GROWING TREES WITH SEQUENTIAL REDIRECTION

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Abstract: A model for growing trees is introduced where nodes receive new links through some redirection process. When a new node enters the network it connects to a randomly selected node, then, after flipping a biased coin, it could stay there or go to its ancestor, and the process is repeated up to c times.

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1 Introduction

Growing random networks are interesting objects studied in the last twenty years by physicists and mathematicians, since they appear in several applications, see [1], [2], [3]. In its most basic form, a growing tree with redirection is defined as follow: a group of initial nodes is given, a new node enters the system, first connects to a target node (chosen randomly in the whole network) and then, with some probability p, redirects its link to the ancestor of the target node. This model was proposed in [4], and generalized in [5], where the entering node flips a coin and follows $j \neq 1$ links before connecting to a node. This model has an interesting feature, a phase transition between unbounded growth and condensation at some p_j.

We propose here a variant of this model: a node enters the system, connects to a random node, and then, with some probability p redirects its link to the ancestor of the target node, and with probability $1-p$ stays there; if it stays, the process ends, if not, we repeat this process up to c times. Our model present a secondary phase transition when c goes to infinity, which is not the limit of the critical values $\{p_c\}$, and it will be analyzed in a forthcoming work.

2 Redirection with c=2

Let us focus on the case $c = 2$, in order to fix ideas. Initially, at $t = 0$, the network is composed of one node, the seed. At each time step t a new node enters the network and connects to a random node, chosen uniformly among the existing nodes. With some probability p redirects its link to the ancestor; then, with probability p redirects again to the ancestor of the ancestor. The total number of nodes is $N = 1 + t$, and the number of links is $L = t$. We will define the *height* of a node as the minimum number of links to the seed. The probability that a node at the depth g in the directed network receives a link is:

$$P_g = (1-p)N_g + p(1-p)N_{g+1} + p^2 N_{g+2}, \tag{1}$$

except for the seed, where:

$$P_0 = N_0 + pN_1 + p^2 N_2, \tag{2}$$

and the following normalization holds:

$$N_0 + pN_1 + p^2 N_2 + \sum_{i=1}^{\infty} \left[(1-p)N_i + p(1-p)N_{i+1} + p^2 N_{i+2}\right] = N \tag{3}$$

2.1 Condensation

When $t \gg 1$, the rate equation for N_g and for the average depth $G = \sum gN_g$, are respectively:

$$\partial_t N_{1,t} = \frac{1}{N}\left[p^2 N_2 + pN_1 + N_0\right]$$
$$\partial_t N_{g,t} = \frac{1}{N}\left[(1-p)N_{g-1} + p(1-p)N_g + p^2 N_{g+1}\right]$$

and

$$\partial_t G = p^2 n_1 + (1 - p - p^2) + \frac{G}{t}, \tag{4}$$

where $n_g = N_g/N$ is the proportion of nodes at depth g.

Now, there are two possible cases:

1. If n_1 goes to zero in the long time limit, 4 simplifies into

$$\partial_t G = (1 - p - p^2) + \frac{G}{t}, \tag{5}$$

whose solution is $G \sim (1 - p - p^2)t \ln t$. This solution suggest that a phase transition occurs around $p_c = \frac{\sqrt{5}-1}{2}$, since the coefficient cannot be negative.

2. If n_1 does not vanish in the long time limit, we must take it into account. Let us highlight that a value of n_1 with these characteristics implies the formation of a condensate in the network, the seed attracts a non-vanishing fraction of the links in the network.

Now, let us find the values of p and the corresponding value of n_1 for which such a condensate exists. To do so, we need to find the stationary solutions of the equations for n_g:

$$\begin{aligned}(1+t)\partial_t n_1 &= (n_0 + p n_1 + p^2 n_2) - n_1 \\ (1+t)\partial_t n_g &= \left[(1-p)n_{g-1} + p(1-p)n_g + p^2 n_{g+1}\right] - n_g\end{aligned} \tag{6}$$

The stationary solution are obtained by recurrence and by using the fact that n_0 is negligible in the long time limit. Indeed, N_0 remains equal to 1 by construction, so that $n_0 \to 0$. It is easy to get the general stationary solution:

$$n_g = \frac{1}{C}\left(\frac{1-p}{p^2}\right)^{g-1}, \tag{7}$$

whose normalization constant is

$$C = \sum_{g=1}^{\infty}\left(\frac{1-p}{p^2}\right)^{g-1}.$$

Consequently, the system reaches a stationary solution when $\frac{1-p}{p^2} < 1$ and hence $p > \frac{\sqrt{5}-1}{2}$, so $C = \frac{p^2}{p^2+p-1}$.

