Relative entropy of an interval for a massless boson at finite temperature

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We compute Araki's relative entropy associated with a bounded interval $I = (a, b)$ between a thermal state and a coherent excitation of itself in the bosonic U(1)-current model, namely the (derivative of the) chiral boson. For this purpose we briefly review some recent results on the entropy of standard subspaces and on the relative entropy of nonpure states such as thermal states. In particular, recently Bostelmann, Cadamuro, and Del Vecchio have obtained the relative entropy at finite temperature for the unbounded interval $(-\infty, t)$, using previous results of Borchers and Yngvason, mainly a unitary dilation that provides the modular evolution in the negative half-line. Here we find a unitary rotation in order to make use of the full $\mathrm{PSL}(2,\mathbb{R})$ symmetries and obtain the modular group, the modular Hamiltonian, and the relative entropy S of a bounded interval at finite temperature. Such relative entropy entails both a Bekenstein-like bound and a QNEC-like bound, but violates $S'' \geq 0$ (derivative with respect to the length of the interval, with its center fixed). Finally, we extend the results to the free massless boson in $1+1$ dimensions with analogous bounds.

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I. INTRODUCTION

The relative entropy is a measure of the distinguishability of two states. In quantum mechanics, for two density matrices ρ and σ it is defined as

$$S(\rho||\sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma). \tag{1}$$

If the system is bipartite, one can reduce each density matrix to one of the partitions and compute the relative entropy between the reduced density matrices. The counterpart in Quantum Field Theory (QFT) would be to reduce the states to a spacetime region. Intuitively, the smallest this region is, the fewer operators one has at hand to characterize the states, and therefore the relative entropy decreases. However, generic bounded regions of spacetime such as causal diamonds are assigned a von Neumann algebra which is a Type III factor, and these algebras have no trace-class operators and there are no density matrices [1]. Nevertheless, the definition of the relative entropy (1) can be suitably extended [2]: given two (normal and faithful) states ω and ω' of the von Neumann algebra A associated with some region, with both states represented on a Hilbert space by Ω and Ω', the relative entropy is

$$S_A(\omega||\omega') = -(\Omega, \log \Delta_{\Omega',\Omega}\Omega), \tag{2}$$

where $\Delta_{\Omega',\Omega}$ is the relative modular operator (which is defined in the next section). This is in sharp contrast with the entanglement entropy that necessarily diverges due to the generic UV behavior of correlations through the boundary of the corresponding region [3].

Araki's relative entropy (2) can be hard to compute in general cases. Recently it has been computed in a number of specific situations for coherent states acting on the vacuum of the free scalar QFT: on a Rindler wedge [4–6], on a causal diamond in spacetime dimension greater than 2 [7], and later on the two-dimensional case [8]. In addition, in the bosonic vacuum U(1)-current model (namely, a massless boson on a light ray) the relative entropy of coherent states was computed for a half-line in [9] and used to obtain the analogous expression for (unbounded) regions of the null plane for a free scalar in [10]. For similar results in free fermionic Conformal Field Theories (CFTs), see [11]. For interacting theories, as far as we know the only results available correspond to coherent states in chiral CFTs [12,13], while for free QFT in curved spacetimes see [14].

All of the above computations compare the vacuum state to a coherent excitation of itself (with the exception

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of [14]). Recently in [15], among other things, the relative entropy for a half-line in the *thermal* U(1)-current model was computed. That is, a Kubo-Martin-Schwinger (KMS) state of inverse temperature β was compared to a coherent excitation of itself, when restricted to a half-line. The fact that the relative entropy was computed on the half-line is an important point, since by general arguments its modular operator had already been obtained in [16]. Even in the vacuum case, one can see that to obtain the modular operator for a bounded interval it is necessary to make use of the full $\mathrm{PSL}(2,\mathbb{R})$ group of conformal transformations of the chiral boson theory (we will discuss this later in detail).

The main goal of this article is to compute the relative entropy of coherent states on a bounded interval $I \subset \mathbb{R}$ for the thermal U(1)-current model and the free massless boson at finite temperature in $1+1$ dimensions restricted to a causal diamond. We will find two bounds for each theory, a Bekenstein-like bound and mainly a controlled violation of the quantum null energy condition (QNEC), which was already anticipated in [15].

II. PRELIMINARIES ON THE MODULAR STRUCTURE OF THE WEYL ALGEBRA

Given a symplectic space $(\mathcal{K},\sigma)$ we get a canonical commutation relations (CCR) algebra with relations

$$W(f)W(g) = e^{-i\sigma(f,g)}W(f+g), \qquad W(f)^* = W(-f), \tag{3}$$

where $f, g \in \mathcal{K}$. We will refer to this algebra as $\mathrm{CCR}(\mathcal{K},\sigma)$.

A positive symmetric bilinear form τ such that

$$\sigma(f,g)^2 \leq \tau(f,f)\tau(g,g) \tag{4}$$

defines a quasifree state by [17]

$$\omega(W(f)) = e^{-\frac{1}{2}\tau(f,f)} \tag{5}$$

with two-point function $w_2(f,g) = \tau(f,g) + i\sigma(f,g)$. However, the state may not be pure. It is pure if and only if w_2 is a complex inner product. More precisely, from (4) it can be shown that a contraction D exists such that

$$\sigma(f,g) = \tau(f,Dg). \tag{6}$$

Because of the nondegeneracy of σ, D is invertible and $D = C|D|$ is its polar decomposition. C is a complex structure and the state ω is pure if and only if w_2 is a complex inner product, bilinear with respect to the complex structure C. This is actually equivalent to the statement that ω is pure if and only if $|D| = 1$ (see [18] for further details). Let us defer how to construct a purification of a nonpure state to the end of this section.

A. Modular theory and relative entropy

We assume we have a complex Hilbert space $\mathcal{H}$ with inner product $\langle f,g\rangle = w_2(f,g)$. In other words, we are assuming for now that the state is pure. Let us call its (bosonic) Fock space $\Gamma(\mathcal{H})$. We have a representation of the $\mathrm{CCR}(\mathcal{K},\sigma)$ algebra on the Fock space $\Gamma(\mathcal{H})$. Indeed, $W(f)$ acts on $\Gamma(\mathcal{H})$ as $V(f)$:

$$V(f)e^0 = e^{-\frac{1}{2}\langle f,f\rangle}e^f, \qquad f \in \mathcal{H}, \tag{7}$$

with

$$e^f \coloneqq 1 \oplus f \oplus \frac{1}{\sqrt{2!}} f\otimes f \oplus \cdots. \tag{8}$$

Calling $\Omega \coloneqq e^0$ the vacuum vector,

$$(\Omega, V(f)\Omega) = e^{-\frac{1}{2}\langle f,f\rangle} = \omega(W(f)). \tag{9}$$

We can define the local algebras associated with a given real-linear subspace $H \subset \mathcal{H}$:

$$R(H) \coloneqq \{V(f); f \in H\}''. \tag{10}$$

It turns out that Ω is cyclic (respectively, separating) for $R(H)$ if and only if H is cyclic (respectively, separating). H is cyclic if $\overline{H + iH} = \mathcal{H}$ while separating if $H \cap iH = 0$. If H is also closed, it is called a *standard* subspace.

