Renormalization and Short Distance Singular Structure.

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The relation between renormalization and short distance singular divergencies in quantum field theory is studied. As a consequence a finite theory is presented. It is shown that these divergencies are originated by the multiplication of distributions (and worse defined mathematical objects). Some of them are eliminated defining a multiplication based in dimensional regularization while others disappear considering the states as functionals over the observables space. Non renormalizable theories turn to be finite, but anyhow they are endowed with infinite arbitrary constants.

I. INTRODUCTION.

Quantum Field Theory can be reduced to the knowledge of Wightman functions (or T-ordered Feynman functions or retarded functions or euclidean functions, etc.) [1], [2]. These functions are short distance singular mathematical objects (i. e. they diverge in the so called "coincidence limits" namely when some of their variables coincide) , e. g.: the symmetric part of the two points Wightman function has a Hadamard singularity, precisely:

$$w^{(2)}(x,x') = u\sigma^{-1} + v\ln|\sigma| + w \tag{1}$$

where $\sigma = \frac{1}{2}(x-x')^2$, and u, v, w are smooth functions [1]. These local singularities originate the infinite ultraviolet results of Quantum Field Theory [5][2]. To eliminate these infinities the theory must be renormalized in such a way that meaningless divergent expressions become meaningful. This technique is well known but not completely satisfactory because by using it "...we learned to peacefully coexist with alarming divergencies... but these infinities are still with us, even though deeply buried in the formalism." [1]. On the other hand, as we know that the short distance singularities are the cause of renormalization, if we somehow remove these singularities we will directly obtain a finite and exact Quantum Field Theory from the scratch. Phrased in another way: in this paper we will find the short distance singularity in two quantum field theory models and we will show that if these singularities are substracted the theory turns out to be finite. The substraction of short distance singularities has being essentially used for many years, e. g., in Quantum Field Theory in Curved Space-Time, [4], [6], [7] (and other chapters of quantum field theory e. g. [5] chapter 5), but it was not considered as a general method with a rational motivation, as we are now trying to prove.

We hope that the study of the singular short distance structure will lead us, in the future, either to find lagragians free of this sickness (may be superstring or membrane lagrangians) or to find more elaborated ways to remove this structure. Moreover, since the quantum field theory equations can be highly not linear it will be clear that, in a general case, the singular structure cannot be just removed by adding similar terms to those of the bare lagrangian. The mechanism must be more general. Here we are presenting the physical basis of this mechanism. Essentially we believe that, since the origin of the problem are the short distance singularities, philosophically it is wrong to put the blame on the old good lagrangian and to torture it until it yields a finite theory. The cure must be provided where the sickness is located[3].

We will find the singular structure using usual dimensional regularization [8] and, in the cases where it is possible, Hadamard regularization [4] and we will removed it by *two different ways* at two different level of comprehension that we will discuss below.

[1]For Wightman functions see [2], cap. VII, eq. (3.11). For Feynman functions see [3] eqs. (17.61) and (16.72). For symmetrical functions see [4].

[2]There is also another kind of potentially dangerous singularities as we will see in section VI.

[3]Renormalization would be like an electroshock. It works but we do not know why it is so. We are looking for something like brain surgery, where the disease is cured in the place where it is located.

A. Simple substraction method. Detection of the local singularities.

In sections II to V we will review this well known method with three purposes. i.- To introduce the main equations. ii.- To detect the local singularities (as in eqs. (28), (43), (80), (104), and (116)). iii.- To show the modification of the roles played by the coupling constant when we go from the usual method to the new one and to obtain renormalization group equations with the new method. We will study the theory in a space of dimensions n. Generically the theory will be finite for $n \neq 4$, but it will present short range singularities for $n \to 4$. E. g.: any two point function will have the structure:

$$w^{(2)}(x-x') = w^{(2)(s)}(x-x') + w^{(2)(r)}(x-x') \tag{2}$$

where $w^{(2)(s)}(x-x')$ is the singular component (in a sense that we will precise below), that diverges when $n \to 4$ or $x \to x'$, and $w^{(2)(r)}(x-x')$ is the regular one. The substraction method, for these functions, is just to make the singular part equal to zero or to subtract the singular part from $w^{(2)}(x-x')$. We will give two examples of this procedure:

i.-Scalar quantum field theory in a curved space-time (a theory invariant under the group of general coordinates transformations, with no self interaction and therefore with linear equations with variable coefficients) in section II. In this case we only need two points functions as those of eq. (2).

ii.-$\lambda\phi^4$ theory (a theory invariant under the Poincaré group with self interaction and therefore with non-linear equations with constant coefficients) studied in sections III, IV, and V. In the second example we will need N-point functions.

These example are chosen not only because they are the simplest but also because the two theories are quite different and cover a large range of phenomena[4]

Then, let us precise how we will define the singular and the regular components in the general case of N-point functions, in complete agreement with the usual procedures of dimensional regularization. If $w^{(N)}(x_1, x_2, ..., x_N)$ are some (symmetric) N-point functions (like Feynman or Euclidean functions) we can define the corresponding functional generator ([2], eq. (II.2.21), [5] eq. (3.2.11)) as:

$$Z[\rho] = \exp i \left\{ \frac{1}{N!} \sum_{N=0}^{\infty} \int w^{(N)}(x_1, x_2, ..., x_N)\rho(x_1)\rho(x_2)...\rho(x_N) dx_1 dx_2 ... dx_N \right\} \tag{3}$$

where [5]:

$$w^{(N)}(x_1, x_2, ..., x_N) \sim \langle 0|\phi(x_1)\phi(x_2)...\phi(x_N)|0\rangle \tag{4}$$

But, in a realistic field theory (namely a theory with interaction) these functions are badly defined (as the two-point function of eq. (2)) since they are objects with mathematical properties that are *worse than those of the distributions,* because if these objects were distributions all the integrals:

$$\int w^{(N)}(x_1, x_2, ..., x_N)\rho(x_1)\rho(x_2)...\rho(x_N) dx_1 dx_2 ... dx_N$$

would be well defined (if, e. g.: $\rho(x) \in \mathcal{S}$ the Schwarz space). But this is not the case, as we will see, so $Z[\rho]$ and its derivatives are not well defined[6].

As we have already said in the case of quantum filed theory in curved space time we only deal with two point functions. But for the $\lambda\phi^4$-theory we will deal with the two, four and six point functions, in the coincidence limit where some points go to 0 and some points go to an arbitrary value z, because these are the only relevant functions in the perturbation expansion of this theory up to λ^2 order. So we will be only interested in defining the singular and regular parts of the functions $w^{(2)}(x_1, x_2)$, in the coincidence limit $x_1 = x_2 = 0$, function $w^{(4)}(x_1, x_2, x_3, x_4)$, in the coincidence limit $x_1 = x_2 = 0$, $x_3 = x_4 = z$, and function $w^{(6)}(x_1, x_2, x_3, x_4, x_5, x_6)$, in the coincidence

[4] E. g. conformal or trace anomaly, conservation of the energy momentum tensor, etc. in example i.

[5] The symbol $\sim$ means that the r.h.s. of the next equation can also be truncated ([2], eqs. (II.2.18) and (II.2.23).

[6] Namely, axiom B of [2], page 58, is only valid for free theories, since from this axiom and Schwartz "nuclear theorem" it is shown that (4) is a distribution. Moreover, not only it is necessary that $Z[\rho]$ would be well defined but also its $\partial/\partial\rho-$derivatives. So all $w^{(N)}(x_1, x_2, ..., x_N)$ must be well defined functions after renormalization.

limit $x_1 = x_2 = x_3 = 0$, $x_4 = x_5 = x_6 = z$. We will see that these coincidence limits have the general form $[w^{(2)}(0)]^{\beta}[w^{(2)}(z)]^{\alpha}$, namely the product of the power of an infinite quantity multiplied by the power of a distribution (or a worse mathematical object). In fact, these powers appear in the higher order point functions (see [2] eq. (II.2.18)). So we have two problems that we will solve using dimensional regularization:

i.- To obtain the regular part of $w^{(2)}(0)$. It is an easy problem since via dimensional regularization $w^{(2)}(0)$ reads:

$$w^{(2)}(0) = \sum_{\gamma=0}^{C} \frac{d^{(\gamma)}}{(n-4)^{\gamma}} \tag{5}$$

where C is a natural number and $d^{(\gamma)}$ are some coefficients. Then the singular and regular components will be defined as:

$$[w^{(2)}(0)]^{(s)} = \sum_{\gamma=1}^{C} \frac{d^{(\gamma)}}{(n-4)^{\gamma}} \tag{6}$$

and

$$[w^{(2)}(z)]^{(r)} = d^{(0)} \tag{7}$$

Then the regular part of $[w^{(2)}(0)]^{\beta}$ is simply $[d^{(0)}]^{\beta}$.

ii.- To obtain the regular part of $[w^{(2)}(z)]^{\alpha}$. This is a more difficult problem since we must *multiply the ill defined function* $w^{(2)}(x_1, x_2)$ *by itself.* But function $w^{(2)}(x_1, x_2)$ is worse than a distribution, so it cannot be multiplied by itself in a unique and well defined way[7]. Thus we will be forced to define the multiplication procedure for, e. g.: $[w^{(2)}]^2$ and $[w^{(2)}]^3$, in an ad hoc way based on *dimensional regularization* (see [5] pages. 162 to 167 and 207 to 214). To stress this fact we will call them $[w^{(2)}]^{(d)2}$ and $[w^{(2)}]^{(d)3}$ respectively (where the susperindex "d" comes from "dimensional regularization"). Then the *multiplication procedure* will be the following:

a.-Using dimensional regularization we will find that the powers are regular when $n \neq 4$, but when $n \to 4$ they behave as:

$$[w^{(2)}(z)]^{(d)\alpha} = \sum_{\delta=0}^{D} \frac{d^{(\alpha,\delta)}(z)}{(n-4)^{\delta}} \tag{8}$$

where D is a natural number and $d^{(\alpha,\delta)}(z)$ are distributions (showing that, in effect, the objects we are dealing with are worse than distributions).

b.- The singular and regular component will be defined as:

$$[w^{(2)}(z)]^{(d)\alpha(s)} = \sum_{\delta=1}^{D} \frac{d^{(\alpha,\delta)}(z)}{(n-4)^{\delta}} \tag{9}$$

and

$$[w^{(2)}(z)]^{(d)\alpha(r)} = d^{(\alpha,0)}(z) \tag{10}$$

Moreover, the multiplication (ii) and the procedure to take the regular part for $z = 0$ (i) are *not commutative.* After these definitions we can substitute $[w^{(2)}(0)]^{\beta}$ and $[w^{(2)}(z)]^{\alpha}$ by $[w^{(2)}(0)^{(r)}]^{\beta}$ and $[w^{(2)}(z)]^{(d)\alpha(r)}$. Then if we consider only these regular parts, which are in general distributions (but regular functions in the two examples below), the functional generator $Z[\rho]$ and its derivatives (eq. (3)) turns out to be well defined as well as the theory that it generates. The existence of singularities like those of the above equations is proved by the examples below (see also section V). The decompositions (6), (7) and (9), (10) are not unique, since $\infty = \infty + c$ or $\infty = c.\infty$, for any finite c. This ambiguity will be present in our method, as in ordinary renormalization theory, and it yields the running coupling constants and the renormalization group, as we will see.

[7]Here is where one type of the divergencies is "deeply buried in the formalism" [1]. We will find another type of potentially dangerous divergencies in section VI.

