# Penetration of waves in global stochastic conducting media

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(Received 14 September 2022; revised 30 December 2022; accepted 12 April 2023; published 3 May 2023)

The attenuation in the propagation of a plane wave in conducting media has been studied. We analyzed a wave motion suffering dissipation by the Joule effect in its propagation in a medium with global disorder. We solved the stochastic telegrapher's equation in the Fourier-Laplace representation allowing us to find the space penetration length of a plane wave in a complex conducting medium. Considering fluctuations in the loss of energy, we found a critical value $k_c$ for Fourier's modes, thus if $|k| < k_c$ the waves are localized. We showed that the penetration length is inversely proportional to $k_c$. Thus, the penetration length $L = k_c^{-1}$ becomes an important piece of information for describing wave propagation with Markovian and non-Markovian fluctuations in the rate of the absorption of energy $\tau^{-1}$. In addition, intermittent fluctuations in this rate have also been studied.

DOI: 10.1103/PhysRevE.107.054107

## I. INTRODUCTION

### A. Damped wave propagation

The study of the propagation of electromagnetic waves in a continuous medium and in wave guides (cavity of infinite length) are important theoretical and experimental problems. That is, *the* electromagnetic field in the presence of an electric current produces damping in the wave evolution of the electric $E$ and magnetic $B$ fields, interestingly, the symmetry between these fields is preserved. To understand this wave propagation, *the* telegrapher's equation (TE) helps to describe the evolution of the fields [1]. A complete analysis in terms of the $E$ wave, $H$ wave and *principal* waves, as a function of the boundary conditions in the wave guides is given in Ref. [2]. In addition, the derivation and analysis of the TE in electric circuits and transmission lines can be found in Refs. [3,4]. In a recent paper [5] it was shown that, if the continuous medium has a time-varying electric permittivity $\epsilon$, magnetic permeability $\mu$ and/or conductivity $\sigma_c$, the electric and magnetic fields do not satisfy the same wave equation, and this symmetry breaking may have important consequences of physical interest [6]. Therefore, the analysis of time fluctuations in macroscopic coefficients $\{\epsilon, \mu, \sigma_c\}$ is an important issue to be studied in the electromagnetic wave propagation problem. The stochastic TE approach can also be related to the problem of electromagnetic wave propagation in the presence of a fluctuating conductivity, wave penetration in lossy media [7], and conductivity perturbation on the ionospheric potential [8].

The TE was also used to characterize electromagnetic wave packets, with the aim of understanding superluminal and subluminal values of the group velocity appearing as relevant kinematic characteristics [9] as well as in the analysis of spatially attenuated and temporally attenuated harmonic solutions of the TE, with the aim to compare the phase, energy, and group velocities [10]. Also the TE was used in the analysis of damping and propagation of surface gravity waves on an irregular bottom [11,12];

### B. Hyperbolic diffusion

In addition to the study of the TE in the context of damped wave propagation, the hyperbolic diffusion theory through the Cattaneo-Fick approach also leads to the same TE. The interest in the hyperbolic diffusion is based on the fact that this transport description introduces a finite-velocity diffusion for the propagation of a positive normalized packet [13–19]. This analysis was also extended to heterogeneous media using fractal descriptions [20]; recently, the two-dimensional (2D) and three-dimensional (3D) versions of the TE were studied in engineering analysis with boundary problems [21,22]. Contributions relative to the TE in connection with stochastic resetting processes are given in Refs. [23–27], and for understanding experimental contributions on the modulation transfer between microwave beams [28]. In the context of transport in heterogeneous media, finite-velocity diffusion in random media [29] was previously tackled as well as in the presence of global disorder [30]. To end the large variety of applications of the stochastic TE in hyperbolic diffusion, let us comment that in biology, this stochastic TE can also be used to tackle the flagella motion [31] and stochastic resetting problems [24,32]; this is true in biomedical optics as well [33,34]. Also, hyperbolic diffusion can be used in the analysis of random Bolztmann-Lorentz models [35].

### C. Attenuation of wave-like packets

The TE has two important parameters: the wave velocity of propagation in the conducting medium $v$ and the rate of

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absorption of energy $\tau^{-1}$. In heterogeneous media, we can consider that the velocity is a random function $v \to v(x)$, this problem was tackled in the past [29], while if the medium is complex we can also consider studying global disorder in the velocity of propagation $v \to v(t)$ (i.e., $v(t)$ is a noise or considering a randomized time $r(t)$ [18,28,30]). In particular, in biophysics (run-and-tumble models [25–27,31]) it is important to consider that the rate $\tau^{-1}$ is a time-dependent random function. Notably, some of these problems can be mapped to the present stochastic TE.

Due to the large number of applications of the TE, it is important to have a good description of the penetration of a wave packet in a complex continuous medium. Recently, it was shown that the TE admits two types of harmonic representations which show temporally or spatially attenuated solutions. While the two harmonic wave solutions give different wave velocities (for example, phase and group), depending on the type of problems one of the two forms of solutions is always assumed [10]. In the present research we are concerned with space periodicity (i.e., finite systems), therefore we focus on temporally attenuated and spatially periodic solutions (called TASP waves in Ref. [10]).

To understand the space attenuation of a pulse, in a complex stochastic medium, we studied the TE with a time-fluctuating rate of absorption of energy $\tau^{-1}$. To describe these fluctuations we use a general binary noise (we use binary noise with exponential waiting-time, the Markovian case, and biexponential waiting-time, the non-Markovian case). In this manner, we can also study, analytically, the penetration of the pulse for the case of having an intermittent stochastic absorption of energy. Our analysis is carried out considering the pulse moving to the right direction. The exact solution of the stochastic TE is obtained using the Fourier-Laplace transform and Fourier-localized modes are presented for all cases studied.

The organization of the paper is as follows. In Sec. II we obtain an attenuated plane-wave solution of the TE associated to the electromagnetic field without fluctuations in the rate of energy. In Sec. III we solve the stochastic TE with a fluctuating absorption of energy using the Fourier-Laplace representation. Thus, with this purpose we define a mean *effective* plane wave. We find a critical value $k_c$ for obtaining Fourier-localized modes, considering Markovian and non-Markovian fluctuations in the absorption of energy. In Sec. IV we show numerical results of the stochastic TE in space and time for a pulse moving in one direction. We analytically study the attenuation length of a pulse in a fluctuating (complex) continuous medium. The agreement with numerical solutions concerning the energy transfer is shown. Finally, conclusions of the present work are introduced in Sec. VI. The Appendices are dedicated to mathematical details of our work.

## II. DAMPED PLANE WAVE IN THE TELEGRAPHER'S EQUATION

### Perturbative wave solution in a continuous medium

The TE was obtained in transport theory [13–20], engineering [4,21], and biological models [36]. In the electromagnetic context, the TE is [1], see Appendix A,

$$\left[\partial_t^2 + \frac{1}{\tau}\partial_t - v^2\partial_x^2\right]\psi(x,t) = 0,$$
$$\text{with suitable initial conditions,} \tag{1}$$

where $\tau^{-1}$ is the attenuation parameter (rate of absorption of energy) of the electromagnetic field, therefore, $v$ is the velocity of the wave in the continuous media. For the electromagnetic field these parameters can be given in terms of the Maxwell equations in the presence of currents, thus $\tau = \epsilon/4\pi\sigma_c$ and $v^2 = c^2/\mu\epsilon$, where $\sigma_c$ is the conductivity, $\epsilon$ the dielectric, and $\mu$ the permeability parameters [1]. An alternative derivation of $\tau$ and $v$ for an electric circuit in a waveguide can also be obtained [2,4]. Rewriting (1) as

$$\left[\partial_t^2 - v^2\partial_x^2\right]\psi(x,t) = -\frac{1}{\tau}\partial_t\psi(x,t), \tag{2}$$

the left-hand side of this equation represents the wave motion, so if the parameter $\tau$ is bigger compared with the timescale of the system; i.e., $t/2\tau \ll 1$, it is possible to write the solution in the form $\psi(x,t) \approx e^{-t/2\tau} f(x - vt)$, where $f(x - vt)$ represents the solution of the wave equation, so we can write [1]

$$\psi(x,t) \approx e^{-t/2\tau} f(x - vt) \propto e^{-x/2v\tau} g(x - vt), \tag{3}$$

where a simple transformation between $f(x - vt)$ and $g(x - vt)$ can be invoked. Therefore, it is possible to define the attenuation length (penetration length) as

$$L = 2v\tau, \tag{4}$$

this approximation is valid if the rate of absorption of energy $\tau^{-1}$ is small, which corresponds to space-time scales

$$\frac{t}{2\tau} \ll 1 \Rightarrow vt \ll L, \tag{5}$$
$$\frac{x}{2v\tau} \ll 1 \Rightarrow x \ll L. \tag{6}$$