The Tomita operator S associated with $R(H)$ (and Ω) is defined by (the closure of)

$$SV(f)\Omega = V(f)^*\Omega, \qquad V(f) \in R(H). \tag{11}$$

The *relative* Tomita operator $S_{\Omega',\Omega}$ associated with $R(H)$ is defined by (the closure of)

$$S_{\Omega',\Omega}V(f)\Omega = V(f)^*\Omega' \tag{12}$$

with polar decomposition

$$S_{\Omega',\Omega} = J_{\Omega',\Omega}\Delta^{1/2}_{\Omega',\Omega}. \tag{13}$$

For some algebra $R(H)$ and cyclic and separating states ω and ω', Araki's relative entropy is defined as

$$S_A(\omega||\omega') = -(\Omega, \log\Delta_{\Omega',\Omega}\Omega). \tag{14}$$

This is hard to compute for generic cases; however, it simplifies considerably for coherent states [19], namely when $\Omega' = V(f)\Omega$, and we shall call the corresponding algebraic state ω_f. To see this, we first need to introduce a modular theory for $H \subset \mathcal{H}$, with H *standard*, following [20]. The analogous Tomita operator is defined by the closure of $S_H(f+ig) = f - ig$, and its polar decomposition is

$$S_H = J_H \Delta_H^{1/2}. \tag{15}$$

Then, the entropy of a vector $f \in H$ with respect to H is defined by

$$S_H(f) \coloneqq -\langle f, \log \Delta_H f\rangle. \tag{16}$$

Actually, in [5] it was generalized for $f \in \mathcal{H} \cap \mathrm{Dom}(K_H)$ that

$$S_H(f) \coloneqq -\langle f, P_H \log \Delta_H f\rangle = \sigma(f, P_H i K_H f), \tag{17}$$

where

$$K_H \coloneqq -\log \Delta_H = i \frac{d}{du} \Delta_H^{iu}\Big|_{u=0} \tag{18}$$

is the one-particle modular Hamiltonian and where $P_H\colon H + H' \to H$ is the cutting projector and we are assuming that H is *factorial*, namely $H \cap H' = 0$. Here H' is the symplectic complement of H. In that work they showed that Araki's relative entropy (14), between a coherent state $V(f)\Omega$ and a *pure* state Ω, is nothing but the entropy of the vector $f \in \mathcal{H} \cap \mathrm{Dom}(K_H)$:

$$S(\omega_f || \omega) = \sigma(f, P_H i K_H f). \tag{19}$$

We will take advantage of this result to compute the relative entropy by working exclusively at the level of the one-particle Hilbert space $\mathcal{H}$.

B. Purification of ω

Since we are interested in computing the relative entropy in the case that ω is a thermal state, and since (19) is valid for pure states, we need to work with a purification of ω. This can be achieved by a procedure we recall in this subsection, following [15] (see also [18]).

By means of (6) we can get [21] a complex structure on $\mathcal{K}^{\oplus} \coloneqq \mathcal{K} \oplus \mathcal{K}$:

$$i^{\oplus} = \begin{pmatrix} -D & C\sqrt{1+D^2} \\ C\sqrt{1+D^2} & D \end{pmatrix}. \tag{20}$$

Let us call $\mathcal{H}^{\oplus}$ the complexification of $\mathcal{K}^{\oplus}$. The complex inner product is given by

$$\langle \cdot, \cdot \rangle^{\oplus} = \tau^{\oplus}(\cdot,\cdot) + i\sigma^{\oplus}(\cdot,\cdot), \tag{21}$$

with $\tau^{\oplus} \coloneqq \tau \oplus \tau$ and $\sigma^{\oplus}(\cdot,\cdot) = \tau^{\oplus}(\cdot, -i^{\oplus}\cdot)$. This inner product reduces to w_2 on $\mathcal{K} \simeq \mathcal{K} \oplus 0$:

$$\langle f \oplus 0, g \oplus 0\rangle^{\oplus} = \tau(f,g) + i\sigma(f,g) = w_2(f,g). \tag{22}$$

We have

$$\mathrm{CCR}(\mathcal{K},\sigma) \subset \mathrm{CCR}(\mathcal{K}^{\oplus}, \sigma^{\oplus}). \tag{23}$$

Similarly, for closed subspaces $H \subset \mathcal{K}$,

$$\mathrm{CCR}(H,\sigma) \subset \mathrm{CCR}(\mathcal{K},\sigma). \tag{24}$$

And most importantly on the $\mathrm{CCR}(\mathcal{K},\sigma)$ algebra,

$$\omega^{\oplus}(W(f\oplus 0)) = e^{-\frac{1}{2}\tau^{\oplus}(f\oplus 0, f\oplus 0)} = \omega(W(f)), \tag{25}$$

which justifies why the pure state $\omega^{\oplus}$ of the $\mathrm{CCR}(\mathcal{K}^{\oplus}, \sigma^{\oplus})$ algebra is a purification of ω.

The relative entropy for nonpure states associated with $R(H)$, with $H \simeq H \oplus 0$ the standard and factorial subspace of $\mathcal{H}^{\oplus}$ reads [15]

$$S_{R(H)}(\omega_f || \omega) = \sigma^{\oplus}(f, P_H i^{\oplus} K_H f), \qquad f \in \mathcal{K} \oplus 0 \cap \mathrm{Dom}(K_H). \tag{26}$$

In [15] a more general expression was obtained for other subspaces. We will not need this since for the $U(1)$ model the subspace associated with the bounded interval is standard and factorial as we will show. Note that in (26) the modular Hamiltonian K_H is associated with the modular operator that acts on the larger space $\mathcal{H}^{\oplus}$ while P_H is the real-linear cutting projector onto $H \oplus 0$.

III. THE CHIRAL BOSON CURRENT

We are interested in applying all the above to the case of a free chiral boson. More precisely, we consider the current usually denoted $\phi'(x)$. In the smeared version, the classical theory is defined by the symplectic space of compactly supported real functions $\mathcal{K} = C_c^{\infty}(\mathbb{R})$ with symplectic structure

$$\sigma(f,g) = \int_{\mathbb{R}} f(x) g'(x) dx. \tag{27}$$

As reviewed in the previous section, we have a $\mathrm{CCR}(\mathcal{K},\sigma)$ algebra associated with $(\mathcal{K},\sigma)$. To proceed, we need to define a quasifree state by means of a positive symmetric bilinear form τ. We start with the vacuum state that we will denote ω and then move on to a thermal state ω_β.

A. The vacuum case

The vacuum state is defined by

$$\tau(f,g) = -\frac{1}{\pi} PV \int_{\mathbb{R}^2} dxdy \frac{f(x)g(y)}{(x-y)^2} = (f, \mathfrak{H} g')_{L^2}. \tag{28}$$

Here PV denotes the principal value integral and $\mathfrak{H}$ the Hilbert transform. It is pure since $D = -\mathfrak{H}$, which is unitary or equivalently $D = -i\mathrm{sgn}(p)$ in momentum space [22]; therefore the complexification described in the previous section can be applied directly to $\mathcal{K}$, and we obtain the complex inner product

$$\begin{aligned}\langle f, g\rangle &= -\frac{1}{\pi}\int_{\mathbb{R}^2} dxdy \frac{f(x)g(y)}{(x-y-i\epsilon)^2} \\ &= (f, \mathfrak{H}g')_{L^2} + i\sigma(f,g) \\ &= \frac{1}{\pi}\int_0^\infty \hat{f}(p)^*\hat{g}(p)pdp, \end{aligned} \tag{29}$$

which enables one to establish an isomorphism with $\mathcal{H} \simeq L^2(\mathbb{R}_+, pdp)$. This is the vacuum one-particle Hilbert space. We will mostly work in the coordinate space.

An important role in this work is played by the $\mathrm{PSL}(2,\mathbb{R})$ symmetry of the model. The unitary representation on $\mathcal{H}$ is given by

$$U(g)f(x) = f(g^{-1}\cdot x) \tag{30}$$

with an element of the group g acting on the coordinate x by linear fractional transformations:

$$g\cdot x = \frac{ax+b}{cx+d}, \qquad g\in \mathrm{PSL}(2,\mathbb{R}). \tag{31}$$

We have three one-parametric subgroups related to the KAN decomposition of $\mathrm{PSL}(2,\mathbb{R})$:

$$r(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}, \qquad \delta(s) = \begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{-\frac{s}{2}}\end{pmatrix},$$
$$\tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}, \tag{32}$$

which we will refer to as rotation, dilation, and translation subgroups, respectively. For example, the dilation-translation subgroup acts as

$$U(\tau(t))f(x) = f(x-t), \quad U(\delta(s))f(x) = f(e^{-s}x). \tag{33}$$

Note that the generator of translations P defined by $U(\tau(t)) = e^{itP}$ is positive. This is most easily seen in momentum-space with the convention of footnote V.

