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On the Quasi-Stationary Solutions in $1D$ Non-hermitian Systems

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Abstract

In this paper we show that some $1D$ non-Hermitian lattices can be easily transformed into Hermitian ones. This fact clearly facilitates the analysis of the former. Our theoretical results suggest that the results and conclusions derived earlier by other authors may not be correct.

1. INTRODUCTION

There is great interest in simple one-dimensional non-Hermitian lattices. For example, in a recent paper Yuce[1] discussed the skin effect and quasi-stationary solutions by means of such models. From the analysis of an exactly solvable model and numerical results for a non-solvable one he concluded that the eigenvalues are real for open boundary conditions and complex in the case of periodic ones. Yuce also argued about the existence of a particular solution when the number of lattice sites is infinite and of zero-energy quasi-stationary states.

The purpose of this paper is to put forward a simple and straightforward technique that enables one to obtain results for such lattice models that may otherwise pass unnoticed. In section 2 we show that some tridiagonal non-Hermitian lattices are isospectral (have the same spectrum) to Hermitian ones. In section 3 we apply this method to an exactly solvable lattice. In section 4 we analyse a nonsolvable model. Finally, in section 5 we summarize our main results and draw conclusions.

2. GENERAL TRIDIAGONAL MATRIX REPRESENTATION

Consider a Hamltonian operator H and an orthonormal basis set $\{|j\rangle,\ j=1,2,\ldots,N\}$ such that $H_{i,j}=\langle i|H|j\rangle=0$ for all $|i-j|>1$. If we write the solution to the eigenvalue equation $H|\psi\rangle=E|\psi\rangle$ as

$$|\psi\rangle=\sum_{j=1}^{N}\psi_j|j\rangle, \tag{1}$$

then we obtain the three-term recurrence relation (TTRR)

$$H_{j,j-1}\psi_{j-1}+\left(H_{j,j}-E\right)\psi_j+H_{j,j+1}\psi_{j+1}=0,\ j=1,2,\ldots,N, \tag{2}$$

for the coefficients (amplitudes) ψ_j. We consider open boundary conditions (OBC) $\psi_0=\psi_{N+1}=0$ and periodic boundary conditions (PBC) $\psi_{j+N}=\psi_N$ (following Yuce's nomenclature[1]). In the

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former case we do not bother about the matrix elements $H_{1,0}$ and $H_{N,N+1}$ but in the latter it is assumed that $H_{1,0} = H_{1,N}$ and $H_{N,N+1} = H_{N,1}$.

In what follows we resort to a mathematical argument used earlier by Child et al[2] and Amore and Fernández[3] for the truncation of some particular TTRR. Although the problems studied here are completely different, that approach may still lead to revealing results. It consists of the transformation of the solution to the TTRR as $\psi_j = Q_j d_j$ that leads to

$$\tilde{H}_{j,j-1}d_{j-1} + \left(\tilde{H}_{j,j} - E\right) d_j + \tilde{H}_{j,j+1}d_{j+1} = 0,\ j = 1, 2, \ldots, N,$$
$$\tilde{H}_{j,j-1} = \frac{Q_{j-1}}{Q_j}H_{j,j-1},\ \tilde{H}_{j,j+1} = \frac{Q_{j+1}}{Q_j}H_{j,j+1},\ \tilde{H}_{j,j} = H_{j,j}. \quad (3)$$

If we require that $\tilde{H}^*_{j+1,j} = \tilde{H}_{j,j+1}$ then we find that

$$\left|\frac{Q_{j+1}}{Q_j}\right|^2 = \frac{H_{j,j+1}}{H^*_{j+1,j}}. \quad (4)$$

In other words: if $H_{j,j+1}/H^*_{j+1,j}$ is real and positive then the matrix $\mathbf{H} = \left(H_{i,j}\right)$ is isospectral to the Hermitian matrix $\tilde{\mathbf{H}} = \left(\tilde{H}_{i,j}\right)$. If $\mathbf{H}$ is Hermitian then $\left|Q_{j+1}/Q_j\right|^2 = 1$.

The results just developed apply directly to the OBC because $d_0 = d_{N+1} = 0$ but not to the PBC that lead to $d_N = Q_0 d_0/Q_N$.

3. EXACTLY-SOLVABLE NON-HERMITIAN LATTICE

Yuce[1] studied the one-dimensional non-Hermitian lattice

$$\psi_{j+1} - E\psi_j + \gamma\psi_{j-1} = 0,\ j = 1, 2, \ldots, N, \quad (5)$$

where N is the number of lattice sites, E the energy and $0 \leq \gamma < 1$ the "non-Hermitian degree". He obtained a solution of the form

$$\psi_j = c_1\Delta_+ + c_2\Delta_-,\ \Delta_\pm = \frac{E \pm \sqrt{E^2 - 4\gamma}}{2}, \quad (6)$$

where c_1 and c_2 are arbitrary constants that one determines, together with the energy, from the boundary conditions.