B. Functional method.

In section VI we will present a mathematical structure that naturally yields the elimination of the singularities. We will follow the line of thought of papers [9] and [10] where a formalism to deal with systems with continuous spectrum was introduced. It proves to be useful in the study of decaying, equilibrium, and decoherence (where we have defined a final intrinsically consistent set of histories). So we claim that perhaps it is a *general formalism* that can also be used in the problem of this paper. This mathematical structure would also be the rational justification of the somehow dictatorial or childish substraction method. This is the main contribution of the paper. The idea is the following: Coarse-graining is a well known technique where some features of a system are considered relevant while others are not[8]. The functional method of papers [9], [10] is a generalization of coarse-graining [9], where the states are considered as functional over a certain space of observables[10]. Using this philosophy we will postulate that physical observables are such that cannot see the singular components of the states, because these components are irrelevant for these observables. Symmetrically, singularities could be contained in the observables and we can postulate that physical states cannot see the singular part of the observables[11]. In this way we will obtain the automatic substraction of all kinds of the singularities. There is a good physical reason for this postulate: the singularities (either of states or observables) are just mathematical artifacts originated in the oversimplified lagrangian that we usually choose. Then, clearly physical observables or states cannot see these mathematical unphysical objects. In a more intuitive language: the physical observables or states do not see the singularities because they are too small (point-like). Possibly the physical observables and states just see up to Planck's length [12].

Using the Jaynes philosophy [11] we can say that if *physical* observables can not see *mathematical* singularities (which in fact is a very reasonable position) then the (singular) states of the usual theory are really biased objects because they contain *arbitrary unphysical information* (i. e. the singularities) that cannot be measured by the physical apparatuses that we have in our laboratory i. e. our physical observables (and really this is an experimental fact: since apparatuses measure the values given by the finite renormalized theory). Then the (rough material) singular states, observables, and the mean values obtained with them are *biased over-informed* objects containing dubious information, because in fact "we have a basic ignorance of the nature of infinite energies or infinitesimal distances" ([5], page. 63), while renormalized (or free of any kind of singularities) states, observables, and mean values are *unbiased* objects containing just the physical information available. In fact, to suppose that we know and measure everything would be an "inexcusable hubris" ([5], page. 64). Moreover the resulting theory turns out to be insensitive to our degree of knowledge (originated in the more or less precision of our measurement apparatuses), thus we simply *postulate that this degree of knowledge is, and cannot be, infinite.* All this philosophy is embodied in the mathematical structure studied in section VI.

We will discuss our conclusions in section VII.

II. FIRST METHOD: SCALAR QUANTUM FIELD THEORY IN CURVED SPACE-TIME[13] TO BE READ AFTER SECTION VII. BUT WE CONSIDER THAT THIS DIDACTICAL DISCUSSION IS ESSENTIAL IN ORDER TO CONVINCE THE READER THAT THE NEW FORMALISM ALSO WORKS IN PRACTICE..

This theory is the simplest non-trivial example of the method, the theory of a scalar neutral massive fields in a curved space-time (of dimension n, since we need a formalism prepared for dimensional regularization) with metric

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[8] Or, in observables language, the observables of theory measure only the relevant features.

[9] E. g.: classically coarse-graining is just the particular case where the functionals are built using the characteristic functions of lattices in phase space (see [9]).

[10] Moreover, this is the natural way to face the problem since the observables are more primitive objects than the states [2].

[11] Really this will be the case since observables are products like $\phi(x_1)\phi(x_2)$...of field $\phi(x)$, which are distributions or worse defined mathematical objects.

[12] We could as well postulate that the singular part of the observables see the singular part of the state. Even if there are physical reasons to introduce this postulate in the case of decoherence, this reasons are absent in the case of renormalization (see section VI.B)

[13] THE EXPERT READER MAY GO DIRECTLY TO SECTION V AND CONSIDER SECTIONS II TO IV AS A *DIDACTICAL APPENDIX*

$g_{\mu\nu}(x)$. Let us consider the action [14]:

$$(6.9) \qquad S = S_g + S_m \tag{11}$$

where:

$$(6.11) \qquad S_g = \int (-g)^{\frac{1}{2}} (16\pi G_0)^{-1} (R - 2\Lambda_0) d^n x \tag{12}$$

and:

$$S_m = \int (-g)^{\frac{1}{2}} L_m d^n x \tag{13}$$

where L_m is the matter lagrangian:

$$(3.24) \qquad L_m(x) = \frac{1}{2} \left\{ g^{\mu\nu}(x) \phi_{,\mu}(x) \phi_{,\nu}(x) - [m^2 + \xi R(x)] \phi^2 \right\} \tag{14}$$

G_0 and Λ_0 are the bare Newton and cosmological constants respectively, m is the scalar field mass, $g^{\mu\nu}$ the inverse metric tensor (signature +,-,-,-), g its determinant, ξ a numerical factor, and $R(x)$ the Ricci scalar. For an in-out scattering we can define the functional generator $Z[\rho]$ such that:

$$(6.15) \qquad Z[0] = \langle out, 0 | in, 0 \rangle = e^{iW} \tag{15}$$

so:

$$(6.19) \qquad W = -i \ln \langle out, 0 | in, 0 \rangle \tag{16}$$

Then W can be computed using the effective lagrangian L_{eff}, defined by:

$$(6.36) \qquad W = \int [-g(x)]^{\frac{1}{2}} L_{eff}(x) d^n(x) \tag{17}$$

where L_{eff} reads:

$$(6.37) \qquad L_{eff}(x) = \frac{i}{2} \lim_{x' \to x} \int_{m^2}^{\infty} dm^2 \Delta_F^{DS}(x, x') \tag{18}$$

where $\Delta_F^{DS}(x, x')$ is the De-Witt-Schwinger-Feynman-Green function:

$$(3.138) \qquad \Delta_F^{DS}(x, x') = -i \Delta^{\frac{1}{2}}(x, x') (4\pi)^{-\frac{n}{2}} \times$$

$$\int_0^{\infty} ids (is)^{-\frac{n}{2}} \exp[-im^2 s + \frac{\sigma}{2is}] F(x, x'; is) \tag{19}$$

The $\sigma(x, x')$ is half the square of the geodesic distance between x and x', $\Delta(x, x')$ is the van Vleck-Morette determinant, and

$$(3.137) \qquad F(x, x'; is) = a_0(x, x') + a_1(x, x') is + a_2(x, x') (is)^2 + ... \tag{20}$$

where the a coefficients can be obtained from ref. [6] eqs. (3.131, 2, 3) and corresponds to an expansion in the metric $g_{\mu\nu}(x)$ and its derivatives, precisely to orders 0, 2, 4,...in these derivatives. The coefficients are biscalars, namely all the formalism is covariant under general coordinates transformation.

Eq. (18) is the simple non trivial example of the relation between L_{eff} and the two points function $\Delta_F^{DS}(x, x')$ in the limit $x \to x'$, where in fact $\Delta^{DS}(x, x')$ has a short distance singularity that makes L_{eff} a divergent quantity as we will see. If we want to retain the $n = 4$ dimension of L_{eff} as $(lenght)^{-4}$ when $n \neq 4$, we must introduce an arbitrary mass μ. Then L_{eff} reads:

$$(6.42) \qquad L_{eff} = \frac{1}{2} (4\pi)^{-\frac{n}{2}} \left(\frac{m}{\mu} \right)^{n-4} \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma \left(j - \frac{n}{2} \right) \tag{21}$$

where $a_j(x) = a_j(x, x)$ are functions of the curvatures and its derivatives, and the Γ function diverges when $n \to 4$.

[14] For the sake of conciseness we do not demonstrate the basic equations of quantum field theory in curved space-time. We just quote the number of the equation of reference [6] at the beginning of each of them. In sections III, IV, and V we will use reference [5] for the same purpose in the $\lambda\phi^4$ case.

A. Renormalization using dimensional regularization

By the dimensional regularization method everything is now prepared to renormalize the theory. When $n \to 4$ the first three terms (those that correspond to orders 0, 2, 4) diverge and we obtain the divergent or singular component of L_{eff} that reads (we have dropped the $O(n-4)$ terms):

$$(6.44) \qquad L^{(s)}(x) = -(4\pi)^{-\frac{n}{2}} \left\{ \frac{1}{n-4} + \frac{1}{2} \left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right) \right] \right\} \times$$

$$\left(\frac{4m^4 a_0(x)}{n(n-2)} - \frac{2m^2 a_1(x)}{n-2} + a_2(x) \right) \tag{22}$$

where:

$$(6.46) \qquad a_0(x) = 1$$

$$(6.47) \qquad a_1(x) = \left(\frac{1}{6} - \xi\right) R$$

$$(6.48) \qquad a_2(x) = \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right) \Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 \tag{23}$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor and $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$. The usual renormalization procedure is to absorb this singular component in the bare S_g, so we can renormalize G_0 and Λ_0 as:

$$(6.50) \qquad \Lambda_{phys} = \Lambda_0 + \frac{32\pi m^2 G_0}{(4\pi)^{\frac{n}{2}} n(n-2)} \left\{ \frac{1}{n-4} + \frac{1}{2} \left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right) \right] \right\} \tag{24}$$

$$(6.51) \qquad G_{phys} = G_0/1 + 16 G_0 \frac{2m^2 \left(\frac{1}{6} - \xi\right)}{(4\pi)^{\frac{n}{2}} (n-2)} \left\{ \frac{1}{n-4} + \frac{1}{2} \left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right) \right] \right\} \tag{25}$$

(where we have neglected the squares terms in the bare constants) so we choose G_0 and Λ_0 in such a way that G_{phys} and Λ_{phys} turn out to be finite when $n = 4$. But this is not enough since the divergence of the $a_2(x)$ term cannot be eliminated in this way, so the theory with action S_g is not renormalizable. But, if we add three "H" terms to the gravitational lagrangian which are linear combinations of the three terms of eq. (23): i. e., linear combinations of R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, and $\Box R$, (there are only three "H" terms because there is a relation among the last four terms) and renormalize the three corresponding coefficients (known as α, β, γ) the theory becomes renormalizable and finite (see [6] eqs. (6.52) to (6.56)). So from now on we will consider that these "H" terms are added to the gravitational lagrangian (12).

But let us observe that essentially what we have done with this standard renormalization recipe is to define, as proved in ref. [6], a regular-substracted lagrangian [15], that for $n = 4$ reads:

$$(6.59) \qquad L^{(r)} = L_{eff} - L^{(s)} = \frac{1}{32\pi^2} \int_0^{\infty} \sum_{j=3}^{\infty} a_j(x) (is)^{j-3} e^{-im^2 s} i ds \tag{26}$$

which turns out to be finite and can be used instead of the divergent L_{eff} [16]. Thus we can foresee that both the standard renormalization recipe and the substraction recipe coincide. What we have really made is a substraction

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[15] We are using a particular criterion to define the singular component. This criterion is neither unique not irrelevant [12]. It is clear that the singular term must have the form $\infty \times$ *geometrical object* (namely invariant under general coordinates transformations) But this object can be chosen in a variety of ways, since, as we have said, we know that $\infty = \infty + c$ or $\infty = c.\infty$ for any finite c.

[16] Eq. (26) shows that already in ref. [6] substraction was used in quantum field theory in curved space-time, as we have said in the introduction.

using dimensional regularization. Making the same substraction in $\Delta_F^{DS}(x,x')$ (eq. (20)) we will obtain the regular $\Delta_F^{DS(r)}(x,x')$ [17]. We will make this calculation in the next section using Hadamard regularization [4] because using this method we can better show the presence and nature of the local singularities.