Now we can discuss the modular theory of standard subspaces of $\mathcal{H}$. Consider the subspaces $H(I) = \overline{C_c^\infty(a,b)} \in \mathcal{H}$, which are standard and factorial [20]. Since the above representation has positive P, the modular evolution on $H(\mathbb{R}_+)$ is (Theorem 3.3.1 of [20])

$$\Delta^{iu}_{(0,\infty)} = U(\delta(-2\pi u)), \qquad u\in\mathbb{R}. \tag{34}$$

This can be seen by checking that $F(u) \coloneqq \langle g, \Delta^{iu} f\rangle$ admits an analytic continuation to the strip $-1 < \mathrm{Im}(u) < 0$ and that the KMS property at temperature -1 is satisfied:

$$\langle \Delta^{iu}_{(0,\infty)} f, g\rangle = \langle g, \Delta^{i(u-i)}_{(0,\infty)} f\rangle. \tag{35}$$

By covariance $U(g)H(I) = H(g\cdot I)$ we have

$$\Delta_I^{iu} = U(\bar{g})^{-1}\Delta^{iu}_{(0,\infty)}U(\bar{g}), \quad u\in\mathbb{R}, \quad \bar{g}\cdot I = \mathbb{R}_+. \tag{36}$$

Note that $\bar{g}$ is defined modulo multiplication by a dilation on the left, but this does not affect Δ_I^{iu}.

1. The interval $(-\infty, t)$

For instance, by considering $\bar{g}$ a rotation in π followed by a translation of t,

$$\Delta^{iu}_{(-\infty,t)} f(x) = f(e^{-2\pi u}(x-t)+t). \tag{37}$$

We can compute then the modular Hamiltonian

$$K_{(-\infty,t)} = i\frac{d}{du}\Delta^{iu}_{(-\infty,t)}\bigg|_{u=0} = 2\pi i(t-x)\frac{d}{dx}. \tag{38}$$

And from here the relative entropy of a coherent state

$$\begin{aligned} S_{R((-\infty,t))}(\omega||\omega_f) &= S_{(-\infty,t)}(f) \\ &= 2\pi\int_{-\infty}^t (t-x)f'(x)^2 dx. \end{aligned} \tag{39}$$

Here f need not be localized in the interval $(-\infty, t)$ [15].

2. The interval (a,b)

We can repeat what we have just done for an interval $I = (a,b)$. The first step is to find $\bar{g}$ such that $\bar{g}\cdot I = \mathbb{R}_-$. The idea is to first project I to the circle, forming an arc (θ_a, θ_b). Then, we employ two symmetries (see Fig. 1). The first one consists of a rotation that maps I to $(-\infty, b')$, which can be achieved by noticing that in the circle this is just a rotation of magnitude $-\pi-\theta_a$. Then, $(-\infty, b')$ is mapped by a translation of magnitude b' to $\mathbb{R}_-$.

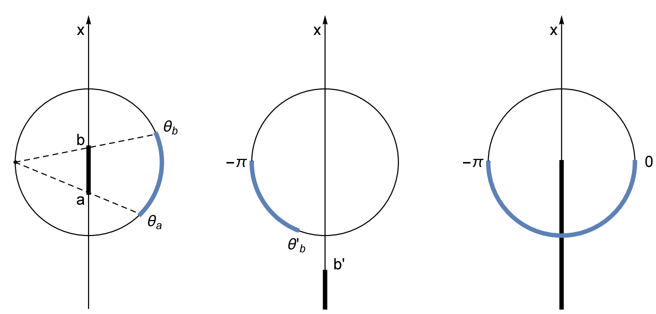

FIG. 1. In the left, a generic interval (a,b) projected to the circle. It is first mapped to $(-\infty, b')$ by a rotation of magnitude $-\pi-\theta_a$, as shown in the middle diagram. Then it is mapped by a translation by b' to $\mathbb{R}_-$, on the right.

Having obtained such $\bar{g}$, we can compute the modular evolution using (36),

$$\Delta^{iu}_{(a,b)} f(x) = f\left(\frac{ae^{-2\pi u}(b-x)+b(x-a)}{e^{-2\pi u}(b-x)+x-a}\right). \quad (40)$$

The modular Hamiltonian is

$$K_{(a,b)} = \frac{2\pi i(x-a)(b-x)}{b-a}\frac{d}{dx}. \quad (41)$$

The last ingredient to compute the relative entropy is the cutting projector P_I associated with the interval $I=(a,b)$. Following the same lines in the proof of Proposition 4.2 in [15] one can see that $P_I K_I f(x) = \chi_I(x) K_I f(x)$ where χ_I is the characteristic function of the interval.

Proposition III.1. Let $f \in C_c^\infty(\mathbb{R})$, then it holds that $P_I K_I f(x) = \chi_I(x) K_I f(x)$.

Proof.—Given $f \in C_c^\infty(\mathbb{R})$ define the functions $g(x) = \chi_I(x) K_I f(x)$ and $g_c(x) = \chi_{I^c}(x) K_I f(x)$, which are piecewise-differentiable functions so their Fourier transforms decay at least as p^{-2} for large p and

$$\|g\|_\tau^2 = \frac{1}{\pi}\Re\int_0^\infty |\hat{g}(p)|^2 p dp < \infty;$$

then $g \in \mathcal{H}$ and also $g_c \in \mathcal{H}$. Moreover, $\sigma(g_c, \varphi) = 0$ for all $\varphi \in C_c^\infty(I)$ because $\mathrm{supp}(g_c) \subseteq I_c$, and then by continuity of σ (with respect to the topology induced by τ), $g_c \in H(I)'$. Similarly one can see that $g \in H(I)$, and therefore, as $H(I)$ is factorial, $P_I K_I f(x) = P_I(g(x) + g_c(x)) = g(x) = \chi_I(x) K_I f(x)$, which completes the proof. ∎

Finally the relative entropy is

$$\begin{aligned} S_{R(I)}(\omega\|\omega_f) &= S_I(f) \\ &= 2\pi\int_a^b \frac{(x-a)(b-x)}{b-a} f'(x)^2 dx. \end{aligned} \quad (42)$$

It is translation invariant, in the sense that $S_I(f) = S_{\tau \cdot I}(f\circ\tau^{-1})$. Is is also immediate to see that, keeping the center $c=(a+b)/2$ of the interval fixed, it is increasing [23] with $L=(b-a)$: $\frac{d}{dL}S_{I(L)}(f) > 0$.

Interestingly, the relative entropy (42) satisfies a Bekenstein-like bound [24]

$$S_I(f) \le \pi\frac{L}{2}\int_a^b f'(x)^2 dx =: \pi\frac{L}{2}E(f) \quad (43)$$

and a QNEC-like bound [25]

$$\begin{aligned} S_I'' &:= \frac{d^2}{dL^2} S_{I(L)}(f) \\ &= \frac{\pi}{2}(f'(b)^2 + f'(a)^2) - \frac{4\pi}{L^3}\int_a^b \left(x - \frac{a+b}{2}\right)^2 f'(x)^2 dx \\ &\ge -\frac{4\pi}{L^3}\int_a^b \left(x-\frac{a+b}{2}\right)^2 f'(x)^2 dx. \end{aligned} \quad (44)$$

The QNEC, when stated in terms of the relative entropy, reads $S''(\lambda) > 0$, with the understanding that λ continuously labels nested spacetime regions. However, in (44) we see a violation of the QNEC, which was anticipated in [15]. What (44) says is that to have a large violation of the QNEC, a considerable amount of energy must be concentrated near the boundaries of the interval (note also that this negative bound can be saturated). Similarly, the Bekenstein-like bound (43) implies that to make a coherent state largely distinguishable from the vacuum, a considerable amount of energy needs to be placed in the interval.