By inserting the solution $n_1 = \frac{p^2+p-1}{p^2}$ into 4 we arrive at the evolution equation

$$\partial_t G = \frac{G}{t}, \tag{8}$$

so the average depth $\frac{G}{N} \sim \frac{G}{t}$ asymptotically goes to a constant, in agreement with the observed formation of condensates. Let us stress that the existence of non-vanishing stationary values of n_g is not possible in the 1-redirection model. In contrast, the formation of condensates take place for any other value $c > 1$. This result follows after generalizing 4 into:

$$\partial_t G = \frac{G}{t+1} + \left(2 - \frac{1-p^{c+1}}{1-p}\right) + \sum_{g=1}^{c-1}\left(n_g \sum_{i=g+1}^{c} p^i\right), \tag{9}$$

and the transition occurs at p_c, p_c being a root of $2 - \frac{1-p^{c+1}}{1-p}$.

2.2 $p < \frac{\sqrt{5}-1}{2}$ CASE

Let us introduce the time scale $d\tau = dt/(1+t)$ in which the set of equations to solve read as:

$$\partial_\tau N_1 = N_0 + pN_1 + p^2 N_2$$
$$\partial_\tau N_g = (1-p)N_{g-1} + p(1-p)N_g + p^2 N_{g+1} \tag{10}$$

This is a linear and homogeneous set of equations and we can expect the solutions to have a time dependence $e^{\beta\tau} \sim t^\beta$, where β is an eigenvalue of the dynamics.In the case $p > \frac{\sqrt{5}-1}{2}$ we have shown that $\beta = 1$ is a proper eigenvalue and we found the eigenvector 7. In the following we look for the solution $\beta(p)$ that is reached when $p < \frac{\sqrt{5}-1}{2}$. Solving the whole spectrum of eigenvalues of the dynamics matrix is out of question. Instead, we introduce the ansatz $N_i = A_i t^\beta$ and look for the solutions A_i for $t \gg 1$:

$$\beta A_1 = pA_1 + p^2 A_2$$
$$\beta A_g = (1-p)A_{g-1} + p(1-p)A_g + p^2 A_{g+1} \tag{11}$$

which can be solved by recurrence.

A priori, any value of $\beta \in (0,1)$ is available, except those ones for which any of the amplitudes A_i becomes negative. In order to find the values of β that fulfill this condition we have studied the recurrence in a numerical way, at a fixed value of p, by looking at $F(\beta)$ where F is the index of first amplitude A_F that becomes negative. By construction, $F(\beta)$ should go to infinity for allowed values of β. Numerical analysis in figure 1 show that a whole region of $\beta < \beta_c$ are excluded due to this non-negative constraint, and, on the other side, any value of $\beta > \beta_c$ makes all values of A_i stay positive and it is, a priori, susceptible to be chosen. In the limiting case, $p \to \frac{\sqrt{(5)}-1}{2}$, the value $\beta = 1$ is recovered.

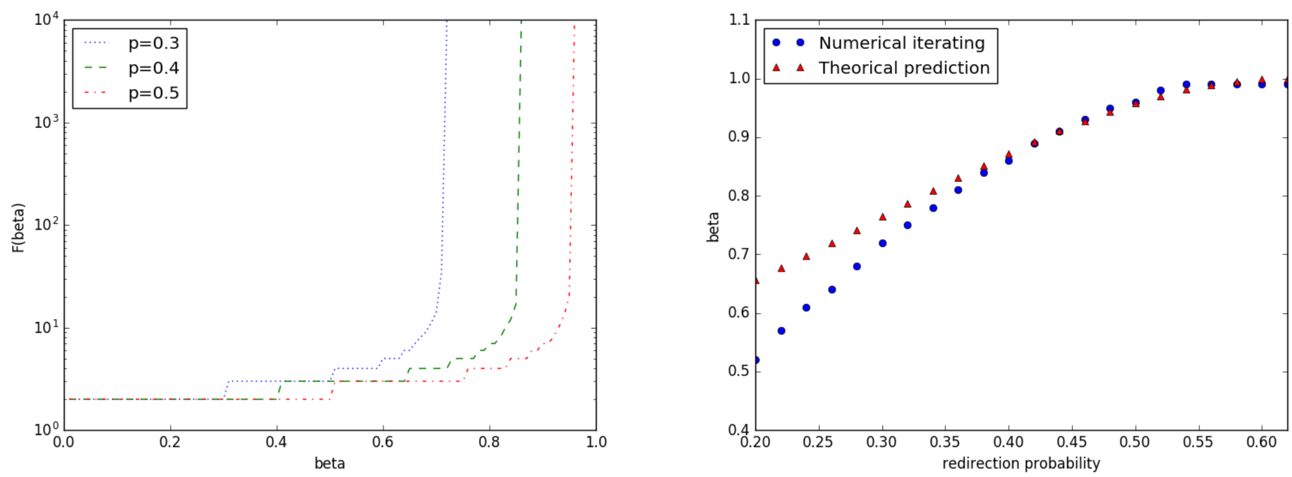

Figure 1: Left panel: relationship between the possible eigenvalue β and the index of the first negative amplitude, $F(\beta)$. The results are obtained by numerically integrating 11 up to $g_{max} = 10000$. Right panel: observed value of β_c obtained with the first negative amplitude approach, results are compared with those obtained with the theoretical prediction 13.