In the next section, we will propose an alternative definition of $L$ for the case of considering time fluctuations in absorption of energy in the TE ($\tau^{-1}$). This will be done using the concept of localization of the mean plane wave [37]. In addition, the case of intermittent absorption of energy will be considered using a particular non-Markov noise in $\tau^{-1}$.

## III. FOURIER-LAPLACE SOLUTION OF THE STOCHASTIC TELEGRAPHER'S EQUATION

### A. Ordered case (no fluctuating $\tau^{-1}$)

To solve the ordinary TE, we use the Fourier and Laplace transformation

$$\psi(k,s) = \int_{-\infty}^{\infty} dx\, e^{ikx} \int_0^{\infty} dt\, e^{-st}\psi(x,t), \tag{7}$$

in (1). We consider a free Dirichlet boundary value problem $\psi(x=\pm\infty,t)=0$ with initial conditions

$$\psi(x,t=0)=\delta(x) \quad \text{and} \quad \partial_t\psi(x,t=0)=0, \tag{8}$$

which corresponds to studying the evolution of a symmetric initially static pulse. Therefore we obtain

$$\psi(k,s)=\frac{s+1/\tau}{s(s+1/\tau)+v^2k^2}. \tag{9}$$

We note that it is possible to rewrite (9) in the form

$$\psi(k,s)=\frac{1}{s+D_{\mathrm{TE}}(s)k^2}, \tag{10}$$

where $D_{\mathrm{TE}}(s)=v^2/(s+1/\tau)$ is a generalized diffusion coefficient. It can be noticed from (10) and for $\tau s\gg 1$ that the plane-wave mode is recovered, i.e., $\psi(k,s)=s/(s^2+v^2k^2)$. In the opposite regimen ($\tau s\ll 1$) a diffusive packet is obtained $\psi(k,s)=s/(s+Dk^2)$, with $D_{\mathrm{TE}}(s)\to\tau v^2=D$.

The solution of the TE (9) can be written as [37]

$$\psi(k,s)=\frac{s+1/\tau}{(s-s_-)(s-s_+)}, \tag{11}$$

with $s_\pm\equiv s_\pm(k)=(-1\pm\sqrt{1-(2v\tau k)^2})/2\tau$. If $2v\tau<|k|^{-1}$ Fourier modes are localized; in the opposite case Fourier modes are delocalized, i.e., $|k|>k_c$ with

$$k_c^{-1}\equiv 2v\,\tau, \tag{12}$$

characterizing the localized gap, notably, $k_c$ is a critical value which can also be obtained in the stochastic TE [37].

Comparing (4) with (12), we can define the attenuation length $L$ of the plane wave ($L$ is the inverse of the border of the localized gap). We will show that this conclusion will also be valid for a general stochastic TE

$$L=1/k_c. \tag{13}$$

This proposal will be checked with the numerical results in Sec. V.

### B. Markovian noise in the rate of absorption of energy $\tau^{-1}$

The stochastic TE with fluctuating absorption of energy is [30]

$$\left[\partial_t^2+\frac{1}{\tau}\partial_t-v^2\partial_x^2\right]\psi(x,t)=-\theta\xi(t)\partial_t\psi(x,t), \tag{14}$$

where $\theta$ is the amplitude of noise $\xi(t)$ (here we are concerned with studying the strong fluctuating case $\theta=\tau^{-1}$). That is, we changed $\tau^{-1}>0$ in (1) to $\tau^{-1}\to[\tau^{-1}+\theta\xi(t)]\geqslant 0$ in (14), thus $\xi(t)$ is a bounded stochastic process [18]. The case of a binary Markovian noise $\xi(t)=\pm1$, $\forall t\geqslant 0$ corresponds to consider an exponential stationary correlation function $\langle\xi(t_1)\xi(t_2)\rangle=\exp[-|t_1-t_2|/T]$. As in the ordered case [$\theta=0$ in (14)], the stochastic TE can be solved in the Fourier-Laplace representation. Averaging over all realizations of the noise [30,35] and using the same initial condition (8) we obtain

$$\langle\psi(k,s)\rangle=\frac{s\tau[k^2T^2v^2+(sT+1)^2]+T[k^2Tv^2+s(2sT+3)]+1}{\tau(k^2v^2+s^2)[k^2T^2v^2+(sT+1)^2]+2k^2sT^2v^2+k^2Tv^2+2s^3T^2+3s^2T+s}. \tag{15}$$

As before, we can write $\langle\psi(k,s)\rangle$ as a function of a dispersive effective diffusion coefficient, similar to (10), with

$$D_{\mathrm{eff}}(k,s)=\frac{v^2[T^2(k^2\tau v^2+s^2\tau+s)+2s\tau T+\tau+T]}{s\tau[k^2T^2v^2+(sT+1)^2]+T[k^2Tv^2+s(2sT+3)]+1}. \tag{16}$$

To analyze Fourier-localized modes, we study a mean plane-wave-like mode taking the contributions of $D_{\mathrm{eff}}(k=0,s)$ into $\langle\psi(k,s)\rangle$, thus, we define a mean effective plane-wave function [37]

$$\langle\overline{\psi(k,s)}\rangle=\frac{1}{s+D_{\mathrm{eff}}(k=0,s)k^2}. \tag{17}$$

Using (16) in (17) we get (in the strong disorder regime $\theta=\tau^{-1}$)

$$\langle\overline{\psi(k,s)}\rangle=\frac{s^2\tau T+s\tau+2sT+1}{s^3\tau T+s^2\tau+2s^2T+s+(s\tau Tv^2+Tv^2+\tau v^2)k^2}. \tag{18}$$

If $\tau\to\infty$, we get $\langle\overline{\psi(k,s)}\rangle\to s/(s^2+k^2v^2)$, this approach assures the wave-like character of $\langle\overline{\psi(k,s)}\rangle$. We can write (18) as a fraction of polynomials such as

$$\langle\overline{\psi(k,s)}\rangle=\frac{Q(s)}{R(s)}=\frac{\tau Ts^2+(\tau+2T)s+1}{\tau Ts^3+(\tau+2T)s^2+(k^2\tau Tv^2+1)s+(k^2Tv^2+k^2\tau v^2)}. \tag{19}$$

