Quantum correlations of a two-dimensional electron gas with Rashba spin-orbit coupling

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We study the correlations of a two-dimensional electron gas with Rashba spin-orbit coupling (SOC). We obtain the two-particle density matrix and use it to derive the exchange hole. We find a nontrivial correlation for electrons with opposite spin projections that does not occur without Rashba SOC. The two-particle density matrix allows us to further study the quantum correlations of the system. We use it to obtain the concurrence and the entanglement of formation in order to quantify the entanglement of the electron spins. Additionally, we calculate the quantum discord and compare it with the entanglement and classical correlations.

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I. INTRODUCTION

Entanglement is a key concept in quantum mechanics that has been extensively studied in the context of quantum information and computing [1,2]. It has also emerged as a significant phenomenon in many-body physics [3], encompassing various aspects such as quantum spin systems [4–6], the Kondo effect [7,8], the fractional quantum Hall effect [9–11], and spins of a noninteracting electron gas [12,13], among others. Correlation functions are essential in describing the physical phenomena of many-body systems; therefore, it is logical to investigate the connection between entanglement and correlation functions.

Quantum discord [14,15] is another type of quantum correlation that measures the difference between quantum mutual information and classical correlations. This correlation has been demonstrated to be useful for certain quantum technology tasks [16,17], and it is also of theoretical interest, as it characterizes quantum correlations using a different approach than the traditional entangled versus separable state classification. It has also been useful for studying the degree of correlation in some many-body systems [18–20].

Another topic of great interest is the Rashba effect [21–27], which is a type of spin-orbit coupling (SOC) that occurs in nanostructures that lack structural inversion symmetry. In the growing research field of spintronics [28] the Rashba SOC is a fundamental tool which allows precise control of the electron spin using electric fields. Since this system shares the many-body nature of electron gases, it is of basic interest to study the correlations in this context.

An important concept in many-body physics is the exchange hole for fermions, which arises due to the Pauli exclusion principle. This basic type of correlation exists even in the absence of interaction between particles. The exchange hole can be obtained from the two-particle density matrix of the system, as it results from the correlation between two electrons. More recently, several correlation measures have been developed in quantum information theory to study bipartite two-level states [29]. In this paper, we investigate the traditional concept of the exchange hole in many-body physics, as well as modern measures of correlation used in quantum information theory, for a two-dimensional electron gas with Rashba SOC.

This paper is organized as follows. In Sec. II we obtain the two-particle density matrix of a two-dimensional electron gas (2DEG) with Rashba SOC. In Sec. III we derive and calculate the spin pair correlation function and the Hamiltonian eigenbasis pair correlation function, and we discuss the possible experimental detection of the described effects. In Sec. IV, we further study the two-particle spin density matrix using concurrence and the entanglement of formation in order to quantify the entanglement between the spins of two electrons. We also calculate the quantum discord of the electron spins and compare it with entanglement and classical correlations. Finally, Sec. V is devoted to the conclusions.

II. TWO-ELECTRON SPIN DENSITY MATRIX

Our system of interest is a two-dimensional electron gas with Rashba SOC in the **x**-**y** plane. The one-particle Hamiltonian without Coulomb interaction is

$$\hat{\mathcal{H}} = -\frac{\hbar^2\nabla^2}{2m^*} + \alpha(k_y\sigma_x - k_x\sigma_y), \tag{1}$$

where α is the Rashba coupling constant and m^* is the effective mass of the electrons. The eigenenergies and eigenfunctions of this Hamiltonian are given by

$$E(\mathbf{k},\gamma) = \frac{\hbar^2k^2}{2m^*} + \gamma k\alpha, \tag{2}$$

$$\psi_{\mathbf{k}\gamma}(\mathbf{r}) = \frac{1}{\sqrt{2A}}e^{i\mathbf{k}\cdot\mathbf{r}}\begin{pmatrix}\gamma ie^{-i\varphi}\\ 1\end{pmatrix}. \tag{3}$$

As expected, the Rashba SOC breaks spin degeneracy in the energy [Eq. (2)], as two separate branches arise depending

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on the spin projection γ. Another characteristic feature of the Rashba SOC is that the direction of spin quantization is perpendicular to the momentum vector $\mathbf{k}$.

To treat the many-particle problem it is convenient to use the second quantization formalism. The diagonal second-quantized Hamiltonian of a 2D electron gas with Rashba SOC is given by

$$\hat{\mathcal{H}}_R = \sum_{\mathbf{k}\gamma} \left[\frac{\hbar^2 k^2}{2m^*} + \gamma k\alpha \right] \hat{c}^\dagger_{\mathbf{k}\gamma} \hat{c}_{\mathbf{k}\gamma}, \tag{4}$$

where $\hat{c}^\dagger_{\mathbf{k}\gamma}$ creates a particle in an eigenstate of the one-particle Hamiltonian [Eq. (1)].