B. The thermal case

We now turn our attention to thermal states. The underlying symplectic space is again $(C_c^\infty(\mathbb{R}), \sigma)$ with (27). The thermal state is defined by

$$\tau_\beta(f,g) = -\pi PV\int_{\mathbb{R}^2} dxdy \frac{f(x)g(y)}{\beta^2\sinh^2\left(\frac{\pi}{\beta}(x-y)\right)}. \quad (45)$$

This gives the two-point function [16]

$$w_2^{(\beta)}(f,g) = -\pi\int_{\mathbb{R}^2}\frac{f(x)g(y)}{\beta^2\sinh^2\left(\frac{\pi}{\beta}(x-y) - i\epsilon\right)}dxdy. \quad (46)$$

Such a two-point function satisfies being translation invariant and the KMS condition with respect to translations. The real Hilbert space $\mathcal{K} = L^2(\mathbb{R}_+, \frac{pdp}{1-e^{-\beta p}})$ is obtained after completion of $C_c^\infty(\mathbb{R})$ with τ_β [15]. Note that this thermal state is the geometric KMS state of [26–28].

Since the state is not pure, we first proceed to "purify." In momentum space $D = -i(1-e^{-\beta p})$, which we use to construct $i^\oplus$ [as given by (20)] and then $\mathcal{H}^\oplus$, the complexification of $\mathcal{K}\oplus\mathcal{K}$.

There are operators acting as the dilation-translation group [29] on the half-lines. For example, on $H(\mathbb{R}_-)$ [16],

$$\begin{aligned} U_\beta(\delta(s))f(x) &= f\left(-\frac{\beta}{2\pi}\log\left(1+e^{-s}(e^{-\frac{2\pi x}{\beta}}-1)\right)\right), \\ U_\beta(\tau(t))f(x) &= f\left(x - \frac{\beta}{2\pi}\log\left(1+\frac{2\pi t}{\beta}e^{\frac{2\pi x}{\beta}}\right)\right), \end{aligned} \quad (47)$$

which satisfy

$$U_\beta(\delta(s))U_\beta(\tau(t))U_\beta(\delta(-s)) = U_\beta(\tau(e^s t)).$$

These operators leave $w_2^{(\beta)}$ invariant, which implies that

$$U_\beta^\oplus(f\oplus 0 + i^\oplus g\oplus 0) := U_\beta f\oplus 0 + i^\oplus U_\beta g\oplus 0, \quad f,g\in\mathcal{K}, \tag{48}$$

are unitaries of $\mathcal{H}^\oplus$ (we show this later on). Because of this, the modular operator associated with $H(\mathbb{R}_-)$ is given by [15,30]

$$\Delta^{iu}_{H(\mathbb{R}_-)} = U_\beta^\oplus(\delta(2\pi u)). \tag{49}$$

Note that this reduces to (37) for $\beta\to\infty$.

1. The interval $(-\infty,t)$

From this last expression, and conjugating with the (vacuum) translation $U(t)$ as in (36), one can compute the modular Hamiltonian associated with $(-\infty,t)$,

$$K_{H((-\infty,t))}f(x) = \beta i^\oplus\left(1-e^{\frac{2\pi}{\beta}(x-t)}\right)f'(x), \tag{50}$$

and the relative entropy

$$S_{R((-\infty,t))}(\omega_f||\omega) = \beta\int_{-\infty}^t\left(1-e^{\frac{2\pi}{\beta}(x-t)}\right)f'(x)^2dx, \tag{51}$$

which was first computed in [15]. Note that in Proposition 5.6 of that reference it is shown that the subspace $H((-\infty,t))$ is both standard and factorial, so the relative entropy can be computed with (19).

2. The interval (a,b)

Now we would like to approach the computation of the relative entropy for the bounded interval $I=(a,b)$. The strategy is analogous to the vacuum case of the previous subsection, but three issues are worth mentioning. First, the subspaces $H(I)$ must be shown to be standard and factorial (which we do at the end). Second the assignment $I\mapsto H(I)$ is not $\mathrm{PSL}(2,\mathbb{R})$-covariant anymore, namely $U_\beta(g)H(I)\neq H(g\cdot I)$. However, we only need to find a $\bar g$ such that

$$U_\beta(\bar g)H(I) = H(\mathbb{R}_-), \tag{52}$$

and then we conjugate with this unitary the modular operator of the negative real line (49) [in complete analogy with (36)]. Explicitly,

$$\Delta_I^{iu} = U(\bar g)^{-1}\Delta^{iu}_{(-\infty,0)}U(\bar g), \qquad u\in\mathbb{R}. \tag{53}$$

Third, the attempt to construct $\bar g$ as described in the vacuum case (see Fig. 1) is not immediate to generalize, since the unitary rotation is no longer available (the vacuum rotation does not leave $w_2^{(\beta)}$ invariant). In [16] the authors find the unitary dilations and translations (47). We need to find a unitary operator that works as a rotation, meaning that *it does not fix* ∞ (in the real-line picture). We propose that there exists $\alpha(\theta,x)$ such that

$$U_\beta(r(\theta))f(x) = f(\alpha(\theta,x)). \tag{54}$$

This means $\alpha(\theta,x)$ should obey the following three conditions:

(1) Identity: $\alpha(0,x)=x$.
(2) One-parameter group: $\alpha(\theta_1,\alpha(\theta_2,x)) = \alpha(\theta_1+\theta_2,x)$.
(3) $w_2^{(\beta)}$ compatibility:

$$\frac{\partial\alpha(\theta,x)}{\partial x}\frac{\partial\alpha(\theta,y)}{\partial y}\sinh\left(\frac{\pi}{\beta}(\alpha(\theta,x)-\alpha(\theta,y))\right)^{-2} = \sinh\left(\frac{\pi}{\beta}(x-y)\right)^{-2}.$$

Of course, $\alpha(\theta,x)$ also depends on β. The third condition, together with (48), assures that the operator $U_\beta^\oplus$ induced by $U_\beta(r(\theta))$ is unitary. Let us see why:

$$\begin{aligned}
&\langle U_\beta^\oplus(f_1\oplus 0+i^\oplus g_1\oplus 0), U_\beta^\oplus(f_2\oplus 0+i^\oplus g_2\oplus 0)\rangle^\oplus = \langle U_\beta(r(\theta))f_1\oplus 0 + i^\oplus U_\beta(r(\theta))g_1\oplus 0,\\
&U_\beta(r(\theta))f_2\oplus 0+i^\oplus U_\beta(r(\theta))g_2\oplus 0\rangle^\oplus\\
&\quad = w_2^{(\beta)}(f_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot)) + w_2^{(\beta)}(g_1\circ\alpha(\theta,\cdot),g_2\circ\alpha(\theta,\cdot)) + \tau_\beta(f_1\circ\alpha(\theta,\cdot),-Dg_2\circ\alpha(\theta,\cdot))\\
&\qquad + i\tau_\beta(f_1\circ\alpha(\theta,\cdot),g_2\circ\alpha(\theta,\cdot)) + \tau_\beta(-Dg_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot)) - i\tau_\beta(g_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot))\\
&\quad = w_2^{(\beta)}(f_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot)) + w_2^{(\beta)}(g_1\circ\alpha(\theta,\cdot),g_2\circ\alpha(\theta,\cdot)) - \sigma(f_1\circ\alpha(\theta,\cdot),g_2\circ\alpha(\theta,\cdot))\\
&\qquad + i\tau_\beta(f_1\circ\alpha(\theta,\cdot),g_2\circ\alpha(\theta,\cdot)) + \sigma(g_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot)) - i\tau_\beta(g_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot))\\
&\quad = w_2^{(\beta)}(f_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot)) + w_2^{(\beta)}(g_1\circ\alpha(\theta,\cdot),g_2\circ\alpha(\theta,\cdot)) + iw_2^{(\beta)}(f_1\circ\alpha(\theta,\cdot),g_2\circ\alpha(\theta,\cdot))\\
&\qquad - iw_2^{(\beta)}(g_1\circ\alpha(\theta,\cdot),f_2\circ\alpha(\theta,\cdot))\\
&\quad = w_2^{(\beta)}(f_1,f_2) + w_2^{(\beta)}(g_1,g_2) + iw_2^{(\beta)}(f_1,g_2) - iw_2^{(\beta)}(g_1,f_2)\\
&\quad = \langle f_1\oplus 0+i^\oplus g_1\oplus 0, f_2\oplus 0+i^\oplus g_2\oplus 0\rangle^\oplus,
\end{aligned} \tag{55}$$

where in the fifth equality we used the $w_2^{(\beta)}$-compatibility condition.

