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Understanding turbulence through numerical simulations

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Abstract

Turbulence is one of the important problems in classical physics that still remain unsolved. The Navier–Stokes equations have been studied for almost two centuries now, and although they seem to properly describe the dynamics of fluids, we still do not have a clear understanding of even the simplest turbulent flows.

We present numerical simulations of three dimensional, homogeneous and isotropic turbulence at moderate Reynolds numbers. We externally drive the fluid with either helical or non-helical forces. In both cases we find that the externally driven system relaxes to a stationary turbulent regime, which is compatible with the Kolmogorov spectrum $E_k \approx k^{-5/3}$. In the helical case, we confirm that the kinetic helicity also cascades directly, along with energy and displaying a Kolmogorov spectrum as well. We find that the dissipation scale of both ideal invariants is also consistent with the Kolmogorov scale (i.e., $k_\nu \approx (\varepsilon/\nu^3)^{1/4}$).

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1. Introduction

Incompressible hydrodynamics has become a paradigmatic model for the theoretical study of homogeneous turbulence. In three-dimensional incompressible flows described

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by a velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, there are two known ideal invariants: the kinetic energy

$$E(t) = \int \mathrm{d}^3x |\boldsymbol{u}(\boldsymbol{x},t)|^2 \tag{1}$$

and the kinetic helicity

$$H(t) = \int \mathrm{d}^3x \ \boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{\omega}(\boldsymbol{x},t) , \tag{2}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ is the vorticity field of the flow. We can define the corresponding isotropic power spectra $E(k,t)$ and $H(k,t)$ such that

$$E(t) = \int_0^\infty \mathrm{d}k \, E(k,t) \tag{3}$$

and

$$H(t) = \int_0^\infty \mathrm{d}k \, H(k,t) . \tag{4}$$

Since the pioneering contribution made by Kolmogorov [1], many studies have assumed flows displaying reflexional symmetry, for which the kinetic helicity is obviously zero ($H(t) = 0$) and therefore plays no role in the dynamics. Within this assumption and for flows being externally driven at large scales ($k \simeq k_F$), the system is expected to relax to a stationary regime characterized by an energy dissipation rate ε and the well known Kolmogorov's energy power spectrum

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3} , \tag{5}$$

where C_K is the Kolmogorov's constant. This universal energy spectrum arises at intermediate wavenumbers between the forcing scale at $k \simeq k_F$ and the energy dissipation wavenumber

$$k_\nu = \left(\frac{\varepsilon}{\nu^3} \right)^{1/4} \tag{6}$$

as a result of a direct energy cascade flowing from k_F to k_ν.

In flows with a lack of reflexional symmetry, the kinetic helicity is in general non-zero and it is expected to play a role in the dynamics [2]. In astrophysics, helical flows are crucial for the generation of large scale magnetic fields as the result of turbulent dynamos. For helical stationary turbulence, Brissaud et al. [3] conjectured that the kinetic helicity injected at a rate ε_H by the external force also cascades directly, and displays a Kolmogorov spectrum $H(k) \simeq \varepsilon_H \varepsilon^{-1/3} k^{-5/3}$. This assertion was confirmed by an EDQNM (Eddy Damped Quasi Normal Markovian) closure calculation [4] and by direct numerical simulations [5], but assuming hyperviscosity as the dissipation process.

Using dimensional arguments, Ditlevsen and Giuliani [6] propose that the dissipation wavenumber for the kinetic helicity ($k_{\nu,H}$) is always smaller than the energy dissipation wavenumber k_ν, scaling with the Reynolds number like $k_{\nu,H}/k_\nu \simeq R^{-9/28}$. In a subsequent paper [7] they support their proposition with calculations from a shell model. This scenario implies that the small scale structures in helical turbulence (i.e., those within the range $k_{\nu,H} \ll k \ll k_\nu$) are always non-helical, posing a serious restriction for turbulent dynamo models.