We consider the case where the electronic density is sufficiently high ($n > m^{*2}\alpha^2/\pi\hbar^4$) so that both energy branches are populated with electrons. In this case the noninteracting many-particle ground state is given by

$$|\Phi_0\rangle = \prod_{|\mathbf{k}|}^{k_F^+} \hat{c}^\dagger_{\mathbf{k}+} \prod_{|\mathbf{k}|}^{k_F^-} \hat{c}^\dagger_{\mathbf{k}-} |0\rangle. \tag{5}$$

Notice that this ground state has two Fermi surfaces, which are two circles with radii k_F^+ and k_F^-.

We will now calculate the two-particle density matrix. By definition, its matrix element, up to a normalization constant, is

$$\rho_{ss';tt'} = \langle \Phi_0 | \hat{\Psi}^\dagger_{t'}(\mathbf{r}') \hat{\Psi}^\dagger_t(\mathbf{r}) \hat{\Psi}_s(\mathbf{r}) \hat{\Psi}_{s'}(\mathbf{r}') | \Phi_0 \rangle, \tag{6}$$

where $\hat{\Psi}^\dagger_s(\mathbf{r})$ is the field operator that creates a particle at $\mathbf{r}$ with spin projection s in the $\hat{\mathbf{x}}$ direction. The field operators can be written as

$$\begin{aligned}
\hat{\Psi}_s(\mathbf{r}) &= \sum_{\mathbf{k}=0}^{\infty} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2A}} \hat{a}_{\mathbf{k}s} = \sum_{\mathbf{k},\mathbf{k}',\gamma'} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2A}} \langle \phi_{\mathbf{k},s} | \psi_{\mathbf{k}',\gamma'} \rangle \hat{c}_{\mathbf{k}'\gamma'} \\
&= \sum_{\mathbf{k},\mathbf{k}',\gamma'} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2A}} \int_0^\infty dr'^2 \phi_{\mathbf{k}s}(\mathbf{r}')^* \psi_{\mathbf{k}'\gamma'}(\mathbf{r}') \hat{c}_{\mathbf{k}'\gamma'} \\
&= \sum_{\mathbf{k},\gamma'} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{2\sqrt{2A}} (\gamma' i e^{i\phi_{\mathbf{k}}} + s) \hat{c}_{\mathbf{k}\gamma'},
\end{aligned} \tag{7}$$

where $\hat{a}^\dagger_{\mathbf{k}s}$ creates a particle in the state $\phi_{\mathbf{k},s}$, which is an eigenstate of an electron without Rashba SOC with momentum $\mathbf{k}$ and spin projection s in $\hat{\mathbf{x}}$. In Eq. (7) we express the operator $\hat{a}_{\mathbf{k}s}$ in terms of the Rashba operators $\hat{c}_{\mathbf{k}'\gamma'}$ and their eigenstates $\psi_{\mathbf{k}',\gamma'}$. Replacing the field operators in Eq. (6) using Eq. (7), we obtain

$$\begin{aligned}
\rho_{ss';tt'} = &\sum_{\mathbf{k}\gamma} \sum_{\mathbf{k}'\gamma'} \sum_{\mathbf{l}'\sigma'} \sum_{\mathbf{l}\sigma} \frac{1}{2^6 A^2} e^{-i(\mathbf{k}-\mathbf{l})\cdot\mathbf{r}} e^{-i(\mathbf{k}'-\mathbf{l}')\cdot\mathbf{r}'} \\
&\times (t' - \gamma i e^{i\phi_k})(t - \gamma' i e^{i\phi_{k'}}) \\
&\times (s + \sigma' i e^{-i\phi_{-l'}})(s' + \sigma i e^{-i\phi_l}) \\
&\times (\delta_{\mathbf{kl}}\delta_{\mathbf{k}'\mathbf{l}'}\delta_{\gamma\sigma}\delta_{\gamma'\sigma'} - \delta_{\mathbf{kl}'}\delta_{\mathbf{k}'\mathbf{l}}\delta_{\gamma\sigma'}\delta_{\gamma'\sigma}).
\end{aligned} \tag{8}$$

Here we used the anticommutation relations of the creation and annihilation operators $\{\hat{c}^\dagger_{\mathbf{k},\gamma}, \hat{c}_{\mathbf{k}',\gamma'}\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\gamma\gamma'}$. Taking the sums to the integral limit and normalizing with $\mathrm{Tr}(\rho) = 1$, we obtain

$$\rho_{\mathrm{R}} = \frac{1}{4-2c} \begin{pmatrix} 1-f_1^2 & f_1 f_2 & -f_1 f_2 & f_2^2 \\ f_1 f_2 & 1-f_2^2 & -f_1^2 & f_1 f_2 \\ -f_1 f_2 & -f_1^2 & 1-f_2^2 & -f_1 f_2 \\ f_2^2 & f_1 f_2 & -f_1 f_2 & 1-f_1^2 \end{pmatrix}, \tag{9}$$

with $c = f_1^2 + f_2^2$, and the spin basis $B = \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, with the spin direction of quantization $\hat{\mathbf{x}}$. Here f_1 and f_2 are defined by