To find $\alpha(\theta,x)$, there is a hint coming from the $\mathrm{PSL}(2,\mathbb{R})$ product rules:

$$\delta(s)r(\theta)\delta(-s)=r(2\arctan(e^{-s}\lambda))\delta\left(\log\left[\frac{1+e^{-2s}\lambda^2}{1+\lambda^2}\right]\right)\times\tau\left(\frac{2\sinh(s)\lambda}{1+e^{-2s}\lambda^2}\right),$$

where $\lambda=\tan\frac{\theta}{2}$. This translates, by means of (47) and (54), into a functional equation

$$(e^s+e^{-s}\lambda^2)(e^{\frac{2\pi}{\beta}\alpha(\theta,\phi(s,x))}-1)=(1+\lambda^2)(e^{\frac{2\pi}{\beta}\alpha(2\arctan(e^{-s}\lambda),x)}-1)-\frac{4\pi}{\beta}\sinh(s)\lambda \tag{56}$$

with

$$\phi(s,x)=\frac{\beta}{2\pi}\log\left(1+e^{-s}(e^{\frac{2\pi x}{\beta}}-1)\right). \tag{57}$$

It is convenient to work with $A(\lambda,x)$ defined by

$$\alpha(\theta,x)=\frac{\beta}{2\pi}\log\left[1+A(\lambda,x)\right]. \tag{58}$$

Differentiating with respect to s and setting $s=0$ we get a partial differential equation

$$\frac{\lambda^2-1}{\lambda^2+1}A(\lambda,x)+\frac{\beta}{2\pi}\left(1-e^{-\frac{2\pi}{\beta}x}\right)\partial_x A(\lambda,x)=\lambda\partial_\lambda A(\lambda,x)+\frac{4\pi}{\beta}\frac{\lambda}{\lambda^2+1}, \tag{59}$$

which has infinite solutions of the form

$$A(\lambda,x)=\frac{2\pi}{\beta}\left[-\lambda+\frac{1+\lambda^2}{\lambda}B\left(\lambda\left(e^{\frac{2\pi}{\beta}x}-1\right)\right)\right] \tag{60}$$

for any function B. From $\alpha(0,x)=x$ we get

$$B(z)\sim\frac{\beta}{2\pi}z,\qquad z\to 0. \tag{61}$$

From this and condition 2 above (group property) evaluated at $x=0$ we get

$$B(z)=\frac{z}{z+\frac{2\pi}{\beta}}. \tag{62}$$

Now plugging this form of B into (60) and taking into account (54) and (58), $U_\beta(r(\theta))f=f(\alpha(\theta,\cdot))$ can be shown to be compatible with $w_2^{(\beta)}$ (condition 3 above) where

$$\alpha(\theta,x)=\frac{\beta}{2\pi}\log\left[1+A(\lambda,x)\right],$$
$$A(\lambda,x)=\frac{2\pi}{\beta}\frac{e^{\frac{2\pi}{\beta}x}-1-\frac{2\pi}{\beta}\lambda}{\lambda\left(e^{\frac{2\pi}{\beta}x}-1\right)+\frac{2\pi}{\beta}}. \tag{63}$$

Let us see this: first of all we rewrite the $w_2^{(\beta)}$-compatibility condition

$$\sinh^2\left(\frac{\pi}{\beta}(\alpha(\theta,x)-\alpha(\theta,y))\right)=\frac{\partial\alpha(\theta,x)}{\partial x}\frac{\partial\alpha(\theta,y)}{\partial y}\sinh^2\left(\frac{\pi}{\beta}(x-y)\right).$$

A straightforward computation [using $e^{\frac{\pi}{\beta}\alpha(\theta,x)}=(1+A(\lambda,x))^{\frac{1}{2}}$] of the square root of the left-hand side gives

$$\sinh\left(\frac{\pi}{\beta}(\alpha(\theta,x)-\alpha(\theta,y))\right)=\frac{1}{2}\left[\frac{(1+A(\lambda,x))^{\frac{1}{2}}}{(1+A(\lambda,y))^{\frac{1}{2}}}-\frac{(1+A(\lambda,y))^{\frac{1}{2}}}{(1+A(\lambda,x))^{\frac{1}{2}}}\right].$$

Squaring this expression and with (63),

$$\sinh^2\left(\frac{\pi}{\beta}(\alpha(\theta,x)-\alpha(\theta,y))\right)=\frac{1}{4}\left[\frac{1+A(\lambda,x)}{1+A(\lambda,y)}-\frac{1+A(\lambda,y)}{1+A(\lambda,x)}-2\right]=Y(\lambda,x)Y(\lambda,y)\sinh^2\left(\frac{\pi}{\beta}(x-y)\right), \tag{64}$$

where

$$Y(\lambda,x)=\frac{4\pi^2\beta(\lambda^2+1)e^{\frac{2\pi x}{\beta}}}{(\beta\lambda(e^{\frac{2\pi x}{\beta}}-1)+2\pi)(\beta(\beta\lambda+2\pi)e^{\frac{2\pi x}{\beta}}-(\beta^2+4\pi^2)\lambda)}.$$

But it turns out that a straightforward computation gives

$$\frac{\partial\alpha}{\partial x}(\theta,x)=Y(\lambda,x),$$

which means that the $w_2^{(\beta)}$-compatibility condition holds.

Having found a unitary rotation, we can implement the first transformation of Fig. 1 with $U_\beta(r(\tilde{\theta}))$ where

$$\tilde{\theta}=2\arctan\left(-\frac{2\pi}{\beta}\frac{e^{\frac{2\pi a}{\beta}}}{e^{\frac{2\pi a}{\beta}}-(\frac{2\pi}{\beta})^2-1}\right), \tag{65}$$

and taking into account that the corresponding unitary $U_\beta^\oplus(r(\theta))$ on $\mathcal{H}^\oplus$ is defined by (48). This rotation sends a to $-\infty$ and b to $b' = \alpha(\tilde{\theta}, b)$, so it maps $H(I)$ to $H((-\infty, b'))$. Then, by a unitary vacuum translation $U(-b')$, $H((-\infty, b'))$ is mapped to $H(\mathbb{R}_-)$ as desired. The unitary $U(\bar{g})$ is the composition of these two unitary transformations.

From (53), the modular evolution on $H(I) \oplus 0$ is

$$\Delta_{(a,b)}^{iu} f(x) \oplus 0 = f\left[\frac{\beta}{2\pi}\log\left(\frac{\sinh(\pi u)e^{\frac{\pi}{\beta}(a+b-x)} - \sinh(\frac{\pi}{\beta}(b-a)+\pi u)e^{\frac{\pi x}{\beta}}}{-\sinh(\pi u)e^{-\frac{\pi}{\beta}(a+b-x)} + \sinh(-\frac{\pi}{\beta}(b-a)+\pi u)e^{-\frac{\pi x}{\beta}}}\right)\right] \oplus 0. \tag{66}$$

By differentiating, the modular Hamiltonian is

$$K_{(a,b)} f(x) \oplus 0 = 2\beta i^\oplus \frac{\sinh(\frac{\pi}{\beta}(x-a))\sinh(\frac{\pi}{\beta}(b-x))}{\sinh(\frac{\pi}{\beta}(b-a))} f'(x) \oplus 0. \tag{67}$$

It coincides with (50) in the limit $a \to -\infty$ and with (41) for $\beta \to \infty$. Again, as in the vacuum case now one can still see that $P_I K_I f(x) = \chi_I(x) K_I f(x)$. The proof is similar to the one we showed above; the only difference is that this time we have to see that g has a finite τ_β norm, but the same argument works. Given $f \in C_c^\infty(\mathbb{R})$, define once again the functions $g(x) = \chi_I(x) K_I f(x)$ and $g_c(x) = \chi_{I^c}(x) K_I f(x)$. These are piecewise-differentiable functions so its Fourier transform is bounded, decays at least like p^{-2} for large p, and then

$$\|g\|_{\tau_\beta}^2 = \frac{1}{\pi}\Re\int_0^\infty \frac{|\hat{g}(p)|^2 p}{1-e^{-\beta p}} dp < \infty.$$