B. Hadamard regularization and the substraction recipe.

Let us now see how we can directly work in the $n=4$ case. The divergencies now appear when $x \to x'$ (not when $n \to 4$ as in the previous section). In this section we will see how the two singular behaviors are related. The effective lagrangian (21) reads:

$$(6.38) \qquad L_{eff} = -\lim_{x' \to x} \frac{\Delta^{1/2}(x,x')}{32\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-(m^2 s - \sigma/2s)} \times$$

$$\left[a_0(x,x') + a_1(x,x')is + a_2(x,x')(is)^2 + ...\right] \tag{27}$$

From eqs. (19) we may compute:

$$\Delta_F^{DS}(x,x') = -i\frac{\Delta^{1/2}(x,x')}{(4\pi)^2} \int_0^\infty ids(is)^{-2} e^{-(m^2 s - \sigma/2s)} \times$$

$$\left[a_0(x,x') + a_1(x,x')is + a_2(x,x')(is)^2 + ...\right] = \overline{\Delta_F^{DS}(x,x')} + \frac{1}{2} i \Delta_F^{DS(1)}(x,x') \tag{28}$$

where (([3], eqs. (17.61), (17.62)):

$$\overline{\Delta_F^{DS}(x,x')} = \frac{\Delta^{1/2} a_0}{8\pi}\delta(\sigma) - \frac{\Delta^{1/2}}{8\pi}\theta(\sigma)\left[\frac{1}{2}(m^2 a_0 - a_1) - \frac{2\sigma}{2^2.4}(m^4 a_0 - 2m^2 a_1 + 2a_2)\right.$$

$$\left. + \frac{(2\sigma)^2}{2^2.4^2.6}(m^6 a_0 - 3m^4 a_1 + 6m^2 a_2 - 6a_3) + ...\right] \tag{29}$$

and

$$\Delta_F^{DS(1)}(x,x') = -\frac{\Delta^{1/2} a_0}{4\pi^2 \sigma} + \frac{\Delta^{1/2}}{2\pi^2}\log\frac{e^\gamma}{2}|2m^2\sigma|\left[\frac{1}{2}(m^2 a_0 - a_1)\right.$$

$$\left. - \frac{2\sigma}{2^2.4}(m^4 a_0 - 2m^2 a_1 + a_2\right]$$

$$-\frac{\Delta^{1/2}}{2\pi^2}\left[\frac{1}{4}m^2 a_0 - \frac{2\sigma}{2^2.4}\left(\frac{5}{4}m^4 - 2m^2 a_1 - a_2\right)\right.$$

$$\left. + \frac{(2\sigma)^2}{2^2.4^2.6}\left(\frac{5}{3}m^6 a_0 - \frac{9}{2}m^4 a_1 + \frac{15}{2}m^2 a_2 - \frac{9}{2}a_3\right) + ...\right]$$

$$\frac{\Delta^{1/2}}{2\pi^2}\left[\left(\frac{a_2}{4m^2} + \frac{a_3}{4m^4} + \frac{a_4}{8m^6} + ...\right)\right.$$

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[17] Since this is the only non vanishing truncated point function in the theory, all ordinary point functions of the theory are finite and they can be directly computed.

$$-\frac{2\sigma}{2^2.4}\left(\frac{a_3}{m^2}+\frac{a_4}{m^4}+...\right)+...\Bigg] \tag{30}$$

According to dimensional regularization the singular part of $\Delta_F^{DS}(x,x')$ corresponds to the one with coefficients $a_0, a_{1,}a_2$ (see (22)). The remaining terms are the regular part (see (26)). Then:

$$\Delta_F^{DS(s)}(x,x') = \frac{\Delta^{1/2}a_0}{8\pi}\delta(\sigma) - \frac{\Delta^{1/2}}{8\pi}\theta(\sigma)\left[\frac{1}{2}(m^2a_0-a_1) - \frac{2\sigma}{2^2.4}(m^4a_0-2m^2a_1+2a_2)\right]+$$

$$+\frac{i}{2}\left\{\frac{\Delta^{1/2}a_0}{4\pi^2\sigma}+\frac{\Delta^{1/2}}{2\pi^2}\log\frac{e^\gamma}{2}|2m^2\sigma|\left[\frac{1}{2}(m^2a_0-a_1)\right.\right.$$

$$\left.-\frac{2\sigma}{2^2.4}(m^4a_0-2m^2a_1+a_2)\right]$$

$$-\frac{\Delta^{1/2}}{2\pi^2}\left[\frac{1}{4}m^2a_0-\frac{2\sigma}{2^2.4}\left(\frac{5}{4}m^4-2m^2a_1-a_2\right)\right.$$

$$\left.\left.+\frac{(2\sigma)^2}{2^2.4^2.6}\left(\frac{5}{3}m^6a_0-\frac{9}{2}m^4a_1+\frac{15}{2}m^2a_2\right)+...\right]+\frac{\Delta^{1/2}}{2\pi^2}\frac{a_2}{4m^2}\right\} \tag{31}$$

This $\Delta_F^{DS(s)}(x,x')$ contains all the terms that diverges when $\sigma\to 0$ (like $\delta(\sigma), 1/\sigma, \log\sigma$) plus the terms with a divergent first derivative when $\sigma\to 0$ (like $\theta(\sigma)$, $\sigma\theta(\sigma), \sigma\log\sigma$) plus some convergent terms when $\sigma\to 0$ (like 1, σ, σ^2). In this way we arrive to the first important conclusion of this section: *The poles of* $\Gamma\left(j-\frac{n}{4}\right)$ *which originate the three coefficients* $a_0, a_{1,}a_2$ *correspond to the divergent terms or the terms with divergent derivative when* $\sigma\to 0$. There also are convergent terms in $\Delta_F^{DS(s)}(x,x')$ but they are physically irrelevant as we will soon see. The regular part of $\Delta_F^{DS}(x,x')$ reads

$$\Delta_F^{DS(r)}(x,x') = -\frac{\Delta^{1/2}}{8\pi}\theta(\sigma)\left[\frac{(2\sigma)^2}{2^2.4^2.6}(-6a_3)+...\right]+\frac{i}{2}\left\{\frac{\Delta^{1/2}}{2\pi^2}\log\frac{e^\gamma}{2}|2m^2\sigma|[\sigma^2+...\right.$$

$$-\frac{\Delta^{1/2}}{2\pi^2}\left[\frac{(2\sigma)^2}{2^2.4^2.6}\left(-\frac{9}{2}a_3\right)\right]+\frac{\Delta^{1/2}}{2\pi^2}\left[\left(\frac{a_3}{4m^4}+\frac{a_4}{8m^6}+...\right)-\frac{2\sigma}{2^2.4}\left(\frac{a_3}{m^2}+\frac{a_4}{m^4}+...\right)+\right] \tag{32}$$

and contains terms that are convergent and with first derivative also convergent when $\sigma\to 0$.

Then we can define the "Hadamard regularization" as the prescription that the singular part of $\Delta_F^{DS}(x,x')$ contains all the terms divergent or with first derivative divergent when $\sigma\to 0$ while the regular part of $\Delta_F^{DS(r)}(x,x')$ contains the terms which are convergent and with convergent first derivative when $\sigma\to 0$. At first sight the dimensional regularization and the Hadamard regularization do not coincide since in $\Delta_F^{DS(s)}(x,x')$ there are convergent terms with all their derivatives, namely those like $1,\sigma,\sigma^2$. Nevertheless the difference is physically irrelevant since these terms are multiplied by $a_0(x,x'), a_1(x,x'), a_2(x,x')$ that when $\sigma\to 0$ have the limits:

$$\lim_{x'\to x} a_i(x,x') = a_i(x), \qquad i=1,2,3 \tag{33}$$

From eq. (23) we see that these terms are proportional to the linear combinations of I, R, R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, and $\Box R$, contained in the terms of the gravitational lagrangian. Therefore terms we are discussing can be absorbed in the gravitational action S_g supplemented by the H terms. Then to unify the two regularizations $\Delta_F^{DS(r)}(x,x')$ must be defined modulo some terms with arbitrary coefficients corresponding to the undefined terms $1,\sigma,\sigma^2$. In the effective lagrangian these terms will produce finite terms that can be added to $\Lambda, G, \alpha, \beta, \gamma$ ([6] eq.(6.60)). The coefficients of these terms will be called l, g, a, b, c. Dropping these terms for the moment, we can compute the regular lagrangian corresponding to $\Delta_F^{DS(r)}(x,x')$ that in the coincidence limit reads:

$$\lim_{x\to x'}\Delta_F^{DS(r)}(x,x')=\lim_{x'\to x}\frac{i\Delta^{\frac{1}{2}}}{4\pi^2}\left[\frac{a_3}{4m^4}+\frac{a_4}{8m^6}+...\right] \tag{34}$$

Then as $\lim_{x'\to x}\Delta=1$ ([3], eq. (17.86)) we have that:

$$\lim_{x\to x'}\Delta_F^{DS(r)}(x,x')=\frac{i}{4\pi^2}\left\{\frac{a_3}{4m^4}+\frac{a_4}{8m^6}+...\right\} \tag{35}$$

We may now add the arbitrary coefficient terms and we obtain

$$\lim_{x\to x'}\Delta_F^{DS(r)}=\frac{i}{(4\pi)^2}\left\{4lm^2+ga_1+\frac{a_2}{m^2}+\frac{a_3}{m^4}+...\right\} \tag{36}$$

where l and g are the already defined arbitrary coefficients and those a,b,c, corresponding to α,β,γ are hidden in a_2. Using eq. (18) we obtain [18]:

$$L^{(r)}(x)=\frac{1}{32\pi^2}\left[2lm^4+gm^2a_1+a_2\log m^2+\frac{a_3}{m^2}+...\right] \tag{37}$$

which turns out to be equal to eq. (26) (except that in the quoted equation the three first terms are missing since they are absorbed in S_g supplemented by the "H" terms), showing the coincident of the two methods.

Therefore the substracted $S^{(r)}$ reads:

$$S^{(r)}=\int(-g)^{\frac{1}{2}}\left[-\frac{2\Lambda_0}{16\pi^2G_0}+\frac{m^4l}{16\pi^2}+\frac{R}{16\pi^2G_0}+\frac{1}{6}\frac{gm^2R}{32\pi^2}+\frac{\log m^2a_2}{32\pi^2}+\frac{a_3}{32\pi^2m^2}+...\right] \tag{38}$$

where the quantities $-\frac{2\Lambda_0}{16\pi^2G_0}+\frac{m^4l}{16\pi^2}$ and $\frac{1}{16\pi^2G_0}+\frac{1}{6}\frac{gm^2}{32\pi^2}$ must be determined by physical measurements (as the α,β,γ that are hidden in a_2).

So using Hadamard regularization and the substraction recipe the result is, somehow, simpler since eqs. (24) and (25) just read.

$$G_{phys}=G_0/1+\frac{1}{6}G_0gm^2,\qquad \Lambda_{phys}=\Lambda_0-\frac{1}{2}G_0m^2l \tag{39}$$

so the bare constants are finite and would coincide, from the very beginning, with the physical ones for the choice $l=g=0$ of the arbitrary coefficients l and g. Thus using Hadamard regularization and the substraction recipe: "*we must remove* $\Delta^{(s)}$ *from* Δ *and use* $\Delta^{(r)}$" we have obtained the same result of section II.A: all the infinities are removed and substituted by finite quantities. Thus "substraction recipe" works as the standard renormalization. The new recipe just consist in the elimination of the singular (or with singular first derivative) short distance components of the two point function $\Delta_F^{DS}(x,x')$, the only relevant truncated two point function in this theory. If we would have a $\lambda\phi^4$ interaction more truncated point functions must be substracted, as we will see in the next example.

III. FIRST METHOD. $\lambda\phi^4$ THEORY IN THE LOWEST ORDER.

In this section we will use the substraction method in the $\lambda\phi^4$ theory with lagrangian[19]:

$$(3.3.1)\qquad L=-\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2-\frac{1}{4!}\lambda\phi^4+\Lambda \tag{40}$$

Dimensional regularization and minimal substraction will be done following ref. [5].

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[18] In the first two terms , instead of $\int_{m^2}^{\infty}$ we use $-\int_0^{m^2}$ and in the third term $-\int_1^{m^2}$, because they work in these terms as $\int_{m^2}^{\infty}$ in the rest of the terms (see [6], pag. 157).

[19] In sections III, IV, and V the numbers before the equations correspond to ref. [5]. Moreover, comparing eqs. (14) with (40) we see that there is a change of convention in the sign of the norm, so in the following sections we change this convention in order that our equations would coincide with those of the corresponding references. Also, in order to comply with ref. [5] we will use, sometimes, $\Delta_F(x)$ and, sometimes, $\Delta_E(x)$.