$$f_1 = \frac{2[k_F^+ J_1(k_F^+|\mathbf{r}-\mathbf{r}'|) + k_F^- J_1(k_F^-|\mathbf{r}-\mathbf{r}'|)]}{|\mathbf{r}-\mathbf{r}'|\left[k_F^{+2} + k_F^{-2}\right]}, \tag{10}$$

$$\begin{aligned}
f_2 = &\frac{\pi\left[k_F^+ (H_0 J_1 - H_1 J_0)|_{k_F^+|\mathbf{r}-\mathbf{r}'|}\right]}{|\mathbf{r}-\mathbf{r}'|\left[k_F^{+2} + k_F^{-2}\right]} \\
&- \frac{\pi\left[k_F^- (H_0 J_1 - H_1 J_0)|_{k_F^-|\mathbf{r}-\mathbf{r}'|}\right]}{|\mathbf{r}-\mathbf{r}'|\left[k_F^{+2} + k_F^{-2}\right]}.
\end{aligned} \tag{11}$$

J_n are the Bessel functions of the first kind, and H_n are the Struve functions of the first kind. Notice that if we set $\alpha = 0$, then $k_F^+ = k_F^- = k_F$, and

$$\begin{aligned}
\tilde{f}_1 &= \frac{2}{|\mathbf{r}-\mathbf{r}'|k_F} J_1(k_F|\mathbf{r}-\mathbf{r}'|), \\
\tilde{f}_2 &= 0.
\end{aligned} \tag{12}$$

Using this in Eq. (9), we recover the known two-electron density matrix with no Rashba SOC ρ_{NR} [12]:

$$\rho_{\mathrm{NR}} = \frac{1}{4-2\tilde{f}_1^2} \begin{pmatrix} 1-\tilde{f}_1^2 & 0 & 0 & 0 \\ 0 & 1 & -\tilde{f}_1^2 & 0 \\ 0 & -\tilde{f}_1^2 & 1 & 0 \\ 0 & 0 & 0 & 1-\tilde{f}_1^2 \end{pmatrix}. \tag{13}$$

III. PAIR CORRELATION FUNCTIONS

A. Spin pair correlation function

The two-particle density matrix is related to the pair correlation function $R_{ss'}$ by $\rho_{ss';ss'} = (4-2c)R_{ss'}$, where the factor $(4-2c)$ is included to keep the normalization. The usual interpretation of $R_{ss'}$ is that of a conditional probability of finding an electron at $(\mathbf{r}, s)$ given that there is another one at $(\mathbf{r}', s')$ [30,31]. Clearly, this conditional probability $R_{ss'}$ depends on the distance between the electrons $|\mathbf{r}-\mathbf{r}'|$. When $s = s'$, the electrons avoid each other due to the Pauli exclusion principle. Each electron is surrounded by a region where the probability of finding another electron with equal spin is smaller than 1, which is known as the exchange hole. This correlation produces the first-order energy correction for the interacting electron gas, which can be interpreted as the Coulomb interaction of the electrons with the positive charge of the hole, reducing the overall energy of the system.

As an example to illustrate the effects studied in this work, we use the values of Rashba coefficient and effective mass corresponding to the surface alloy Bi/Ag(111) [32]. They are $\alpha = 3.05 \times 10^{-11}$ eV m and $m^* = 0.35 m_0$, and we take an electronic density of $n = 6.25 \times 10^{11}\,\mathrm{cm}^{-2}$, with an associated radius $r_0 = (\sqrt{\pi n})^{-1} = 7.14 \times 10^{-7}$ cm.

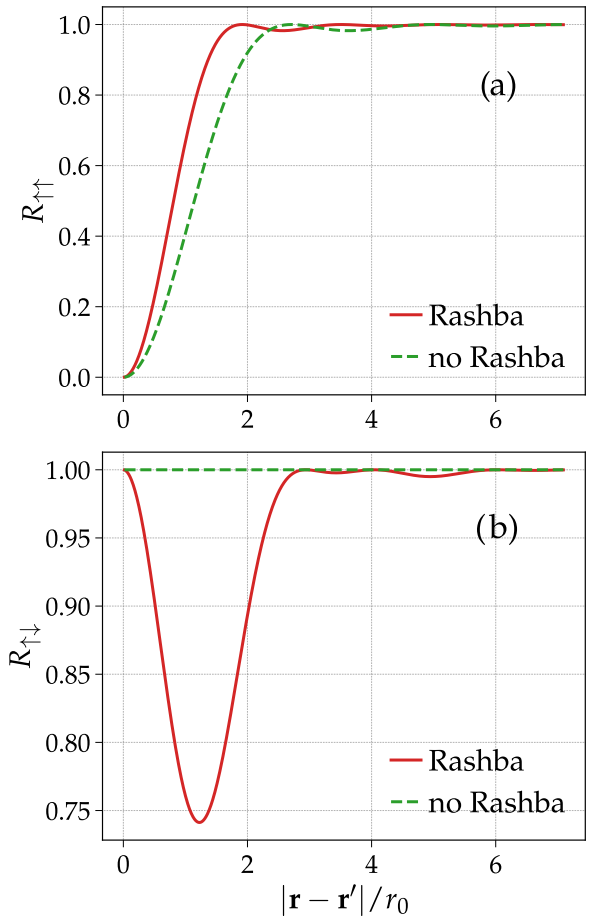

FIG. 1. Pair correlation function $R_{ss'}$ versus the relative distance of electrons for a 2DEG with (solid line) and without (dashed line) Rashba SOC. In (a) the spins of electrons are parallel, while in (b) the spins are antiparallel.