To find a plane-wave-like regimen, we demand the discriminant of $R(s)$ to be positive:

$$-g_6k^6+g_4k^4-g_2k^2+g_0>0, \tag{20}$$

with

$$g_6 = 4\tau^4 T^4 v^6, \tag{21}$$

$$g_4 = \tau^2 T^2 v^4(-8\tau^2 + 13T^2 - 8\tau T), \tag{22}$$

$$g_2 = 2v^2(2\tau^4 + 16T^4 + 18\tau T^3 + 11\tau^2 T^2 + 4\tau^3 T), \tag{23}$$

$$g_0 = \tau^2 + 4T^2. \tag{24}$$

In this form it is possible to obtain $k_c$ from (20), with this value it is possible to define the gap with localized Fourier modes $\mathcal{G} \equiv [-k_c, k_c]$, see Ref. [37]. Note that in that reference dimensionless units were used. So, for $|k| > k_c$ Fourier modes are delocalized (wave-like regimen). With $k_c$ in (13), we can calculate the penetration length, in the space, of a mean plane wave. This critical value is given by

$$k_c^2 = \frac{6\sqrt[3]{2}g_2 g_6 - 2\sqrt[3]{2}g_4^2 + 2g_4 r - 2^{2/3}r^2}{6g_6 r}, \tag{25}$$

with

$$r = \sqrt[3]{\sqrt{\left(27g_0g_6^2 - 9g_2g_4g_6 + 2g_4^3\right)^2 - 4\left(g_4^2 - 3g_2g_6\right)^3} - 27g_0g_6^2 + 9g_2g_4g_6 - 2g_4^3}. \tag{26}$$

The result (25) was previously found using the Terwiel's cumulants technique [30]. In that opportunity the behavior of the localized gap $\mathcal{G} \equiv [-K_c, K_c]$ was studied as a function of $T$ the correlation time of Markovian binary fluctuations in the rate of energy. Therefore, the phenomena of *delocalization* were measured explicitly, see Fig. 1 in Ref. [37] (there $K_c = K_c(\beta)$ in dimensionless units: $K_c = \tau v k_c$, $\beta = \tau/T$). Nevertheless, Terwiel's technique is not very useful if the binary noise is intermittent because an infinite series of Terwiel's diagrams will appear. This is the reason why we will introduce the enlarged master equation (EME) technique to tackle this problem analytically. In the next section, we will generalize this result for the case when the stochastic absorption of energy is a biexponential non-Markovian binary noise. Thus, we can study the intermittent case in particular.

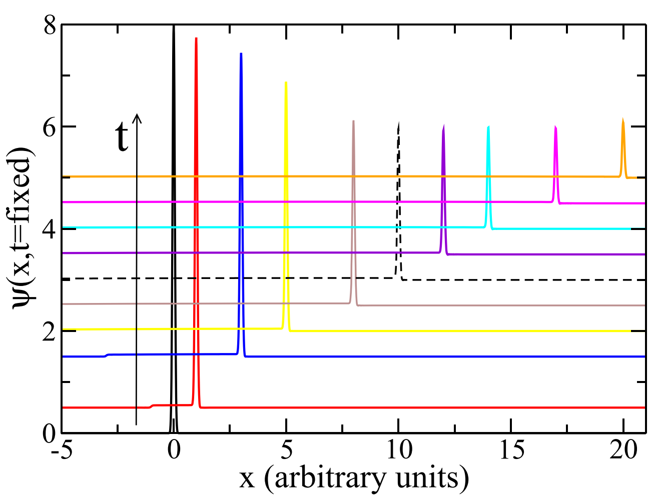


FIG. 1. Amplitude of $\psi(x,t)$ as a function of space for the ordered case $\theta = 0$ and fixed values of times $t = 0, 1, 3, 5, 8, 10, 12, 14, 17, 19$. The pulse spreads with time and it suffers an attenuation in its amplitude. The parameters are $\tau = 5$, $v = 1$ so $L = 10$ [see (4)], using (36) with $\sigma = 0.05$.

### C. Non-Markovian noise in the rate of absorption of energy $\tau^{-1}$

If the fluctuations in the rate of energy are intermittent, the Markovian binary noise cannot be used; we need to generalize its correlation structure to be nonexponential. A possible way to introduce an intermittent timescale separation for the transition of the events $\xi(t) = \pm 1$ is to change the waiting time of the noise.

Here we study a symmetric non-Markovian binary process $\xi(t) = \pm 1$, with $\forall t \geqslant 0$, where two characteristic timescales appear in the waiting-time function for time increments $\{t_j - t_{j-1}\}$ [38]. In this manner the noise can emulate an intermittent stochastic absorption of energy in the wave propagation. The waiting time for the random increments of this binary noise is given by

$$\varphi(\Delta_{ij}) = \alpha p e^{-\alpha\Delta_{ij}} + \beta q e^{-\beta\Delta_{ij}}, \quad \Delta_{ij} = t_i - t_j, \tag{27}$$

with $p + q = 1$ and $\{\alpha, \beta, p\} \geqslant 0$.

In general, the stochastic TE can be written introducing an auxiliary function $\phi(x,t)$:

$$\partial_t \psi(x,t) = \phi(x,t), \tag{28}$$

$$\partial_t \phi(x,t) = v^2 \partial_x^2 \psi(x,t) - \tau^{-1}\phi(x,t) - \theta\xi(t)\phi(x,t). \tag{29}$$

Alternatively, we can write the stochastic TE in the Fourier representation as

$$\partial_t \psi(k,t) = \phi(k,t), \tag{30}$$

$$\partial_t \phi(k,t) = -k^2 v^2 \psi(k,t) - \tau^{-1}\phi(k,t) - \theta\xi(t)\phi(k,t). \tag{31}$$

This can be solved in an analytical way, using the enlarged technique for the master equation [18,35]. The exact mean-value solution of the system (30) to (31) can be represented as $\langle\psi(k,t)\rangle$. Therefore, taking the Markov limit (i.e., $\alpha = \beta = 1/2T$, or $p = 1$ or $q = 1$) we recover the solution of the previous section, that is,

$$\lim_{\alpha \to \beta = (2T)^{-1}} \langle\psi(k,s)\rangle \to (15). \tag{32}$$

As before, we can calculate the determinant of the denominator of the *mean effective* plane wave

$$\overline{\langle\psi(k,s)\rangle} = \frac{Q(s)}{R(s)} \equiv \frac{1}{s + D_{\text{eff}}(k=0,s)k^2}, \tag{33}$$

and so we calculate the characteristic polynomial (as a function of $s$ and $k$)

$$\text{Det} = \sum_{l=0}^{9} f_{2l} k^{2l}, \tag{34}$$

where the coefficients $f_{2l} = f_{2l}(s)$ are not given because they have a huge mathematical expression.

To obtain the localized gap, we use Det = 0 to obtaint the critical value of $k_c$, and so the gap $\mathcal{G} \equiv [-k_c, k_c]$ characterizes the localized Fourier modes (Det > 0). If $|k| > k_c$ Fourier-damped wave-like modes are obtained.

### D. General remarks

The Fourier-Laplace representation allows us to find the localized gap $\mathcal{G} \equiv [-k_c, k_c]$ for the ordered case (12), as well as in the case when there are Markovian binary fluctuations in the absorption of energy (25) (by introducing the *effective* plane-wave approach). In addition to these results we also find the polynomial (34) characterizing the localized gap $\mathcal{G}$ for the case when the fluctuations maybe intermittent.

As we commented before, we propose that the behavior for the attenuation of the mean-value wave packet is characterized by the wave function

$$\langle\psi(x,t)\rangle \approx e^{-x/L} g(x - vt), \text{ with } L = 1/k_c, \tag{35}$$

so the problem of the penetration of a wave in a complex medium is reduced to the calculation of the critical Fourier number $k_c$ [from Eq. (34)]. In Sec. V we are going to test this conjecture by comparing our theoretical predictions against numerical results considering fluctuation in the absorption of energy.

## IV. SIMULATIONS OF THE STOCHASTIC TE

Here, we numerically study the evolution of an initially moving (symmetric) pulse, this can be called the "bullet initial condition." To solve the stochastic TE (14) numerically we use a narrow centered Gaussian function as the initial condition for all cases analyzed in the paper. This narrow Gaussian simulates a pulse moving to the right from the origin ($x = 0$):

$$\begin{aligned} \psi(x,t)|_{t=0} &= \delta(x - vt)|_{t=0} = F(x), \\ \partial_t\psi(x,t)|_{t=0} &= -v\partial_x F(x), \\ F(x)|_{t=0} &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}, \end{aligned} \tag{36}$$

with $\sigma = 0.05$; the initial pulse is moving with the same $v$ as the propagation velocity in the TE. This initial condition will be used in all simulations.