It must be clear that, as we will isolate the divergent parts and then substract them, the theory will necessarily turns out to be finite. Thus our only aim, in sections III, IV, and V, is to detect the local divergencies and to compare our method with the usual one to see how the results are obtained and to show that they are similar, (so in each paragraph "i" we will see how we can find the singular and regular parts of the objects appearing in the theory, in "ii" we will review usual renormalization but using our notation, and in "iii" we will see how substraction recipe handles the divergence problem and compare the results)

A. Singular and regular parts of $\Delta_E(0)$ and mass renormalization.

i.- From eq. (1) we know that $\Delta_E(x)$ is one of the main characters of the play. It is divergent when $x \to 0$. So we will define the singular and the regular parts of $\Delta_E(0)$, first using dimensional diagonalization and then the Hadamard one [20]. In n dimensions it reads (just computing the tadpole graph and neglecting non connected graphs that will be taken into account in III.B and IV.B) :

$$(4.3.8) \qquad \lim_{x \to x'} \Delta_E(x - x') = \Delta_E(0) = \frac{m^2}{(4\pi)^2} \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{n}{2}-2} \Gamma\left(1 - \frac{n}{2}\right) \tag{41}$$

where μ is an arbitrary mass. We can now define $\Delta_E^{(s)}(0)$, the divergent component of $\Delta_E(0)$. As the $\Gamma\left(1 - \frac{n}{2}\right)$ behaves as:

$$\Gamma\left(1 - \frac{n}{2}\right) \approx \frac{2}{n-4} + \gamma \tag{42}$$

when $n \to 4$, (where $\gamma = \pi^2/12$ is the Euler-Mascheroni constant), using the minimal substraction we find the singular part of $\Delta_E(0)$:

$$\Delta_E^{(s)}(0) = \frac{2m^2}{(4\pi)^2} \frac{1}{n-4} \tag{43}$$

In this way we have detected the local divergency. So we reach to a decomposition (as (6)-(7)):

$$\Delta_E(0) = \Delta_E^{(r)}(0) + \Delta_E^{(s)}(0) = \frac{m^2}{(4\pi)^2} \left[\left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{n}{2}-2} \frac{1}{2}\Gamma\left(1 - \frac{n}{2}\right) - \frac{2}{n-4} \right] + \frac{2m^2}{(4\pi)^2} \frac{1}{n-4} \tag{44}$$

Then:

$$\Delta_E^{(r)}(0) = \frac{m^2}{(4\pi)^2} \left[\left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{n}{2}-2} \frac{1}{2}\Gamma\left(1 - \frac{n}{2}\right) - \frac{2}{n-4} \right] \tag{45}$$

Precisely when $n \to 4$ we have:

$$\Delta_E^{(r)}(0) = \frac{m^2}{(4\pi)^2} \left[\log\left(\frac{m^2}{4\pi\mu^2} \right) + \gamma - 1 \right] \tag{46}$$

where μ is the arbitrary mass, so essentially $\Delta_E^{(r)}(0)$ has an arbitrary value.

ii.-Let us now see how $\Delta_E^{(r)}(0)$ is related with mass renormalization. In order to correct the divergency of $\langle \phi_0(x)\phi_0(x') \rangle$ we must correct the divergency of its Fourier transform:

$$(4.3.2) \qquad G_0(p) = \frac{1}{p^2 + m_0^2 + \Sigma_0(p)} \tag{47}$$

---

[20]In both cases the singular component will have the form $\infty \times$ *geometrical object* (in this case invariant under a Lorentz transformation). Of course there are many possible substractions, as in the previous section. In section III.B we will pick the minimal one as in ref. [5]. In section III.C the Hadamard one and we will show the finite difference between the two choices.

The computation of the tadpole graph (in the first $\lambda-$order) yields :

$$(4.3.7) \qquad \Sigma_0^{(1)}(p) = \frac{1}{2}\lambda_0 \Delta_E(0) \tag{48}$$

that makes the term $m_0^2 + \Sigma_0^{(1)}(p)$ divergent. Precisely:

$$m_0^2 + \Sigma_0^{(1)}(p) = m_0^2 + \frac{1}{2}\lambda_0 \left[\Delta_E^{(r)}(0) + \Delta_E^{(s)}(0)\right] \tag{49}$$

In usual renormalization we consider that the (bare) mass m_0 is divergent. Then to compensate this divergency we define a (dressed) mass m such that:

$$m_0^2 + \Sigma_0^{(1)}(p) = m^2 + \Sigma^{(1)}(p) \tag{50}$$

where both terms in the r.h.s. are finite, precisely:

$$m_0^2 = m^2 \left[1 - \frac{\lambda_0}{2}\Delta_E^{(s)}(0)\right] \tag{51}$$

and

$$\Sigma^{(1)}(p) = \frac{1}{2}\lambda_0 \Delta_E^{(r)}(0) \tag{52}$$

Then the physical mass is:

$$(4.3.15) \qquad m^2_{phys.} = m^2 + \Sigma^{(1)}(p) \tag{53}$$

where m_{phys} is a constant while $\Sigma^{(1)}(p)$ and m^2, are finite functions of μ (cf. eq. (46)) satisfying the renormalization group equations.

iii.- Using the substraction recipe we would directly say that in eq. (49) really $\Delta_E^{(s)}(0) = 0$ and we will obtain:

$$m^2_{phys.} = m_0^2 + \frac{1}{2}\lambda_0 \Delta_E^{(r)}(0) \tag{54}$$

which is equivalent to (53) and where:

a.- m_0 plays the role of m. It is therefore finite.

b.-Since $\Delta^{(r)}(0)$ is a function of μ, m_0 must also be a function of μ in such a way that $m^2_{phys.}$ turns out to be a constant. Then m_0^2 satisfies the same renormalization group equation as the m^2 of eq. (53).

This will be a common feature of substraction recipe for all physical constants: *there is no need to introduce a dressed quantity since the bare quantity takes its role, then the bare quantity becomes a function of μ satisfying the renormalization group equations.*

B. The cosmological constant and the Hadamard regularization for $\Delta_E(0)$ in the case $\lambda = 0$.

i.- Let us begin making an identification. $\Delta_E^{(r)}(0)$ in flat space time can also be obtained in the case $n = 4$ (but using Hadamard substraction not minimal substraction) making all the curvatures zero in eq. (34), (namely making all the "a" zero but $a_0 = 1$) and multiplying by $-i$ (since $\Delta_F \rightarrow i\Delta_E$, [5] page. 194). So we obtain for $\lambda = 0$:

$$\Delta_E^{(r)}(0) = \lim_{x,x' \rightarrow 0} \Delta_E^{(r)}(x,x') = \frac{4lm^2}{(4\pi)^2} \tag{55}$$

so essentially in this case $\lim_{x,x'\rightarrow 0} \Delta_E^{(r)}(x,x')$ is just an arbitrary finite constant as in the case of (46). For the case $\lambda \neq 0$ some corrections will appear in eq. (34) ([6] page 301) but the r.h.s. of eq. (55) will always be an arbitrary constant. The origin of this arbitrary is the usual one: a infinite singularity can only be defined modulo a finite undefined constant. So the arbitrary singularity coefficient $\lim_{x,x'\rightarrow 0} \Delta_E^{(r)}(x,x')$ defined for $n = 4$ plays the same role that μ in the case $n \neq 4$. Both parameters are related, when $\lambda = 0$, by:

$$4l = \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 1 \qquad (56)$$

Thus, this preliminary consideration leads us to suppose that there must be something like a cosmological constant also in $\lambda\phi^4$ theory. In fact: in traditional quantum field theory the additional infinite term that appears, due to the addition of infinite ground energy terms $\frac{1}{2}\omega$, can be considered as an unrenormalizable cosmological constant. This term is eliminated using normal ordering. But this renormalization is better understood introducing the just mentioned cosmological constant ([5], section 4.2) that must be renormalized. Using our equation we can define a cosmological constant Λ for this flat space-time theory, if we just add to the usual lagrangian a term Λ as we have done in eq. (40). This term reads (see (38) and (56) in the case $n = 4$):

$$\Lambda = \frac{m^4}{16\pi^2} l = \frac{m^4}{4(4\pi)^2}\left[\log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 1\right] \qquad (57)$$

ii.- Let us see how renormalization method introduces the cosmological constant. When $\lambda = 0$ the vacuum to vacuum expectation (corresponding to the vacuum one-loop graph) reads:

$$(4.2.1) \qquad \langle 0+|0-\rangle = \int [d\phi] \exp\left\{-\int (d_E^n x)\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m_0^2\phi^2 - \Lambda_0\right]\right\} \qquad (58)$$

Thus:

$$(4.2.2) \qquad \frac{\partial}{\partial m^2}\langle 0+|0-\rangle = -\frac{1}{2}\int (d_E^n x)\langle 0+|\phi(x)^2|0-\rangle = -\frac{1}{2}\langle 0+|0-\rangle \int (d_E^n x)\Delta_E(0) \qquad (59)$$

and

$$(4.2.4) \qquad \langle 0+|0-\rangle = \exp\left[-\frac{1}{2}\int dm^2 \int (d_E^n x)\Delta_E(0)\right] \qquad (60)$$

But if $\mathcal{E}$ is the cosmological energy density of the universe it also is:

$$(4.2.9) \qquad \langle 0+|0-\rangle = \exp\left[-\int d_E^n x \mathcal{E}\right] \qquad (61)$$

So:

$$\mathcal{E} = \frac{1}{2}\int dm^2 \Delta_E(0) - \Lambda_0 \qquad (62)$$

where Λ_0 can be considered as an integration constant. Then from eq. (44) we have:

$$\mathcal{E} = \frac{1}{2}\int dm^2 \Delta_E^{(r)}(0) + \frac{1}{2}\int dm^2 \Delta_E^{(s)}(0) - \Lambda_0 \qquad (63)$$

so we can consider that Λ_0 is infinite in such a way as to cancel the infinite in $\Delta_E^{(s)}(0)$, namely:

$$\Lambda_0 = \frac{1}{2}\int dm^2 \Delta_E^{(s)}(0) - \mu^{4-n}\Lambda = \frac{1}{2}\frac{m_0^2}{(4\pi)^2}\frac{1}{n-4} - \mu^{4-n}\Lambda \qquad (64)$$

where Λ is the finite cosmological constant. So finally:

$$\mathcal{E} = \frac{1}{2}\int dm^2 \Delta_E^{(r)}(0) + \mu^{n-4}\Lambda \qquad (65)$$

and when $n \to 4$ we have:

$$(4.2.20) \qquad \mathcal{E} = \frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\ln\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - \frac{3}{2}\right] - \Lambda \qquad (66)$$

where $\mathcal{E}$ is finite and it is not a function of μ but Λ is a function of this mass. Using the Hadamard method of point i we can directly see these facts using eq. (57), since the μ-variation is cancelled in (66). It remains a finite constant which is unimportant since we can add an arbitrary constant to the lagrangian (40). As usual, the condition $\mathcal{E} = const.$ originates the renormalization group equation for Λ.

iii.- Directly from (62) using substraction recipe we would have

$$\mathcal{E} = \frac{1}{2}\int dm^2 \Delta_E^{(r)}(0) + \Lambda_0 \tag{67}$$

that for $n \to 4$ gives:

$$\mathcal{E} = \frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 1\right] - \Lambda_0 \tag{68}$$

namely (66) with the finite merely unimportant difference $\Delta\mathcal{E} = -\frac{m^4}{8(4\pi^2)}$, already discussed and Λ_0 playing the role of Λ. Now both terms in the r. h. s. are finite and functions of μ while $\mathcal{E} = \mathcal{E}_{phys}$ is a physical constant, yielding the renormalization group equation for Λ_0 as in the usual renormalization case.

From now on we will only use the dimensional regularization since the singular structure of the higher point function is not so well studied as the one of the two point function.