Finally, the relative entropy is given by

$$S_{R(I)}(\omega_f||\omega) = 2\beta \int_a^b \frac{\sinh(\frac{\pi}{\beta}(x-a))\sinh(\frac{\pi}{\beta}(b-x))}{\sinh(\frac{\pi}{\beta}(b-a))} f'(x)^2 dx, \tag{68}$$

which is our main result. This relative entropy coincides, modulo some factor, with the modular Hamiltonian in the cutoff theory [Eq. (4.2) in [31]]. This can be formally understood by first noticing that the relative entropy can be related to a difference of mean values of the modular Hamiltonian K and a difference of entanglement entropies,

$$S(\omega_2||\omega_1) = (\langle K_1\rangle_2 - \langle K_1\rangle_1) - (S_2 - S_1). \tag{69}$$

In our case the last parentheses is zero since one state is a unitary applied to the other state (in the vector representation). This explains the connection of (68) to the modular Hamiltonian of [31]. Since the arguments of [31] are of general validity within CFTs, and taking into account the above discussion, it is reasonable to expect that in general (68) will hold with $f'(x)^2$ replaced by the classical energy density $T_{00}(x)$ of the theory. We will confirm this expectation in the next section for the massless scalar QFT in $1+1$ dimensions.

Identically to the vacuum case (42), the relative entropy (68) is translation invariant and with a positive derivative with respect to the length L of the interval, keeping the center $(a+b)/2$ fixed. There is also a Bekenstein-like bound

$$\begin{aligned} S_{(a,b)}(f) &\leq \pi\frac{L}{2}\left(\frac{\tanh(\frac{\pi}{\beta}\frac{L}{2})}{\frac{\pi}{\beta}\frac{L}{2}}\right)\int_a^b f'(x)^2 dx \\ &\leq \pi\frac{L}{2}\int_a^b f'(x)^2 dx, \end{aligned} \tag{70}$$

and a QNEC-like bound [32]

$$\begin{aligned} \frac{d^2}{dL^2} S_{I(L)}(f) \geq -\frac{\pi^2}{\beta\sinh^3(L\frac{\pi}{\beta})}\int_a^b \Bigg[\left(\cosh\left(\frac{\pi}{\beta}L\right) - 1\right)^2 \\ + 2\sinh^2\left(\frac{\pi}{\beta}(x-c)\right)\left(1+\cosh^2\left(\frac{\pi}{\beta}L\right)\right)\Bigg] \\ \times f'(x)^2 dx, \end{aligned} \tag{71}$$

where $c = (a+b)/2$. We shall discuss this expression later on.

Before concluding this section we have to show that $H(I)$ is standard and factorial so the machinery we have been using, and in particular (19), is valid. We do this in the following Proposition.

Proposition III.3. $H(I)$ is standard and factorial.

Proof.—The condition of separability $H(I) \cap i^\oplus H(I) = 0$ follows exactly as in Proposition 5.6 of [15] (or with the logic for what follows). The cyclicity, $(H(I) + i^\oplus H(I))^\perp = 0$, can be shown to hold using the unitary rotation (54). Given any $H((a,b))$ there is an associated subspace $H_{b'} \coloneqq H((-\infty, b')) = U_\beta^\oplus(r(\tilde{\theta}))H(I)$ obtained by a rotation in $\tilde{\theta}$ given by (65) and explained after that equation. The subspace $H_{b'}$ is, by Proposition 5.6 of [15], standard and factorial. It is immediate to show that $0 = (H_{b'} + i^\oplus H_{b'})^\perp = U_\beta^\oplus(r(\tilde{\theta}))(H(I) + i^\oplus H(I))^\perp$ which implies that $H(I)$ is cyclic. Similarly, we conclude that $H(I)$ is factorial since $0 = H_{b'} \cap H'_{b'} = U_\beta^\oplus(r(\tilde{\theta}))(H(I) \cap H(I)')$. ■

IV. THE FREE MASSLESS BOSON IN 1 + 1 DIMENSIONS AT FINITE TEMPERATURE

In this section we take advantage of the quantities we have computed for the chiral boson and combine the two chiralities to obtain the modular flow, modular Hamiltonian, and relative entropy on the interval (a, b) for the massless free boson in two dimensions Φ.

First of all, let us define $x^\pm = t \pm x$, and $j^\pm(x^\pm) = \partial_\pm \phi^\pm(x^\pm)$ are the (nonsmeared) chiral currents of the previous section (below we give further details). In this section we will use $\pm$ symbols to denote copies of the objects of the chiral case (with the exception of the symplectic structure σ and the bilinear form τ). So, for example, $\mathcal{H}$ now refers to a Hilbert space of the two-dimensional model, and $\mathcal{H}_\pm$ are Hilbert spaces of the chiral case.

The symplectic space of the massless boson in two dimensions is [8,33]

$$\mathcal{K} = C_c^\infty(\mathbb{R}) \oplus \dot{C}_c^\infty(\mathbb{R}) \tag{72}$$

with symplectic structure

$$\sigma_{2D}((f_1, g_1), (f_2, g_2)) = \frac{1}{2}\int_{\mathbb{R}} dx (g_1(x) f_2(x) - f_1(x) g_2(x)). \tag{73}$$

Here the pair $(f, g) \in \mathcal{K}$ should be thought as the initial conditions $\Phi(0, x) = f(x)$, $\dot{\Phi}(0, x) = g(x)$ of a solution $\Phi(t, x)$ of the Klein-Gordon equation. In general,

$$\Phi(t, x) = \phi_+(x^+) + \phi_-(x^-), \qquad \phi^\pm \in C_c^\infty(\mathbb{R}). \tag{74}$$

Then, the symplectic structure (73) can be written as

$$\begin{aligned}\sigma_{2D}&((f_1, g_1), (f_2, g_2)) \\ &= -\int_{\mathbb{R}} dx (\phi_1^+(x) \phi_2^{+\prime}(x) + \phi_1^-(x) \phi_2^{-\prime}(x)).\end{aligned} \tag{75}$$

The lack of mixing between the chiralities implies that there is a symplectic isomorphism [34] χ that maps $(\mathcal{K}, \sigma_{2D})$ to $(\mathcal{K}_- \oplus \mathcal{K}_+, -(\sigma \oplus \sigma))$, with the inverse given by

$$\chi^{-1}\begin{pmatrix}\phi_+ \\ \phi_-\end{pmatrix} = \begin{pmatrix}\phi_+(x) + \phi_-(-x) \\ \phi'_+(x) + \phi'_-(-x)\end{pmatrix} = \begin{pmatrix} f(x) \\ g(x)\end{pmatrix}. \tag{76}$$

In turn, this implies that the $\mathrm{CCR}(\mathcal{K}, \sigma_{2D})$ algebra is equivalent to the tensor product

$$\mathrm{CCR}(\mathcal{K}_-, -\sigma) \otimes \mathrm{CCR}(\mathcal{K}_+, -\sigma),$$

with σ as in (27). More precisely, we identify these CCR-algebras by

$$\begin{aligned}&W(\phi_-(x)) \otimes W(\phi_+(x)) \\ &\quad \mapsto W((\phi_+(x) + \phi_-(-x), \phi'_+(x) + \phi'_-(-x))).\end{aligned} \tag{77}$$

This is, in fact, a $*$ isomorphism of the algebras. The change in sign in the symplectic structure σ with respect to the previous section requires a change in sign in the complex structure [35], and these two signs end up compensating each other in the relative entropy [36] (19).