In Fig. 1(a), we present the traditional exchange hole for two electrons with equal spin projection $R_{\uparrow\uparrow}$ for a 2DEG without Rashba SOC, $R^{\mathrm{NR}}_{\uparrow\uparrow} = 1 - f_1^2$, and with Rashba SOC, $R^{\mathrm{Rashba}}_{\uparrow\uparrow} = 1 - f_1^2$. Note that the Rashba SOC seems to reduce the size of the exchange hole. More striking is the correlation $R_{\uparrow\downarrow}$ [Fig. 1(b)], which in the case without Rashba is constant and equal to 1 (no correlation). With Rashba SOC we find a nontrivial correlation for two electrons with opposite spins $R^{(\mathrm{Rashba})}_{\uparrow\downarrow} = 1 - f_2^2$. This strange correlation appears because the spin part of the one-particle solution of the Rashba Hamiltonian (4) depends on the direction of the momentum vector $\mathbf{k}$. This dependence on $\mathbf{k}$ prevents the field operator in Eq. (6) from fully localizing the spin projection. That is because when $\hat{\Psi}_s(\mathbf{r})$ acts on the ground state, the sum in momentum goes up to k_F^γ since $\hat{c}_{\mathbf{k}\gamma}|\Phi_0\rangle = 0$ for $|\mathbf{k}| > k_F^\gamma$ [see the last line of Eq. (7)]. The truncated sum produces a state which does not have complete uncertainty in momentum and, as a consequence, does not have a completely determined position and spin. The same happens without Rashba SOC, but in this case the spin direction is always well defined, and thus, the effect of this delocalization appears in only $R_{\uparrow\uparrow}$. If we consider a higher electronic density n, now k_F^γ is also larger to accommodate those new electrons. As a consequence the wave function of the electrons will be more localized, and for both cases the exchange hole will be smaller. In the limit of infinite density $R^{(\mathrm{Rashba})}_{\uparrow\downarrow} = R^{(\mathrm{NR})}_{\uparrow\downarrow} = 1$, and $R_{\uparrow\uparrow}$ will be 1 everywhere except at the point where $\mathbf{r} = \mathbf{r}'$, respecting the Pauli exclusion principle.

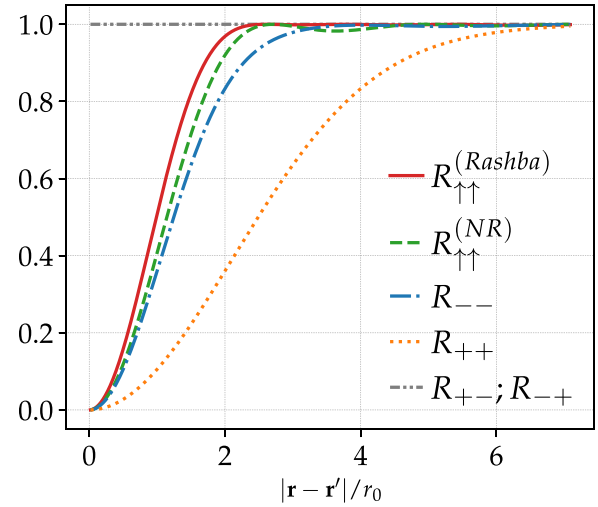

FIG. 2. Hamiltonian eigenbasis pair correlation function $R_{\gamma\gamma'}$ versus the relative distance of electrons for all combinations of γ and γ' and also, for comparison, spin pair correlation function $R_{\uparrow\uparrow}$ for a 2DEG with (solid line) and without (dashed line) Rashba SOC.

B. Hamiltonian eigenbasis pair correlation function

Another pair correlation function which at first sight seems natural is

$$R_{\gamma\gamma'} = \langle\Phi_0|\hat{\Psi}^\dagger_{\gamma'}(\mathbf{r}')\hat{\Psi}^\dagger_{\gamma}(\mathbf{r})\hat{\Psi}_{\gamma}(\mathbf{r})\hat{\Psi}_{\gamma'}(\mathbf{r}')|\Phi_0\rangle, \qquad (14)$$

where $\hat{\Psi}^\dagger_\gamma(\mathbf{r}) = \sum_k^\infty \psi^\dagger_{k\gamma}(\mathbf{r})c^\dagger_{k\gamma}$. Here γ is the quantum number that is used in Eq. (2) to label the two energy branches. Calculating this function, we obtain

$$R_{\gamma\gamma'} = 1 - \delta_{\gamma\gamma'}\frac{\pi^2 H_0^2 J_1^2 - 2\pi^2 H_0 H_1 J_0 J_1 + \pi^2 H_1^2 J_0^2 + 4J_1^4}{k_F^{\gamma 2}|\mathbf{r}-\mathbf{r}'|^2}, \qquad (15)$$