In the present paper, we study the role of kinetic helicity in incompressible turbulent flows with the aid of three-dimensional direct numerical simulations. In Section 2 we briefly describe our numerical code, our results are described in Section 3, and our conclusions are summarized in Section 4.

2. Description of the code

We numerically integrate the Navier–Stokes equation

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f} \tag{7}$$

in a cubic box, assuming periodic boundary conditions and incompressibility (i.e., $\nabla \cdot \boldsymbol{u} = 0$). We performed simulations of 128^3 and 256^3 spatial grid points, using a pseudo-spectral scheme for the spatial derivatives and second order Runge–Kutta for the time integration [8]. We verified that the balance of energy and helicity is satisfied with a high degree of accuracy. We developed a parallel version of the code, which compiles with the MPI library (Message Passing Interface). The runs were performed in a 40 nodes Beowulf cluster at the Department of Physics of the University of Buenos Aires (Argentina).

The external force $\boldsymbol{f}$ consists of a stationary *ABC* flow with a wavenumber $k_F = 3$ (for further details on the code and the external force, see Ref. [9]), which is an eigenfunction of the curl operator with eigenvalue k_F. We used $\nu = 0.02$ in all the simulations and $|\boldsymbol{f}| = 6$. To develop non-helical turbulence, we apply an external forcing which is a superposition of sinusoidal functions of all Fourier modes satisfying $|\boldsymbol{k}| = k_F$. In the next section we show the results arising from both helical and non-helical simulations.

3. Power spectra

We performed 128^3 and 256^3 simulations with both helical and non-helical forcing, integrating Eq. (7) for several turnover times, to make sure that the system relaxes to a stationary regime. In Fig. 1a we show the time-averaged energy (full line) and helicity (dotted line) compensated power spectra (i.e., $k^{5/3}E(k)$ and $k^{5/3}H(k)$) for the non-helical case. Only the positive part of this spectrum has been displayed, while the total helicity remains approximately zero (see Fig. 1b). In Fig. 1b we see the total energy $E(t)$ and the total kinetic helicity (multiplied by 10 to allow its visualization and divided by k_F, i.e., $10\ H(t)/k_F$) as a function of time. We see in both plots that the content of helicity is negligibly small, as expected. The compensated energy spectrum is consistent with Kolmogorov, since at intermediate wavenumbers the spectrum remains approximately horizontal.

In Fig. 2a we display the time-averaged energy and helicity (divided by k_F) compensated power spectra for the case with helical forcing. We verify that indeed both spectra remain virtually identical throughout all wavenumbers

$$H(k) \approx k_F E(k) \tag{8}$$

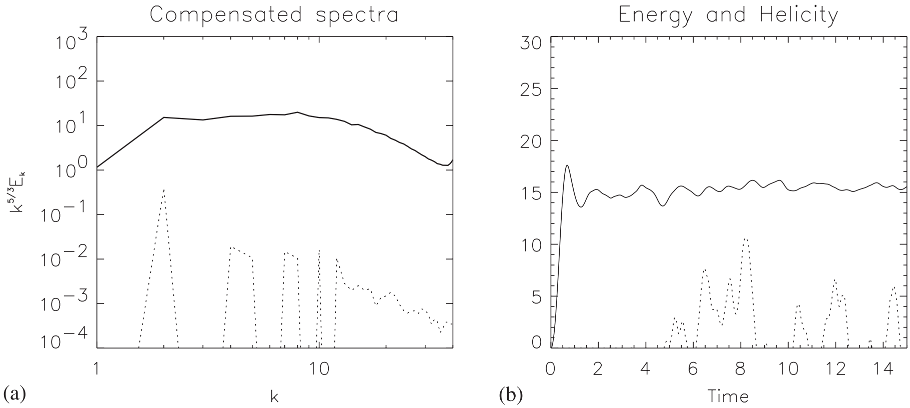

Fig. 1. (a) Time-averaged energy (full line) and helicity (dotted line) compensated power spectra corresponding to a non-helical stationary regime. Only the positive part of the helicity spectrum is plotted. (b) Total energy (full line) and total helicity (divided by k_F, dotted line) as functions of time. Helicity is multiplied by 10 to allow its visualization.