### A. Ordered case (without fluctuations in the rate $\tau^{-1}$)

Taking $\theta = 0$ in (14), we get the ordered case $[\partial_t^2 + \frac{1}{\tau}\partial_t - v^2\partial_x^2]\psi(x,t) = 0$, this can be solved numerically using (36). In Fig. 1, we show the evolution solution of the TE as a function of space, the vertical shift in $\psi$ represents different values of time $t$. The time evolution of $\psi$ represents a pulse moving to the right, as predicted by the theory, this evolution has a factor of attenuation given by $\sim e^{-x/L}$; for this case $\tau = 5$ and $v = 1$, so $L = 10$ [see (4)], which represents the penetration length of the pulse (spatial attenuation). The dashed line represents when the initial pulse has been attenuated by a factor $e^{-1}$, thus, the wave behavior is expected for times $vt \ll L$ and space $x \ll L$, see (5) and (6).

### B. Markov binary noise (fluctuations in the rate $\tau^{-1}$)

The Markovian binary noise (also called dichotomic noise) is recovered if, in the waiting-time function (27), we take $p = 1$ or $q = 1$ or $\beta = \alpha = 1/2T$. In the context of the EME [18,35] equations (28) and (29) can be written as a system of differential equations, using auxiliary functions $\psi^{\pm}$ and $\phi^{\pm} = \partial_t\psi^{\pm}$, in space time

$$\partial_t \begin{pmatrix} \psi^+(x,t) \\ \phi^+(x,t) \\ \psi^-(x,t) \\ \phi^-(x,t) \end{pmatrix} = \begin{pmatrix} -a & 1 & a & 0 \\ v^2\partial_x^2 & -(\tau^{-1} + \theta + a) & 0 & a \\ a & 0 & -a & 1 \\ 0 & a & v^2\partial_x^2 & -(\tau^{-1} - \theta + a) \end{pmatrix} \begin{pmatrix} \psi^+(x,t) \\ \phi^+(x,t) \\ \psi^-(x,t) \\ \phi^-(x,t) \end{pmatrix}. \tag{37}$$

In accordance with the "bullet initial condition" (36) the system of equations (37) is solved numerically, using the initial condition

$$\begin{aligned} \psi^+(x,t) &= F(x)/2, \\ \phi^+(x,t) &= -v\partial_x F(x)/2, \\ \psi^-(x,t) &= F(x)/2, \\ \phi^-(x,t) &= v\partial_x F(x)/2, \end{aligned}$$

where $F(x) = e^{-x^2/2\sigma^2}/\sqrt{2\pi\sigma}$ is a narrow Gaussian function with $\sigma = 0.05$ [see (36)]. The mean-value plane-wave evolution of the pulse moving to the right is given by

$$\langle\psi(x,t)\rangle = \psi^+(x,t) + \psi^-(x,t). \tag{38}$$

In Fig. 2, it is shown that the evolution of the pulse for different values of $t = 0, 4, 8, 12, 16, 20, 22.09, 26, 30, 34$ and $v = 1, \tau = 10$. Now the factor of attenuation is $e^{-x/L}$, with $L = 1/k_c$ given in (25). In this case, it gets $L = 22.09 > v\tau$.

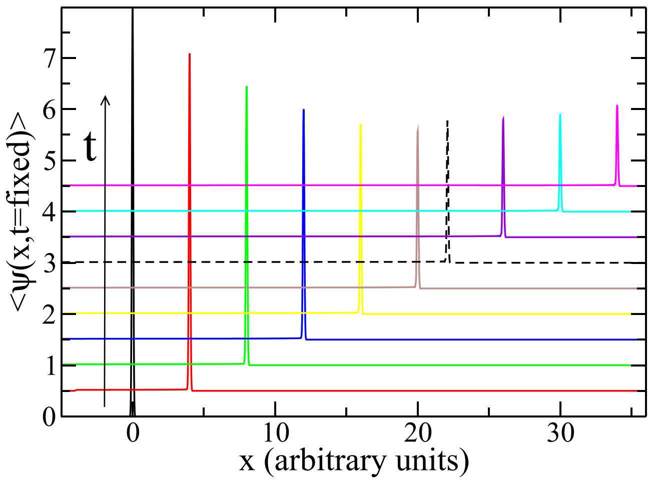


FIG. 2. Amplitude of $\langle\psi(x,t)\rangle$ (averaged over all noise realizations) as a function of space for the Markovian binary noise $\xi(t)$ with $\theta=\tau^{-1}$ and fixed values of times $t=0,4,8,12,16,20,22.09,26,30,34$. The pulse spreads and it suffers an attenuation in its amplitude. The parameters are $\tau=10$, $v=1$, $T=1$, thus $L=1/k_c=22.09$, and $k_c$ is given by (25).

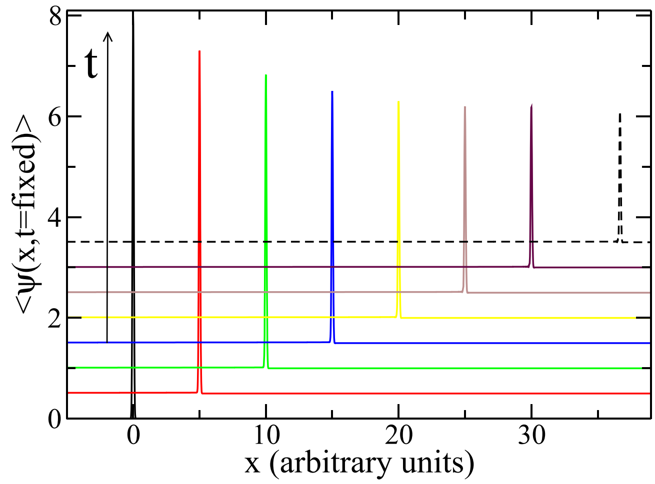


FIG. 3. Amplitude of $\langle\psi(x,t)\rangle$ (averaged over all noise realizations) as a function of space for the non-Markovian binary noise $\xi(t)$ with $\theta=\tau^{-1}$, for $t=0,5,10,15,20,25,30,36.68$. The parameters used are $\tau=15$, $v=1$, $\alpha=1$, $p=0.95$, $\beta=0.1$ (corresponding to the intermittent case). Thus $L=1/k_c=36.68$ and $k_c$ is given by setting Det $=0$, see (34).

### C. Non-Markov binary noise (fluctuations in the rate $\tau^{-1}$)

To study a non-Markov binary noise with biexponential waiting time (27), we can also use the EME approach [18]. In this case, the noise $\xi(t)$ is represented by four states $\{\xi_\alpha^+,\xi_\beta^+,\xi_\alpha^-,\xi_\beta^-\}$. Similar to the Markovian case, now we need the auxiliary functions $\psi^\pm_{\alpha(\beta)}$ and $\phi^\pm_{\alpha(\beta)}=\partial_t\psi^\pm_{\alpha(\beta)}$. The EME is

$$\partial_t\vec{V}=\mathbf{A}\cdot\vec{V}, \tag{39}$$

with $\vec{V}=\{\psi_\alpha^+,\phi_\alpha^+,\psi_\alpha^-,\phi_\alpha^-,\psi_\beta^+,\phi_\beta^+,\psi_\beta^-,\phi_\beta^-\}$, here $\mathbf{A}=\mathbf{H}+\mathbf{L}$ where $\mathbf{L}$ includes information on the TE and $\mathbf{H}$ is the master Hamiltonian associated to the biexponential waiting-time noise (non-Markovian); for details see Ref. [35].