where the Struve and Bessel functions are evaluated at $k_F^\gamma|\mathbf{r} - \mathbf{r}'|$. It is clear that there is no correlation for opposite values of γ and γ' ($R_{+-} = R_{-+} = 1$). Additionally, there are two different nontrivial correlations, R_{++} and R_{--}, as shown in Fig. 2. The density used for Fig. 2 was larger than the one used in Fig. 1 in order to show them all together. The difference in size between R_{++} and R_{--} arises from the fact that to construct R_{++}, one uses field operators that sum all the states with $\gamma = +$, corresponding to all states in momentum space inside a disk of radius k_F^+, which is smaller than the disk used to construct R_{--} ($k_F^+ < k_F^-$). Having fewer momentum states in the sum corresponds to a more spread out state in position, resulting in a larger overlap in the wave function of the electrons and thus a larger exchange hole. This correlation function exhibits a behavior similar to the one for a 2DEG without Rashba SOC in the sense that it does not show correlations for opposite values of γ. However, while it is mathematically acceptable, it does not have the physical interpretation of the

spin pair correlation function, which represents the probability of finding an electron at a specific position and spin, given that there is another electron at a different position with equal or opposite spin. This is because the quantum number γ does not contain any information about the spin state of an electron created with $\hat{\Psi}^{\dagger}_{\gamma}(\mathbf{r})$.

There is also an issue with $R_{\gamma\gamma'}$ concerning an important sum rule that the pair correlation function must satisfy (see Appendix 4 in Ref. [31]). The sum rule is given by

$$\int n(\mathbf{r})[R(\mathbf{r})-1]d\mathbf{r} = \int h(\mathbf{r})d\mathbf{r} = -1, \tag{16}$$

with $h = n(R-1)$ and

$$R = \sum_{ss'} \frac{R_{ss'} n_s n_{s'}}{n^2}. \tag{17}$$

We calculated this sum rule numerically for the pair correlation function obtained from the density matrix in Eq. (9) and for the one defined in Eq. (15); the results were -0.99993 and -2.0279, respectively.

C. Experimental detection

In principle, the pair correlation function has measurable effects. Although measuring R directly is difficult, one can measure the static structure factor of the electron gas, which is related to R by

$$S(q) = \frac{\langle \hat{n}_{-\mathbf{q}} \hat{n}_{\mathbf{q}} \rangle}{N} = 1 + n \int (R(r)-1) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}, \tag{18}$$

and its spin-resolved counterpart,

$$\begin{aligned} S_{ss'}(q) &= \frac{\langle \hat{n}_{-\mathbf{q},s} \hat{n}_{\mathbf{q},s'} \rangle}{N} \\ &= \frac{n_s}{n} + \frac{n_s n_{s'}}{n} \int (R_{ss'}(r)-1) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}. \end{aligned} \tag{19}$$

Both versions would differ from the 2DEG without Rashba SOC; however, the spin-resolved one would have a significant difference as $S^{(\mathrm{NR})}_{\uparrow\downarrow} = 0$, meaning there are no features, in contrast to the case with Rashba SOC, for which we would find some nontrivial structure in $S^{(\mathrm{Rashba})}_{\uparrow\downarrow}$. The problem with measuring this quantity is that we have not accounted for the correlations induced by the Coulomb interaction. These correlations would considerably affect both $S_{\uparrow\uparrow}$ and $S_{\uparrow\downarrow}$. This effect has to be taken into account in order to properly discriminate the exchange and Coulomb correlations. They can be calculated using quantum Monte Carlo, as done in Ref. [33]. The spin-resolved structure factor for a Fermi gas was measured using Bragg spectroscopy [34]. However, to the best of our knowledge, no experiment of this kind has been carried out for a 2DEG with Rashba interaction. An experimental measurement of the spin-resolved static structure factor could, in principle, show the effects presented here.

IV. QUANTUM CORRELATION MEASURES

The pair correlation function is useful for understanding the consequences of the exchange symmetry of electrons and how correlations can arise even without interactions. However, it does not distinguish between classical and quantum correlations, like entanglement or quantum discord. Understanding the nature of the correlations of identical fermions is still a topic of active research [35].

In this section we apply modern quantum information measures in order to explore the nature of the correlations produced by the exchange symmetry. Correlation measures in quantum information theory have been well developed for two-qubit states. These states are described by a density matrix mathematically equivalent to the one we obtained with Eq. (6). Effectively, we trace the entire system except for two electrons, which can be viewed as two qubits [3,12,36]. Thus, we can apply all the machinery developed in quantum information to our system.

A. Entanglement

In order to calculate the entanglement of the density matrix given in Eq. (9), we use the partial transposition criterion [37,38]. This criterion states that a density matrix ρ of two qubits is entangled iff $\rho^{\mathrm{T}_2} \geqslant 0$, or, equivalently, ρ^{T_2} have non-negative eigenvalues. The partial transpose is defined as $\rho^{\mathrm{T}_2}_{ss';tt'} = \rho_{st';ts'}$. The result of applying this criterion is that $f_1^2 + f_2^2 \geqslant 1/2$. As long as this relationship holds, entanglement is present. This condition is valid for a region where $0 \leqslant |\mathbf{r}-\mathbf{r}'| \leqslant r_e$. Consequently, any two electrons within a disk of radius r_e will exhibit entanglement in spin.

