa(z), \theta(z) \rangle = 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, then (by a parametrized version of [32, theorem 2.4]) there corresponds to an appropriately chosen isometry from $\operatorname{span}\{\kappa(z)\}$ onto $\operatorname{span}\{\kappa(\overline{z})\}$ a nonself-adjoint symmetric extension $\widetilde{B}$ of $B_0$ such that $\operatorname{dom}(\widetilde{B})$ is not dense and $\ker(\widetilde{B}^* - zI) = \operatorname{span}\{\theta(z)\}$. We claim that $\widetilde{B}$ is a nondensely defined entire operator. Indeed, $\widetilde{B} \in \mathcal{S}(\mathcal{H})$ (the simplicity follows from the properties of the associated polynomials [1, chapter 1, addenda and problem 7]). Moreover, since

$$\langle \theta(\overline{z}), \delta_2 \rangle = b_1^{-1} \qquad \forall z \in \mathbb{C},$$

$\delta_2$ is an entire gauge.

### *2.2. On de Branges spaces with zero-free associated functions*

Let $\mathcal{B}$ denote a nontrivial Hilbert space of entire functions with inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}}$. $\mathcal{B}$ is said to be a de Branges space when, for every function $f(z)$ in $\mathcal{B}$, the following conditions hold.

(A1) For every $w \in \mathbb{C} \setminus \mathbb{R}$, the linear functional $f(\cdot) \mapsto f(w)$ is continuous.
(A2) For every non-real zero $w$ of $f(z)$, the function $f(z)(z - \overline{w})(z - w)^{-1}$ belongs to $\mathcal{B}$ and has the same norm as $f(z)$.
(A3) The function $f^{\#}(z) := \overline{f(\overline{z})}$ also belongs to $\mathcal{B}$ and has the same norm as $f(z)$.

In view of the Riesz lemma, (A1) is equivalent to the existence of a reproducing kernel $k(z, w)$ that belongs to $\mathcal{B}$ for every non-real $w$ such that $\langle k(\cdot, w), f(\cdot)\rangle_{\mathcal{B}} = f(w)$ for every $f(z) \in \mathcal{B}$. Also, for any $w \in \mathbb{C}$, $k(w, w) = \langle k(\cdot, w), k(\cdot, w)\rangle_{\mathcal{B}} \geqslant 0$, where, as a consequence of (A2), the positivity is strict for every non-real $w$ unless $\mathcal{B} \cong \mathbb{C}$; see the proof of [6, theorem 23]. Note that $k(z, w) = \langle k(\cdot, z), k(\cdot, w)\rangle_{\mathcal{B}}$ whenever $z$ and $w$ are both non-real; therefore, $k(w, z) = \overline{k(z, w)}$. Furthermore, due to (A3), it can be proven (again using [6, theorem 23]) that $\overline{k(\overline{z}, w)} = k(z, \overline{w})$ for every non-real $w$. Also note that $k(z, w)$ is entire with respect to its first argument and, by (A3), it is anti-entire with respect to the second one (once $k(z, w)$, as a function of its second argument, has been extended to the whole complex plane [6, problem 52]).

There is an alternative definition of a de Branges space. Its starting point is an entire function $e(z)$ of the Hermite–Biehler class, i.e. an entire function without zeros in the upper half-plane $\mathbb{C}^+$ that satisfies the inequality $|e(z)| > \left|e^{\#}(z)\right|$ for $z \in \mathbb{C}^+$. On the basis of this function, one first defines $\mathcal{B}(e)$ to be the linear manifold of all entire functions $f(z)$ such that both $f(z)/e(z)$ and $f^{\#}(z)/e(z)$ belong to the Hardy space $H^2(\mathbb{C}^+)$ and, second, equips it with the inner product

$$\langle f(\cdot), g(\cdot)\rangle_{\mathcal{B}(e)} := \int_{-\infty}^{\infty} \frac{\overline{f(x)}g(x)}{|e(x)|^2}\, \mathrm{d}x.$$

Then, $\mathcal{B}(e)$ turns out to be a Hilbert space of entire functions.

Now, according to [6, chapter 2], every space $\mathcal{B}(e)$ obeys (A1)–(A3), and conversely, given a space $\mathcal{B}$, there exists an Hermite–Biehler function $e(z)$ such that $\mathcal{B}$ coincides with $\mathcal{B}(e)$ as sets and the respective norms satisfy the equality $\|f(\cdot)\|_{\mathcal{B}} = \|f(\cdot)\|_{\mathcal{B}(e)}$. Thus, both definitions of de Branges spaces are equivalent.

**Remark 2.6.** For an entire function $f(z)$, the condition that $f(z)/e(z)$ and $f^{\#}(z)/e(z)$ are in $H^2(\mathbb{C}^+)$ is equivalent to

$$\int_{-\infty}^{\infty} \left|\frac{f(x)}{e(x)}\right|^2 \mathrm{d}x < \infty$$

and the functions $f(z)/e(z)$ and $f^{\#}(z)/e(z)$ being of bounded type and nonpositive mean type in the upper half-plane [27, proposition 2.1].

The function $e(z)$ is not uniquely determined by the de Branges space $\mathcal{B}$. However, if one sets

$$e(z) = -\mathrm{i}\sqrt{\frac{\pi}{k(w_0, w_0)\mathrm{im}(w_0)}}\,(z - \overline{w_0})\, k(z, w_0),$$

where $w_0$ is some fixed complex number in $\mathbb{C}^+$, then $\mathcal{B} = \mathcal{B}(e)$ in the sense given above.

An entire function $g(z)$ is said to be associated with a de Branges space $\mathcal{B}$ if for all $f(z) \in \mathcal{B}$ and $w \in \mathbb{C}$,

$$\frac{g(z)f(w) - g(w)f(z)}{z - w} \in \mathcal{B}.$$

The set of associated functions is denoted by $\operatorname{assoc}\mathcal{B}$. It can be shown that

$$\operatorname{assoc}\mathcal{B} = \mathcal{B} + z\mathcal{B}; \tag{2.17}$$

see [6, theorem 25] and [13, lemma 4.5] for alternative characterizations. Incidentally, let us note that $e(z) \in \operatorname{assoc}\mathcal{B}(e) \setminus \mathcal{B}(e)$; this fact straightforwardly follows from [6, theorem 25].

The space $\operatorname{assoc}\mathcal{B}(e)$ contains a distinctive family of entire functions:

$$s_\beta(z) := \frac{\mathrm{i}}{2}[\mathrm{e}^{\mathrm{i}\beta}e(z) - \mathrm{e}^{-\mathrm{i}\beta}e^{\#}(z)], \quad \beta \in [0, \pi).$$

These real entire functions are related to the self-adjoint extensions of the multiplication operator $S$ defined by

$$\operatorname{dom}(S) := \{f(z) \in \mathcal{B} : zf(z) \in \mathcal{B}\}, \quad (Sf)(z) = zf(z). \tag{2.18}$$

The operator $S$ is closed, symmetric with deficiency indices $(1, 1)$, and its domain is not necessarily dense in $\mathcal{B}$ [13, proposition 4.2]. Furthermore, $S$ is regular [13, corollary 4.7] and hence simple. It turns out that $\overline{\operatorname{dom}(S)} \neq \mathcal{B}$ if and only if there exists $\gamma \in [0, \pi)$ such that $s_\gamma(z) \in \mathcal{B}$. Moreover, $\operatorname{dom}(S)^\perp = \operatorname{span}\{s_\gamma(z)\}$ [6, theorem 29] and [13, corollary 6.3]; compare with (i) of proposition 2.1.

