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Tsallis' maximum entropy ansatz leading to exact analytical time dependent wave packet solutions of a nonlinear Schrödinger equation

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HIGHLIGHTS

- We discuss solutions of a nonlinear Schrödinger equation.
- This is related to Tsallis' nonextensive thermostatistics.
- Our solutions are time dependent q-Gaussian wave-packets.
- The free particle and harmonic oscillator cases are considered.

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ABSTRACT

Tsallis maximum entropy distributions provide useful tools for the study of a wide range of scenarios in mathematics, physics, and other fields. Here we apply a Tsallis maximum entropy ansatz, the q-Gaussian, to obtain time dependent wave-packet solutions to a nonlinear Schrödinger equation recently advanced by Nobre, Rego-Monteiro and Tsallis (NRT) [F.D. Nobre, M.A. Rego-Monteiro, C. Tsallis, Phys. Rev. Lett. 106 (2011) 140601]. The NRT nonlinear equation admits plane wave-like solutions (q-plane waves) compatible with the celebrated de Broglie relations connecting wave number and frequency, respectively, with energy and momentum. The NRT equation, inspired in the q-generalized thermostatistical formalism, is characterized by a parameter q and in the limit $q \to 1$ reduces to the standard, linear Schrödinger equation. The q-Gaussian solutions to the NRT equation investigated here admit as a particular instance the previously known q-plane wave solutions. The present work thus extends the range of possible processes yielded by the NRT dynamics that admit an analytical, exact treatment. In the $q \to 1$ limit the q-Gaussian solutions correspond to the Gaussian wave packet solutions to the free particle linear Schrödinger equation. In the present work we also show that there are other families of nonlinear Schrödinger-like equations, besides the NRT one, exhibiting a dynamics compatible with the de Broglie relations. Remarkably, however, the existence of time dependent Gaussian-like wave packet solutions is a unique feature of the NRT equation not shared by the aforementioned, more general, families of nonlinear evolution equations.

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1. Introduction

A new nonlinear Schrödinger equation has been recently advanced by Nobre, Rego-Monteiro and Tsallis [1,2]. The NRT proposal constitutes an intriguing contribution to a line of enquiry that has been the focus of continuous research activity for several years: the exploration of nonlinear versions of some of the fundamental equations of physics [3,4]. In particular, nonlinear versions of the Schrödinger equation have found important applications in various areas [4]. The most studied nonlinear Schrödinger equation involves a cubic nonlinearity in the wave function. In classical contexts this nonlinear equation has been applied to fibre optics and also to the study of water waves. In quantum mechanical scenarios this kind of equations usually govern the behaviour of a single-particle wave function that provides an effective, mean-field description of a quantum many-body system. A celebrated example of this approach is given by the Gross–Pitaevskii equation, used in the study of Bose–Einstein condensates [5]. The cubic nonlinear term appearing in the Gross–Pitaevskii equation describes the short-range interactions between the atoms constituting the condensate. In this case, the nonlinear Schrödinger equation for the system's (effective) single-particle wave function can be obtained assuming a Hartree–Fock-like form for the global many-body wave function and a Dirac's delta form for the interaction potential between the particles.

The NRT equation is inspired in the thermostatistical formalism based upon the Tsallis S_q non-additive, power-law entropic functional, whose applications to the study of diverse physical system and processes have attracted considerable attention in recent years (see, for instance, [6–10] and references therein). In particular, the S_q entropy constitutes a useful tool for the analysis of diverse problems in quantum physics [11–19].

The NRT nonlinear Schrödinger equation governing the field $\Phi(x,t)$ ("wave function") corresponding to a particle of mass m reads [1,2]

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\frac{\Phi(x,t)}{\Phi_0}\right]^{2-q}, \tag{1}$$

where the scaling constant Φ_0 guarantees an appropriate physical normalization for the different terms appearing in the equation, i is the imaginary constant, $\hbar$ is Planck's constant, and q is a real parameter formally associated with Tsallis' entropic index in nonextensive thermostatistics [7]. It was shown in Ref. [1] that the wave equation (1) admits time dependent solutions having the "q-plane wave" form,

$$\Phi(x,t) = \Phi_0\left[1+(1-q)i(kx-wt)\right]^{\frac{1}{1-q}}, \tag{2}$$

with k and w real parameters having, respectively, dimensions of inverse length and inverse time (that is, k can be regarded as a wave number and w as a frequency). In the limit $q \to 1$, the q-plane waves (2) reduce to the plane wave solutions $\Phi_0\exp(-i(kx-wt))$ of the standard, linear Schrödinger equation describing a free particle of mass m.

The q-plane wave solutions (2) propagate at a constant velocity $c = w/k$ without changing shape, thus exhibiting a soliton-like behaviour. Moreover, and in contrast to the $q \to 1$ case yielding standard plane waves, the solutions corresponding to $q \neq 1$ do not have a spatially constant modulus. In fact (defining $\psi = \Phi/\Phi_0$) we have

$$|\psi(x,t)|^2 = \left[1+(1-q)^2(kx-wt)^2\right]^{\frac{1}{1-q}}, \tag{3}$$

which corresponds, for $1 < q < 3$ to a normalizable q-Gaussian centred at $x = wt/k$. Therefore, in this case the q-plane wave solution describes a phenomenon characterized by a certain degree of spatial localization. A field-theoretical approach to the NRT equation was developed in Ref. [2], where it was shown that this equation can be derived from a variational principle. The nonlinear NRT equation is formally related to the nonlinear Fokker–Planck equation (NLFP) with a diffusion term depending on a power of the density. These kind of evolution equations, and their relations with the nonextensive thermostatistical formalism, have been the focus of an intensive research recently [20–29]. In spite of the formal resemblance between the NRT Schrödinger equation and the nonlinear Fokker–Planck, there are profound differences between these two types of equations. For instance, the nonlinear Fokker–Planck equation does not admit q-plane wave solutions of the form (2) that propagate without changing their shape.