Given a positive symmetric bilinear form τ_{2D} on $\mathcal{K}$ and its corresponding quasifree state on $\mathrm{CCR}(\mathcal{K}, \sigma_{2D})$, by the isomorphisms mentioned above we get a quasifree product state on $\mathrm{CCR}(\mathcal{K}_+, \sigma) \otimes \mathrm{CCR}(\mathcal{K}_-, \sigma)$ with the same τ for each chiral copy. Therefore the vacuum one-particle Hilbert space is

$$\mathcal{H} \simeq \mathcal{H}_- \oplus \mathcal{H}_+, \tag{78}$$

where $\mathcal{H}_\pm$ are copies of the chiral boson Hilbert space $L^2(\mathbb{R}_+, pdp)$. The isomorphism (76) is antilinear, since in momentum space (or coordinate space, using properties of the Hilbert transform $\mathfrak{H}$) it is direct to show that

$$\chi^{-1} i_1 = -i_2 \chi^{-1}, \tag{79}$$

where i_1 is the complex structure of the chiral boson and i_2 is the complex structure in [7,8]

$$i_2 \coloneqq \begin{pmatrix} 0 & |p|^{-1} \\ -|p| & 0\end{pmatrix}. \tag{80}$$

Therefore,

$$\begin{aligned}\tau_{2D}(\Phi, \Psi) &= \sigma_{2D}(\Phi, i_2\Psi) \\ &= -\sigma(\phi_+, (\chi i_2 \Psi)_+) - \sigma(\phi_-, (\chi i_2 \Psi)_-) \\ &= -\tau(\phi_+, -i_1(\chi i_2 \Psi)_+) - \tau(\phi_-, -i_1(\chi i_2 \Psi)_-) \\ &= \tau(\phi_+, i_1(\chi i_2 \Psi)_+) + \tau(\phi_-, i_1(\chi i_2 \Psi)_-) \\ &= \tau(\phi_+, \psi_+) + \tau(\phi_-, \psi_-).\end{aligned} \tag{81}$$

Analogously, for the thermal state we have

$$\mathcal{H}^\oplus \simeq \mathcal{H}_-^\oplus \oplus \mathcal{H}_+^\oplus \tag{82}$$

with $\mathcal{H}_\pm^\oplus$ two copies of the purified Hilbert space that we constructed in the previous section (which was called $\mathcal{H}^\oplus$, and we hope there is no confusion). The Fock spaces are related as

$$\Gamma(\mathcal{H}^\oplus) \simeq \Gamma(\mathcal{H}_-^\oplus) \otimes \Gamma(\mathcal{H}_+^\oplus). \tag{83}$$

From now on we identify all these spaces with the appropriate isomorphisms.

A. Modular flow and modular Hamiltonian

Let us consider a causal diamond with base (a,b) on the time-zero surface. Its corresponding standard subspace is $H(\Diamond)$ of pairs $(f,g)\in\mathcal{K}$ supported on the interval (a,b) or equivalently Klein-Gordon (KG) fields Φ with initial conditions given by (f,g). Note that such a diamond is described in null coordinates as $(x^-,x^+)\in((-b,-a),(a,b))$. Right wedges are obtained in limit $b\to\infty$ and similarly $a\to-\infty$ for left wedges.

At the one-particle level, we have

$$K_{H(\Diamond)}\simeq K_{H((-b,-a))}\oplus K_{H((a,b))}. \tag{84}$$

This follows from the fact that for $\Phi,\Psi\in H(\Diamond)$,

$$\begin{aligned}S_{H(\Diamond)}(\Phi+i\Psi)&=\Phi-i\Psi\\&=\phi_+-i\psi^++\phi_--i\psi^-\\&=S_{H((-b,-a))}(\phi_-+i\psi^-)\\&\quad+S_{H((a,b))}(\phi_++i\psi^+),\end{aligned}\tag{85}$$

implying that $S_{H(\Diamond)}\simeq S_{H((-b,-a))}\oplus S_{H((a,b))}$ and then $\Delta_{H(\Diamond)}\simeq\Delta_{H((-b,-a))}\oplus\Delta_{H((a,b))}$. The modular evolution in the diamond,

$$\Delta^{iu}_{H(\Diamond)}\simeq\Delta^{iu}_{H((-b,-a))}\oplus\Delta^{iu}_{H((a,b))}, \tag{86}$$

which explicitly reads

$$\left[\Delta^{iu}_{H(\Diamond)}\begin{pmatrix}f\\g\end{pmatrix}\oplus\begin{pmatrix}0\\0\end{pmatrix}\right](x)=\begin{pmatrix}[\Delta^{iu}_{H((-b,-a))}\phi_-](-x)+[\Delta^{iu}_{H((a,b))}\phi_+](x)\\ [\Delta^{iu}_{H((-b,-a))}\phi_-]'(-x)+[\Delta^{iu}_{H((a,b))}\phi_+]'(x)\end{pmatrix}\oplus\begin{pmatrix}0\\0\end{pmatrix}, \tag{87}$$

where $\phi_\pm$ should be thought as given in terms of (f,g) using the isomorphism χ and the evolution of each chirality is given in (66). A more intuitive presentation of the modular flow is to show the geometric transformation of the coordinates (t,x) inside the diamond, as in Fig. 2.

B. Relative entropies and bounds

The relative entropy in two dimensions is the sum of the relative entropies of the chiral copies, which follows from (75) and (84). Before arriving to an explicit expression of the relative entropies for different cases, we first find the modular Hamiltonians.

1. The wedge

On a right wedge $W_R(a)$ with base (a,∞), given (76) and (84), we have the corresponding vacuum modular Hamiltonian acting on the initial conditions

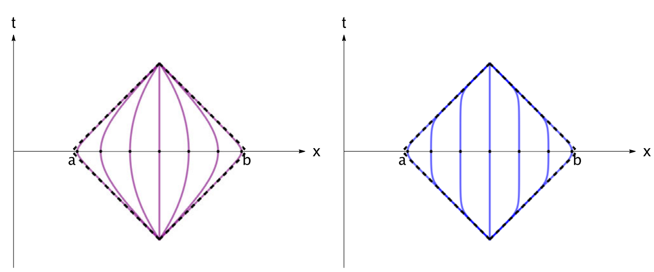

FIG. 2. Modular flow for low temperature (left) and high temperature (right).

$$\begin{aligned}K_{H((W_R(a))}\begin{pmatrix}f\\g\end{pmatrix}&=\begin{pmatrix}(K_{H((-\infty,-a))}\phi_-)(-x)+(K_{H((a,\infty))}\phi_+)(x)\\(K_{H((-\infty,-a))}\phi_-)'(-x)+(K_{H((a,\infty))}\phi_+)'(x)\end{pmatrix}\\&=-2\pi i\begin{pmatrix}(-a+x)\phi_-'(-x)+(x-a)\phi_+'(x)\\((-a-x)\phi_-')'(-x)+((x-a)\phi_+')'(x)\end{pmatrix}\\&=-2\pi i\begin{pmatrix}(-a+x)\phi_-'(-x)+(x-a)\phi_+'(x)\\((a-x)\phi_-'(-x)+(x-a)\phi_+'(x))'\end{pmatrix}\\&=-2\pi i\begin{pmatrix}(x-a)g(x)\\((x-a)f'(x))'\end{pmatrix},\end{aligned}\tag{88}$$

where in the second line we made use of the antilinearity between the chiral spaces $\mathcal{H}_\pm$ and $\mathcal{H}$. Then,

$$K_{H(W_R(a))}=-2\pi i\begin{pmatrix}0 & x-a\\ \frac{d}{dx}(x-a)\frac{d}{dx} & 0\end{pmatrix}. \tag{89}$$

Plugging this modular Hamiltonian in (19),

$$S_{H(W_R(a))}((f,g))=2\pi\int_a^\infty(x-a)T_{00}(x)dx, \tag{90}$$

with

$$T_{00}(x)=\frac{1}{2}(f'(x)^2+g(x)^2) \tag{91}$$

the classical energy density at $t = 0$ of the KG field Φ. This is the same result as that of [4], with a translation by a.

Similarly, for the thermal state we have

$$K^{(\beta)}_{H(W_R(a))}\begin{pmatrix} f \\ g \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} = -\beta i^{\oplus}\begin{pmatrix} (1-e^{-\frac{2\pi}{\beta}(x-a)})g(x) \\ [(1-e^{-\frac{2\pi}{\beta}(x-a)})f'(x)]' \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{92}$$

where we have used (50) and (67) in the limit $b \to \infty$. The relative entropy on the wedge at finite temperature is then

$$S^{(\beta)}_{H(W_R(a))}((f,g)) = \beta\int_a^{\infty}(1-e^{-\frac{2\pi}{\beta}(x-a)})T_{00}(x)dx. \tag{93}$$

Note that this expression is valid even for initial conditions supported outside $x > a$, since the cutting projector in (19) restricts the integral to the wedge [7]. The only restriction on the initial conditions (f, g) is that they belong to the domain of the modular Hamiltonian (92).