Given a self-adjoint extension $S_\sharp$ of $S$, one can find a unique $\beta$ in $[0, \pi)$ such that

$$(S_\sharp - wI)^{-1} f(z) = \frac{f(z) - \frac{s_\beta(z)}{s_\beta(w)} f(w)}{z - w}, \quad w \notin \operatorname{spec}(S_\sharp), \quad f(z) \in \mathcal{B}, \tag{2.19}$$

with $\operatorname{spec}(S_\sharp) = \left\{x \in \mathbb{R} : s_\beta(x) = 0\right\}$ [13, propositions 4.6 and 6.1]. When $S_\sharp$ is a self-adjoint operator extension of $S$, then (2.19) is equivalent to

$$\operatorname{dom}(S_\sharp) = \left\{ g(z) = \frac{f(z) - \frac{s_\beta(z)}{s_\beta(z_0)} f(z_0)}{z - z_0}, \quad f(z) \in \mathcal{B}, \quad z_0 : s_\beta(z_0) \neq 0 \right\}$$

and

$$(S_\sharp g)(z) = zg(z) + \frac{s_\beta(z)}{s_\beta(z_0)} f(z_0).$$

In this context, the function

$$g_n(z) := \frac{s_\beta(z)}{z - x_n}$$

is the eigenfunction of $S_\sharp$ corresponding to $x_n \in \operatorname{spec}(S_\sharp)$. Hence, due to the fact that $S$ is regular and simple, every $s_\beta(z)$ has only real zeros of multiplicity 1 and the zeros of any pair $s_\beta(z)$ and $s_{\beta'}(z)$ always interlace.

The classical notion of associated functions (2.17) has been generalized in [24] as follows. For $n \in \mathbb{Z}^+$, let

$$\operatorname{assoc}_n \mathcal{B} := \mathcal{B} + z\mathcal{B} + \cdots + z^n \mathcal{B}. \tag{2.20}$$

These linear sets of the so-called $n$-associated functions were introduced in the context of intermediate Weyl coefficients and have been thoroughly studied in [24, 40]. Moreover, for any $n \in \mathbb{Z}^+$, one has necessary and sufficient conditions for the existence of a real zero-free entire function in the space $\operatorname{assoc}_n \mathcal{B}$. The statement of this important result (see theorem 2.7) is essentially theorem 3.2 of [40] with a slight modification justified by lemmas 3.3 and 3.4 of [30]. See also [39] for a more elementary (and restricted) version of this theorem.

**Theorem 2.7.** *Suppose $e(x) \neq 0$ for $x \in \mathbb{R}$ and $e(0) = (\sin\gamma)^{-1}$ for some fixed $\gamma \in (0, \pi)$. Let $\{x_j\}_{j\in\mathbb{N}}$ be the sequence of zeros of the function $s_\gamma(z)$. Also, let $\{x_j^+\}_{n\in\mathbb{N}}$ and $\{x_j^-\}_{n\in\mathbb{N}}$ be the sequences of positive, respectively, negative zeros of $s_\gamma(z)$, arranged according to increasing modulus. Then, a zero-free, real entire function belongs to $\operatorname{assoc}_n \mathcal{B}(e)$ if and only if the following conditions hold true.*

*(C1) The limit* $\displaystyle\lim_{r\to\infty} \sum_{0<|x_j|\leqslant r} \frac{1}{x_j}$ *exists.*

*(C2)* $\displaystyle\lim_{j\to\infty} \frac{j}{x_j^+} = -\lim_{j\to\infty} \frac{j}{x_j^-} < \infty.$

*(C3) Assuming that $\{b_j\}_{n\in\mathbb{N}}$ are the zeros of $s_\beta(z)$, define*

$$h_\beta(z) := \begin{cases} \lim_{r\to\infty} \prod_{|b_j|\leqslant r} \left(1 - \frac{z}{b_j}\right) & \text{if 0 is not a root of } s_\beta(z), \\ z \lim_{r\to\infty} \prod_{0<|b_j|\leqslant r} \left(1 - \frac{z}{b_j}\right) & \text{otherwise.} \end{cases}$$

*The series* $\sum_{j\in\mathbb{N}} \left| \frac{1}{x_j^{2n} h_0(x_j) h'_\gamma(x_j)} \right|$ *is convergent.*

**Remark 2.8.** By a simple argument due to Woracek [41], if there is a zero-free function in a de Branges space $\mathcal{B}(e)$, then there is a real zero-free function in $\mathcal{B}(e)$. Indeed, let $f(z) \in \mathcal{B}(e)$ be zero-free. Then, $f(z)/f^{\#}(z)$ is an entire, zero-free function of bounded type in the upper half-plane (see remark 2.6). By [28, theorem 6.17], one has

(a) $f(z)/f^{\#}(z)$ is of exponential type,
(b) $\int_{-\infty}^{\infty} \frac{\log^+ |f(x)/f^{\#}(x)|}{1+x^2} \, dx < \infty.$

In view of (a), the Hadamard factorization theorem yields $f(z)/f^{\#}(z) = C\,e^{(a+ib)z}$ with $a, b \in \mathbb{R}$, but (b) implies that $a = 0$. Thus, $f(z)/f^{\#}(z) = C\,e^{ibz}$. Clearly, it suffices to consider the case $b > 0$ since if $b = 0$, then $f(z)$ is real, and if $b < 0$, then one considers $f^{\#}(z)/f(z)$ instead of $f(z)/f^{\#}(z)$. Now, the entire function $g(z) := f(z)\,e^{-i\frac{a}{2}z}$ is real, zero-free and it is straightforward to verify that

$$\int_{-\infty}^{\infty} \left| \frac{g(x)}{e(x)} \right|^2 dx < \infty$$

and the quotients $g(z)/e(z)$, $g(z)^{\#}/e(z)$ are of bounded type and nonpositive mean type in the upper half-plane. According to remark 2.6, this means that $g(z) \in \mathcal{B}(e)$.

**Remark 2.9.** As discussed in [24], every one of the linear set $\operatorname{assoc}_n \mathcal{B}(e)$ can be turned into a de Branges space. In fact, from corollary 3.4 of [24], it follows that

$$\operatorname{assoc}_n \mathcal{B}(e(z)) = \mathcal{B}((z+w)^n e(z)),$$

as sets, for any $w \in \mathbb{C}^+$. This fact will be used later in section 4.

**Remark 2.10.** The two previous remarks can be used to sharpen theorem 2.7. Namely, if $\operatorname{assoc}_n \mathcal{B}(e)$ contains a (possibly non-real) zero-free function, then conditions (C1), (C2) and (C3) are fulfilled.

### *2.3. A functional model for operators in $\mathcal{S}(\mathcal{H})$*

In this subsection, we construct a functional model following the framework developed in [30], but having adapted it to comprise all the operators in the class $\mathcal{S}(\mathcal{H})$. This functional model is based on Krein's representation theory [18, theorems 2 and 3], [10, section 1.2], but differs from it in a crucial way as commented in remark 2.17. It is worth mentioning that there is an alternative functional model for the same class $\mathcal{S}(\mathcal{H})$ recently developed in [26]. Some of the material in this subsection can also be found in [31].

The functional model described below rests on the properties of the generalized Cayley transform given in proposition 2.2 with the following addition.