A property of the solutions (2) that was highlighted by NRT [1] is that they are consistent with the celebrated de Broglie relations [30]

$$\begin{aligned} E &= \hbar w, \\ p &= \hbar k, \end{aligned} \tag{4}$$

connecting, respectively, energy with frequency and momentum with wave number. Indeed, it can be verified that the q-plane wave (2) satisfies Eq. (1) if and only if the parameters w and k comply with the relation

$$w = \frac{\hbar k^2}{2m}, \tag{5}$$

which, combined with (4), lead to the standard relation between linear momentum and kinetic energy,

$$E = \frac{p^2}{2m}. \tag{6}$$

This suggests that it is conceivable that the q-plane wave (2) represents a particle of mass m with kinetic energy $\hbar w$ and momentum $\hbar k$ [1].

Wave packets (in particular Gaussian wave packets) are of paramount importance in quantum mechanics, both from the conceptual and practical points of view [30,31]. Wave packets also played a distinguished role in the historical development of quantum physics [32]. It is interesting to explore the existence of time dependent wave packet solutions of the NRT nonlinear Schrödinger equation. The original presentation of the NRT equation [1] was strongly focused upon the q-plane wave solutions. However, as a first step towards elucidating the meaning of the NRT equation, it is imperative to explore more general solutions. The aim of the present effort is to investigate a family of exact analytical time dependent solutions to the NRT equation exhibiting the form of q-Gaussian wave packets and corresponding, in the limit $q \to 1$, to the celebrated Gaussian wave packets solutions of the linear Schrödinger equation.

2. Time dependent wave packet solutions

In this work we are going to investigate solutions to the NRT equation based upon the q-Gaussian wave packet ansatz,

$$\psi(x,t) = \frac{\Phi(x,t)}{\Phi_0} = \left[1-(1-q)(a(t)x^2+b(t)x+c(t))\right]^{\frac{1}{1-q}}, \tag{7}$$

where a, b, and c are appropriate (complex) time dependent coefficients. Notice that ψ depends on time only through these three parameters. Inserting the ansatz (7) into the left and the right hand sides of the NRT equation (1) one obtains

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar(\dot{a}(t)x^2+\dot{b}(t)x+\dot{c}(t))\psi^q, \tag{8}$$

and

$$\frac{1}{2-q}\frac{\partial^2\psi^{2-q}}{\partial x^2}\Big/ 2\psi^q = (3-q)a(t)^2x^2+(3-q)a(t)b(t)x-a(t)+(1-q)a(t)c(t)+b(t)^2/2. \tag{9}$$

Combining now Eqs. (1), (8) and (9) one sees that the ansatz (7) constitutes a solution of the NRT equation provided that the coefficients a, b, and c comply with the set of coupled ordinary differential equations,

$$i\dot{a}(t) = \frac{\hbar}{m}(3-q)a(t)^2 \tag{10}$$

$$i\dot{b}(t) = \frac{\hbar}{m}(3-q)a(t)b(t) \tag{11}$$

$$i\dot{c}(t) = \frac{\hbar}{m}\left((1-q)a(t)c(t)-a(t)+\frac{b(t)^2}{2}\right). \tag{12}$$

The above set of differential equations admits the general solution,

$$a(t) = \frac{1}{\frac{(3-q)i\hbar t}{m}+\alpha} \tag{13}$$

$$b(t) = \frac{\beta}{\frac{(3-q)i\hbar t}{m}+\alpha} \tag{14}$$

$$c(t) = \left(\frac{(3-q)i\hbar}{m}t+\alpha\right)^{-\frac{1-q}{3-q}}\left[\frac{\left(\frac{(3-q)i\hbar}{m}t+\alpha\right)^{\frac{1-q}{3-q}}}{1-q}+\frac{\beta^2}{4}\left(\frac{(3-q)i\hbar}{m}t+\alpha\right)^{\frac{1-q}{3-q}-1}+\gamma-\frac{1}{1-q}\right], \tag{15}$$

where α, β, and γ are integration constants determined by the initial conditions $a(0)$, $b(0)$, and $c(0)$,

$$\alpha = \frac{1}{a(0)}, \tag{16}$$

$$\beta = \frac{b(0)}{a(0)}, \tag{17}$$

$$\gamma = a(0)^{\frac{q-1}{3-q}}\left(c(0)-\frac{1}{1-q}-\frac{1}{4}\frac{b(0)^2}{a(0)}\right)+\frac{1}{1-q}. \tag{18}$$

2.1. Limit case $q \to 1$

Let us now briefly consider the limit $q \to 1$ of the evolving q-Gaussian wave packet. In this case the time dependent parameters $a(t)$, $b(t)$, and $c(t)$ are

$$a(t) = \frac{1}{\frac{2i\hbar t}{m} + \alpha} \tag{19}$$

$$b(t) = \frac{\beta}{\frac{2i\hbar t}{m} + \alpha} \tag{20}$$

$$c(t) = \frac{1}{2}\ln\left(\frac{2i\hbar t}{m} + \alpha\right) + \frac{1}{4}\left(\frac{\beta^2}{\frac{2i\hbar t}{m} + \alpha}\right) + \gamma, \tag{21}$$

with α, β and γ integration constants. The general solution can be written as

$$\psi = \lim_{q\to 1}\left[1 - (1-q)(a(t)x^2 + b(t)x + c(t))\right]^{\frac{1}{1-q}}$$
$$= \exp\left(-\frac{1}{2}\ln\left(\frac{2i\hbar t}{m} + \alpha\right) - \frac{(\beta/2 + x)^2}{\frac{2i\hbar t}{m} + \alpha} - \gamma\right). \tag{22}$$