IV. FIRST METHOD. $\lambda\phi^4$ THEORY AT SECOND PERTURBATION ORDER.

A. Singular and regular parts of $[\Delta_F(z)]^{(d)2}$ and the coupling constant renormalization.

i.- Computing the fish graph we found that the scattering amplitude T reads:

$$(3.5.11) \qquad T = \lambda_0 + \frac{1}{2}\lambda_0^2[F(s) + F(t) + F(u)] \tag{69}$$

where λ_0 is the coupling constant and s, t, u the Mandelstam variables, F reads [21]:

$$(3.5.13) \qquad F(-P^2) = -\frac{i}{2}\int (d^4z) e^{iPz} \langle 0|T\phi^2(0)\phi^2(z)|0\rangle \tag{70}$$

(as in ref. [5] we have omitted the disconnected graphs, that were taken into account in section III.B, and we will consider again in IV.B), $\langle 0|T\phi^2(0)\phi^2(z)|0\rangle$ is the four point function divergent coincidence limit mentioned in section I.A that we must study and substracted, precisely:

$$(3.5.9) \qquad \langle 0|T\phi^2(0)\phi^2(z)|0\rangle = -2\Delta_F(z)^2 \tag{71}$$

So we see that the coincidence limit is the (undefined) product of $\Delta_F(z)$ by itself. Using dimensional regularization, as explained in section I.A, we define

$$\langle 0|T\phi^2(0)\phi^2(z)|0\rangle = -2\Delta_F(z)^{(d)2} \tag{72}$$

We will decompose this quantity as:

$$\Delta_F(z)^{(d)2} = \Delta_F(z)^{(d)2(s)} + \Delta_F(z)^{(d)2(r)} \tag{73}$$

---

[21] Really eq. (3.5.13) in ref. [5] reads:

$$(3.5.13) \qquad F(-P^2) = i\int (d^4z) e^{iPz} \Delta_F(z)^2$$

but we must remember that $\Delta_F(z)$ is a singular function (something worse than a distribution) so $\Delta_F(z)^2$ is a meaningless expression unless a multiplication procedure would be prescribed (which is done in eq. (3.5.14) of ref. [5]). Moreover decomposition (3.2.19), of the same reference, which is the base of the equation above, cannot be used when two points coincide, since this decomposition is inspired in the case when these two points are far apart, as in the definition of the truncated functions.

according to the prescription (9)-(10). Then we will obtain the regular $F^{(r)}(-P^2)$ as:

$$F^{(r)}(-P^2) = i\int (d^4z)e^{iPz}\Delta_F(z)^{(d)2(r)} \tag{74}$$

and if we use this $F^{(r)}$ instead of F in eq. (69) the physical T will turn out finite. We can directly make all the procedure on $F(-P^2)$, the Fourier transform of $\Delta_F(z)^2$. Using dimensional regularization we obtain:

$$(3.5.30) \qquad F(-P^2) = -\frac{\mu^{n-4}}{(4\pi)^2}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 d\alpha \left[\frac{m^2+\alpha(1-\alpha)P^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2} \tag{75}$$

This equation can be considered as a way to obtain the square $\Delta_F(z)^2$ i.e. to make this square when it is possible ($n \neq 4$) and then take the limit $n \to \infty$. When $n \to 4$ it is:

$$(3.5.31) \qquad \Gamma\left(2-\frac{n}{2}\right) \to \frac{2}{4-n} + f(n) \tag{76}$$

where $f(n)$ is a regular function such that $\lim_{n\to 4} f(n) = -\gamma$. So, we can find the Fourier transform of the decomposition (73):

$$F(-P^2) = \mu^{n-4}[F^{(s)}(-P^2) + F^{(r)}(-P^2)] \tag{77}$$

where the factor μ^{n-4} has been displayed to make $F^{(s)}(-P^2)$ and $F^{(r)}(-P^2)$ adimensional and where:

$$F^{(s)}(-P^2) = -\frac{1}{(4\pi)^2}\frac{2}{4-n}\int_0^1 d\alpha = -\frac{1}{(4\pi)^2}\frac{2}{4-n} \tag{78}$$

and

$$(3.5.33) \qquad F^{(r)}(-P^2) = -\frac{1}{(4\pi)^2}\int_0^1 d\alpha \left\{\Gamma\left(2-\frac{n}{2}\right)\left[\frac{m^2+\alpha(1-\alpha)P^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2} + \frac{2}{n-4}\right\} \tag{79}$$

Making now the inverse Fourier transforation of eq. (78) we have that:

$$i\Delta_F(z)^{(d)2(s)} = \frac{1}{(4\pi)^2}\frac{2}{n-4}\delta(z) \tag{80}$$

which, in fact, has the form announced in eq. (9). It is singular when $z = 0$ and it shows that only the regular part is relevant when $z \neq 0$. So we have detected the local divergency. Again, the non uniqueness of the result is shown by the presence of μ in equation (77).

ii.- The usual renormalization procedure would be to put the singular and regular parts in (69) to obtain:

$$(3.5.37) \qquad T = \lambda_0 + \lambda_0^2\frac{\mu^{n-4}}{(4\pi)^2}\frac{3}{n-4} + \frac{\mu^{n-4}}{2}\lambda_0^2[F^{(r)}(s) + F^{(r)}(t) + F^{(r)}(u)]\} \tag{81}$$

where the physical quantity T must be finite and $\mu-$independent. This is achieved introducing a renormalized λ such that:

$$(3.5.38) \qquad \lambda_0 = \mu^{4-n}\lambda\left(1-\frac{3\lambda}{(4\pi)^2}\frac{1}{n-4}\right) \tag{82}$$

so λ_0 turns out to be infinite and λ finite: Then

$$(3.5.48) \qquad T = \lambda + \frac{1}{2}\lambda^2[F^{(r)}(s) + F^{(r)}(t) + F^{(r)}(u)] \tag{83}$$

where all the magnitudes are finite. As T is $\mu-$independent we can obtain the renormalization group equation for λ.

iii.-According to the substraction method, we must make zero $\Delta_F(z)^{(d)2(s)}$ or $F^{(s)}(-P^2)$ and we obtain the finite physical value of T :

$$(3.5.48) \qquad T = \lambda_0 + \frac{1}{2}\lambda_0^2[F^{(r)}(s) + F^{(r)}(t) + F^{(r)}(u)] \tag{84}$$

where λ_0 is a finite quantity.

Making the limit $n \to 4$ it turns out that:

$$(3.5.66) \qquad F^{(r)}(s) = \frac{1}{(4\pi)^2}\left\{\log\left(\frac{m^2 e^{\gamma}}{4\pi\mu^2}\right) + \sqrt{1-\frac{4m^2}{s}}\log\left[\frac{\sqrt{s-4m^2}-\sqrt{s}}{\sqrt{s-4m^2}+\sqrt{s}} - 2\right]\right\} \tag{85}$$

We see that with the substitution $\lambda_0 \leftrightarrow \lambda$ eqs. (83) and (84) are the same. In the case of the substraction method λ_0 is a finite $\mu-$function and as T is $\mu-$independent so we can obtain the same renormalization group equation as above.

B. The cosmological corrected constant and $[\Delta_E(0)]^2$.

i.- In the previous section we have neglected non-connected terms, e. g. in eq. (48), because the mass term was a consequence of the equation

$$(4.3.5) \qquad G_0^{(1)}(x-x') = -\frac{1}{2}\lambda_0 \Delta_E(0)\int (d_E^n \overline{x})\Delta_E(x-\overline{x})\Delta_E(x-\overline{x}) \tag{86}$$

that really reads:

$$G_0^{(1)}(x-x') = -\frac{1}{2}\lambda_0 \int (d_E^n \overline{x})\left\{\Delta_E(0)\Delta_E(x-\overline{x})\Delta_E(x-\overline{x}) + \frac{1}{4.3}\Delta_E(x-x')\left[\Delta_E(0)\right]^2\right\} \tag{87}$$

Moreover, the cosmological constant is originated in the equation:

$$(3.3.9) \qquad \langle 0_+|0_-\rangle = \int [d\phi]\exp\left\{-\int L_0(d_E^n x)\right\}\exp\left\{-\frac{\lambda_0}{4!}\int \phi^4(d_E^n x)\right\} =$$

$$\langle 0_+|0_-\rangle^{(0)} - \frac{\lambda_0}{4!}\langle 0_+|\int \phi^4(d_E^n x)|0_-\rangle^{(0)} + ... =$$

$$\langle 0_+|0_-\rangle^{(0)} - \frac{3\lambda_0}{4!}\int (d_E^n \overline{x})\left[\Delta_E(0)\right]^2\langle 0_+|0_-\rangle^{(0)} \tag{88}$$

which corresponds to eq. (60) with an extra term. $\langle 0_+|0_-\rangle^{(0)}$ corresponds to the case $\lambda_0 = 0$ and the second term to the non-connected graphs (the "eight", the "square of the figure eight", etc.). In all these expressions $\left[\Delta_E(0)\right]^2$appears and it must be substituted by $\left[\Delta_E(0)^{(r)}\right]^2$ according to the substraction recipe. As $\Delta_E(0)$ is not a distribution, but just the divergent quantity (41), we must only substitute it by $\Delta_E^{(r)}(0)$, using decomposition (6)-(7), and making eqs. (87) and (88) finite.

ii.- Let us now go to the renormalization method: At order two we have:

$$(4.4.5) \qquad \mathcal{E} = \frac{m^n}{(4\pi)^{\frac{n}{2}}}\frac{1}{n}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{2}\mu^{n-4}\frac{m^4}{(4\pi)^2}\frac{1}{n-4} +$$

$$\frac{1}{2}\mu^{n-4}\frac{\lambda m^4}{(4\pi)^2}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{n-4}\right]^2 +$$

$$\frac{1}{2}\mu^{n-4}\frac{m^4}{(4\pi)^2}\frac{1}{n-4}\left(1-\frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right) - \Lambda_0 \tag{89}$$

It can be checked that the two first lines of this equation are finite when $n \to 4$. So we must define a renormalized cosmological constant Λ such that:

$$(4.4.6) \qquad \Lambda_0 = \mu^{n-4}\left[\frac{1}{2}\frac{m^4}{(4\pi)^2}\frac{1}{n-4}\left(1-\frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right)+\Lambda\right] \tag{90}$$

Then we have the final finite expression:

$$\mathcal{E} = \frac{m^n}{(4\pi)^{\frac{n}{2}}}\frac{1}{n}\Gamma\left(1-\frac{n}{2}\right) - \mu^{n-4}\frac{1}{2}\frac{m^4}{(4\pi)^2}\frac{1}{n-4} - \mu^{n-4}\Lambda +$$

$$+\frac{1}{2}\mu^{n-4}\frac{\lambda m^4}{(4\pi)^2}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right)-\frac{1}{n-4}\right]^2 \tag{91}$$

which is finite when $n \to 4$. In fact, when $\lambda = 0$, we have that:

$$\mathcal{E} = \frac{m^n}{(4\pi)^{\frac{n}{2}}}\frac{1}{n}\Gamma\left(1-\frac{n}{2}\right) - \mu^{n-4}\frac{1}{2}\frac{m_0^4}{(4\pi)^2}\frac{1}{n-4} - \mu^{n-4}\Lambda \tag{92}$$

which is a finite quantity, as we have proved in section III.B (it corresponds to $\langle 0_+|0_-\rangle^{(0)}$) while:

$$\left[\left(\frac{m_0^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right)-\frac{1}{n-4}\right] \tag{93}$$

is finite for (45), so the r. h. s. of eq. (91) is finite. When $n \to 4$ we find:

$$\mathcal{E} = \frac{m^4}{4(4\pi)^2}\left[\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-\frac{3}{2}\right]+\frac{\lambda}{8}\frac{m^4}{(4\pi)^2}\left[\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-1\right]^2-\Lambda \tag{94}$$

The terms of the r.h.s. are $\mu-$functions that originate the renormalization group equation as usual.

iii.-Using directly the substraction method in eq. (89) we would have when $n \to 4$:

$$\mathcal{E} = \frac{m^4}{4(4\pi)^2}\left[\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-1\right]+\frac{\lambda}{8}\frac{m^4}{(4\pi)^2}\left[\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-1\right]^2-\Lambda_0 \tag{95}$$

with all terms finite and Λ_0 a function of μ as usual which is equal to (94) with the exception of the already known unimportant constant. For both methods the renormalization group equation can be obtained prescribing that $\mathcal{E}$ would not be a function of μ.