Let us try to find analytically the location of the transition. To do so, we focus on the equation

$$\beta A_g = (1-p)A_{g-1} + p(1-p)A_g + p^2 A_{g+1}$$

for large values of g, we assume that A_g depends continuously on g, so $A_{g+1} = A_g + A'_g + \frac{1}{2}A''_g$ and $A_{g-1} = A_g - A'_g + \frac{1}{2}A''_g$, replacing this new values on the above equation the recurrence becomes into the following homogeneous differential equation:

$$2(1-\beta)A_g + 2(p^2+p-1)A'_g + (p^2-p+1)A''_g = 0. \tag{12}$$

It is straightforward to show that the solutions of this equation undergo a transition at:

$$\beta_c = \frac{-p^4 - 2p^3 + 3p^2 + 1}{2(p^2 - p + 1)}. \tag{13}$$

Above this value, the amplitude A_g is positive and asymptotically behaves like an exponential $A_g \sim e^{\frac{-2(p^2+p-1)+\sqrt{\triangle}}{2(p^2-p+1)} g}$ where $\triangle = 4(p^2 + p - 1)^2 - 8(p^2 - p + 1)(1 - \beta)$; below this value the solution exhibits an oscillatory behaviour taken negatives solutions, consequently, these values of β are forbidden.

This theoretical prediction is shown in total agreement with the numerical analysis performed, as we can see in the graph, at least for values of $p < \frac{\sqrt{(5)}-1}{2}$.

3 Degree distribution with $c = 2$

It is difficult to derive a closed equation for the degree distribution. This is due to the fact that a $2-variable$ distribution for the degrees of the nodes at the extremes of one link has to be added in order to account for the redirection with $c = 2$. Similarly, once one tries to write an equation for that distribution, this involve three degrees characterizing two adjacent links has to be considered, etc., leading to an infinite hierarchy.

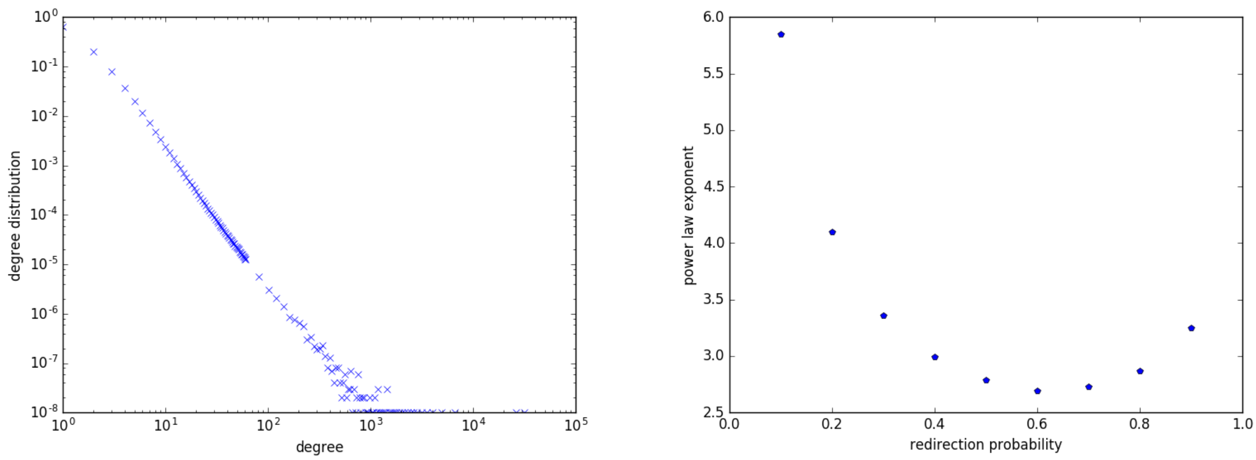

Figure 2: In the left figure, degree distribution measured from simulations with $p = 0.4$ and $t = 2\ 10^6$. In the right figure, power law exponent ν of the values between 10 and $10^{1.5}$ of the distribution $k^{-\nu}$.

To do so, we perform 50 computer realizations of the random process, measure the degree distribution after long time $t = 2\ 10^6$ and average over the many realizations. One can observe that the stationary part of the distribution converges toward a power law $k^{-\nu}$, as we show at the figure 2.

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