2. The interval

Repeating the previous computations for the time-zero interval (a, b), we obtain for the vacuum

$$K_{H((a,b))}\begin{pmatrix} f \\ g \end{pmatrix} = -2\pi i\begin{pmatrix} \frac{(b-x)(x-a)}{b-a}g(x) \\ [\frac{(b-x)(x-a)}{b-a}f'(x)]' \end{pmatrix}. \tag{94}$$

The vacuum relative entropy of a coherent state is

$$S_{H((a,b))}((f,g)) = 2\pi\int_a^b\frac{(b-x)(x-a)}{b-a}T_{00}(x)dx. \tag{95}$$

On the other hand, at finite temperature we have

$$K^{\beta}_{H((a,b))}\begin{pmatrix} f \\ g \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} = -2\beta i^{\oplus}\begin{pmatrix} \frac{\sinh(\frac{\pi}{\beta}(x-a))\sinh(\frac{\pi}{\beta}(b-x))}{\sinh(\frac{\pi}{\beta}(b-a))}g(x) \\ [\frac{\sinh(\frac{\pi}{\beta}(x-a))\sinh(\frac{\pi}{\beta}(b-x))}{\sinh(\frac{\pi}{\beta}(b-a))}f'(x)]' \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{96}$$

The relative entropy of a coherent state in the thermal state representation is

$$S^{(\beta)}_{H((a,b))}((f,g)) = 2\beta\int_a^b\frac{\sinh(\frac{\pi}{\beta}(x-a))\sinh(\frac{\pi}{\beta}(b-x))}{\sinh(\frac{\pi}{\beta}(b-a))} \times T_{00}(x)dx. \tag{97}$$

This expression confirms, at least for this model, the expectation that in a CFT the relative entropy of coherent states on a finite interval has this form, where the dependence on the model enters only in T_{00}. Because of this, the bounds obtained earlier for the chiral model also hold in this case. The Bekenstein-like bound reads

$$S_{(a,b)}((f,g)) \le \pi\frac{L}{2}\left(\frac{\tanh(\frac{\pi}{\beta}\frac{L}{2})}{\frac{\pi}{\beta}\frac{L}{2}}\right)\int_a^b T_{00}(x)dx \le \pi\frac{L}{2}\int_a^b T_{00}(x)dx, \tag{98}$$

while the QNEC-like bound is

$$\frac{d^2}{dL^2}S_{I(L)}((f,g)) \ge -\frac{\pi^2}{\beta\sinh^3(L\frac{\pi}{\beta})}\int_a^b\left[\left(\cosh\left(\frac{\pi}{\beta}L\right)-1\right)^2 + 2\sinh^2\left(\frac{\pi}{\beta}(x-c)\right)\left(1+\cosh^2\left(\frac{\pi}{\beta}L\right)\right)\right] \times T_{00}(x)dx. \tag{99}$$

V. CONCLUSIONS

We have extended the relative entropy on $\mathbb{R}_-$ with $T \ge 0$ of [15] to a bounded interval [see (68)]. To achieve this, we found a unitary in the thermal Hilbert space implementing a rotation. Such a unitary may turn out to be useful for other related computations.

From the relative entropy (68) a Bekenstein-like bound and a QNEC-like bound can be observed. There is, however, a violation of the QNEC $S'' > 0$, and all of this is in agreement with [15]. For the vacuum case, given an energy E we can find a family of functions $f_n \in H(I)$ such that $S''_I(f_n)$ given in (44) goes to zero (just concentrating the energy density closer and closer around the center of the interval), thus making the QNEC violation as small as desired. On the contrary, in the thermal case this is not possible because there is always a bound for the violation of the QNEC given by

$$S''_I(f_n) \to -\frac{\pi^2}{\beta\sinh^3(L\frac{\pi}{\beta})}\left(\cosh\left(\frac{\pi}{\beta}L\right)-1\right)^2 E < 0$$

despite how the energy density is distributed [see (71)].

The computations in the context of a thermal $U(1)$ current left a clear path to analyze the case of a thermal state of the free massless boson in $1+1$ dimensions restricted to a causal diamond. In the last section we obtained the modular Hamiltonian (96) and relative entropy (97) at finite temperature in $1+1$ dimensions, with analogous bounds as in the chiral case. In principle most of these techniques could be used for the massless boson in higher dimensions and also the free massive boson in $d+1$ dimensions with $T > 0$ [37]. In addition, it would be very interesting to extend the formalism to include noncoherent states, although this seems a much more complicated affair.

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[19] The relative entropy between coherent states satisfies $S(\omega_f||\omega_g) = S(\omega_{f-g}||\omega)$, so there is no loss of generality in assuming that one state is not excited by a Weyl unitary.

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[21] We assume that $\mathcal{K}$ is complete with respect to τ and that σ is non-degenerate.

[22] We are taking the Fourier transform as $\hat{f}(p) = \int_{\mathbb{R}} dx e^{ixp} f(x)$. With this convention the Hilbert transform in momentum space is $i\ \mathrm{sgn}(p)$. More importantly, the generator of unitary translations (33) is positive.

[23] This is just a check of the monotonicity of the realtive entropy: Enlarging the von Neumann algebra makes the relative entropy larger, i.e. $S_A \leq S_B$ if $A \subset B$ with A and B von Neumann algebras.

[24] Here $E(f) \coloneqq \int_a^b f'(x)^2 dx$ is the 1-particle energy associated to the interval of $f \in \mathcal{H}$.

[25] This bound (for the interval centered at 0) is the one in Proposition 3.7 of [15], where the appropriate $T_f(s, L)$ is $2\pi \int_{-\min\{s,\frac{L}{2}\}}^{\min\{s,\frac{L}{2}\}} \frac{(x+\frac{L}{2})(\frac{L}{2}-x)}{L} f'(x)^2 dx$. It is straightforward to see that such T_f satisfies the smoothness hypothesis C^1 of the Proposition.

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[29] We are not claiming these operators form a representation on $\mathcal{K}$. For instance, there are values of the parameters where the logarithms are not defined and the function is instead defined to be zero. We will not need to take this into account.

[30] We have a sign difference with respect to [15] in the parameter inside the dilation δ. This translates into a sign difference in the modular Hamiltonian, but then the relative entropy coincides with equation (5.22) of that reference.

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[32] Again, this the same as the result of Proposition 3.7 of [15], this time $T_f(s,L)$ is $2\beta\int_{-\min\{s,\frac{L}{2}\}}^{\min\{s,\frac{L}{2}\}}\frac{\sinh\frac{\pi}{\beta}(x+\frac{L}{2})\sinh\frac{\pi}{\beta}(\frac{L}{2}-x)}{\sinh\frac{\pi L}{\beta}}f'(x)^2dx$. It is straightforward to see that such T_f satisfies the smoothness hypothesis of the Proposition.

[33] Here we are defining $f\in\dot{C}_c^\infty(\mathbb{R})$ if $f\in C_c^\infty(\mathbb{R})$ and $\hat{f}(0)=0$. This is necessary to avoid the well-known IR problem of the massless 2-dimensional field [38].

[34] It is most easily written in Fourier space: $\hat{\phi}_\pm(\pm p)=\frac{1}{2}(\hat{f}(p)\pm\frac{i}{p}\hat{g}(p))$.

[35] In order to see this, note that τ is independent of this change in sign, since it must be positive. Therefore from the defining equation of the complex structure $\tau(\cdot,D\cdot)=\sigma(\cdot,\cdot)$ it is seen that a change in sign in σ translates into a change in sign in D and therefore in the complex structure.

[36] The 1-particle modular Hamiltonian K is not affected by this sign change, since S is not affected as seen by its definition and neither is S^*, therefore $\Delta=S^*S$ is not affected.

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