C. The $[\Delta_E(z)]^{(d)3}$ and the wave function renormalization.

i.- Really mass renormalization of section III.A is based in the Green function:

$$(4.3.4) \qquad G_0^{(1)}(x-x') = -\frac{1}{4!}\lambda_0\int(d_E^n\overline{x})\langle\phi(x)\phi(x')\phi^4(\overline{x})\rangle^{(0)} \tag{96}$$

that can be written as:

$$(4.3.5) \qquad G_0^{(1)}(x-x') = -\frac{1}{4!}\lambda_0\Delta_E(0)\int(d_E^n\overline{x})\Delta_E(x-\overline{x})\Delta_E(x'-\overline{x}) \tag{97}$$

In the next order we must compute:

$$(4.5.2) \qquad G_0^{(2)}(x-x') = \frac{1}{2}\left(-\frac{\lambda_0}{4!}\right)\int(d_E^n y)(d_E^n z)\langle\phi(x)\phi(x')\phi^4(y)\phi^4(z)\rangle^{(0)} \tag{98}$$

Computing this Green function, as we have done with the previous one, we find:

a.- Vacumm disconnected graphs: They are the "eight", the "square eight", etc. which are removed by ordinary renormalization of the cosmological constant or by the corresponding substraction that makes this constant finite but undefined, as shown in eqs. (57) or (95), ([5], pages. 205 and 206).

b.- Disconnected two legs graph: It is the product of the "tadpole" by the "eight". Both graphs have already been considered either by renormalization or substraction.

c.- Connected two legs graphs: Namely:

c_1.- The "double scoop" or "double bubble" graph, with an integral:

$$(4.5.3) \qquad \Sigma_0^{(2,1)}(p) = -\frac{1}{4}\lambda_0^2 \int (d_E^n y)\Delta_E(y)^2\Delta_E(0) \tag{99}$$

(which really is not a function of p). It has two factors:

- $\Delta_E(0)$ that was considered in section III.A and made finite by both recipes.

- $\Delta_E(y)^2 = \Delta_E(y)^{(d)2}$ which was considered in section IV.A, since the integral in eq. (99) is just the integral in eq. (74) with $P = 0$, which also was made finite by both recipes. So $\Sigma_0^{(2,1)}$ turns out to be finite either way. Finally let us observe that in $\Sigma_0^{(2,1)}$the typical expression $[w^{(2)}(0)]^\beta [w^{(2)}(z)]^\alpha$, of section I.A, appears for the first time in its complete version.

c_2.- The "setting sun" graph,which is a function of p

$$(4.5.4) \qquad \Sigma_0^{(2,2)}(p) = -\frac{1}{6}\lambda_0^2 \int (d_E^n x)\Delta_E(x)^3 e^{-ipx} \tag{100}$$

To deal with this integral we must first compute $\Delta_E^{(d)3}(x)$ multiplying $\Delta_E(x)$ three times, then make its dimensional regularization, and finally its Fourier transform $\Delta_E^{(d)3}(p)$. We obtain:

$$(4.5.37) \qquad \Sigma_0^{(2,2)}(p) = -\frac{1}{6}\left(\frac{\lambda}{(4\pi)^2}\right)^2 p^2 \left(\frac{p^2}{4\pi\mu^2}\right)^{n-4} \frac{\Gamma\left(\frac{n}{2}-1\right)^3 \Gamma(3-n)}{\Gamma\left(\frac{3n}{2}-3\right)} \tag{101}$$

As when $n \to 4$:

$$\frac{\Gamma\left(\frac{n}{2}-1\right)^3 \Gamma(3-n)}{\Gamma\left(\frac{3n}{2}-3\right)} \to \frac{1}{2}\frac{1}{n-4} \tag{102}$$

then:

$$\left[\Sigma_0^{(2,2)}(p)\right]^{(s)} = -\frac{1}{12}\left(\frac{\lambda}{(4\pi)^2}\right)^2 p^2 \frac{1}{n-4} \tag{103}$$

Then, we conclude that:

$$\Delta_E^{(d)3(s)}(x) = \frac{1}{2}\left(\frac{1}{2\pi}\right)^n \frac{1}{(4\pi)^2}\frac{1}{n-4}\int p^2 e^{ipx}(d_E^n p) = \frac{1}{2}\frac{1}{(4\pi)^2}\frac{1}{n-4}\nabla^2\delta(x) \tag{104}$$

which, in fact, has the form announced in eq. (9). It is local, since it is singular when $z = 0$ and vanishing for $z \neq 0$. We have detected another local singularity. In the finite limit $n \to 4$ we obtain

$$(4.5.38) \qquad \Sigma_0^{(2,2)}(p) = -\frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 p^2 \left(\log\frac{p^2}{4\pi\mu^2} + const.\right) \tag{105}$$

Substracting all singularities the propagator $G_0^{(2)}(x-x')$ turns out to be finite to the second $\lambda-$order. But, of course, an ambiguity appears in the constant of eq. (105) that must be fixed by a measurement.

ii.- In the renormalization theory we must add all the results of the connected graphs to obtain:

$$(4.5.39) \qquad G_0(p)^{-1} = p^2\left\{1 - \frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \frac{1}{n-4}\right\} + m_0^2 + m^2\left\{\frac{\lambda}{(4\pi)^2}\left[\frac{1}{n-4} + finite\right]\right\} -$$

$$-\left[\frac{\lambda}{(4\pi)^2}\right]^2\left\{m^2\left[\frac{2}{(n-4)^2} + \frac{1}{2}\frac{1}{n-4}\right] + finite\ function\ of\ p^2\right\} \tag{106}$$

To eliminate the infinities via renormalization a renormalized $G(p)$ is defined as:

$$(4.5.40) \qquad G_0(p) = z_1^2 G(p) \tag{107}$$

where:

$$(4.5.41) \qquad z_1^2 = 1 + \frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \frac{1}{n-4} \tag{108}$$

then up to the order λ^2 we have:

$$(4.5.42) \qquad G(p)^{-1} = p^2 + m_0^2 + m^2 \left\{ \frac{\lambda}{(4\pi)^2}\left[\frac{1}{n-4} + finite\right]\right\} -$$

$$- \left[\frac{\lambda}{(4\pi)^2}\right]^2 \left\{ m^2 \left[\frac{2}{(n-4)^2} + \frac{5}{12}\frac{1}{n-4}\right] + finite \right\} \tag{109}$$

and the renormalized the mass reads:

$$(4.5.43) \qquad m_0^2 = m^2 \left\{ 1 - \frac{\lambda}{(4\pi)^2}\frac{1}{n-4} + \left[\frac{\lambda}{(4\pi)^2}\right]^2 \left[\frac{2}{(n-4)^2} + \frac{5}{12}\frac{1}{n-4}\right]\right\} \tag{110}$$

Then:

$$G(p)^{-1} = p^2 + m^2 + finite(\mu) \tag{111}$$

All terms are finite and $G(p)^{-1}$, m^2, and $finite(\mu)$are $\mu-$functions. p^2 is the physical constant quantity that originates the renormalization group. The renormalization of eqs. (107) and (108) is usually considered as a wave function renormalization:

$$(4.5.46) \qquad \phi_0(x) = z_1 \phi(x) \tag{112}$$

where $\phi(x)$ is the renormalized field.

iii.- Using the substraction recipe eq. (111) reads:

$$G(p)^{-1} = p^2 + m_0^2 + finite(\mu) \tag{113}$$

since all the infinities disappear from eqs. (109) and (110), but a finite undeterminate constant remains that must be fixed by a measurement that corresponds to the one of the wave function renormalization. As usual $G(p)^{-1}$, m_0^2 and $finite(\mu)$ are $\mu-$functions that originate the renormalization group. Moreover, from eq. (108) with no infinity we have $z_1 = 1$ and there is no need of the wave function renormalization[22]. Then using our recipe, the result is the same.

V. FIRST METHOD. $\lambda\phi^4$ THEORY AT ANY ORDER. MORE GENERAL $\lambda\phi^L$ THEORIES. SPECULATIONS ON NON-RENORMALIZABLE THEORIES

We can now follow a well known path. For the $\lambda\phi^4$ theory the superficial divergence is:

$$(5.2.21) \qquad D = 4 - N \tag{114}$$

where N is the number of external legs of the graph. Then, only graphs with $N = 2$ and $N = 4$ have basic divergencies. Moreover the convergence of all the graphs, to λ^2 can be reduced to prove the convergence of the primitive divergent graph ([13], page. 144), namely the tadpole and the fish graphs, the double scoop, and, the setting sun (and the non connected graphs) which were studied in the previous sections. These graphs are finite under ordinary renormalization (or if the substraction recipe are used). So, repeating these calculations to any order

[22]This fact must be most welcome since now both the "bare" and "renormalized" fields satisfy the same equal time commutation relations.

all graphs of the renormalized theory are finite and the theory turns out to be finite to all orders [14]. $\lambda\phi^4$ theory can be considered as renormalizable since it has a finite number of primitive divergent graphs and therefore a finite number of relevant singular point functions, namely three. So we now know that *using substraction method the theory is directly finite to any order.*

To complete the panorama we can study the problem in more general scalar field theories. Theories with interactions $\lambda\phi^l$ with $l > 4$ turn out to be non renormalizable because they have an infinite number of primitive divergent graphs and therefore a infinite number of relevant singular point functions, that cannot be compensated with the finite number of terms of the bare lagrangian. But the substraction recipe can anyhow be used making all these singular functions finite, and these theories would become also finite. So *all theories can be made finite if we use the substraction recipe.*

In fact, let us consider what we know about this kind of theories:

i.-In order to make the theory finite we must make finite (by renormalization or substraction) all the superficially divergent subgraphs ($D \geq 0$). The mass dimension in each term is the superficial divergency.

ii.- The divergent terms are polynomials of finite order in the external momentum. Using dimensional regularization with minimal substraction the coefficients of these polynomials contain positive integer powers of the parameters of the theory multiplied by poles in $n-4$ ([5], page 235). So the typical *divergent* term reads:

$$P(p_1, p_2, ..., p_N) = \sum A^{\gamma}_{\alpha\beta\delta_1,...\delta_N} \frac{m^{\alpha}\lambda^{\beta}...}{(n-4)^{\gamma}} p_1^{\delta_1} p_2^{\delta_2} ... p_N^{\delta_N} \tag{115}$$

that, under a Fourier transform: $w_N^{(s)}(x_1, x_2, ..., x_N) \approx$
$\int dp_1 \int dp_2 ... \int dp_N P(p_1, p_2, ..., p_N) e^{-ix_1p_1} e^{-ix_2p_2} ... e^{-ix_Np_N}$ corresponds to the local singularity:

$$w_N^{(s)}(x_1, x_2, ..., x_N) \approx \sum A^{\gamma}_{\alpha\beta\delta_1,...\delta_N} \frac{m^{\alpha}\lambda^{\beta}...}{(n-4)^{\gamma}} \nabla^{\delta_1}\delta(x_1)\nabla^{\delta_2}\delta(x_2)...\nabla^{\delta_N}\delta(x_N) \tag{116}$$

as in eq. (104), i. e. singularities of the (6) type. All the singularities $\nabla^{\delta i}\delta(x_i)$ are well defined distributions in variable x_i (there are no meaningless expressions as $\delta(0)\int_0^{\infty} d\omega$ that we will consider and eliminate in the next section) multiplied by infinite poles $1/(n-4)^{\gamma}$.