Let β_0 and β_1 be the real and imaginary parts of β, respectively, such that $\beta = \beta_0 + i\beta_1$. Taking now $\exp(-4\gamma) = \frac{2\alpha}{\pi}\exp\left(-\frac{\beta_1^2}{\alpha}\right)$, defining $k_0 = -\beta_1/\alpha$, and $\tan 2\theta = \frac{2\hbar t}{m\alpha}$, and shifting the origin of the x-axis according to $x + \beta_0/2 \to x$, (22) can be cast under the guise,

$$\psi = \left(\frac{2\alpha/\pi}{\frac{4\hbar^2 t^2}{m^2} + \alpha^2}\right)^{1/4} \exp\left(-i\,\left(\theta + \hbar k_0^2 t/2m\right)\right)\exp(ik_0 x)\exp\left(-\frac{(x - \hbar k_0 t/m)^2}{\frac{2i\hbar t}{m} + \alpha}\right), \tag{23}$$

recovering the well known Gaussian wave packet solution of the standard linear Schrödinger equation.

2.2. Illustrative example of wave packet evolution for $q = 2$

The evolution of the time dependent q-Gaussian solution is illustrated in Fig. 1, where the square modulus $|\psi|^2$ of an initially localized solution is depicted against the (nondimensional) time variable $\bar{t} = \left(\frac{\hbar a_0}{m}\right)t$ and spatial coordinate $\bar{x} = \frac{1}{\sqrt{a_0}}x$, for $q = 2$ (here $a_0 = |a(0)|$ stands for the modulus of the initial value of the parameter a). It is interesting that in the nonlinear ($q = 2$) case, as time progresses, $|\psi(x,t)|^2$ develops two peaks that depart from each other. This behaviour exhibits some qualitative similarity with the evolution of an initially localized particle in the tight binding model [33].

2.3. The case $q = 3$ and "Frozen" solutions

When $q = 3$ the set of coupled differential equations governing the evolution of the parameters a, b, and c, admit the solution

$$a = a_c,$$
$$b = b_c,$$
$$c = \frac{b_c^2 - 2a_c}{4a_c} + c_1\exp\left(2i\frac{\hbar}{m}a_c t\right), \tag{24}$$

where a_c, b_c, and c_1, are time independent constants. The concomitant solution to the NRT equation reads

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left[a_c x^2 + b_c x + \frac{b_c^2}{4a_c} + c_1\exp\left(2i\frac{\hbar}{m}a_c t\right)\right]^{-\frac{1}{2}}. \tag{25}$$

It is interesting that the solution (25) is periodic (with period, in the dimensionless time variable $\bar{t} = \frac{\hbar}{m}a_c t$, equal to π) even though the particle is not subjected to an external confining potential. Indeed, this solution describes a quasi-stationary scenario where the shape of $|\psi(x,t)|^2$ "pulsates" with the above mentioned period. In the extreme case given by $c_1 = 0$, the amplitude of the "pulsations" vanishes and we obtain the stationary solution,

$$\psi(x) = \frac{1}{\sqrt{2}}\left[a_c x^2 + b_c x + \frac{b_c^2}{4a_c}\right]^{-\frac{1}{2}}. \tag{26}$$

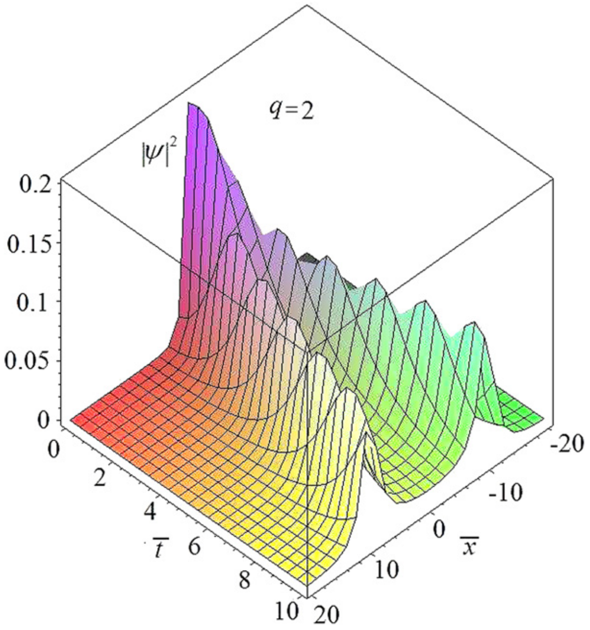

Fig. 1. Behaviour of $|\psi(x,t)|^2$ determined by the NRT nonlinear Schrödinger equation with $q=2$ and initial conditions given by $\alpha=1$, $\beta=1$, and $\gamma=1$. All depicted quantities are dimensionless.

The norm $N=\int|\psi|^2dx$ of this last solution is finite provided that $a_c\neq 0$ and b_c/a_c is not a real number.

It is interesting that there are "frozen" solutions like (26) also for instances of the NRT equation characterized by other values of q. Indeed, it can be verified that the NRT equation admits stationary solutions of the form

$$\psi(x)=[bx+ic]^{\frac{1}{2-q}}, \tag{27}$$

with $b\neq 0$ and $c\neq 0$ real constants. The square modulus $|\psi|^2$ of the "frozen" solutions (27) has a q-Gaussian profile,

$$|\psi(x)|^2=\left[b^2x^2+c^2\right]^{\frac{1}{2-q}} \tag{28}$$

and a finite norm for $2<q<4$. Notice that the solutions (27) can be cast under the guise of a q-exponential, but with a q-value given by $\tilde{q}=q-1$, which is different from the q-value associated with the concomitant nonlinear NRT evolution equation. Therefore, strictly speaking, the solutions (27) are not comprised within those corresponding to the ansatz (7).