For the OBC $\psi_0 = \psi_{N+1} = 0$ Yuce obtained the eigenvalues

$$E_n = 2\sqrt{\gamma}\cos\left(\frac{n\pi}{N+1}\right),\ n = 1, 2, \ldots, N. \quad (7)$$

He argued that in the limit $N \to \infty$ there is another solution when $\Delta_\pm < 1$, where the energies form a continuous band inside an ellipse in the complex plane:

$$\frac{E_R^2}{(1+\gamma)^2} + \frac{E_I^2}{(1-\gamma)^2} < 1, \quad (8)$$

where E_R and E_I are the real and imaginary parts of E, respectively. He stated that "this novel solution is unique to non-Hermitian systems" and added a further analysis and interpretation of this solution that we do not discuss here.

There is something amiss in Yuce's arguments that passed unnoticed. The condition $\Delta_\pm < 1$ is based on the assumption that $\Delta_\pm$ are real; therefore $E = \Delta_+ + \Delta_-$ is also real. Consequently, Yuce's condition is inconsistent with the results he derived from it and it is not clear where the complex eigenvalues that give rise rise to equation (8) come from. Unfortunately, Yuce did not indulge in an explicit derivation of his results.

In what follows we apply the procedure outlined above with $Q_j = \gamma^{j/2}$ that leads to the Hermitian lattice

$$d_{j+1} - \epsilon d_j + d_{j-1} = 0,\ \epsilon = \frac{E}{\sqrt{\gamma}},\ j = 1, 2, \ldots, N. \tag{9}$$

Since the lattice (9) is Hermitian for all $\gamma > 0$ we conclude that the eigenvalues ϵ are real and that $E(\gamma) = \sqrt{\gamma}E(1)$. This equation is reminiscent of the Hückel model for polyatomic molecules[4] and we know that the solutions are given by

$$c_{j,n} = A \sin\left(\frac{jn\pi}{N+1}\right), \tag{10}$$

where A is a normalization factor, for the eigenvalues (7). This analysis already shows that one does not expect complex eigenvalues for the OBC and that the energy band is restricted to $-2\sqrt{\gamma} < E < 2\sqrt{\gamma}$ for all $\gamma > 0$ when $N \to \infty$.

4. NONSOLVABLE MODEL

In this section we consider a Hamiltonian operator of the form

$$H = \sum_{j=1}^{N-1} \left(u_j |j\rangle \langle j+1| + v_j |j+1\rangle \langle j|\right) + \sum_{j=1}^{N} w_j |j\rangle \langle j|. \tag{11}$$

We assume that the parameters u_j, v_j and w_j are real and $u_j \neq v_j$ that contains some of Yuce's models as particular cases. It is clear that we can apply the results of section 2 with $u_j = H_{j,j+1}$, $v_j = H_{j+1,j}$ and $w_j = H_{j,j}$. The coefficients ψ_j satisfy the three-term recurrence relation

$$u_j\psi_{j+1} + \left(w_j - E\right)\psi_j + v_{j-1}\psi_{j-1} = 0,\ j = 1, 2, \ldots, N, \tag{12}$$

and for the time being we restrict ourselves to the OPC $\psi_0 = \psi_{N+1} = 0$.

The tridiagonal matrix that gives rise to this secular equation is symmetric if

$$Q_{j+1}^2 = \frac{v_j}{u_j} Q_j^2,\ j = 1, 2, \ldots, N-1. \tag{13}$$

Therefore, we conclude that if $u_j v_j > 0$ then the secular equation (12) is isospectral to the symmetric one

$$\sqrt{u_j v_j} d_{j+1} + \left(w_j - E\right) d_j + \sqrt{u_{j-1} v_{j-1}} d_{j-1} = 0. \tag{14}$$

It is clear that we are in the presence of what has been called generalized Hermiticity[5] or quasi-Hermiticity[6]. The argument above shows that the non-Hermitian Hamiltonian operator (11) is isospectral to an Hermitian one and, consequently, its eigenvalues are real[7]. This fact clearly explains why Yuce found that the eigenvalues in his open-chain models were real. However, we cannot extend that reasoning to periodic boundary conditions ($\psi_{N+1} = \psi_1$) because $d_{N+1} = d_1$ requires $Q_1/Q_{N+1} = 1$ that is not always consistent with equation (13) as argued also in section 2. In the model given by Yuce's equation (9) the parameters $u_n = T_n = 1 + (-1)^n\delta$, $-1 < \delta < 1$, are the alternating hopping amplitudes, $v_n = \gamma T_n$, $0 \leq \gamma < 1$, and $w_n = 0$. Consequently, the sufficient condition for real eigenvalues developed above becomes $\gamma T_n^2 > 0$; that is to say, it is sufficient that $T_n \neq 0$ and $\gamma > 0$.