So let us compare the two methods:

i.- Renormalization: In this case the divergent (115) terms must be compensated by counterterms like:

$$\frac{\delta m^{\alpha'}\delta\lambda^{\beta'}...}{(n-4)^{\gamma}} p_1^{\delta_1} p_2^{\delta_2} ... p_N^{\delta_N} \tag{117}$$

where $\alpha \neq \alpha', \beta \neq \beta'$ but $\alpha + \beta = \alpha' + \beta'$ in such a way to have the same dimension (or the same superficial divergence D). It is clear that in general such counter terms must be infinite and will be only finite in particular cases (renormalizable theories). Moreover non-renormalizable theories are considered non-controllable, since they must have an infinite number of counter terms, implying new interaction terms of growing power.

ii.- Divergencies will disappear using the substraction recipe and the theory will turn out finite anyhow. In fact, as in our method the lagrangian remains untouched, and we can make the theory finite simply substracting the divergent terms. Then all the $\gamma > 0$ terms will disappear and $w_N^{(r)}(x_1, x_2, ..., x_N)$ will be a well defined functions [23]. As, from the general formalism of quantum theory [2], we are used to deal with a host of infinite divergent point functions[24], to deal with a similar host of finite point functions, obtained via the substraction recipe, it cannot be a major theoretical problem. So under our method both renormalizable and non- renormalizable theories are finite. Nevertheless, in renormalizable theories the ambiguous terms are combined in such a way that the unknown parameters of the theory can be computed with a finite number of physical data, while in the case of non-renormalizable theories this number is infinite[25].

[23] Really we also have an infinite set of counter terms, but not in the lagrangian, they are the singular terms of the point functions that must we substracted from these functions to obtain the regular terms, so they are precisely located in the place where they are needed.

[24] Like those listed in footnote 1.

[25] E. g.: in the $\lambda\phi^4$ theory the renormalization group shows that all the residues of the poles depend on those of the first order poles ([5]. page 241). Namely all the ambiguities corresponding to higher divergencies depend on the first order ambiguities, and therefore all these ambiguities can be computed with just some measurements. In the general $\lambda\phi^l$ case there is not such a miracle and we must deal with infinite ambiguities.

Then using our method, non-renormalizable theories most likely make some sense and, if they have small coupling constants, probably would yield good results, using a few terms of the perturbation expansion and a few physical data, but of course we do not know yet if they have any physical relevance. Moreover, in recent years it has become increasingly apparent that the usual renormalization is not a fundamental physical requirement ([15], vol. 1, page 518). We stop our speculation here, since this will be the subject of forthcoming researches.

VI. SECOND METHOD.

In this section we will try to find a theoretical justification for the substraction method, following the authors' ideas of the references: [2], [16], [17], and, [18]. We will also find new potentially dangerous divergencies hidden in the formalism that will also be eliminated. The quoted authors consider that the first object that must be taken into account in quantum fields theory is the set of observables O that we will use (belonging to the space of the relevant observables $\mathcal{O}$). Then the states ρ can be considered as the functionals over these observables yielding the mean values $(\rho|O)$. If the spectra of the observables of the problem are discrete we have that $(\rho|O) = Tr(\rho O)$. If one or many of these spectra are continuous the problem is more difficult because the last symbol is ill-defined. This happens, e. g., when the energy spectrum is continuous. In papers [9] we solve this problem (based in the mathematical structure introduced in paper [19]), finding good results for many physical problems. In the present paper we deal with short distance divergences, related with the position operators, which also have a continuous spectrum. So we will try to adapt the method of paper [9] to this new problem. But first let us review the formalins of this paper.

A. Van Hove formalism.

Let us consider a system with an hamiltonian H with continuous energy spectrum $0 \leq \omega < +\infty$. In the simple case at least some generalized observables reads:

$$O = \int\int d\omega d\omega' \left[O_{\omega}\delta(\omega-\omega') + O_{\omega\omega'}\right] |\omega\rangle\langle\omega'| \tag{118}$$

where O_{ω} and $O_{\omega\omega'}$ are regular functions (with properties we will discuss below). These observables are contained in a space $\mathcal{O}$. The introduction of distributions like $\delta(\omega-\omega')$ is necessary because the "singular term" $O_{\omega}\delta(\omega-\omega')$ appears in observables that cannot be left outside the space $\mathcal{O}$, like the identity operator, the hamiltonian operator, or the operators that commute with the hamiltonian. So, even in this simple case the observables contain δ functions (while in more elaborated cases they will also contain other kind of distributions). Symmetrically a generalized state reads:

$$\rho = \int\int d\omega d\omega' \left[\rho_{\omega}\delta(\omega-\omega') + \rho_{\omega\omega'}\right] |\omega\rangle\langle\omega'| \tag{119}$$

where ρ_{ω} and $\rho_{\omega\omega'}$ are also regular functions (with properties to be defined). These states are contained in a convex set of states $\mathcal{S}$. The introduction of distributions like $\delta(\omega-\omega')$ is also necessary in this case because the "singular term" $\rho_{\omega}\delta(\omega-\omega')$ appears in generalized states that cannot be left outside the set $\mathcal{S}$, like the equilibrium state[26]. With this mathematical structure it is impossible to calculate something like $Tr(\rho O)$ because meaningless $\delta(0)\int_0^{\infty} d\omega$ appear. *This is the main problem (if* $O_{\omega} \neq 0$ *and* $\rho_{\omega} \neq 0$*).* Let us keep in mind that with the old philosophy we are just considering the mean value $Tr(\rho O)$ as a simple *inner product* (and in doing so we have the problem of the $\delta(0)\int_0^{\infty} d\omega$).

The problem is solved if we consider the characteristic algebra of the operators $\mathcal{A}$ (see the complete version in [20]) containing the space of the self-adjoints observables $\mathcal{A}_S$ which contains the minimal subalgebra $\widehat{\mathcal{A}}$ of the observables that commute with the hamiltonian H (that we can consider as the typical "diagonal" operators). Then we have:

$$\widehat{\mathcal{A}} \subset \mathcal{A}_S \subset \mathcal{A} \tag{120}$$

Now we can make the quotient

---

[26]Usually this state is not considered in the scattering theory. So it is only *potentially dangerous* for more general theories

$$\frac{\mathcal{A}}{\widehat{\mathcal{A}}} = \mathcal{V}_{nd} \tag{121}$$

where $\mathcal{V}_{nd}$ would represent the vector space of equivalent classes of operators that do not commute with H (the "non-diagonal operators"). These equivalence classes reads

$$[a] = a + \widehat{\mathcal{A}}, \qquad a \in \mathcal{A},\ [a, H] \neq 0 \tag{122}$$

So we can decompose $\mathcal{A}$ as:

$$\mathcal{A} = \widehat{\mathcal{A}} + \mathcal{V}_{nd} \tag{123}$$

(this decomposition corresponds to the one in eq. (118). But neither the two + of the last two equation is a direct sum, since we can add and substract an arbitrary $a \in \widehat{\mathcal{A}}$ from each term of the r. h. s. of the last equation.

At this point we can ask ourselves which are the measurement apparatuses that really matter in the case of decoherence under a evolution e^{-iHt}. Certainly the apparatuses that measure the observables that commute with H which are contained in $\widehat{\mathcal{A}}$ that correspond to diagonal matrices $\sim \delta(\omega - \omega')$. Also the apparatuses that measure observables that do not commute with H that corresponds to the off-diagonal terms that are contained in $\mathcal{V}_{nd}$. The terms corresponding to the latter kind of apparatuses (either in the observables or in the corresponding states) must vanish when $t \to \infty$, so they must be endowed with mathematical properties adequated to produce this limit. Riemann-Lebesgue theorem tell us that this fact take places if functions $O_{\omega\omega'}$ are regular (and also the $\rho_{\omega\omega'}$ below). So we define a sub algebra of $\mathcal{A}$, that can be called a van Hove algebra, as:

$$\mathcal{A}_{vh} = \widehat{\mathcal{A}} \oplus \mathcal{V}_r \subset \mathcal{A} \tag{124}$$

where the vector space $\mathcal{V}_r$ is the space of observables with $O_\omega = 0$ and $O_{\omega\omega'}$ a regular function. Now the $\oplus$ is a direct sum because $\widehat{\mathcal{A}}$ contains $\delta(\omega - \omega')$ and $\mathcal{V}_r$ just regular functions and a kernel cannot be both a δ and a regular function. Moreover, as our observables must be selfadjoint the space of observables must be

$$\mathcal{O} = \mathcal{A}_{vhS} = \widehat{\mathcal{A}} \oplus \mathcal{V}_{rS} \subset \mathcal{A}_S \tag{125}$$

where $\mathcal{V}_{rS}$ contains only self-adjoint operator (namely $O^*_{\omega\omega'} = O_{\omega'\omega}$). Restriction (125) is just the choice (coarse-graining) of the relevant measurement apparatuses for our problem, those that measure the diagonal terms in $\widehat{\mathcal{A}}$ and those that measure the non diagonal terms that vanish when $t \to \infty$ in $\mathcal{V}_{rS}$. Moreover $\mathcal{O} = \mathcal{A}_{vhS}$ is dense in $\mathcal{A}_S$ (because any distribution can be approximated by regular functions) and therefore essentially it is the minimal possible coarse-graining. Let us call $|\omega) = |\omega\rangle\langle\omega|$ to the vectors of the basis of $\widehat{\mathcal{A}}$ and $|\omega, \omega') = |\omega\rangle\langle\omega'|$ to those of $\mathcal{V}_{rS}$. Then a generic observable of $\mathcal{O}$ reads

$$O = \int d\omega O_\omega |\omega) + \int \int d\omega d\omega' O_{\omega'} |\omega, \omega') \tag{126}$$

namely is a vector in the basis $\{|\omega), |\omega, \omega')\}$ where O_ω and $O_{\omega\omega'}$ are regular functions (with properties precised in the paper [9] and omitted here, as we will do with all the functions that will appear in this brief review).

The states must be considered as linear functional over the space $\mathcal{O}$ ($\mathcal{O}'$ the dual of space $\mathcal{O}$ [16], [17], [18]):

$$\mathcal{O}' = \mathcal{A}'_{vhS} = \widehat{\mathcal{A}}' \oplus \mathcal{V}'_{rS} \subset \mathcal{A}'_S \tag{127}$$

Therefore the states read:

$$\rho = \int d\omega \rho_\omega (\omega| + \int \int d\omega d\omega' \rho_{\omega\omega'} (\omega, \omega'| \tag{128}$$

where ρ_ω and $\rho_{\omega\omega'}$ are regular functions and $\{(\omega|, (\omega, \omega'|\}$ is the cobasis of $\{|\omega), |\omega, \omega')\}$. The set of these generalized states is the convex set $\mathcal{S} \subset \mathcal{O}'$. Now the mean value:

$$(\rho|O) = \int d\omega \rho_\omega O_\omega + \int \int d\omega d\omega' \rho_{\omega\omega'} O_{\omega'\omega} \tag{129}$$

is well defined and yields reasonable physical results [9][27]. In the last equation terms like $\delta(0) \int_0^\infty d\omega$ have disappeared. *This is the simple trick that allows as to deal with the singularities in a rigorous mathematical way and to obtain*

[27] Moreover, the introduction of the singular observables automatically yield the introduction of the singular states [9].

correct physical results in papers [9] *and* [10]. Essentially we have defined a new observable space $\mathcal{O}$ that contains the observables O of eq. (126) that are adapted to solve our problem. In this way we have found a method to deal with the singular terms containing Dirac's deltas. We are now considering the mean value $(\rho|O)$ not as an inner product but as a the action *functional* ρ acting on the vector O (and the $\delta(0)\int_0^\infty d\omega$ have disappeared). Decoherence is a consequence of Riemann-Lebesgue theorem in the time evolution of the last equation, namely:

$$(\rho(t)|O) = \int d\omega \rho_\omega O_\omega + \int\int d\omega d\omega' e^{-i(\omega-\omega')t}\rho_{\omega\omega'}O_{\omega'\omega} \tag{130}$$

B. The formalism in the simplest case.

Let us now use the same technique to deal with the singularities of quantum field theory. But first let us remember that in quantum field theory there coexist at least two different mathematical structures:

- The abstract Hilbert space $\mathcal{H}$ where the field $\phi(x)$ is an operator and the vacuum state $|0\rangle$ a vector. The multiplication in the characteristic algebra $\mathcal{A}$ is the multiplication of these operators. This is not the place where divergencies are produced. Therefore we will not modify this structure.