In accordance with the "bullet initial condition" (36), the enlarged system of equations [associated to the four-states noise $\xi(t)$] must be solved using the initial condition

$$\begin{aligned}
\psi_\alpha^+(x,t)&=F(x)/4,\\
\phi_\alpha^+(x,t)&=-v\partial_xF(x)/4,\\
\psi_\alpha^-(x,t)&=F(x)/4,\\
\phi_\alpha^-(x,t)&=v\partial_xF(x)/4,\\
\psi_\beta^+(x,t)&=F(x)/4,\\
\phi_\beta^+(x,t)&=-v\partial_xF(x)/4,\\
\psi_\beta^-(x,t)&=F(x)/4,\\
\phi_\beta^-(x,t)&=v\partial_xF(x)/4,
\end{aligned}$$

where $F(x)=e^{-x^2/2\sigma^2}/\sqrt{2\pi\sigma}$ is a narrow Gaussian function with $\sigma=0.05$ [see Ref. (36)]. Solving the EME $\partial_t\vec{V}=\mathbf{A}.\vec{V}$, we finally get the exact result

$$\langle\psi(x,t)\rangle=\psi_\alpha^+(x,t)+\psi_\alpha^-(x,t)+\psi_\beta^+(x,t)+\psi_\beta^-(x,t). \tag{40}$$

In Figs. 3 and 4 we show $\langle\psi(x,t)\rangle$ for different values of $t$, with $\tau=15$, $v=1$, $\alpha=1$, $p=0.95$, and for two values of $\beta=0.1,0.5$, respectively. In Figs. 1, 2, 3, and 4 the plots in dashed lines correspond to the case when the pulse has decayed $e^{-1}$ from the initial condition. It can also be seen that for intermittent fluctuations in the rate of energy the pulse penetrates a longer distance in the medium.

### D. General remarks

The numerical solution of (28) and (29), for the evolution of a symmetric initially moving pulse (bullet initial condition)

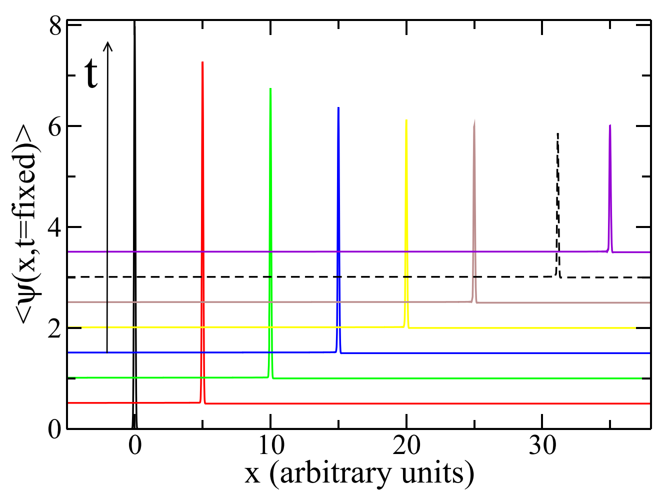


FIG. 4. Amplitude of $\langle\psi(x,t)\rangle$ (averaged over all noise realizations) as a function of space for the non-Markovian binary noise $\xi(t)$ with $\theta=\tau^{-1}$, and fixed values of the time $t=0,5,10,15,20,25,31.15,35$. The parameters used in these numerical results are $\tau=15$, $v=1$, $\alpha=1$, $p=0.95$, $\beta=0.5$. Thus $L=1/k_c=31.15$ and $k_c$ is given by setting $(34)=0$.

allows us to test our conjecture concerning the attenuation of a pulse in a complex medium.

We compared the profile $\psi(x,t)$ corresponding to the ordinary TE, at the some time $t^*$ when the amplitude has been reduced by a factor $e^{-1}$, against the exact mean-value profile $\langle\psi(x,t)\rangle$ corresponding to the case when there are fluctuations in the absorption of energy (with global disorder). It can be seen that the mean-value solution of the stochastic TE changes the time when the amplitude decreases by the factor $e^{-1}$ (compare the dashed lines in Figs. 1 to 4) and so the position of the peak of the PDF. The main shape of the distribution is preserved, while the intrinsic behavior of the mean-value process is quite different than from the ordinary TE. This fact can readily be seen calculating different statistical objects, for example, the velocity autocorrelation function (VAF), the mean-square displacement $\langle x(t)^2\rangle$, the diffusion coefficient $D(\theta,\alpha,\beta,p)$, and so on, all of these objects can be seen in our previous Ref. [35]. Here we prove, in addition, that the penetration length $1/k_c$ is also a relevant quantity that strongly depends on the fluctuations. For completeness we show (using the Green-Kubo formula [18]) the VAF in the Laplace representation for the general case $0\leqslant\theta\leqslant\tau^{-1}$ for Markovian fluctuations in the absorption of energy (Sec. III B)

$$D_{\text{eff}}(s)\equiv\int_0^\infty\langle V(0)V(t)\rangle e^{-st}dt=\frac{s^2}{2}\langle x(s)^2\rangle$$
$$=\frac{\tau v^2(1+sT+T/\tau)}{[(s\tau+1)(sT+1)+(s\tau+1-\tau^2\theta^2)T/\tau]},\quad(41)$$

as can be seen the presence of Markovian fluctuations (of intensity $\theta$ and correlation time $T$) changes the VAF drastically. Taking zero (global) disorder $\theta=0$, we recover the usual exponential result (in the Laplace representation). From the VAF (41) it is simple to calculate the diffusion coefficient $D=D_{\text{eff}}(0)$. The same conclusions happen considering the VAF for non-Markovian fluctuations (Sec. III C), the function $D_{\text{eff}}(s)$ can be obtained from our formulas (33), but in this case it is a huge expression to be written here.

The present results are calculated solving the EME associated to the stochastic evolution of the mean-value wave $\langle\psi(x,t)\rangle$, considering the random transitions coming from the four-state noise $\xi(t)$ [TE written in the forms (28) and (29)]. The ordered case corresponds to $\theta=0$; for Markovian noise $\xi(t)$ the EME is (37), while if the noise has two times the scale (non-Markovian case) the EME is $\partial_t\vec{V}=\mathbf{A}.\vec{V}$, with $\mathbf{A}=\mathbf{L}+\mathbf{H}$ given in terms of a fixed-noise TE and the master Hamiltonian $\mathbf{H}$ of the four-state (biexponential) noise $\xi(t)$ [35]. For all cases, the exact mean-value of the field $\langle\psi(x,t)\rangle$ can straightforwardly be written in terms of auxiliary functions $\psi^{\pm}_{\alpha,\beta}(x,t)$. The agreement between our theoretical predictions (Sec. III) using the *effective* plane-wave $\langle\overline{\psi(k,s)}\rangle$ and the present numerical solutions $\langle\psi(x,t)\rangle$ will be presented by comparing the absorption of energy per unit of area. These figures will be shown in the next section.