To quantify the entanglement of the two electrons, we use two related measures of entanglement: concurrence and the entanglement of formation. The concurrence of a mixed state ρ is given by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{20}$$

where λ_i are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in descending order. Here $\tilde{\rho}$ is the result of applying the spin-flip operation to ρ,

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \tag{21}$$

Concurrence is a monotonic function which is zero iff the state is separable. Also, an entangled state ρ has concurrence greater than zero, and $C(\rho) = 1$ iff ρ is a maximally entangled state.

Using concurrence, it is possible to obtain an exact formula for calculating the entanglement of formation of a two-qubit state,

$$E = h\left(1 + \sqrt{\frac{(1-C^2)}{2}}\right), \tag{22}$$

where h is the Shannon entropy. Entanglement of formation is an entropic measure of entanglement which is more physically motivated than concurrence. It uses the singlet state $|\Phi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ as an entanglement unit. The entanglement of any other state will then be defined by relating it to this entanglement unit. For a complete explanation of these measures see Refs. [38,39].

Now we compute these measures for the two-particle density matrix of the 2DEG with Rashba SOC. First, we calculate the concurrence

$$C = \max\left\{0, \frac{3|c-1| - (c+1)}{2(c-2)}\right\}, \tag{23}$$

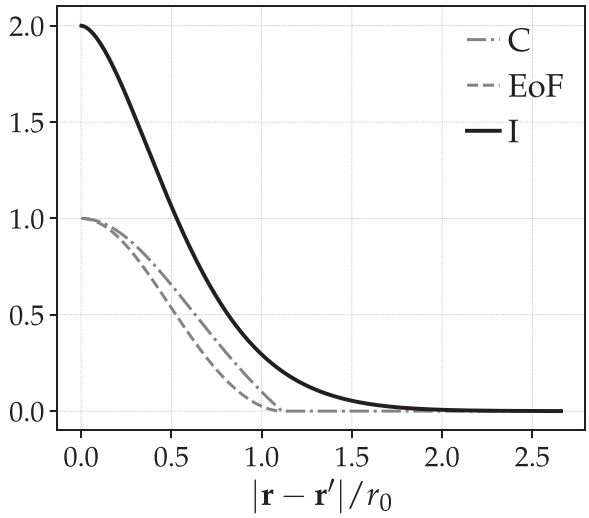

FIG. 3. Quantum mutual information I (solid line), concurrence C (dotted-dashed line), and entanglement of formation (EoF; dashed line) versus the relative distance between electrons for a 2DEG with Rashba SOC.

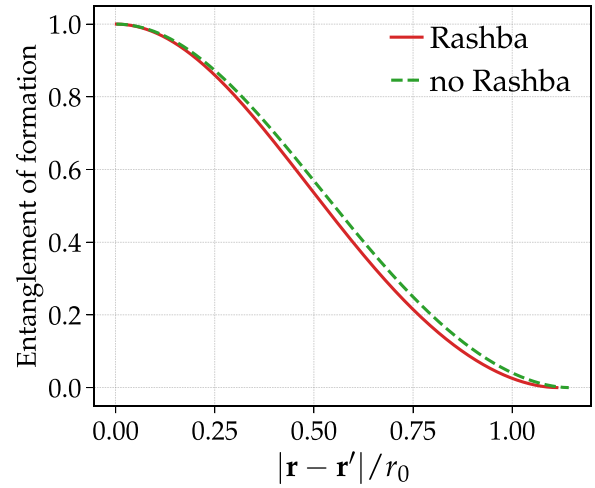

FIG. 4. Entanglement of formation versus the relative distance between electrons for a 2DEG with (solid line) and without (dashed line) Rashba SOC. The Rashba SOC seems to reduce the entanglement of the spins of the electrons.

where $c = f_1^2 + f_2^2$, as defined in Sec. II. With this result, it is easy to obtain the entanglement of formation using Eq. (22). These results are shown in Fig. 3.

Notice that we have obtained a finite value for the entanglement distance r_e for both the concurrence and the entanglement of formation. When the separation between electron pairs becomes larger than this distance, entanglement disappears. On the other hand, the exchange hole does not exhibit a cutoff value; instead, $R_{ss'}$ approaches unity as the distance between electrons becomes infinitely large (see Fig. 1). This observation indicates the presence of other types of correlations besides entanglement.