**Proposition 2.11.** *Let $A$ be an element of $\mathcal{S}(\mathcal{H})$ and $J$ be an involution that commutes with one of the canonical self-adjoint extensions of $A$ (hence with all of them), say, $A_\gamma$. For every $v \in \operatorname{spec}(A_\gamma)$, there exists $\psi_v \in \ker(A^* - vI)$ such that $J\psi_v = \psi_v$.*

**Proof.** Let $\phi_v$ be a nontrivial element of $\ker(A_\gamma - vI)$. It follows from the fact that $J$ commutes with $A_\gamma$ that $J\phi_v \in \ker(A_\gamma - vI)$. But, by our assumption on the deficiency indices of $A$ and its regularity, the subspace $\ker(A^* - vI)$ is one dimensional and it contains $\ker(A_\gamma - vI)$. So $J$, restricted to $\ker(A^* - vI)$, reduces to multiplication by a scalar $\alpha$ and the properties of the involution imply that $|\alpha| = 1$. Now, $\psi_v := (1 + \alpha)\phi_v$ has the required properties. □

For any $A \in \mathcal{S}(\mathcal{H})$ and a fixed involution $J$ that commutes with the self-adjoint extensions of $A$ within $\mathcal{H}$, define

$$\xi_{\gamma,v}(z) := h_\gamma(z)\left[I + (z - v)(A_\gamma - zI)^{-1}\right]\psi_v\,, \tag{2.21}$$

where $v$ and $\psi_v$ are chosen as in the previous proposition, and $h_\gamma(z)$ is a real entire function whose zero set is $\operatorname{spec}(A_\gamma)$ (see part (i) of proposition 2.4). Clearly, up to a zero-free real entire function, $\xi_{\gamma,v}(z)$ is completely determined by the choice of the self-adjoint extension $A_\gamma$ and $v$. In fact, as stated more precisely below, $\xi_{\gamma,v}(z)$ does not depend on $A_\gamma$ nor on $v$.

**Proposition 2.12.** *For the function defined in (2.21), the following holds.*

*(i) The vector-valued function $\xi_{\gamma,v}(z)$ is zero-free and entire. It lies in $\ker(A^* - zI)$ for all $z \in \mathbb{C}$.*
*(ii) $J\xi_{\gamma,v}(z) = \xi_{\gamma,v}(\overline{z})$ for every $z \in \mathbb{C}$.*
*(iii) Given $\xi_{\gamma_1,v_1}(z)$ and $\xi_{\gamma_2,v_2}(z)$, there exists a zero-free real entire function $g(z)$ such that $\xi_{\gamma_2,v_2}(z) = g(z)\xi_{\gamma_1,v_1}(z)$.*

**Proof.** In view of proposition 2.2, the proof of (i) is rather straightforward. In fact, one should only follow the first part of the proof of [30, lemma 4.1]. The proof of (ii) also follows easily from our choice of $\psi_v$ and $h_\gamma(z)$ in the definition of $\xi_{\gamma,v}(z)$. To prove (iii), one first uses proposition 2.2 and the fact that $\dim\ker(A^* - vI) = 1$ to obtain that $\xi_{\gamma_2,v_2}(z)$ and $\xi_{\gamma_1,v_1}(z)$ differ by a nonzero scalar complex function. Then, the reality of this function follows from (ii). □

Due to (iii) of proposition 2.12, from now on, the function $\xi_{\gamma,v}(z)$ will be denoted by $\xi(z)$. Actually, the proof of (iii) leads to the following remark.

**Remark 2.13.** Every vector-valued entire function satisfying (i) and (ii) is unique up to a zero-free real entire function. Moreover, if a vector-valued entire function satisfies (i), then, for the involution constructed in proposition 2.3, it also complies with (ii).

On the basis of the function $\xi(z)$ that we have constructed, let us now define

$$(\Phi\varphi)(z) := \langle \xi(\overline{z}), \varphi\rangle, \qquad \varphi \in \mathcal{H}.$$

$\Phi$ maps $\mathcal{H}$ onto a certain linear manifold $\widehat{\mathcal{H}}$ of entire functions. Since $A$ is simple, it follows that $\Phi$ is injective. A generic element of $\widehat{\mathcal{H}}$ will be denoted by $\widehat{\varphi}(z)$, as a reminder of the fact that it is the image under $\Phi$ of a unique element $\varphi \in \mathcal{H}$. Clearly, the linear space $\widehat{\mathcal{H}}$ is turned into a Hilbert space by defining

$$\langle \widehat{\eta}(\cdot), \widehat{\varphi}(\cdot)\rangle := \langle \eta, \varphi\rangle,$$

and $\Phi$ is an isometry from $\mathcal{H}$ onto $\widehat{\mathcal{H}}$.

**Proposition 2.14.** *$\widehat{\mathcal{H}}$ is a de Branges space.*

**Proof.** It suffices to show that the axioms given at the beginning of section 2.2 hold for $\widehat{\mathcal{H}}$.

It is straightforward to verify that $k(z,w) := \langle \xi(\overline{z}), \xi(\overline{w})\rangle$ is a reproducing kernel for $\widehat{\mathcal{H}}$. This accounts for (A1).

Suppose $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}$ has a zero at $z = w$. Then, its pre-image $\varphi \in \mathcal{H}$ lies in $\operatorname{ran}(A - wI)$. This allows one to set $\eta \in \mathcal{H}$ by

$$\eta = (A - \overline{w}I)(A - wI)^{-1}\varphi = \varphi + (w - \overline{w})(A_\gamma - wI)^{-1}\varphi.$$

Now, recalling (2.21) and applying the resolvent identity, one obtains

$$\langle \xi(\overline{z}), \eta\rangle = \frac{z - \overline{w}}{z - w}\langle \xi(\overline{z}), \varphi\rangle.$$

Since $\eta$ and $\varphi$ are related by a Cayley transform, the equality of norms follows. This proves (A2).

As for (A3), consider any $\widehat{\varphi}(z) = \langle \xi(\overline{z}), \varphi\rangle$. Then, as a consequence of (ii) of proposition 2.12, one has $\widehat{\varphi}^{\#}(z) = \langle \xi(\overline{z}), J\varphi\rangle$. □

**Remark 2.15.** The last part of the proof given above shows that $\# = \Phi J \Phi^{-1}$.

The following statement is obvious, but it gives the indispensable properties of any functional model so we bring it up here for the sake of completeness.

**Proposition 2.16.** *If $S$ is the multiplication operator in $\widehat{\mathcal{H}}$ given by (2.18), then we have the following.*

*(i)* $S = \Phi A \Phi^{-1}$ *and* $\operatorname{dom}(S) = \Phi \operatorname{dom}(A)$.
*(ii) The self-adjoint extensions of $S$ within $\widehat{\mathcal{H}}$ are in one–one correspondence with the self-adjoint extensions of $A$ within $\mathcal{H}$.*

**Remark 2.17.** The functional model we have constructed yields a de Branges space for every operator in $\mathcal{S}(\mathcal{H})$. In contrast, Krein's representation theory yields a de Branges space only when the operator is in $\mathcal{E}_0(\mathcal{H})$.

In the previous subsection, we explained that the operator of multiplication $S$ in a de Branges space $\mathcal{B}$ is in $\mathcal{S}(\mathcal{B})$. Now, the functional model that we have constructed tells us that every element in $\mathcal{S}(\mathcal{H})$ is unitarily equivalent to the multiplication operator in a certain de Branges space. Although this assertion is also present in [26], our functional model is simpler and more straightforward.

## 3. Characterization of *n*-entire operators

This section provides various sets of necessary and sufficient conditions for an operator in $\mathcal{S}(\mathcal{H})$ to be in $\mathcal{E}_n(\mathcal{H})$. We heavily rely on the functional model that we have constructed above for our characterizations.

**Proposition 3.1.** *$A \in \mathcal{S}(\mathcal{H})$ is $n$-entire if and only if* $\operatorname{assoc}_n \widehat{\mathcal{H}}$ *contains a zero-free entire function.*

**Proof.** Let $m(z) \in \operatorname{assoc}_n \widehat{\mathcal{H}}$ be the function whose existence is assumed. Such a function can be written as $m(z) = m_0(z) + z m_1(z) + \cdots + z^n m_n(z)$ for some functions $m_0(z), m_1(z), \ldots, m_n(z) \in \widehat{\mathcal{H}}$, each of them in turn satisfying $m_j(z) = \langle \xi(\overline{z}), \mu_j\rangle$ for some $\mu_j \in \mathcal{H}$. Therefore, $\mu_0 + z\mu_1 + \cdots + z^n \mu_n$ is never orthogonal to $\ker(A^* - zI)$ for all $z \in \mathbb{C}$.