2.4. q-plane waves and related particular solutions

The case $a=0$ leads to a family of particular solution of the set of Eq. (10), given by

$$\begin{aligned} a &= 0,\\ b &= b_c,\\ c &= -i\frac{\hbar b_c^2}{2m}\,t\,+c_0 \end{aligned} \tag{29}$$

where b_c and c_0 are complex constants. A simple but important case is obtained when the constant b_c is purely imaginary and $c_0=0$,

$$\begin{aligned} a &= 0,\\ b &= -ik, \quad k\in\mathbf{R},\\ c &= i\frac{\hbar k^2}{2m}\,t, \end{aligned} \tag{30}$$

which, after setting $w=\hbar k^2/2m$, is clearly seen to correspond to the q-plane wave solutions considered in Ref. [1]. Therefore, the exact solutions to the NRT equation studied in Ref. [1] constitute a particular case of the time dependent q-Gaussian wave packet investigated here.

Another interesting particular instance of the solutions associated with (29) corresponds to the case where the constant b_c is a real number and $c_0 = -i\frac{\hbar b_c^2}{2m} t_0$, with $t_0 > 0$ a real constant with dimensions of time, leading to,

$$\psi(x, t) = \left[1 + (1-q)\left(-b_c x + \frac{i\hbar}{2m} b_c^2 (t+t_0)\right)\right]^{\frac{1}{1-q}}. \tag{31}$$

The squared modulus of the above solution has a q-Gaussian form,

$$|\psi(x, t)|^2 = \left[\frac{(1-q)^2\,\hbar^2}{4m^2} b_c^4 (t+t_0)^2\right]^{\frac{1}{1-q}} \left\{1 + \left[\frac{2m(1-(1-q)b_c x)}{(1-q)\hbar b_c^2 (t+t_0)}\right]^2\right\}^{\frac{1}{1-q}}, \tag{32}$$

leading to a finite norm, $N = \int |\psi|^2 dx$, for $t > -t_0$ and $1 < q < 3$ (see the next subsection). The solution (31) exhibits a finite-time singularity in the past: indeed, its norm diverges at the finite time $t = -t_0$. In the limit $q \to 1$ the solutions (31) go to

$$\psi(x, t) = \exp(-b_c x) \exp\left[\frac{i\hbar}{2m} b_c^2 (t+t_0)\right], \tag{33}$$

which are formal solutions of the standard linear Schrödinger equation but are, evidently, physically unacceptable because they are not normalizable (and the wave function's square modulus $|\psi|^2$ itself diverges when $xb_c \to -\infty$). It is nevertheless interesting that the nonlinearity associated with $q > 1$ not only regularizes (in the sense of leading to a finite norm N) the plane wave solutions of the Schrödinger equation (as stressed by NRT in Ref. [1]) but it also regularizes the exponential solutions (33) (at least for all times $t > -t_0$).

2.5. (Non-)preservation of the norm

It is known that, in general, time dependent solutions to the NRT nonlinear Schrödinger equation do not preserve the norm [2]. The q-plane wave solutions studied in Ref. [2] constitute a remarkable exception: they do preserve the norm. Up to now the q-plane wave solutions were the only known time dependent solutions to the NRT equation. This means that no norm non-preserving explicit solution was known before our present work. Therefore, it is of some interest to explore whether the solutions to the NRT equation investigated here preserve the norm or not. The norm N of the q-Gaussian wave packet is given by

$$N = \int_{-\infty}^{\infty} dx |\psi|^2, \tag{34}$$

where

$$|\psi|^2 = \left[1 - 2(1-q)\Re(a(t)x^2 + b(t)x + c(t)) + (1-q)^2 |a(t)x^2 + b(t)x + c(t)|^2\right]^{\frac{1}{1-q}}. \tag{35}$$

When $a \neq 0$ the q-Gaussian wave packet is normalizable (that is, $N < \infty$) provided that $1 < q < 5$ and the polynomial $P(z) = 1 - (1-q)(a(t)z^2 + b(t)z + c(t))$ does not have real roots. Notice that for $a \neq 0$ the range of q-values admitting normalizable q-Gaussian wave functions of the form (7) is larger than the range of q-values leading to normalizable q-plane wave functions (which is $1 < q < 3$ [1]).

Let us now consider a concrete instance of a solution to the NRT equation that does not preserve the norm. In the case of the solution (31) the norm is

$$\begin{aligned} N &= \frac{1}{|1-q|b_c} \left[\frac{(1-q)^2\,\hbar^2}{4m^2} b_c^4 (t+t_0)^2\right]^{\frac{1}{2}+\frac{1}{1-q}} \int_{-\infty}^{+\infty} du \left[1+u^2\right]^{\frac{1}{1-q}} \\ &= \frac{1}{|1-q|b_c} \left[\frac{(1-q)^2\,\hbar^2}{4m^2} b_c^4 (t+t_0)^2\right]^{\frac{1}{2}+\frac{1}{1-q}} \sqrt{\pi}\, \frac{\Gamma\left(\frac{3-q}{2(q-1)}\right)}{\Gamma\left(\frac{1}{q-1}\right)}. \end{aligned} \tag{36}$$

The norm is finite for $t > -t_0$ and $1 < q < 3$ and proportional to $(t+t_0)^{1+\frac{2}{1-q}}$. Therefore, the norm is not conserved: it vanishes in the limit $t \to +\infty$, it is a finite and monotonously decreasing function of time for all finite times $t > t_0$, and it diverges at $t = -t_0$. Consequently, (31) constitutes an explicit example of a norm non-preserving solution to the NRT equation.