## V. ABSORPTION OF ENERGY PER UNIT OF AREA AND TIME IN A WAVE

To study the penetration of a plane wave in a complex medium let us study the space attenuation of a packet. Let us define the energy of the wave per unit of area and time as $\mathcal{P}(x,t)$ [1]. For instance, for one mechanical wave propagating to the right, the explicit expression is given in Appendix B. Remembering that in the attenuated regime a wave solution of TE, see (3), can be written as $\psi(x,t)\propto e^{-x/L}g(x-vt)$. It is possible to see that the contribution of the dissipative term to the variation $d\mathcal{P}/dx$ can be written as (see Appendix B)

$$\frac{d\mathcal{P}}{dx}\propto\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2.\quad(42)$$

For the stochastic TE with fluctuations in the rate $\tau^{-1}$, we can define a mean flux of energy such as

$$\overline{\frac{d\mathcal{P}}{dx}}=-(\partial_t\langle\psi(x,t)\rangle)^2.\quad(43)$$

Here we are concerned with a spatial dependence of this dissipative measure, whereby we integrate with time

$$\mathcal{I}\equiv\int_0^\infty\overline{\frac{d\mathcal{P}}{dx}}dt=-\int_0^\infty dt(\partial_t\langle\psi(x,t)\rangle)^2.\quad(44)$$

If we replace $\langle\psi(x,t)\rangle=e^{-x/L}g(x-vt)$ in (44), we get the mean power of absorption:

$$\mathcal{I}=-e^{-2x/L}\int_0^\infty dt\,[\partial_t g(x-vt)]^2\propto e^{-2x/L}A.\quad(45)$$

Therefore, as before, we propose $L=1/k_c$ for the stochastic TE. From our numerical results, we plot (44) and compare this with (45) using our theoretical results on $k_c$; this will be our goal.

In Figs. 5(a) and 5(b) the no-fluctuating case is presented, using $v=1$ it is shown that the variation of energy integrated on time is $\int_0^\infty\frac{d\mathcal{P}}{dx}dt$, for $\tau=5$ ($L=2v\tau=10$) and $\tau=0.2$ ($L=2v\tau=0.4$), respectively. Notice that only in the case when $\tau^{-1}$ is small, this variation is well fitted by an exponential curve $Ae^{-2x/L}$ (the small rate of absorption of energy $\tau^{-1}$ in the TE).

In Fig. 6, we show the mean power of absorption $\mathcal{I}$ as a function of space $x$, for binary Markovian noise $\xi(t)$ in the TE, that is, considering the term $-\theta\xi(t)\partial_t\psi(x,t)$ in (14). We fitted the simulated results with an exponential function $Ae^{-2x/L}$, with $A=-1060$, and the corresponding $L=1/k_c=22.09$ from (25), as before $k_c$ is the border of the localized gap; i.e., only for $|k|>k_c$ the mean plane-wave is damped but propagating. Here in this plot we take $v=1$, $\tau=10$, and $T=1$ (correlation-time of the Markov binary noise).

Now we show the non-Markovian noise case (27); as before considering the term $-\theta\xi(t)\partial_t\psi(x,t)$ in (14). In Figs. 7 and 8 we show $\mathcal{I}$ as a function $x$ for two values of $\beta=0.1,\ 0.5$, and $v=1$, $\tau=15$, $\alpha=1$, $p=0.95$. For both cases it is possible to fit with an exponential function as $Ae^{-2x/L}$, for $\beta=0.1$ (intermittent case) we get $A=-1070$, and the corresponding $L=1/k_c=36.68$ from (34), while for $\beta=0.5$, the values are $A=-1146$, with $L=1/k_c=31.15$ from (34). It is noted that for intermittent fluctuations in the rate of absorption of energy, the wave is safer in its propagation.

Thus, we conclude that for binary noise in the TE (Markovian and non-Markovian cases), our approach of the mean plane wave $\langle\psi(x,t)\rangle\backsim e^{-x/L}g(x-vt)$ is appropriated. In

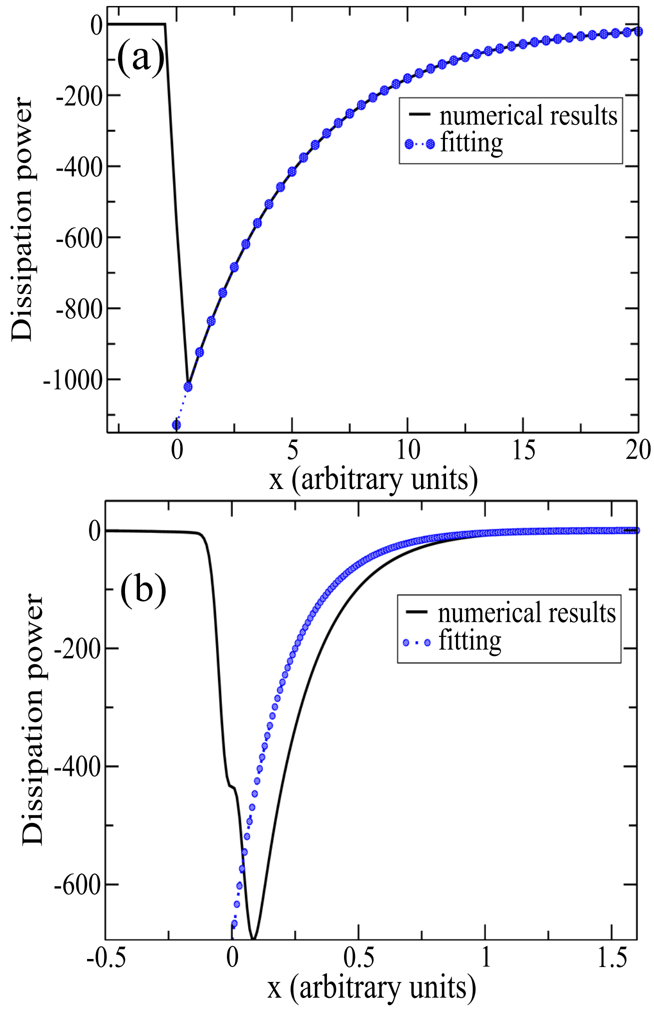


FIG. 5. Power of absorption $\int_0^\infty \frac{d\mathcal{P}}{dx}\, dt$ as a function of space $x$ for the ordered case, i.e., with $\theta = 0$ in (14). (a) With $\tau = 5$ and (b) $\tau = 0.2$, the plots in circles with lines is the exponential fitting $Ae^{-2x/L}$ where $A$ is a constant value, with $L = 1/k_c = 2v\tau$, for both plots $v = 1$.

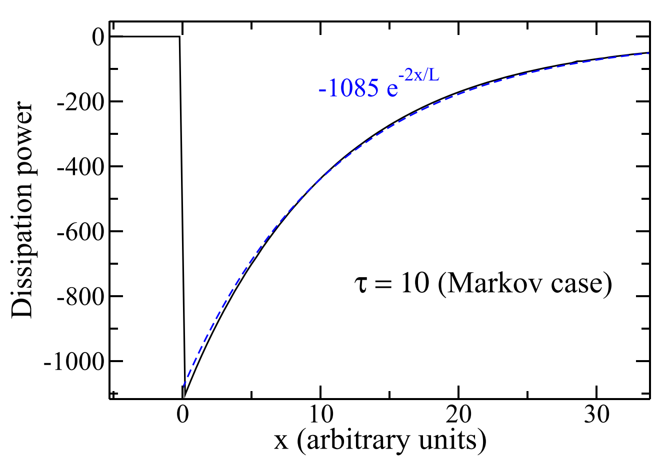


FIG. 6. Mean power of absorption $\mathcal{I}$ as a function of space $x$ for binary Markovian noise $\xi(t)$. The full line is the numerical result and the dotted line represents the fitting curve with exponential function $Ae^{-2x/L}$, with $A = -1085$, with $L = 1/k_c = 22.09$ [we obtain $k_c$ from (25)]. Setting parameters are $v = 1$, $\tau = 10$ and $T = 1$.