The proof of the necessity is rather obvious hence omitted. □

**Remark 3.2.** Krein asserted without proof that if a densely defined operator is in $\mathcal{E}_0(\mathcal{H})$, then one can always find a gauge $\mu$ that commutes with the involution $J$ of proposition 2.3 ($\mu$ is a *real* entire gauge) [19, theorem 8]. The proof actually follows directly from our construction by means of remarks 2.8 and 2.15 since the image under $\Phi$ of an entire gauge is a zero-free function.

**Example 3.3.** In $\mathcal{H} = L^2[-a, a]$, $0 < a < +\infty$, consider the operator

$$\mathrm{dom}(A) = \{\varphi(x) \in \mathrm{AC}[-a, a] : \varphi(a) = 0 = \varphi(-a)\}, \quad A := \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}.$$

Clearly, $A$ is closed and symmetric. Moreover,

$$\mathrm{dom}(A^*) = \mathrm{AC}[-a, a], \quad A^* = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x},$$

from which it is straightforward to verify that the deficiency indices of $A$ are $(1, 1)$. The canonical self-adjoint extensions of $A$ can be parametrized as

$$\mathrm{dom}(A_\gamma) = \{\varphi(x) \in \mathrm{AC}[-a, a] : \varphi(a) = \mathrm{e}^{-\mathrm{i}2\gamma}\varphi(-a)\}, \quad A_\gamma = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x},$$

for $\gamma \in [0, \pi)$. These self-adjoint extensions correspond to different realizations of the linear momentum operator within the interval $[-a, a]$. By a straightforward calculation,

$$\mathrm{spec}(A_\gamma) = \left\{\frac{\gamma + k\pi}{a} : k \in \mathbb{Z}\right\}. \tag{3.1}$$

Clearly, the spectra are interlaced and their union equals $\mathbb{R}$ so it follows that $A$ is regular, hence simple (see subsection 2.1).

Let us define $\xi(x, z) := \mathrm{e}^{-\mathrm{i}zx}$, $x \in [-a, a]$, $z \in \mathbb{C}$. This zero-free entire function belongs to $\ker(A^* - zI)$ for all $z \in \mathbb{C}$. By remark 2.13 and proposition 3.1, for proving that $A$ is 1-entire, it suffices to find $\mu_0(x), \mu_1(x) \in L^2[-a, a]$ such that

$$\int_{-a}^{a} \mathrm{e}^{-\mathrm{i}yx}\mu_0(x)\,\mathrm{d}x + y\int_{-a}^{a} \mathrm{e}^{-\mathrm{i}yx}\mu_1(x)\,\mathrm{d}x = 1 \tag{3.2}$$

for all $y \in \mathbb{R}$ (and then use analytic continuation to the whole complex plane). Our searching will be guided by formally taking the inverse Fourier transform of (3.2) and switching without much questioning the order of integration, obtaining in that way the differential equation

$$\mu_0(x) - \mathrm{i}\mu_1'(x) = \delta(x).$$

This equation suggests to set

$$\mu_0(x) = \frac{1}{2a}\chi_{[-a,a]}(x), \tag{3.3}$$

$$\mu_1(x) = -\mathrm{i}\frac{a + x}{2a}\chi_{[-a,0]}(x) + \mathrm{i}\frac{a - x}{2a}\chi_{[0,a]}(x), \tag{3.4}$$

where $\chi_S(x)$ denotes the characteristic function of the set $S$. A simple computation shows that indeed (3.3) and (3.4) satisfy (3.2). Thus, it has been proven that $A \in \mathcal{E}_1(\mathcal{H})$, and below, in example 3.8 it will be shown that $A \notin \mathcal{E}_0(\mathcal{H})$.

**Example 3.4.** In $\mathcal{H} = L^2[0, a]$, $0 < a < +\infty$, we consider the operator

$$D := -\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

with the domain

$$\mathrm{dom}\,(D) = \{\varphi(x) \in \mathrm{AC}^2[0, a] : \varphi'(0) = 0, \varphi(a) = \varphi'(a) = 0\}.$$

This operator is symmetric and has deficiency indices $(1, 1)$. The adjoint operator $D^*$ is given by the same differential expression as $D$ but with the domain

$$\operatorname{dom}(D^*) = \{\varphi(x) \in \mathrm{AC}^2[0, a] : \varphi'(0) = 0\}.$$

The self-adjoint restriction of $D^*$ can be parametrized by $\beta \in [0, \pi)$ and are given by

$$D_\beta := -\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

with the domain

$$\operatorname{dom}(D_\beta) = \{\varphi(x) \in \mathrm{AC}^2[0, a] : \varphi'(0) = 0, \varphi(a)\sin\beta + \varphi'(a)\cos\beta = 0\}.$$

That is, the operators $D_\beta$ are the (self-adjoint) realizations of the Laplacian operator in the interval $[0, a]$ with the Neumann boundary condition at $x = 0$. The spectra of these operators are simple and discrete. Moreover, they are pairwise interlaced, so $A$ is regular and therefore simple.

The function $\xi(x, z) := \cos(\sqrt{z}x)$ is the (unique) solution of the equation

$$-\xi''(x, z) = z\xi(x, z), \quad z \in \mathbb{C},$$

with the boundary conditions $\xi(0, z) = 1$ and $\xi'(0, z) = 0$. Hence, this entire function belongs to $\ker(D^* - zI)$ for every $z \in \mathbb{C}$. We will show that there exist functions $\mu_0(x), \mu_1(x) \in L^2[0, a]$ such that

$$\int_0^a \cos(yx)\mu_0(x)\,\mathrm{d}x + y^2 \int_0^a \cos(yx)\mu_1(x)\,\mathrm{d}x = 1, \quad y \in \mathbb{R}^+. \tag{3.5}$$

By identifying $y = \sqrt{z}$, and then by analytic continuation from $z \in \mathbb{R}^+$ to $\mathbb{C}$, this will prove that $D$ is 1-entire. To find the functions $\mu_0(x)$ and $\mu_1(x)$, we use the same heuristic approach of the previous example.

We assume that $\mu_0(x)$ and $\mu_1(x)$ are the even functions on the interval $[-a, a]$. Then, (3.5) is equivalent to

$$\frac{1}{2}\int_{-a}^a \mathrm{e}^{-\mathrm{i}yx}\mu_0(x)\,\mathrm{d}x + y^2\frac{1}{2}\int_{-a}^a \mathrm{e}^{-\mathrm{i}yx}\mu_1(x)\,\mathrm{d}x = 1,$$

where now this equation can be considered valid for all $y \in \mathbb{R}$. Then, we take the Fourier transform to obtain the formal differential equation

$$\mu_0(x) - \mu_1''(x) = 2\delta(x),$$

a solution of which is given by (the even extension of)

$$\mu_0(x) = \frac{1}{a}\chi_{[0,a]}(x), \quad \mu_1(x) = \frac{1}{2a}(x - a)^2\chi_{[0,a]}(x).$$

A straightforward computation shows that these functions indeed fulfil (3.5).

The following may be considered as an alternative definition of a densely defined $n$-entire operator.