Let us now briefly consider the behaviour of the norm N in the case of more general solutions of the q-Gaussian form (7) with $1 < q < 5$. We are going to consider the case $q \neq 3$ so that the expressions (13)–(14) hold (when $q = 3$ the solutions of the NRT equation are not given by (13)–(14)). See Section 2.3. We shall also assume that $a(0) \neq 0$, leaving out the limit case where the q-Gaussian solutions reduce to the q-plane wave ones. It is possible, after some algebra, to express the norm in the form,

$$N = (q-1)^{\frac{2}{1-q}} |R|^{\frac{2}{q-3}} \int_{-\infty}^{+\infty} \left| |R|^{\frac{2}{q-3}} \left(x + \frac{\beta}{2}\right)^2 + \left(\frac{1}{q-1} + \gamma\right) v \right|^{\frac{2}{1-q}} dx, \tag{37}$$

where

$$R = \frac{(3-q)i\hbar t}{m} + \alpha, \qquad v = \frac{R}{|R|}. \tag{38}$$

Making now the change of variables (we assume β real),

$$y = |R|^{\frac{1}{q-3}} \left(x + \frac{\beta}{2}\right) \tag{39}$$

the expression (37) can be recast as

$$N = (q-1)^{\frac{2}{1-q}} |R|^{\frac{1}{q-3}} \int_{-\infty}^{+\infty} \left| y^2 + \left(\frac{1}{q-1} + \gamma\right) v \right|^{\frac{2}{1-q}} dy. \tag{40}$$

Now, it is evident that for large values of t we have $v \to i$ and the asymptotic behaviour of N is given by

$$(q-1)^{\frac{2}{1-q}} |R|^{\frac{1}{q-3}} \int_{-\infty}^{+\infty} \left[y^4 + \left(\frac{1}{q-1} + \gamma\right)^2 \right]^{\frac{1}{1-q}} dy = \frac{\pi (q-1)^{\frac{2}{1-q}}}{\sqrt{2}\Gamma\left(\frac{3}{4}\right) \left(\frac{1}{q-1} + \gamma\right)^{\frac{1}{q-1} - \frac{1}{4}}} \frac{\Gamma\left(\frac{1}{q-1} - \frac{1}{4}\right)}{\Gamma\left(\frac{1}{q-1}\right)} |R|^{\frac{1}{q-3}}. \tag{41}$$

The integral appearing in the above expression is time independent. The time dependence of N, for a given value of q and for large values of t, is only through the factor $|R|^{\frac{1}{q-3}}$. This implies that for large values of t the asymptotic time behaviour of N is given by $t^{\frac{1}{q-3}}$, meaning that,

$$\begin{aligned} &\lim_{t\to\infty} N = 0, \quad (1 < q < 3), \\ &\lim_{t\to\infty} N = \infty, \quad (3 < q < 5). \end{aligned} \tag{42}$$

Notice that the above asymptotic regimes do not include the limit case $q = 1$, which requires a separate treatment (see Section 2.1). In fact, the limits $q \to 1$ and $t \to \infty$ of (40) do not commute. It is, however, of some interest to see how the conservation of N follows from the limit $q \to 1$ of that equation. Taking into account that $|v|^2 = vv^* = 1$, the expression (40) for N can be cast as

$$N = |R|^{\frac{1}{q-3}} \int_{-\infty}^{+\infty} \left| \left[1 - (1-q)\left(v^* y^2 + \gamma\right)\right]^{\frac{1}{1-q}} \right|^2 dy, \tag{43}$$

leading to

$$\begin{aligned} \lim_{q\to 1} N &= |R|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \left| \exp\left[-\left(v^* y^2 + \gamma\right)\right] \right|^2 dy \\ &= |R|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp\left[-\frac{2\alpha}{|R|} y^2 - 2\gamma\right] dy \\ &= \sqrt{\frac{\pi}{2\alpha}} \exp(-2\gamma), \end{aligned} \tag{44}$$

which is clearly time independent.

It is interesting that in the marginal case $q = 3$, which separates the two regimes indicated in (42), generic q-Gaussian solutions have an oscillating norm, instead of a constant one (see Section 2.3). These results motivate us to formulate the following conjecture: *the only time-dependent solutions of the NRT equation* ($q \neq 1$) *preserving the norm during a finite time interval* $[t_0, t_1]$ *are those exhibiting the q-plane wave form.*

The non-preservation of the norm suggests that the NRT equation might provide an effective description of a many-body dynamics akin to a reaction–diffusion process, where particles are created or annihilated by some mechanism while they propagate or diffuse. In this sense, the NRT equation exhibits some similarities with the Fisher one-dimensional reaction–diffusion equation [34]. This equation governs the evolution of a (real) probability density $u(x, t)$ associated with a population undergoing both diffusion and Verhulst growth. In general the norm $\int u(x, t)dx$ is not preserved. However, the

Fisher equation admits (particular) travelling wave solutions of the form $u(x-ct)$ that propagate without change of shape at a constant speed c. Within this analogy between the NRT and the Fisher equations, the NRT q-plane wave solutions seem to play a role similar to that of the travelling wave solutions of the Fisher equation.