To distinguish between entanglement and other correlations, we utilized the quantum mutual information, defined as $I(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB})$. In general, the quantum mutual information is a positive value, representing the difference between the sum of the uncertainties of each subsystem and the uncertainty in the complete system. The quantum mutual information becomes zero only when the joint state can be expressed as a product state of each subsystem, i.e., when $\rho^{AB} = \rho^A \otimes \rho^B$. In such cases, there are no correlations between the subsystems. This positive quantity serves as a measure of the shared information stored in the joint state ρ^{AB}, indicating the amount of correlation between the subsystems. The quantum mutual information results in our system:

$$I(\rho) = 2 + \left(\frac{1+c}{4-2c}\right)\log_2\left(\frac{1+c}{4-2c}\right) + 3\left(\frac{1-c}{4-2c}\right)\log_2\left(\frac{1-c}{4-2c}\right), \tag{24}$$

and it is shown as a solid line in Fig. 3. We see that the quantum mutual information vanishes as $|\mathbf{r}-\mathbf{r}'| \to \infty$, similar to what happens to the correlations associated with the exchange hole (see Fig. 1). Indeed, in the case of the exchange hole, as $|\mathbf{r}-\mathbf{r}'| \to \infty$, the pair correlation functions $R_{ss'}$ approach a value of 1, which indicates the decay of correlations between the two electrons at very large separations.

The concurrence and the entanglement of formation show a maximum at $|\mathbf{r}-\mathbf{r}'| = 0$ (see Fig. 3). The state that corresponds to this maximum can be obtained directly from the density matrix [Eq. (9)] and is $|\phi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, the singlet state, which is a maximally entangled state for two spins. When the distance between the electrons increases, the triplet states start to appear in the density matrix, reducing the overall entanglement, up to r_e, where entanglement disappears. However, there are still correlations in the density matrix after r_e, until $|\mathbf{r}-\mathbf{r}'| \to \infty$. In this limit, the density matrix becomes proportional to the identity; all states of the basis are equally likely, and therefore, the correlations disappear.

In the case without Rashba SOC, the condition for entanglement of electron spins is given by $\tilde{f}_1^2 \geqslant 1/2$. The entanglement of formation was calculated, and a comparison of this measure was made for both systems, as shown in Fig. 4. The entanglement is smaller in the system with Rashba SOC, and also the radius of entanglement r_e decreases. Like what happens for the exchange hole, the difference between entanglement for the two systems disappears when $n \to \infty$ since $f_2 \to 0$, $f_1 \to \tilde{f}_1$, and $\rho_{\mathrm{R}} \to \rho_{\mathrm{NR}}$.

B. Quantum discord

Quantum discord $\mathcal{Q}(\rho^{AB})$ is a kind of quantum correlation that arises from the nature of measurements in quantum mechanics, which in general alter the state of the system. The mutual information quantifies the correlations between the components of a system and can be computed in two related ways, both of which yield the same result in classical systems. However, in quantum mechanics, the natural generalizations of these two definitions often lead to conflicting results, giving rise to quantum discord. The crucial distinction lies in the fact that one extension, known as the quantum mutual information $I(\rho^{AB})$, does not involve any measurement, whereas the other

generalization, referred to as classical correlation $\mathcal{C}(\rho^{AB})$, requires a measurement in one subsystem.

Quantum discord is thus defined as $\mathcal{Q}(\rho^{AB}) = I(\rho^{AB}) - \mathcal{C}(\rho^{AB})$. The classical correlations [14,40,41] are defined by

$$\mathcal{C}(\rho^{AB}) = \sup_{\{B_k\}} I(\rho^{AB}|\{B_k\}), \tag{25}$$

where $\{B_k\}$ describe a von Neumann measurement for subsystem B and $I(\rho^{AB}|\{B_k\})$ is the quantum mutual information in terms of the conditional entropy of having measured k, that is,

$$\begin{aligned} I(\rho|\{B_k\}) &= S(\rho^A) - S(\rho|\{B_k\}), \\ S(\rho|\{B_k\}) &= \sum_k p_k S(\rho_k), \end{aligned} \tag{26}$$

which depends on the density operator associated with the measurement result k:

$$\rho_k = \frac{1}{p_k}(\mathrm{I}\otimes B_k)\rho(\mathrm{I}\otimes B_k), \tag{27}$$

where $p_k = \mathrm{Tr}[(\mathrm{I}\otimes B_k)\rho(\mathrm{I}\otimes B_k)]$ to keep normalization.

The maximization involved in calculating the classical correlations [Eq. (25)] turns out to be complicated for a general bipartite two-level state; however, some simpler cases have been solved analytically [42,43]. For our system the density matrix [Eq. (9)] has two parameters, f_1 and f_2. We already have the quantum mutual information [Eq. (24)], so we need to calculate the classical correlations [Eq. (25)] to get the quantum discord.