**Proposition 3.5.** *A densely defined operator $A$ is in $\mathcal{E}_n(\mathcal{H})$ if and only if there exists a collection $\mu_0, \ldots, \mu_n \in \mathcal{H}$ such that*

$$\sum_{j=0}^n \left\langle (A^*)^j \xi(\bar{z}), \mu_j \right\rangle \neq 0 \tag{3.6}$$

*for all $z \in \mathbb{C}$.*

**Proof.** Since $A^*$ is an operator, part (i) of proposition 2.12 becomes $A^*\xi(z) = z\xi(z)$ and hence $(A^*)^j\xi(z) = z^j\xi(z)$, $j \in \mathbb{Z}^+$. Given $\mu_0, \ldots, \mu_n \in \mathcal{H}$, one has the identity

$$\langle \xi(\overline{z}), \mu_0 + z\mu_1 + \cdots + z^n\mu_n \rangle = \sum_{j=0}^{n} \left\langle \overline{z}^j \xi(\overline{z}), \mu_j \right\rangle = \sum_{j=0}^{n} \left\langle (A^*)^j \xi(\overline{z}), \mu_j \right\rangle .$$

The statement then follows. □

**Remark 3.6.** The previous proposition can be extended to operators with a non-dense domain provided that (3.6) is written in terms of the operator part of the adjoint relation. See [3] for more details.

**Proposition 3.7.** *For* $A \in \mathcal{S}(\mathcal{H})$, *consider the self-adjoint extensions (within* $\mathcal{H}$*)* $A_0$ *and* $A_\gamma$, *with* $0 < \gamma < \pi$. *Then, A is n-entire if and only if* $\operatorname{spec}(A_0)$ *and* $\operatorname{spec}(A_\gamma)$ *obey conditions (C1), (C2) and (C3) of theorem 2.7.*

**Proof.** Apply theorem 2.7 and remark 2.10 along with proposition 3.1. □

**Example 3.8.** Consider the operator $A$ and its self-adjoint extensions $A_\gamma$, with $\gamma \in [0, \pi)$, given in example 3.3. Taking into account (3.1), we now direct our attention to conditions (C1), (C2) and (C3) of proposition 3.7. (C1) and (C2) are trivially fulfilled. As for (C3), we choose $\gamma = \pi/2$ and note that

$$h_{\pi/2}(z) = \lim_{m\to\infty} \prod_{k=1}^{m} \left(1 - \frac{4a^2z^2}{\pi^2(4k^2 - 4k + 1)}\right) = \cos(az),$$

while a similar computation shows that $h_0(z) = \sin(az)$. This implies that (C3) is satisfied only for $n \geqslant 1$. That is, $A \in \mathcal{E}_1(\mathcal{H}) \setminus \mathcal{E}_0(\mathcal{H})$.

**Example 3.9.** Let us return to the Laplacian operator $D$ given in example 3.4. For any $\beta \in [0, \pi)$, an eigenvalue $b$ of the self-adjoint extension $D_\beta$ satisfies the identity

$$\sqrt{b}\frac{\sin\sqrt{ba}}{\cos\sqrt{ba}} = \frac{\sin\beta}{\cos\beta}.$$

In particular,

$$\operatorname{spec}(D_0) = \left\{\frac{\pi^2k^2}{a^2} : k \in \mathbb{N}\right\}, \quad \operatorname{spec}(D_{\pi/2}) = \left\{\frac{\pi^2}{a^2}\left(\frac{2k-1}{2}\right)^2 : k \in \mathbb{N}\right\}.$$

Conditions (C1) and (C2) of proposition 3.7 are clearly fulfilled. As for (C3), a computation shows that

$$h_0(z) = \lim_{m\to\infty} \prod_{k=1}^{m} \left(1 - \frac{a^2z}{\pi^2k^2}\right) = \frac{\sin a\sqrt{z}}{a\sqrt{z}},$$

and similarly, $h_{\pi/2}(z) = \cos a\sqrt{z}$. Hence, the series that defines (C3) is convergent as long as $n \geqslant 1$. That is, $D \in \mathcal{E}_1(\mathcal{H}) \setminus \mathcal{E}_0(\mathcal{H})$.

The result of the last example can be extended to the canonical self-adjoint extensions of the Schrödinger operator

$$H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{3.7}$$

where $V(x) \in L^1[0, a]$, with the domain

$$\operatorname{dom}(H) = \left\{\varphi(x) \in \mathrm{AC}^2[0, a] : \varphi'(0) = 0, \varphi(a) = \varphi'(a) = 0\right\}. \tag{3.8}$$

In a suitable sense, $V(x)$ is a small perturbation of the Laplacian operator. Due to this fact, it is shown in theorem 4.1 of [27] that the de Branges space associated with the operator $H$ is as a set equal to the one associated with the Laplacian operator. As a consequence of this, one can formulate the following assertion.

**Corollary 3.10.** *In $\mathcal{H} = L^2[0, a]$, $a > 0$, every Schrödinger operator given by (3.7) with $V(x) \in L^1[0, a]$ and domain (3.8) belongs to $\mathcal{E}_1(\mathcal{H}) \setminus \mathcal{E}_0(\mathcal{H})$.*

With some additional little work, this result can further be extended to all Schrödinger operators arising from regular differential expressions. In connection with this, see theorem 10.7 of [27].

**Proposition 3.11.** *Assume that, for $A \in \mathcal{S}(\mathcal{H})$, one can find a collection $\eta_0, \eta_1, \ldots, \eta_n \in \mathcal{H}$ such that (1.1) is fulfilled for all $z \in \mathbb{C}$ except a finite set of points. Then, $A$ is $n$-entire.*

**Proof.** In view of the functional model introduced above, noting that

$$\Phi(\eta_0 + z\eta_1 + \cdots + z^n\eta_n) = \widehat{\eta_0}(z) + z\widehat{\eta_1}(z) + \cdots + z^n\widehat{\eta_n}(z),$$

and recalling (2.20), it suffices to consider the case of a de Branges space $\mathcal{B}$ such that $\operatorname{assoc}_n \mathcal{B}$ contains a non-trivial entire function having a finite number of roots. Suppose such a function $g(z) \in \operatorname{assoc}_n \mathcal{B}$ exists. Let $z_1, z_1, \ldots, z_k$ be its zeros whose respective (necessarily finite) multiplicities are $m_1, m_1, \ldots, m_k$. Since $\operatorname{assoc}_n \mathcal{B}$ is division invariant [40, lemma 2.11], one has

$$f(z) := \frac{g(z)}{(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k}} \in \operatorname{assoc}_n \mathcal{B}$$

and it is zero-free. This completes the proof. □

**Remark 3.12.** For every $A \in \mathcal{S}(\mathcal{H})$ and every $n \in \mathbb{Z}^+$, one can always find a set $\eta_0, \eta_1, \ldots, \eta_n \in \mathcal{H}$ such that (1.1) is fulfilled for all $z \in \mathbb{C}$ except a countable set of points. However, in view of proposition 3.7, it is clear that there are operators in $\mathcal{S}(\mathcal{H})$ that are not $n$-entire (just consider an operator having a canonical self-adjoint extension whose spectrum does not satisfy one of the conditions (C1) or (C2) stated there; see the example given below). We therefore conclude that the statement of proposition 3.11 is sharp.