As a final comment, let us mention that the non-conservation of the norm suggests that, when considering pairs of time dependent solutions ψ_1 and ψ_2 to the NRT equation, the overlap $\int \psi_1^* \psi_2 dx$ is not conserved either. However, there may still be some conserved measure of the "distance" or "fidelity" between pairs of time dependent solutions. The search for such a measure constitutes a line of enquiry that may shed new light on the nature of the NRT dynamics. A possible direction to explore in this regard would be the one suggested by Yamano and Iguchi in Ref. [35], where non-Csiszar f-divergence measures were considered in connection with nonlinear Liouville-type equations.

3. Wave packet solutions in the presence of an harmonic confining potential

Now we are going to use the q-Gaussian ansatz (7) to investigate time dependent solutions of nonlinear Schrödinger equation given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2 \psi^{2-q}}{\partial x^2} + V(x)\psi^q, \tag{45}$$

where $V(x) = \frac{1}{2}Kx^2$. In the limit $q \to 1$ the above equation reduces to the time dependent Schrödinger equation of the quantum harmonic oscillator. Inserting the ansatz (7) into the right hand side of Eq. (45) we get

$$\begin{aligned} &-\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2 \psi^{2-q}}{\partial x^2} + V(x)\psi^q \\ &= \frac{\hbar^2}{2m}\left[2a(t)(1-(1-q)(a(t)x^2+b(t)x+c(t))) - (2a(t)x+b(t))^2\right]\psi^q + \frac{1}{2}Kx^2\psi^q \\ &= \frac{\hbar^2}{2m}\left[\left(\frac{m}{\hbar^2}K - 2(3-q)a(t)^2\right)x^2\right]\psi^q - \frac{\hbar^2}{2m}\left[2(3-q)a(t)b(t)x - 2a(t) + 2(1-q)a(t)c(t) + b(t)^2\right]\psi^q. \end{aligned}$$

Comparing Eq. (8) with (9) we obtain the following set of coupled differential equations for the evolution of the parameters appearing in the ansatz

$$\begin{aligned} i\dot{a}(t) &= \frac{\hbar}{m}(3-q)a(t)^2 - \frac{K}{2\hbar} \\ i\dot{b}(t) &= \frac{\hbar}{m}(3-q)a(t)b(t) \\ i\dot{c}(t) &= \frac{\hbar}{m}\left((1-q)a(t)c(t) - a(t) + \frac{b(t)^2}{2}\right). \end{aligned} \tag{46}$$

It is clear that in the limit $K \to 0$ the above equations reduce to the ones corresponding to the free particle case. It can be verified after some algebra that for $1 < q < 3$ the first of the above equations admits the general solution,

$$a = \sqrt{\frac{mK}{2\hbar^2(3-q)}}\left\{\frac{\exp\left[2\sqrt{\frac{mK}{2\hbar^2(3-q)}}\left(\frac{i\hbar t(3-q)}{m}+\alpha\right)\right]+1}{\exp\left[2\sqrt{\frac{mK}{2\hbar^2(3-q)}}\left(\frac{i\hbar t(3-q)}{m}+\alpha\right)\right]-1}\right\}, \tag{47}$$

where α is an integration constant. As expected for solutions to an harmonic oscillator system, it is clear from (47) that $a(t)$ exhibits a periodic time behaviour. The corresponding period is $\tau = \pi\sqrt{\frac{2m}{(3-q)K}}$ which, of course, diverges in the limit $K \to 0$. In fact, expanding $a(t)$ in a power series of $z = \sqrt{\frac{mK}{2\hbar^2(3-q)}}$ and keeping the zeroth order term, it is straightforward to verify that

$$\lim_{K\to 0} a(t) = \left(\frac{i\hbar t(3-q)}{m}+\alpha\right)^{-1}, \tag{48}$$

which coincides with the expression (13) corresponding to the free particle solution. For $3 < q < 5$ the general solution for $a(t)$ reads

$$a = i\sqrt{\frac{mK}{2\hbar^2(q-3)}}\left\{\frac{\exp\left[2\sqrt{\frac{mK}{2\hbar^2(q-3)}}\left(\frac{\hbar t(q-3)}{m}+i\alpha\right)\right]+1}{\exp\left[2\sqrt{\frac{mK}{2\hbar^2(q-3)}}\left(\frac{\hbar t(q-3)}{m}+i\alpha\right)\right]-1}\right\}, \tag{49}$$

which again, by recourse to a procedure akin to the one followed in the case of Eq. (47), can be shown to reduce to the free particle solution in the limit $K \to 0$. Finally, in the marginal case $q = 3$ we have

$$a(t) = a_c + \frac{iKt}{2\hbar}, \tag{50}$$

where a_c is an integration constant. It is clear that when $K = 0$ we recover the solution for a free particle discussed in Section 2.3.

3.1. Quasi-stationary solutions

A particularly interesting solution of the set of equations (46) is given, for $1 < q < 3$, by

$$\begin{aligned} a &= a_c = \frac{1}{\hbar}\sqrt{\frac{mK}{2(3-q)}}, \\ b &= 0, \\ c &= \frac{1}{1-q}\left[1 - \exp\left(-i(1-q)\frac{\hbar a_c t}{m}\right)\right], \end{aligned} \tag{51}$$

leading, in turn, to the following solution for the nonlinear Schrödinger equation:

$$\psi = \left[\exp\left(-i(1-q)\frac{\hbar a_c t}{m}\right) - (1-q)a_c x^2\right]^{\frac{1}{1-q}}. \tag{52}$$

It is interesting to consider now the $q \to 1$ limit of the above solution,

$$\begin{aligned} \lim_{q\to 1}\psi &= \exp\left(-i\frac{\hbar a_c t}{m}\right)\lim_{q\to 1}\left[1 - (1-q)a_c \exp\left(i(1-q)\frac{\hbar a_c t}{m}\right)x^2\right]^{\frac{1}{1-q}} \\ &= \exp\left(-i\frac{\hbar a_c t}{m}\right)\exp\left(-a_c x^2\right) \\ &= \exp\left(-i\frac{\omega t}{2}\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right), \end{aligned} \tag{53}$$

which is the (unnormalized) wave function associated with the ground state of a standard harmonic oscillator with natural frequency $\omega = \sqrt{\frac{K}{m}}$ and zero point energy $E_0 = \frac{1}{2}\hbar\omega$.