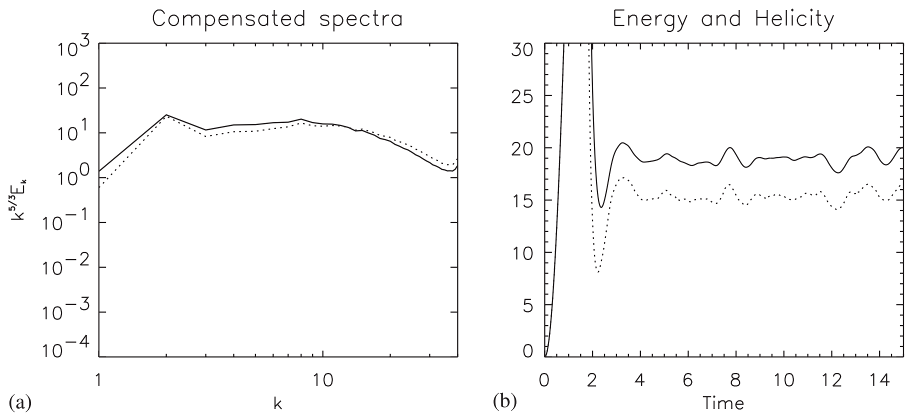

Fig. 2. (a) Time-averaged energy (full line) and helicity (dotted line) compensated power spectra corresponding to a helical stationary regime. (b) Total energy (full line) and total helicity (divided by k_F, dotted line) as functions of time.

i.e., not only within the inertial range, but also in the energy-containing and dissipative region of the spectrum. Therefore, our numerical simulations clearly show that dissipative scales of energy and helicity are the same (a similar result has also been obtained by Chen et al. [10]), indicating that the proposition of Ditlevsen and Giuliani [6] is probably incorrect. Fig. 2b shows the behavior of the total energy and total helicity (divided by k_F vs. time). We see that $H(t) \approx k_F E(t)$. In Fig. 3 we show a slice of the cubic box, displaying the spatial distribution of kinetic energy (Fig. 3a) and kinetic helicity (Fig. 3b). Only the patterns of positive helicity are shown. Note that the patterns

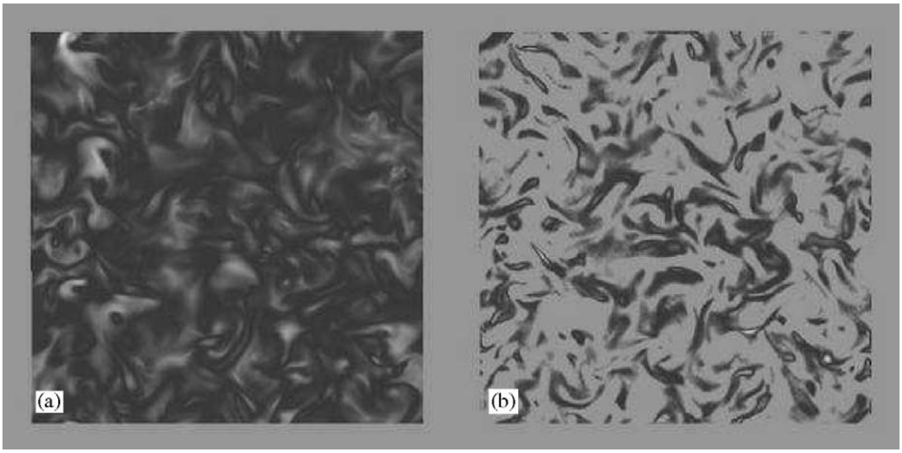

Fig. 3. (a) Slice of 256^2 displaying the intensity of kinetic energy for the helically driven case at $t = 10$. (b) Same slice of fluid displaying the intensity of positive kinetic helicity (the regions of negative kinetic helicity are grey).