First, we write the density matrix in terms of Pauli matrices:

$$\begin{aligned} \rho = \frac{I}{4} + \frac{1}{4-2c}\Bigg[& \frac{-f_1^2+f_2^2}{2}(\sigma_x\otimes\sigma_x + \sigma_z\otimes\sigma_z) \\ & - \frac{f_1^2+f_2^2}{2}\sigma_y\otimes\sigma_y + f_1 f_2(-\sigma_x\otimes\sigma_z + \sigma_z\otimes\sigma_x)\Bigg]. \end{aligned} \tag{28}$$

Now we have to calculate the density matrix ρ_k after a measurement on subsystem B and the probability p_k to calculate the classical correlations. Following the procedures in Refs. [42,43], we obtain

$$\mathcal{C}(\rho) = \frac{1-c}{2-c}\log_2\left(\frac{2-2c}{2-c}\right) + \frac{1}{2-c}\log_2\left(\frac{2}{2-c}\right), \tag{29}$$

and the quantum discord is given by

$$\begin{aligned} \mathcal{Q}(\rho) = 2 &+ \left(\frac{1+c}{4-2c}\right)\log_2\left(\frac{1+c}{4-2c}\right) + 3\left(\frac{1-c}{4-2c}\right) \\ &\times \log_2\left(\frac{1-c}{4-2c}\right) - \frac{1-c}{2-c}\log_2\left(\frac{2-2c}{2-c}\right) \\ &- \frac{1}{2-c}\log_2\left(\frac{2}{2-c}\right). \end{aligned} \tag{30}$$

The quantum discord, concurrence, and classical correlations C_{cl} are shown in Fig. 5. In contrast to entanglement (given by the concurrence), quantum discord goes to zero smoothly as a function of $|\mathbf{r}-\mathbf{r}'|$. The classical correlations are always smaller than quantum discord. The relationship between quantum discord and entanglement is less clear, but it is interesting that there are states which are separable (zero concurrence) and still have nonzero quantum discord. The relations between quantum discord, concurrence, and the classical correlations are qualitatively the same as those obtained from a Werner state [42–44], which is the state obtained from the density matrix of a 2DEG without Rashba SOC [Eq. (13)].

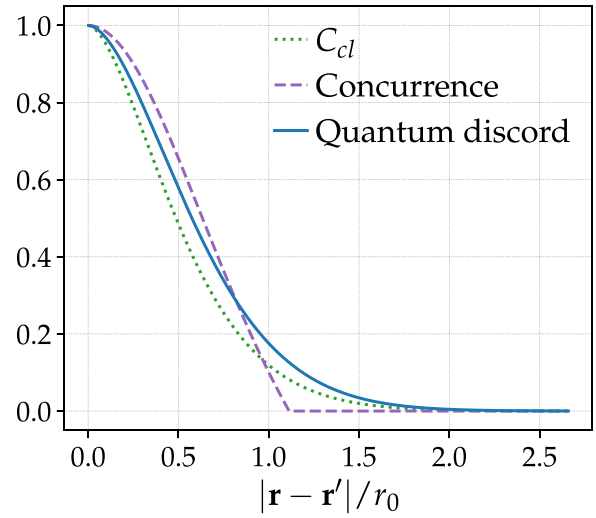

FIG. 5. Quantum discord Q (solid line), concurrence C (dashed line), and classical correlation C_{cl} (dot-dashed line) versus the relative distance between electrons for a 2DEG with Rashba SOC.

V. CONCLUSIONS

We calculated the two-particle density operator for a two-dimensional electron gas with Rashba spin-orbit coupling. From this operator, we derived the pair correlation function and compared it with the case without Rashba SOC. The antisymmetrization of the wave function of electrons creates a correlation between electrons of equal spin projection, which is present in both cases. This correlation can be explained by the fact that for both systems, electrons are not completely localized in position, and if they are sufficiently close to one another, their wave functions have a nonzero overlap. This is known as the exchange hole, which is a basic concept in many-body physics.

More strikingly, we found an additional correlation between electrons of opposite spin that occurs only in the system with Rashba SOC. This correlation is less intuitive and is due to the fact that the spin projection is not well defined. This result could, in principle, be measured via the spin-resolved static structure factor, which is directly related to the pair correlation function.

To further understand the correlation described by the pair correlation function and to quantify the entanglement of the electron pair, we used concurrence and the entanglement of formation, two measures employed in quantum information theory. We obtained a condition for the entanglement of two electron spins ($f_1^2 + f_2^2 \geqslant 1/2$) which defines a disk with a radius r_e. Any two electrons inside this disk will be entangled in spin. We found that the entanglement is maximum when

the relative distance between electrons is zero, and then it decreases as the distance between electrons becomes larger. The entanglement of formation, as shown in Fig. 4, was calculated for a 2DEG with and without Rashba SOC, and it shows a reduction in the entanglement distance in the case with Rashba SOC.

Finally, we calculated the quantum discord of the system, which is a measure of quantum correlations fundamentally different from entanglement. As shown in Fig. 5, concurrence is greater than quantum discord for small separations of electrons, but for larger distances, quantum discord becomes dominant, and it smoothly goes to zero as the electrons get farther apart. The relationship between quantum discord, concurrence, and classical correlations is qualitatively the same as for a Werner state, which is the state obtained by the density operator of a 2DEG without Rashba SOC [Eq. (13)]. We can conclude that Rashba SOC quantitatively changes modern correlation measures. On the other hand, the introduction of Rashba SOC does not qualitatively change the correlations or the relations among them. However, for the more traditional exchange hole, we found an unexpected correlation for electrons of different spin projections that does not appear in a 2DEG without Rashba SOC.

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