**Example 3.13.** The following is an example of an operator in $\mathcal{S}(\mathcal{H})$ but not in $\mathcal{E}_n(\mathcal{H})$ for any $n \in \mathbb{Z}^+$. In this case, $\mathcal{H} = L^2(\mathbb{R})$. Our starting point is the harmonic oscillator operator

$$H_0 := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$$

with its usual domain of self-adjointness $\operatorname{dom}(H_0)$. Let $\kappa(x) \in L^2(\mathbb{R})$ be a cyclic vector for $H_0$:

$$\kappa(x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{1/2}} \phi_n(x) = \frac{1}{\pi^{1/4}} \mathrm{e}^{-\frac{1}{2}(x^2 - 2\sqrt{2}x + 1)},$$

where $\{\phi_n(x)\}_{n=0}^{\infty}$ is the basis of normalized eigenvectors of $H_0$. Consider the family of rank-1 perturbations of $H_0$:

$$H_\beta := H_0 + \beta \langle \kappa(\cdot), \cdot \rangle_{L^2(\mathbb{R})} \kappa(x), \qquad \beta \in \mathbb{R}.$$

Clearly, all these operators are self-adjoint with the domain $\operatorname{dom}(H_\beta) = \operatorname{dom}(H_0)$. Moreover, by [11], these operators are all canonical self-adjoint extensions of

$$H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2, \qquad \operatorname{dom}(H) = \left\{ \varphi(x) \in \operatorname{dom}(H_0) : \langle \kappa(\cdot), \varphi(\cdot) \rangle_{L^2(\mathbb{R})} = 0 \right\}.$$

$H$ is a closed regular symmetric operator with deficiency indices $(1, 1)$ (the regularity follows from the cyclicity of $\kappa(x)$). Observe that $\operatorname{dom}(H)$ is not dense in $L^2(\mathbb{R})$. Thus, there is an exceptional self-adjoint extension of $H$ that is not an operator; this extension corresponds to the 'infinite coupling' $\beta = \infty$. For the purpose of this example, we do not need to describe this extension.

Now, recalling that $\operatorname{spec}(H_0) = \{2n+1 : n \in \mathbb{Z}^+\}$, we have the spectrum of a self-adjoint extension of $H$ (hence all of them due to the interlacing property) that satisfies neither (C1) nor (C2) in proposition 3.7. Therefore, $H$ is not $n$-entire for all $n \in \mathbb{Z}^+$.

## 4. On Gelfand triplets associated with *n*-entire operators

We recall that Krein's notion of operators entire in the generalized sense is formulated in terms of a triplet of Hilbert spaces that arises from the domain of the adjoint operator. In this section, we aim to obtain an analogous result for densly defined operators in any of the classes $\mathcal{E}_n(\mathcal{H})$. Our derivation however will be more convoluted as it will be based on construing the pair $\mathcal{B}$ and $\operatorname{assoc}_n \mathcal{B}$ as part of a Gelfand triplet.

As noted in remark 2.9, the linear space $\operatorname{assoc}_n \mathcal{B}(e)$ becomes a de Branges space when endowed (for instance) with the inner product

$$\langle f(x), g(x)\rangle_{-n} := \int_{\mathbb{R}} \frac{\overline{f(x)}g(x)}{(x^2+1)^n\,|e(x)|^2}\,\mathrm{d}x. \tag{4.1}$$

We remark that the inner product (4.1) is not the only possible choice. In spite of this, we will stick to (4.1) in order to simplify the ongoing discussion. Let us henceforth denote the spaces $\operatorname{assoc}_n \mathcal{B}(e)$ with the inner product (4.1) as $\mathcal{B}_{-n}$. It is clear that

$$\mathcal{B}(e) =: \mathcal{B}_0 \subset \mathcal{B}_{-1} \subset \mathcal{B}_{-2} \subset \cdots,$$

and moreover, $\|f(x)\|_{-n} \leqslant \|f(x)\|_{-n+1}$ for every $f(z) \in \mathcal{B}_{-n+1}$. Let $S_{-n}$ denote the operator of multiplication with the maximal domain in $\mathcal{B}_{-n}$.

**Lemma 4.1.** *Assume that $S_0$ is densely defined in $\mathcal{B}_0$. Then, $S_{-n}$ is densely defined in $\mathcal{B}_{-n}$. Also, $\mathcal{B}_0$ is dense in $\mathcal{B}_{-n}$.*

**Proof.** Let us start by proving the assertion for $n = 1$. Consider $f(z) \in \mathcal{B}_{-1}$. Then, $f(z) = h(z) + zg(z)$ for some $h(z), g(z) \in \mathcal{B}_0$. Since $S_0$ is densely defined in $\mathcal{B}_0$, there exists a sequence $\{g_l(z)\} \subset \operatorname{dom}(S_0)$ converging to $g(z)$ in the $\mathcal{B}_0$ norm. Define $f_l(z) := h(z) + zg_l(z)$. Then, clearly, $\{f_l(z)\} \subset \mathcal{B}_0$. Moreover, since

$$\|xg(x) - xg_n(x)\|_{-1}^2 = \int_{\mathbb{R}} \frac{x^2\,|g(x) - g_n(x)|^2}{(x^2+1)\,|e(x)|^2}\mathrm{d}x \leqslant \|g(x) - g_n(x)\|_0^2\,,$$

$\{f_l(z)\}$ converges to $f(z)$ in the $\mathcal{B}_{-1}$ norm (hence pointwise uniformly on compact subsets).

So far we have proven that $\mathcal{B}_0$ is dense in $\mathcal{B}_{-1}$. Since $\mathcal{B}_0 \subset \operatorname{dom}(S_{-1})$, the latter operator is densely defined. We now proceed by induction assuming that $S_{-n+1}$ is densely defined in $\mathcal{B}_{-n+1}$ and noting that $\mathcal{B}_{-n} = \operatorname{assoc}\mathcal{B}_{-n+1}$.

The fact that $\mathcal{B}_0$ is dense in $\mathcal{B}_{-n}$ follows from the ordering of norms. □

Fix $n \in \mathbb{N}$. We aim to find a linear space $\mathcal{B}_{+n} \subset \mathcal{B}_0$ such that $\{\mathcal{B}_{+n}, \mathcal{B}_0, \mathcal{B}_{-n}\}$ is a Gelfand triplet. Most of the following discussion is based on standard arguments; see [4].

Let $\boldsymbol{D} : \mathcal{B}_{-n} \to \mathcal{B}_0$ be the adjoint of the immersion map from $\mathcal{B}_0$ into $\mathcal{B}_{-n}$. This linear map is well defined as long as $\mathcal{B}_0$ is dense in $\mathcal{B}_{-n}$, i.e. under the condition of lemma 4.1, and it turns out to be one-to-one. By definition, one has

$$\langle g(x), \boldsymbol{D}f(x)\rangle_0 = \langle g(x), f(x)\rangle_{-n}, \tag{4.2}$$

for any $f(z) \in \mathcal{B}_{-n}$ and $g(z) \in \mathcal{B}_0$. Since the immersion map has norm less than 1, the same holds true for $\boldsymbol{D}$.

The map $\boldsymbol{D}$ gives an explicit relation between the reproducing kernels $k_0(z, w)$ and $k_{-n}(z, w)$.