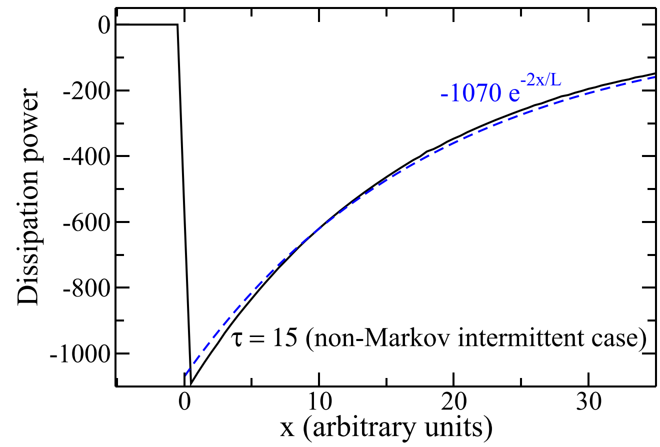


FIG. 7. Mean power of absorption $\mathcal{I}$ as a function of $x$ for binary non-Markovian noise $\xi(t)$ . The full line (black) represents the numerical result, and the dotted line (blue) represents the fitting curve with $Ae^{-2x/L}$, with $A = -1170$. The attenuation length is $L = 1/k_c = 36.68$ [we obtain $k_c$ setting (34) equal to zero]. The parameters are $v = 1, \alpha = 1, \tau = 15, p = 0.95, \beta = 0.1$ (corresponding to intermittent fluctuations in the rate of absorption of energy).

accordance with the result of the delocalization of the plane wave in a fluctuation conducting medium [37].

As a reference, we also calculate the attenuation length $L = 1/k_c$, from (34) to analyze the behavior of $L$ with the variation of some parameters of the stochastic TE. In Fig. 9 it is shown the attenuation length, $L$, for two values of $p = 0.5$ and $p = 0.95$ as a function of $\beta$ with $v = 1$, $\alpha = 1$, $\tau = 10$. This measure shows a decreasing monotonous behavior with $\beta$. Notice that $\alpha = \beta$ corresponds to the Markovian case. The intermittent limit $\beta \ll \alpha$ has higher values of $L$, indicating that the wave motion is more conserved due to the structure of the time correlation in the noise.

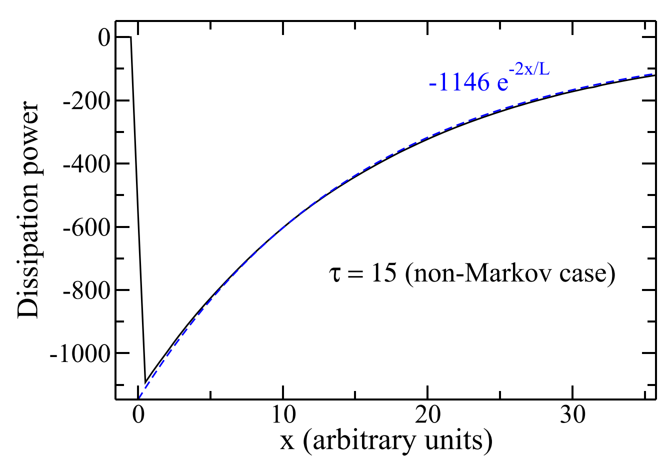


FIG. 8. Mean power of absorption $\mathcal{I}$ as a function of $x$ for binary non-Markovian noise. The full line (black) is the numerical results and the dotted line represents the fitting curve with $Ae^{-2x/L}$, with $A = -1146$. The attenuation length is $L = 1/k_c = 31.15$ [we obtain $k_c$ setting (34) equal to zero]. The parameters are $v = 1, \alpha = 1, \tau = 15$, $p = 0.95, \beta = 0.5$.

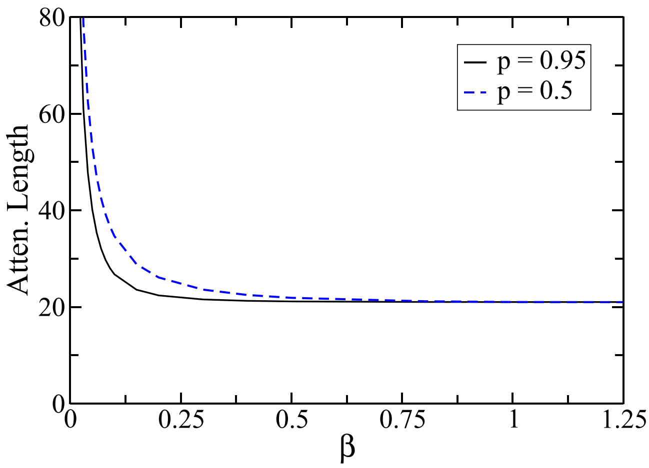


FIG. 9. The attenuation length $L$ for the case of non-Markovian noise $\xi(t)$ in the stochastic TE as a function of $\beta$ for $p = 0.5$, 0.95 with $v = 1, \alpha = 1, \tau = 10$. This result shows a monotonous behavior with $\beta$ (the inverse of one of the timescales in the biexponential waiting time of the noise, the other scale is set to $\alpha^{-1} = 1$ ). Thus, the case $\beta \ll 1$ corresponds to the intermittent limit.

## VI. CONCLUSION

As we commented in the Introduction, the TE has been a source of inspiration to solve many different problems ranging from mathematics and physics to engineering and biology. In addition to these comments, in engineering transport of waves in conducting media, it is important to have control on the space penetration of the wave in the medium [7–10,16,21,22] . Therefore, the analysis of the delocalization by the presence of global disorder in the rate $\tau^{-1}$ is an interesting problem to be tackled [37]. A second important measure to control the wave motion in a complex medium is to study the space attenuation in the pulse evolution. This was the subject in the present paper.

To understand the space penetration of a pulse, we studied the space attenuation of a plane wave. In fact, we analyzed an effective plane wave $\langle\overline{\psi(k,s)}\rangle$. From this effective wave function we were able to define the attenuation length $L$ as the inverse of the border of the localized gap. This measure was calculated analytically from the poles $s = s(k)$ of $\langle\overline{\psi(k,s)}\rangle$ with $k$ and $s$ being the Fourier and Laplace variables, respectively.

Therefore, our task, in the present paper, has been to characterize the penetration length $L$ as a function of the noise and telegrapher parameters. This is a complex task considering the dimension of the phase-space parameters. Nevertheless, we were able to characterize the behavior of $L$, finding that the space attenuation of the mean value of the solution of the TE can be characterized by a function of the form $\sim e^{-x/L} g(x - vt)$, where $L = 1/k_c$, with $k_c$ being the border of the localized gap, which can, in fact, be obtained from the denominator polynomial of $\langle\overline{\psi(k,s)}\rangle$ . We characterized the penetration length for the stochastic TE just by calculating $k_c$. The size of the gap regulates the penetration of the pulse, in principle, and for all fluctuation in the rate of energy $\tau^{-1}$, the critical value $k_c$ can be calculated by perturbation in Terwiel's diagrams [30], for example, using Poisson noise [39]. Nevertheless, only the non-Markovian binary noise, presented here, allows an exact analytical evaluation for $k_c$; this is the reason why the present exact results are of relevance.

To test our proposition $\langle\psi(x,t)\rangle \sim e^{-x/L} g(x - vt)$, we solved the stochastic TE numerically and compared the space-time behavior of $\langle\psi(x,t)\rangle$ against our theoretical predictions. This comparison was carefully tested for several values of the parameters of the model: Telegrapher parameters and noise parameters (to model the fluctuations in the rate $\tau^{-1}$). That is, we adopt $\tau^{-1} \to \ \tau^{-1} + \theta\xi(t)$ with $\xi(t)$ being a noise, and $0 \leqslant \theta \leqslant \tau^{-1}$ is the intensity of the fluctuations. In particular, we carried out an analytic work considering a binary noise $\xi(t) = \pm 1, \forall t \geqslant 0$ for the strong disorder case $\theta = \tau^{-1}$ (see Ref. [30] for details concerning weak versus strong disorder).