The norm of the solution (52) is finite for $1 < q < 3$ and $-t_c < t < t_c$ with $t_c = \frac{\pi m}{(q-1)\hbar a_c}$, and it has finite-time singularities at $t = \pm t_c$. The time derivative of the norm of the solution (52) is

$$\frac{dN}{dt} = 2(1-q)\frac{\hbar a_c^2}{m}\sin\left[(1-q)\frac{\hbar a_c t}{m}\right]\int_{-\infty}^{\infty} x^2\left[1 - 2a_c(1-q)\cos\left[(1-q)\frac{\hbar a_c t}{m}\right]x^2 + (1-q)^2 a_c^2 x^4\right]^{\frac{q}{1-q}} dx. \tag{54}$$

The expression between square brackets appearing in the integrand on the right hand side of the above equation is in general larger than zero and, consequently, the time derivative of the norm is different from zero. Therefore, the time dependent solution (52) constitutes another explicit example of a solution that does not preserve the norm. In the limit $q \to 1$ we get $|t_c| \to \infty$ and, of course, $dN/dt \to 0$.

In the marginal case $q = 3$ the quasi-stationary solutions with the parameter a constant in time do not exist. In this case a is always a linear function of t (see Eq. (50)). For $3 < q < 5$ the quasi-stationary solution is still given by Eqs. (51)–(52), but now the constant a_c is a purely imaginary number,

$$a_c = \frac{i}{\hbar}\sqrt{\frac{mK}{2(q-3)}}. \tag{55}$$

The explicit form of the solution is now,

$$\psi(x,t) = \left[\exp\left((1-q)\frac{t}{m}\sqrt{\frac{mK}{2(q-3)}}\right) - (1-q)\frac{i}{\hbar}\sqrt{\frac{mK}{2(q-3)}}\,x^2\right]^{\frac{1}{1-q}}. \tag{56}$$

The norm of the above solution is

$$N = \int_{-\infty}^{+\infty} \left[\exp\left(2(1-q)\frac{t}{m}\sqrt{\frac{mK}{2(q-3)}} \right) + \frac{(1-q)^2}{q-3}\left(\frac{mK}{2\hbar^2}\right) x^4 \right]^{\frac{1}{1-q}}$$
$$= \exp\left[\frac{5-q}{2}\left(\frac{t}{m}\right)\sqrt{\frac{mK}{2(q-3)}} \right]\left(\frac{mK}{2\hbar^2}\right)^{-\frac{1}{4}} \int_{-\infty}^{+\infty} (1+u^4)^{\frac{1}{1-q}}\, du, \tag{57}$$

which is finite for all values of t. We see that N grows exponentially with time.

Notice that the $K \to 0$ limit of the quasi-stationary solutions considered here corresponds to the trivial ψ-constant free particle solution.

4. Generalizations of the NRT approach and unique features of the NRT equation

The existence of q-plane wave solutions consistent with the de Broglie relations connecting frequency and wave number respectively with energy and momentum was one of the features of NRT equation discussed in Ref. [1]. It is worth noticing that there are other possible nonlinear Schrödinger-like equations exhibiting similar properties. That is, the NRT approach can be substantially generalized. Let us consider a pair of one-variable functions $L(u)$ and $F(u)$ satisfying the functional relation,

$$\frac{d^2}{du^2}\left[L(F(u))\right] = \frac{dF}{du}. \tag{58}$$

It can then be verified after some algebra that the (in general nonlinear) Schrödinger-like equation

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[L\left(\frac{\Phi(x,t)}{\Phi_0}\right)\right] \tag{59}$$

admits exact time dependent plane wave-like solutions of the form,

$$\Phi(x,t) = \Phi_0 F[i(kx - wt)], \tag{60}$$

with $\hbar w = \hbar^2 k^2/2m$. Indeed, inserting the ansatz (60) into the right and left hand sides (59) and setting $u = i(kx - wt)$ we get,

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = \left(\frac{dF}{du}\right)\left(\frac{\partial u}{\partial t}\right) = \hbar w\frac{dF}{du},$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[L\left(\frac{\Phi(x,t)}{\Phi_0}\right)\right] = -\frac{\hbar^2}{2m}\frac{d^2}{du^2}\left[L(u)\right]\left(\frac{\partial u}{\partial x}\right)^2$$
$$= \frac{\hbar^2 k^2}{2m}\frac{d^2}{du^2}\left[L(F(u))\right], \tag{61}$$

where, in the last equation, the fact that $\partial^2 u/\partial x^2 = 0$ was used. It is clear that the functional relation (58) implies (59), provided that $\hbar w = \hbar^2 k^2/2m$. This in turn leads, via the de Broglie relations, to the kinetic energy–momentum relation $E = p^2/2m$. Similarly to what happens within the NRT scenario, the solutions (60) to the nonlinear evolution Eq. (59) propagate without changing shape and with constant velocity $v = w/k$. Furthermore, as we have seen, they comply with a relation between w and k that is consistent with the de Broglie connection between frequency, wave number, energy, and momentum.