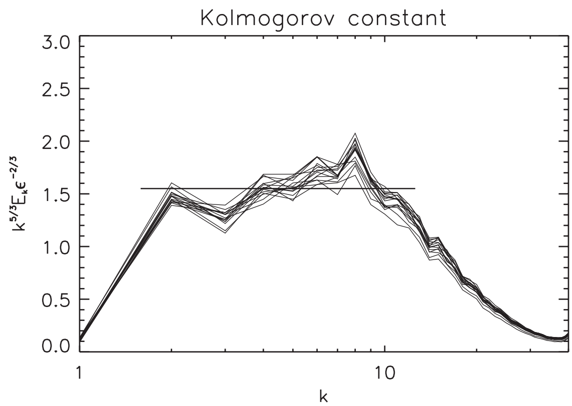

Fig. 4. Compensated energy power spectra divided by $\varepsilon^{2/3}$ for the non-helical forcing, at different times during the stationary turbulent regime. The best horizontal fit corresponds to a Kolmogorov constant of $C_K = 1.55$.

for both ideal invariants show similar contents of large and small scales, as expected from the similarity between their power spectra (see Eq. (8)). Note that the filling factor between both patterns is different, since we are only plotting the positive helicity. Chen et al. [11], running simulations at slightly higher resolution, find that kinetic helicity spatial structures are somewhat finer than those of kinetic energy.

We also obtained the energy dissipation rate as the time average of $\nu \int \mathrm{d}^3x \ |\boldsymbol{\omega}|^2$ in the stationary part of the simulation. For the non-helical case, the dimensionless dissipation rate is $\varepsilon \approx 33$. We determined the value of the Kolmogorov's constant C_K (see Eq. (5)) from Fig. 4, which displays compensated energy power spectra divided

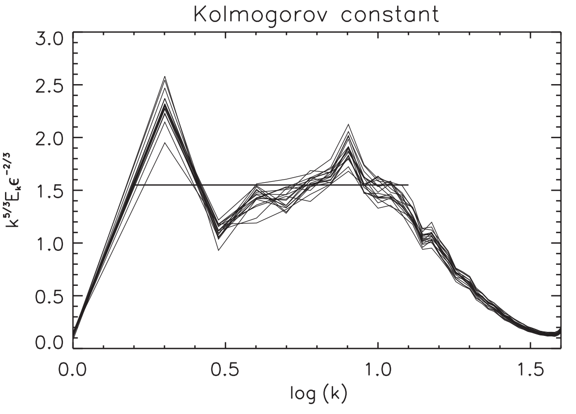

Fig. 5. Compensated energy power spectra divided by $\varepsilon^{2/3}$ for the helical forcing, at different times during the stationary turbulent regime. The best horizontal fit corresponds to a Kolmogorov constant of $C_K = 1.55$.

by $\varepsilon^{2/3}$ at different times during the stationary stage. The best horizontal fit corresponds to a Kolmogorov constant of $C_K = 1.55$.

We also obtained the energy dissipation rate and the Kolmogorov's constant for the helical case. The corresponding values are $\varepsilon \approx 35.2$ and again $C_K = 1.55$, as shown in Fig. 5.

We also estimated the energy dissipation wavenumbers using Eq. (6). The value for both for the non-helical and helical simulations is $k_\nu \approx 42$. Since our pseudo-spectral code uses the $\frac{2}{3}$ rule to dealias, the Nyquist wavenumber for the 128^3 runs is $k_{Ny} = 43$, and therefore we can guarantee that the dissipative scale is well resolved.

4. Conclusions

In the present paper we performed 128^3 and 256^3 direct simulations of the three dimensional incompressible Navier–Stokes equations to study the role of kinetic helicity in turbulent regimes. We find that kinetic helicity cascades along with energy from large to small scales, displaying a power spectrum which is proportional to the energy-power spectrum. The proportionality between both power spectra covers the energy-containing, the inertial and the dissipative ranges. In particular, we obtain that the helicity dissipative scale coincides with the energy dissipative scale.

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