**Lemma 4.2.** *For all $f(z) \in \mathcal{B}_{-n}$, one has*

$$(\boldsymbol{D}f)(z) = \langle k_0(x, z), f(x)\rangle_{-n}.$$

*Moreover, $k_0(z, w) = (\boldsymbol{D}k_{-n})(z, w)$.*

**Proof.** Since $(\boldsymbol{D}f)(z) \in \mathcal{B}_0$, and taking into account (4.2), one has

$$(\boldsymbol{D}f)(z) = \langle k_0(x, z), \boldsymbol{D}f(x)\rangle_0 = \langle k_0(x, z), f(x)\rangle_{-n},$$

thus leading to the first assertion. The second assertion follows by noting that

$$\langle(\boldsymbol{D}k_{-n})(x, w), g(x)\rangle_0 = \langle k_{-n}(x, w), g(x)\rangle_{-n} = g(w) = \langle k_0(x, w), g(x)\rangle_0 ,$$

for all $g(z) \in \mathcal{B}_0$ (as a subset of $\mathcal{B}_{-n}$) and all $w \in \mathbb{C}$. □

Now define the space $\mathcal{B}_{+n}^{\text{pre}} := \boldsymbol{D}\mathcal{B}_0$ equipped with the sesquilinear form

$$\langle\cdot, \cdot\rangle_{+n} := \left\langle\cdot, \boldsymbol{D}^{-1}\cdot\right\rangle_0 . \tag{4.3}$$

**Lemma 4.3.** *The sesquilinear form $\langle\cdot, \cdot\rangle_{+n}$ is an inner product so $\mathcal{B}_{+n}^{\text{pre}}$ is an inner product space which turns out to be not complete. Moreover, $\|g(x)\|_{+n} > \|g(x)\|_0$ for $g(z) \in \mathcal{B}_{+n}^{\text{pre}}$.*

**Proof.** The first assertion follows from the fact that

$$\begin{aligned}\|g(x)\|_{+n}^2 &= \left\langle(\boldsymbol{D}\boldsymbol{D}^{-1}g)(x), (\boldsymbol{D}^{-1})g(x)\right\rangle_0 \\ &= \left\langle(\boldsymbol{D}^{-1}g)(x), (\boldsymbol{D}^{-1}g)(x)\right\rangle_{-n} = \left\|(\boldsymbol{D}^{-1}g)(x)\right\|_{-n}^2 .\end{aligned}$$

This implies that $\boldsymbol{D}^{-1}$ has bounded norm (equal to 1) as a linear map from $\mathcal{B}_{+n}^{\text{pre}}$ to $\mathcal{B}_{-n}$. Since $\mathcal{B}_0$ is not closed as a subset of $\mathcal{B}_{-n}$, the second assertion follows. Finally, since $\|\boldsymbol{D}\| < 1$, $\|g(x)\|_0 < \left\|(\boldsymbol{D}^{-1}g)(x)\right\|_{-n}$ for all $g(z) \in \mathcal{B}_0$, thus implying the last statement. □

Let us denote by $\mathcal{B}_{+n}$ the completion of $\mathcal{B}_{+n}^{\text{pre}}$ with respect to the norm $\|\cdot\|_{+n}$. Let $\boldsymbol{T}^{\text{pre}}$ be the restriction of $\boldsymbol{D}^{-1}$ to $\mathcal{B}_{+n}^{\text{pre}}$. This operator can be seen as a densely defined map in $\mathcal{B}_{+n}$ with range in $\mathcal{B}_{-n}$. Now, denote by $\boldsymbol{T}$ the extension of $\boldsymbol{T}^{\text{pre}}$ by continuity to the whole space $\mathcal{B}_{+n}$. Since for $f(z), g(z) \in \mathcal{B}_{+n}^{\text{pre}}$,

$$\begin{aligned}\langle(\boldsymbol{T}f)(x), (\boldsymbol{T}g)(x)\rangle_{-n} &= \left\langle(\boldsymbol{D}^{-1}f)(x), (\boldsymbol{D}^{-1}g)(x)\right\rangle_{-n} \\ &= \left\langle(\boldsymbol{D}^{-1}f)(x), g(x)\right\rangle_0 = \langle f(x), g(x)\rangle_{+n}\end{aligned}$$

as follows from (4.2) and (4.3), one has

$$\langle(\boldsymbol{T}f)(x), (\boldsymbol{T}g)(x)\rangle_{-n} = \langle f(x), g(x)\rangle_{+n}$$

for all $f(z), g(z) \in \mathcal{B}_{+n}$ due to the continuity of the inner product. Also, $\operatorname{ran}(\boldsymbol{T}) = \mathcal{B}_{-n}$. Suppose there is $h(z) \in \mathcal{B}_{-n}$ orthogonal to $\operatorname{ran}(\boldsymbol{T})$. Then, one has

$$\begin{aligned}\langle h(x), (\boldsymbol{T}g)(x)\rangle_{-n} &= \left\langle h(x), (\boldsymbol{D}^{-1}g)(x)\right\rangle_{-n} \\ &= \left\langle(\boldsymbol{D}h)(x), (\boldsymbol{D}^{-1}g)(x)\right\rangle_0 = 0\end{aligned}$$

for all $g(z) \in \mathcal{B}_{+n}^{\text{pre}}$. Since $\boldsymbol{D}^{-1}\mathcal{B}_{+n}^{\text{pre}} = \mathcal{B}_0$, it follows that $(\boldsymbol{D}h)(z) \equiv 0$, thus the claim.

It follows from the construction above that the spaces $\mathcal{B}_{+n}$, $\mathcal{B}_0$ and $\mathcal{B}_{-n}$ form a Gelfand triplet. The duality bracket $[\cdot, \cdot] : \mathcal{B}_{+n} \times \mathcal{B}_{-n} \to \mathbb{C}$ is given by

$$[h(x), f(x)] = \langle(\boldsymbol{T}h)(x), f(x)\rangle_{-n} , \qquad h(z) \in \mathcal{B}_{+n}, \quad f(z) \in \mathcal{B}_{-n}.$$

Moreover, $\mathcal{B}_{+n}$ is more than just a dense linear manifold within $\mathcal{B}_0$, as shown next.

**Proposition 4.4.** *$\mathcal{B}_{+n}$, equipped with the inner product $\langle\cdot,\cdot\rangle_{+n}$, is a de Branges space.*

**Proof.** Given $w \in \mathbb{C}$, define $k_{+n}(z, w) := (\boldsymbol{D}k_0)(z, w)$. For $g(z) \in \mathcal{B}_{+n}^{\text{pre}}$, one has

$$\langle k_{+n}(x, w), g(x)\rangle_{+n} = \langle (\boldsymbol{D}k_0)(x, w), g(x)\rangle_{+n} = \langle k_0(x, w), g(x)\rangle_0 = g(w).$$

The continuity of the inner product implies that $k_{+n}(z, w)$ is a reproducing kernel for $\mathcal{B}_{+n}$.

Suppose that $g(z) \in \mathcal{B}_{+n}$ has a non-real zero at $z = w$. Then, $g(z) \in \operatorname{ran}(S_{+n} - wI)$, where $S_{+n}$ is the operator of multiplication with the maximal domain in $\mathcal{B}_{+n}$. Let $V_{\overline{w}w}$ denote the Cayley transform that maps $\ker(S_{+n}^* - \overline{w}I)$ onto $\ker(S_{+n}^* - wI)$. By standard results, we obtain

$$(V_{\overline{w}w}g)(z) = \frac{z-\overline{w}}{z-w}g(z) \in \operatorname{ran}(S_{+n} - \overline{w}I) \subset \mathcal{B}_{+n}.$$

Moreover, $\|(V_{\overline{w}w}g)(x)\|_{+n} = \|g(x)\|_{+n}$.

As usual, denote $f^\#(z) := \overline{f(\overline{z})}$. Note that, for $f(z) \in \mathcal{B}_{-n}$ and $h(z) \in \mathcal{B}_0$,

$$\begin{aligned}\left\langle(\boldsymbol{D}f)^\#(x), h(x)\right\rangle_0 = \left\langle h^\#(x), (\boldsymbol{D}f)(x)\right\rangle_0 &= \left\langle h^\#(x), f(x)\right\rangle_{-n}\\ &= \left\langle f^\#(x), h(x)\right\rangle_{-n} = \left\langle(\boldsymbol{D}f^\#)(x), h(x)\right\rangle_0,\end{aligned}$$

thus $(\boldsymbol{D}f)^\#(z) = (\boldsymbol{D}f^\#)(z)$. Therefore, $(\boldsymbol{D}^{-1}g)^\#(z) = (\boldsymbol{D}^{-1}g^\#)(z)$ for all $g(z) \in \mathcal{B}_{+n}^{\text{pre}}$ which, by continuity, implies $(\boldsymbol{T}g)^\#(z) = (\boldsymbol{T}g^\#)(z)$ for all $g(z) \in \mathcal{B}_{+n}$. As a consequence, $g^\#(z) \in \mathcal{B}_{+n}$ whenever $g(z) \in \mathcal{B}_{+n}$ and $\left\|g^\#(x)\right\|_{+n} = \|g(x)\|_{+n}$. □

From the previous discussion, we see that the reproducing kernels associated with each one of the spaces $\mathcal{B}_{+n}$, $\mathcal{B}_0$ and $\mathcal{B}_{-n}$ are related by the identities

$$k_{+n}(z, w) = (\boldsymbol{D}k_0)(z, w) = (\boldsymbol{D}^2k_{-n})(z, w).$$

However, the identity of lemma 4.2 can be sharpened as follows.