We can easily derive another general result about the eigenvalues and eigenvectors of the Hamiltonian operator (11) when $w_j = w$ for all j. In this particular case we can rewrite the secular equation (12) as

$$u_j\psi_{j+1} - \epsilon\psi_j + v_{j-1}\psi_{j-1} = 0, \tag{15}$$

where $\epsilon = E - w$. If we substitute $\psi_j = (-1)^j\tilde{\psi}_j$ then this equation becomes

$$u_j\tilde{\psi}_{j+1} + \epsilon\tilde{\psi}_j + v_{j-1}\tilde{\psi}_{j-1} = 0. \tag{16}$$

We conclude that if ψ_k is a column eigenvector of the matrix representation $\mathbf{H}$ of H with eigenvalue $E_k = \epsilon_k + w$ then $\tilde{\psi}_k$ is an eigenvector with eigenvalue $E_{k'} = -\epsilon_k + w$. More precisely: for N even we have $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{\frac{N}{2}} < 0 < \epsilon_{\frac{N}{2}+1} = -\epsilon_{\frac{N}{2}} < \ldots < \epsilon_N = -\epsilon_1$, while $\epsilon_1 < \epsilon_2 < \ldots < \epsilon_{\frac{N+1}{2}} = 0 < \epsilon_{\frac{N+1}{2}+1} = -\epsilon_{\frac{N+1}{2}-1} < \ldots < \epsilon_N = -\epsilon_1$ for N odd. The eigenvalues in Yuce's FIGURE 2, for $N = 40$ lattice sites exhibit the former distribution for $w = 0$.

We can easily obtain some additional analytical results about Yuce's Hamiltonian operator (9). If $\mathbf{H}$ is the $N \times N$ matrix representation of this Hamiltonian and $\mathbf{I}$ is the $N \times N$ identity matrix we have

$$\begin{aligned}
\det(\mathbf{H}) &= (-\gamma)^{N/2}(\delta - 1)^N, \\
\det(\mathbf{H} - E\mathbf{I}) &= E^2\left(E^2 - 4\gamma\right)^{\frac{N}{2}-1},\ \delta = 1, \\
\det(\mathbf{H} - E\mathbf{I}) &= \left(E^2 - 4\gamma\right)^{\frac{N}{2}},\ \delta = -1,
\end{aligned} \tag{17}$$

for N even and

$$\det(\mathbf{H} - E\mathbf{I}) = -E\left(E^2 - 4\gamma\right)^{\frac{N-1}{2}},\ \delta = \pm 1, \tag{18}$$

for N odd. From equation (17) we draw the following conclusions: first, there are zero-energy states only for $\delta = 1$, second, only two eigenvalues approach zero when $\delta \to 1$, third, at $\delta = \pm 1$ all the nonzero eigenvalues become $E = \pm 2\sqrt{\gamma}$. The two eigenvalues that vanish as $\delta \to 1$ are $E_{\frac{N}{2}}$ and $E_{\frac{N}{2}+1} = -E_{\frac{N}{2}}$. The analysis of the eigenvalues and eigenvectors for $N = 3, 4, 5, 6$ suggests that the matrix representation of H at $\delta = \pm 1$ is normal (diagonalizable). These analytical results are reflected in Yuce's FIGURE 2 (note that $2\sqrt{\gamma} \approx 0.89$ when $\gamma = 0.2$). There is, however, an important discrepancy: present analytical results disagree with Yuce's estimate that zero-energy quasi-stationary states exist for $-0.67 < \delta < 0$ when $\gamma = 0.2$. The analytical expression for $\det(\mathbf{H})$ in equation (17) clearly shows that there cannot be zero eigenvalues unless $\delta = 1$.

5. CONCLUSIONS

In this paper we have shown that some non-Hermitian lattices are isospectral to Hermitian ones which explains why the former exhibit real eigenvalues. Present analysis is relevant for OBC but it is not so useful in the case of PBC unless some additional restrictions on the matrix elements take place. Our theoretical expressions suggest that some results (and, consequently, the physical conclusions derived from them) on non-Hermitian lattices obtained earlier by other authors may not be correct.

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