In the present paper, we analyzed Markovian and non-Markovian binary noises to consider also the intermittent case. Therefore, our general binary noise has three parameters: $\{p, \alpha, \beta\}$, with $p$ being the probabilistic weight for the timescale $\alpha^{-1}$ and $q = 1 - p$ the probabilistic weight for the timescale $\beta^{-1}$. The mean waiting time for transitions in the binary noise is given by $t_{\rm trans} = p/\alpha + q/\beta$, while the correlation function for this non-Markov noise is a complex biexponential function of all parameters, see Ref. [38] for details. The intermittent case can be considered, taking $p \ll 1$ with very different timescales $\alpha^{-1} \ll \beta^{-1}$.

We note that $\tau$ is the important timescale to preserve the wave motion in the ordinary TE, while, in the stochastic TE and if the binary noise has a biexponential waiting time, the timescale characterizing the attenuation of the pulse has an structure which depends on the competition between noise against telegrapher's parameters. The space lengthscale $L = 1/k_c$ has a nontrivial structure on the physical parameters, which was evidenced by the root of our equation (34) while if the noise is Markovian the expression for $k_c$ is much simpler, see (25). In particular we showed that in the intermittent limit the wave motion is safely preserved.

In addition to these results, the group velocity $ds(k)/dk$, which characterizes other aspects in the wave motion can also be studied from the poles of the mean effective plane-wave (33). This subject is in progress.

*Data availability*. The data that support the findings of this study are available from the corresponding author upon reasonable request.

## ACKNOWLEDGMENT

MN and MOC thank the funding provided by Secretaria de Ciencia y Tecnología de la Universidad Nacional de Cuyo.

## APPENDIX A: ELECTROMAGNETIC WAVE EQUATION WITH ABSORPTION

Dissipation in the wave motion occurs in conducting media. We assume electromagnetic waves in an isotropic, homogenous, nonferroelectric, and nonferromagnetic conducting medium, therefore, the presence of currents has to be taken into account. In terms of the conductivity (time-dependent parameter) $\sigma_c = \sigma_c(t)$ by Ohm's law we get

$$\mathbf{J} = \sigma_c \mathbf{E}, \tag{A1}$$

which is regarded as a constitutive relation in addition to $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. Assuming no free charges $\rho = 0$ and substituting (A1) in $\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t\mathbf{D}$ we get

$$\nabla\times\mathbf{B} = \frac{4\pi\mu}{c}\sigma_c\mathbf{E} + \frac{\epsilon\mu}{c}\partial_t\mathbf{E}. \tag{A2}$$

Then, using the identity $\nabla\times(\nabla\times\mathbf{Q}) \equiv \nabla(\nabla\times\mathbf{Q}) - \nabla^2\mathbf{Q}$ for any $\mathbf{Q}$, and the rest of Maxwell's equations

$$\nabla\times\mathbf{E} = 0,$$
$$\nabla\times\mathbf{B} = 0,$$
$$\nabla\times\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{B},$$

we find a damped evolution equation for the field $\mathbf{B}$:

$$\frac{\mu\epsilon}{c^2}\partial_t^2\mathbf{B} + \frac{4\pi\mu}{c^2}\sigma_c\partial_t\mathbf{B} = \nabla^2\mathbf{B}. \tag{A3}$$

In a similar way, we get for the field $\mathbf{E}$ the evolution equation

$$\frac{\mu\epsilon}{c^2}\partial_t^2\mathbf{E} + \frac{4\pi\mu}{c^2}\partial_t(\sigma_c\mathbf{E}) = \nabla^2\mathbf{E}. \tag{A4}$$

As it can be seen comparing (A3) with (A4) the symmetry in the evolution equation is only restored in the case that the conductivity is a constant. We can write a TE for each field

$$\partial_t^2\mathbf{B} + \frac{1}{\tau}\partial_t\mathbf{B} = v^2\nabla^2\mathbf{B}, \tag{A5}$$

$$\partial_t^2\mathbf{E} + \partial_t\left(\frac{1}{\tau}\mathbf{E}\right) = v^2\nabla^2\mathbf{E}, \tag{A6}$$

with

$$v^2 = \frac{c^2}{\mu\epsilon}, \qquad \frac{1}{\tau} = \frac{4\pi\sigma_c}{\epsilon}. \tag{A7}$$

The stochastic TE corresponds to the case when $\sigma_c = \sigma_c(t)$ is a random function of time and so $\tau^{-1} \to \tau^{-1} + \theta\xi(t)$ where $\theta$ is the amplitude of noise $\xi(t)$. Our stochastic TE (14) can therefore be associated to the evolution of the magnetic field in the presence of a fluctuating conductivity. For the electric field we get an extra term proportional to $\dot{\sigma}_c$.

## APPENDIX B: ABSORPTION OF ENERGY IN A DAMPED PLANE WAVE

To study the length of attenuation, we can define the energy of the wave per unit of area and time $\mathcal{P} \equiv \mathcal{P}(x,t)$. For instance, for a one-dimensional mechanical wave propagating to the right (where $\rho_0$ is a density of mass in a cord), $\mathcal{P}$ can be calculated as [1]

$$\mathcal{P} = \mathcal{A}^2\rho_0 v w^2\sin^2(kx - \omega t), \tag{B1}$$

where $\mathcal{A}$ is the amplitude of a trigonometric wave (the power of a wave in a given medium is proportional to the square of the amplitude). Remembering that, if $\tau \gg 1$ (arbitrary units) in the TE, we can approximate its solution by (3), i.e., $\psi(x,t) \simeq \exp(-x/2v\tau)g(x - vt)$. Thus, choosing

$$\psi(x,t) = \mathcal{A}\exp(-x/2v\tau)\cos(k[x - vt]),$$

we can write $\mathcal{P}$ as

$$\mathcal{P} = (\mathcal{A}')^2\rho_0 v\omega^2\sin^2(kx - \omega t), \tag{B2}$$

with $\mathcal{A}' \equiv \mathcal{A}e^{-x/2v\tau}$. Then the variation of this flux will be

$$\frac{d\mathcal{P}}{dx} = (\mathcal{A}')^2\rho_0\omega^2\{\omega\sin 2(kx - \omega t) - \frac{1}{\tau}\sin^2(kx - \omega t)\}.$$

If we take the average in the time of the first term, it becomes zero (this result represents the nondissipative contribution), so the second term comes from the dissipation. Thus, we get

$$\left(\frac{d\mathcal{P}}{dx}\right)_{\text{diss}} = -(\mathcal{A}')^2\rho_0\frac{\omega^2}{\tau}\sin^2(kx - \omega t) = -\frac{\rho_0}{\tau}\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2.$$

For the stochastic TE, we can define a dissipation flux of energy such as

$$\overline{\frac{d\mathcal{P}}{dx}} \equiv -(\partial_t\langle\psi(x,t)\rangle)^2,$$

this proposition will be contrasted with the mean value of the numerical solution of the stochastic TE. Due to the fact that we are concerned with the spatial dependence of this dissipation measure, whereby we integrate with time

$$\mathcal{I} = -\int_0^\infty dt\ (\partial_t\langle\psi(x,t)\rangle)^2. \tag{B3}$$

If we replace $\langle\psi(x,t)\rangle \sim e^{-x/L}g(x - vt)$ in (B3) and owing to the periodic structure of $g(x - vt)$ we get

$$\mathcal{I} = -e^{-2x/L}\int_0^\infty dt\ (\partial_t g(x - vt))^2 \propto e^{-2x/L}A,$$

which is the result presented in (45).

Now our proposition is to consider $L = 1/k_c$ for the stochastic TE. Plotting (44) from our numerical results presented in Sec. IV and comparing it with an exponential behavior will be our goal. So, having fluctuation in the rate of energy $\tau^{-1}$ (using Markovian and non-Markovian fluctuations) we will check that it is correct to consider $\langle\psi(x,t)\rangle \backsim e^{-x/L}g(x - vt)$ in the expression for $\mathcal{I}$. We note that in the ordinary TE, for large $\tau$, the relevant parameter is $L = 2v\tau$.

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