One procedure to generate pairs of functions satisfying (58) is the following. One starts with a function $G(u)$ such that its derivative dG/du admits an inverse. Then we define,

$$F(u) = \frac{dG}{du},$$
$$L(u) = G\left[F^{(-1)}(u)\right], \tag{62}$$

where $F^{(-1)}(u)$ is the inverse function of $F(u)$, satisfying $F^{(-1)}(F(u)) = u$. It can be verified that the functions defined by (62) comply with the required relation (58).

In the case of the NRT equation we have

$$G(u) = \frac{1}{2-q}[1+(1-q)u]^{\frac{2-q}{1-q}},$$
$$F(u) = [1+(1-q)u]^{\frac{1}{1-q}},$$
$$L(u) = \frac{u^{2-q}}{2-q}. \tag{63}$$

An example (different from the NRT one) of a nonlinear Schrödinger-like equation of the form (59) admitting the plane wave-like solutions (60) is given by $F(u) = \sinh(u)$ and $L(u) = \sqrt{1+u^2}$. This case corresponds to the nonlinear evolution equation,

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left\{1+\left[\frac{\Phi(x,t)}{\Phi_0}\right]^2\right\}^{\frac{1}{2}}, \tag{64}$$

having plane wave-like solutions

$$\Phi(x,t) = \Phi_0 \sinh[i(kx - wt)], \tag{65}$$

with $\hbar w = \hbar^2 k^2/2m$.

As mentioned in the introduction, the NRT equation was inspired by the nonextensive generalized thermostatistical formalism based on the constrained optimization of Tsallis' entropy [1]. Indeed, there are formal connections between the NRT scenario and the alluded thermostatistical formalism. An intriguing question, which is beyond the scope of the present work but certainly deserves to be explored, is the possible existence of connections between (some of) the nonlinear Schrödinger equations (59), on the one hand, and other formalisms based on non-standard entropic functionals different from the Tsallis one [10,36], on the other hand.

We have seen that the NRT equation is not the only nonlinear equation of the form (59) admitting plane wave-like solutions consistent with the de Broglie relations. It is then natural to ask which of Eq. (59) also admit solutions of the form

$$\Phi(x,t) = \Phi_0 F[d_2(t)x^2 + d_1(t)x + d_0(t)], \tag{66}$$

with $d_1(t), d_2(t), d_0(t)$ appropriate time dependent coefficients. The solutions (66) would constitute generalizations of the q-Gaussian solutions to the NRT equation previously discussed in the present work. It turns out that the NRT equation is the only member of the family (59) admitting both solutions of the form (60) and of the form (66). If one inserts on the left and right hand sides of the evolution Eq. (59) the expression for $\psi = \frac{\Phi}{\Phi_0}$ given by the ansatz (66) one gets

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar(\dot{d}_2x^2 + \dot{d}_1x + \dot{d}_0)F'(u), \tag{67}$$

and

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Big[L(F(u))\Big] = -\frac{\hbar^2}{2m}\left\{(2d_2x+d_1)^2\frac{d^2}{du^2}\Big[L(F(u))\Big] + 2d_2\frac{d}{du}\Big[L(F(u))\Big]\right\}, \tag{68}$$

where $u = d_2(t)x^2 + d_1(t)x + d_0(t)$. Then, to have solutions of the form (60) (plane wave-like), Eq. (58) must hold. On the other hand, to have solutions of the form (66) one further condition is required,

$$\frac{d}{du}\Big[L(F(u))\Big] = (r_1 + r_2u)F'(u), \tag{69}$$

with r_1 and r_2 appropriate constants. It follows from (69) that

$$L''(F(u))F'(u) = r_2. \tag{70}$$

Combining now Eqs. (58), (69) and (70) one obtains

$$r_2F'(u) + (r_1 + r_2u)F''(u) = F'(u), \tag{71}$$

which leads to

$$F'(u) = F_0(r_1 + r_2u)^{\frac{1-r_2}{r_2}}, \tag{72}$$

where F_0 is an integration constant. Finally, we have,

$$F(u) = F_0(r_1 + r_2u)^{\frac{1}{r_2}} + F_1, \tag{73}$$

involving one more integration constant F_1. Making now the identification $r_2 = 1 - q$ one sees that (up to multiplicative and additive constants) the form of the solution (73) coincides with q-Gaussian wave packet.

5. Conclusions

We have obtained a new family of exact, analytical time dependent wave packet solutions to the nonlinear Schrödinger equation recently proposed by Nobre, Rego-Monteiro and Tsallis [1,2]. Our solutions have the form of a q-exponential evaluated upon a quadratic function of the spatial coordinate x with time dependent coefficients. Therefore, these solutions have a q-Gaussian form. They extend and generalize the previously known solutions to the NRT equation. The solutions investigated here by us correspond, in the limit $q \to 1$ of the parameter q, to the Gaussian wave packet solutions to the standard linear Schrödinger equation. Our present wave packet solutions admit as a special particular case the q-plane wave solutions studied in Ref. [1]. In the present work we also discuss other families of nonlinear Schrödinger-like equations, besides the NRT one, leading to a dynamics compatible with the de Broglie relations. In this regard, it is remarkable that the existence of the Gaussian-like time dependent solutions investigated in this work is a unique feature of the NRT equation not shared by the above mentioned, more general, families of nonlinear evolution equations.

We also obtained q-Gaussian wave packet solutions for the case of a harmonic potential. In the limit $q \to 1$ these latter solutions reduce to Schrödinger's celebrated Gaussian wave packet solutions to the harmonic oscillator. As a particular case of the time dependent q-Gaussian wave packets associated with the harmonic potential we found a quasi-stationary solution yielding in the $q \to 1$ limit the wave function corresponding to the ground state of the quantum harmonic oscillator.

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