**Proposition 4.5.** *For every $w \in \mathbb{C}$,*

$$(\boldsymbol{T}^{-1}k_{-n})(\cdot, w) = k_0(\cdot, w),$$

*and therefore, $k_0(\cdot, w) \in \mathcal{B}_{+n}$.*

**Proof.** Take any $m(z) \in \mathcal{B}_{-n}$, then

$$m(z) = \langle k_{-n}(x, z), m(x)\rangle_{-n} = \left\langle(\boldsymbol{T}\boldsymbol{T}^{-1}k_{-n})(x, z), m(x)\right\rangle_{-n} = \left[(\boldsymbol{T}^{-1}k_{-n})(x, z), m(x)\right].$$

Now, if one assumes that also $m(z) \in \mathcal{B}_0$, then, by using the fact that

$$\left[(\boldsymbol{T}^{-1}k_{-n})(x, z), m(x)\right] = \left\langle(\boldsymbol{T}^{-1}k_{-n})(x, z), m(x)\right\rangle_0,$$

one concludes that

$$m(z) = \left\langle(\boldsymbol{T}^{-1}k_{-n})(x, z), m(x)\right\rangle_0 = \langle k_0(x, z), m(x)\rangle_0, \tag{4.4}$$

since $k_0(z, w)$ is the reproducing kernel in $\mathcal{B}_0$. Thus, the assertion follows from the second equality in (4.4) due to the arbitrary choice of $m(z) \in \mathcal{B}_0$. □

**Remark 4.6.** Since $k_0(z, w)$ satisfies

$$(S_0^*k_0)(z, w) = \overline{w}k_0(z, w),$$

one concludes that $k_0(z, w)$ is in $\mathcal{B}_{+n} \cap \operatorname{dom}((S_0^*)^n)$ for every $n \in \mathbb{N}$.

**Corollary 4.7.** *Assume that $S$ is densely defined on $\mathcal{B}$. Let $\{\mathcal{B}_{+n}, \mathcal{B}, \mathcal{B}_{-n}\}$ be the Gelfand triplet associated with $\mathcal{B}$ as above, with the duality bracket $[\cdot,\cdot]$. Then, $S$ is $n$-entire if and only if there exists an entire function $m(z) \in \mathcal{B}_{-n}$ such that $[k(x,z), m(x)] \neq 0$ for all $z \in \mathbb{C}$.*

**Proof.** By definition $S$ is $n$-entire if and only if $n+1$ entire functions $m_0(z), \ldots, m_n(z) \in \mathcal{B}$ can be found such that

$$\mathcal{B} = \operatorname{ran}(S - zI) \dotplus \operatorname{span}\{m_0(z) + zm_1(z) + \cdots + z^n m_n(z)\}, \text{ for all } z \in \mathbb{C}.$$

Equivalently, $S$ is $n$-entire if and only if there exists a zero-free entire function $m(z) \in \operatorname{assoc}_n(\mathcal{B})$, i.e.

$$\langle k_{-n}(x,z), m(x)\rangle_{-n} \neq 0, \quad \text{for all } z \in \mathbb{C}. \tag{4.5}$$

The assertion then follows from (the proof of) proposition 4.5. □

Let us consider a densely defined operator $A \in \mathcal{S}(\mathcal{H})$. Associated with $A$ we have an isometry $\Phi$ that maps $\mathcal{H}$ to the de Branges space $\widehat{\mathcal{H}} := \Phi\mathcal{H}$. On it, $S := \Phi A \Phi^{-1}$ is densely defined. Then, we can construct, by the way previously discussed, the Gelfand triplet $\{\widehat{\mathcal{H}}_{+n}, \widehat{\mathcal{H}}, \widehat{\mathcal{H}}_{-n}\}$. Define

$$\mathcal{H}_{+n} := \Phi^{-1}\widehat{\mathcal{H}}_{+n},$$

which is a dense linear manifold within $\mathcal{H}$ and itself is a Hilbert space if equipped with the inner product

$$\langle \eta, \omega\rangle_{+n} := \left\langle \eta, \Phi^{-1}\boldsymbol{D}^{-1}\Phi\omega\right\rangle, \quad \eta, \omega \in \mathcal{H}_{+n}.$$

It follows from remark 4.6 that $\xi(z) \in \mathcal{H}_{+n}$ for every $z \in \mathbb{C}$. Now define $\mathcal{H}_{-n}$ as the set of continuous linear functionals on $\mathcal{H}_{+n}$. This linear set is a Hilbert space when equipped with the inner product

$$\langle \phi, \psi\rangle_{-n} := \left\langle \boldsymbol{G}^{-1}\phi, \boldsymbol{G}^{-1}\psi\right\rangle_{+n}, \quad \phi, \psi \in \mathcal{H}_{-n},$$

where $\boldsymbol{G}$ is the standard bijection from $\mathcal{H}_{+n}$ onto $\mathcal{H}_{-n}$ [4]. These considerations along with corollary 4.7 constitute the proof of the following proposition. Here, we denote the duality bracket between $\mathcal{H}_{+n}$ and $\mathcal{H}_{-n}$ also by $[\cdot,\cdot]$.

**Proposition 4.8.** *Given a densely defined operator $A \in \mathcal{S}(\mathcal{H})$, let $\{\mathcal{H}_{+n}, \mathcal{H}, \mathcal{H}_{-n}\}$ be the Gelfand triplet obtained as above. Then, $A \in \mathcal{E}_n(\mathcal{H})$ if and only if there exists $\eta \in \mathcal{H}_{-n}$ such that $[\xi(z), \eta] \neq 0$ for every $z \in \mathbb{C}$.*

## 5. Concluding remarks

In this paper, we introduce a classification of operators within the class $\mathcal{S}(\mathcal{H})$ of regular, closed symmetric operators on a (necessarily) separable Hilbert space. This classification is based on a geometric condition that generalizes a criterion due to M G Krein for his definition of operators entire and entire in the generalized sense. These new classes $\mathcal{E}_n(\mathcal{H})$ of $n$-entire operators have a number of distinctive properties and there are various characterizations apart from the geometric condition used in their definition. Noteworthily, there is a spectral characterization of $\mathcal{E}_n(\mathcal{H})$ that may be useful in several applications. In this respect, the theory exposed here tentatively opens up new directions of research related to the inverse and direct spectral analysis of operators, particularly, one-dimensional Schrödinger operators.

We also have studied the $\mathcal{E}_n(\mathcal{H})$ class by means of associated Gelfand triplets, following the way Krein defined and studied the operators entire in the generalized sense. There are several aspects of this approach (discussed in section 4) that deserve further investigation. For instance, it seems insightful to define the Gelfand triplet for an operator in $\mathcal{E}_n(\mathcal{H})$ in a more intrinsic way (i.e. without resorting to a functional model). In any case, the results discussed here shed some light on the theory of de Branges spaces and may be of interest for those studying it.

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