- The vector space of functions of N ($N \to \infty$) variables $x_1, x_2, ..., x_N$ where the functions $\phi(x_1)\phi(x_2)...\phi(x_N)$ can be considered as the coordinates of the vectors of a vector space $\mathcal{N}$ in a basis $|x_1, x_2, ..., x_N)$. Since we have proved that really the "functions" $\phi(x_1)\phi(x_2)...\phi(x_N)$ are distributions or worse we will give to this space the mathematical structure that we explained in the previous subsection[28].

The characteristic algebra is $\mathcal{A} = \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{N}$.

Let us begin with the case of just two variables to see the analogy with the previous section. Then, as the observables like $\phi(x)\phi(x')$ are distributions (or worse) it is reasonable to consider that all the observables are singular[29]. Let us begin with the simplest case i. e.: with just the singularity (80). Then our observables would read (like in (118) or (80) with $z = x - x'$):

$$O = \int\int dx dx' \left[\frac{O_x}{n-4}\delta(x-x') + O_{xx'}\right]|x, x') \tag{131}$$

where O_x and $O_{xx'}$ are regular functions. But if we continue the road of eqs. (118) and (119) we will find the same problems as above. On the other hand using the philosophy just explained[30] we can define the space of observables

$$\mathcal{O} = \mathcal{A}_{vhS} = \widehat{\mathcal{A}} \oplus \mathcal{V}_{rS} \subset \mathcal{A}_S \tag{132}$$

where $\widehat{\mathcal{A}}$ is now the space of the $\delta(x-x')-$singularity with pole $(n-4)^{-1}$ and $\mathcal{V}_{nS}$ is the space of regular observables measured by physical apparatuses. O_x and $O_{xx'}$ are regular function and $\mathcal{O} = \mathcal{A}_{vhS}$ is dense in $\mathcal{A}_S$. Then we may transform the eq. (131) to make it similar to (126), namely:

$$O = \int dx \frac{O_x}{n-4}|x) + \int\int dx dx' O_{xx'}|x, x') \tag{133}$$

so now the observables are vectors of a space $\mathcal{O} \subset \mathcal{N}\otimes\mathcal{H}\otimes\mathcal{H}$ of basis $\{|x), |x, x')\}$. Then the states of this system are just some linear functional over the space $\mathcal{O}$.

$$\mathcal{O}' = \mathcal{A}'_{vhS} = \widehat{\mathcal{A}}' \oplus \mathcal{V}'_{rS} \subset \mathcal{A}'_S \tag{134}$$

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[28] Mathematically speaking this would be the one of a "nuclear" space $\mathcal{N}$, namely the generalization of the ordinary N-rank tensor space to the case where the N indices are continuous. In the future we will base an axiomatic quantum field theory using this mathematical structure.

[29] We may say that we are using the continuous spectrum of the position operator [21] which is $-\infty < x < +\infty$ and define the basis $|x, x')$ as $|x\rangle\langle x|$. But this is not necessary since we can directly say that the space $\mathcal{N}$ of vectors with coordinates $\phi(x)\phi(x')$ has a basis $|x, x')$.

[30] But now referred to the measurement apparatuses, i. e. those that measure variable x that now take the role of variable ω.

For a moment let us postulate that also the singularities in the states do exist [31]. In this perspective the state must be linear combinations in the basis $\{(x|, (x, x'|\}$ (where $\{(x|, (x, x'|\}$ is the cobasis of $\{|x), |x, x')\}$), so they must read:

$$\rho = \int dx \rho_x (x| + \int \int dx dx' \rho_{xx'} (x, x'| \tag{135}$$

where ρ_x and $\rho_{xx'}$ are regular function. With these definitions the action of functional $(\rho|$ over the vector $|O)$ reads:

$$(\rho|O) = \int dx \frac{\rho_x O_x}{n-4} + \int \int dx dx' \rho_{xx'} O_{x'x} \tag{136}$$

and it will be well defined when $n = 4$ only if the first term of the r.h.s. vanishes. But this is precisely the case since, based in the arguments of subsection I.B, we know that either the real physical observables must be such that $O_x = 0$, namely they cannot see the singularities of the states (because really they only are mathematical artifacts, etc.) or $\rho_x = 0$ (namely the states cannot see the singularities of the observables, etc.). Then either $O_x = 0$ or $\rho_x = 0$ and the last equation reads:

$$(\rho|O) = \int \int dx dx' \rho_{xx'} O_{x'x} \tag{137}$$

and therefore we have eliminated the singular term $\frac{\rho_x O_x}{n-4}$ of eq. (136) which now have no physical effect. *In this way we can justified the elimination of* **all** *singular terms as we have done with* (80) *as we will see* [32].

C. The formalism in the general case.

To generalize this idea let us go back to eq. (3). We know that the functional $Z[\rho]$ and its derivatives define the whole theory. Moreover, following the above ideas it must be written as[33]:

$$Z[\rho] = \exp i(\rho|O) \tag{138}$$

where

$$|O) = |\phi(x_1)\phi(x_2)...\phi(x_N)) \tag{139}$$

being $\phi(x_1)\phi(x_2)...\phi(x_N)$ the components of a vector $|O) \in \mathcal{A} = \mathcal{N} \otimes \mathcal{H} \otimes \mathcal{H}$ for any N and

$$(\rho| = \rho(x_1)\rho(x_2)...\rho(x_N)|0\rangle\langle 0| \tag{140}$$

where $(\rho| \in \mathcal{A}' = \mathcal{N}' \otimes \mathcal{H} \otimes \mathcal{H}$. Remember that what really matters for our analysis are the "functions" $\phi(x_1)\phi(x_2)...\phi(x_N)$ and $\rho(x_1)\rho(x_2)...\rho(x_N)$ are in spaces $\mathcal{N}$ and $\mathcal{N}'$ while the way to operate with $|0\rangle\langle 0|$ over the field $\phi(x)$ remains the usual one since it takes place in space $\mathcal{H}$. Moreover, these are the observables and states that really matters since they define $Z[\rho]$. The observable $|O)$ is the generalized version of eq. (133) thus:

$$O = \sum_N \int dx_1 \int dx_2 ... \int dx_N O^{(r)}_{x_1 x_2 ... x_N} |x_1, x_2, ... x_N) +$$

[31] This is not really the case as we will see in the next subsection.

[32] Of course we can also directly say that the term $\int dx \rho_x O_x/(n-4)$ is unphysical. But there is a difference between eq. (129) and the last equation. In the former the singular observables see the singular states and therefore it has two terms. In the latter there are either singular observables or singular states and it has only one term. Therefore the two coarse-graining use in sections VI. A and VI.B are different. This fact is no surprising since the singular terms (in ω) are necessary in the case of decoherence to represent the diagonal final state but these singular terms (in x) must disappear in the case of quantum field theory since this is the way divergent poles disappear. The two different coarse-graining are introduced to explain two different observed physical facts.

[33] The next symbol contains a sum over the indices $N = 0, 1, 2, ...$

$$\sum_{N,\alpha_i,i}\int dx_1\int dx_2...\int dx_{N-i}\frac{O^{(\alpha_i,s)}_{N,x_1x_2...x_{N-i}}}{(n-4)^{\alpha_i}}|N,\alpha_i,x_1,x_2,...x_{N-i})\Bigg] \tag{141}$$

for all possible N and all possible coincidence limits symbolized by i. As before we can define an observable space

$$\mathcal{O}=\mathcal{A}_{vhS}=\widehat{\mathcal{A}}\oplus\mathcal{V}_{rS}\subset\mathcal{A}_S \tag{142}$$

where:

i.-The first term of the r.h.s. of eq. (141) belongs to the space $\mathcal{V}_{rS}$ with basis $\{|x_1,x_2,...x_N)\}$ and regular functions $O^{(r)}_{x_1x_2...x_N}$.

ii.-The second term of the r.h.s. of eq. (141) belongs to the space $\widehat{\mathcal{A}}$, the algebra of the singularities of eq. (116) with basis $\{|N,\alpha_i,x_1,x_2,...x_{N-i})\}$ and regular functions $O^{(\alpha_i,s)}_{N,x_1x_2...x_{N-i}}$. Then the singular terms are like those of eq. (116).

$(\rho|$ is the generalized version of the state $\rho(x_1)\rho(x_2)...\rho(x_N)|0\rangle\langle0|$. Then, if we repeat the reasoning of eq. (135), these generalized states would read:

$$\rho=\sum_N\int dx_1\int dx_2...\int dx_N\rho^{(r)}_{x_1x_2...x_N}(x_1,x_2,...x_N|+$$

$$\sum_{N,\alpha_i,i}\int dx_1\int dx_2...\int dx_{N-i}\rho^{(\alpha_i,s)}_{N,x_1x_2...x_{N-i}}(N,\alpha_i,x_1,x_2,...x_{N-i}| \tag{143}$$

As above we can defined the state space as:

$$\mathcal{O}'=\mathcal{A}'_{vhS}=\widehat{\mathcal{A}}'\oplus\mathcal{V}'_{rS}\subset\mathcal{A}'_S \tag{144}$$

where:

i.-The first term of the r.h.s. of eq. (143) belongs to the space $\mathcal{V}'_{rS}$ with basis $\{(x_1,x_2,...x_N|\}$ and regular functions $\rho^{(r)}_{x_1x_2...x_N}$.

ii.-The second term of the r.h.s. of eq. (143) belongs to the space $\widehat{\mathcal{A}}$, with basis $\{(N,\alpha_i,x_1,x_2,...x_{N-i}|\}$ and regular functions $\rho^{(\alpha_i,s)}_{N,x_1x_2...x_{N-i}}$.

Then:

$$(\rho|O)=\sum_N\int dx_1\int dx_2...\int dx_N\rho^{(r)}_{x_1x_2...x_N}O^{(r)}_{x_1x_2...x_N}+$$

$$\sum_{N,\alpha_i,i}\int dx_1\int dx_2...\int dx_{N-i}\rho^{(\alpha_i,s)}_{N,x_1x_2...x_{N-i}}O^{(\alpha_i,s)}_{N,x_1x_2...x_{N-i}}(n-4)^{-\alpha_i} \tag{145}$$

which is only a mathematically well defined object when $n\rightarrow4$ if only the coordinates $\rho^{(r)}_{x_1x_2...x_N}$ and $O^{(r)}_{x_1x_2...x_N}$ do not vanish. But this is the case since either

i.- the physical observables really read

$$O=\sum_N\left[\int dx_1\int dx_2...\int dx_NO^{(r)}_{x_1x_2...x_N}|x_1,x_2,...x_N)\right] \tag{146}$$

since they have only the regular part (because they do not see the singularities of the states, etc.) so they have no singular $(n-4)^{-\alpha_i}$ terms or

ii.- the states really read

$$\rho=\sum_N\left[\int dx_1\int dx_2...\int dx_N\rho^{(r)}_{x_1x_2...x_N}(x_1,x_2,...x_N|\right] \tag{147}$$

since they have only the regular part (because they do not see the singularities of the observables, etc.) so they have no singular $(n-4)^{-\alpha_i}$ terms. But here we have a better argument: they have only regular part since the functions $\rho(x_1)\rho(x_2)...\rho(x_N)$ of eq. (140) *are usually consider regular and with no singularity.*

Therefore if we use the functional idea embodied in eq. (145), or better eq. (140), and the regular state of eq. (146) or regular observables in (147) we just have:

$$Z[\rho] = \exp i \sum_N \int dx_1 \int dx_2 ... \int dx_N O^{(r)}_{x_1 x_2 ... x_N} \rho^{(r)}_{x_1 x_2 ... x_N} \tag{148}$$

which is finite and the same happens with the $\partial/\partial\rho$ derivatives of $Z[\rho]$. Thus the theory is finite. *So the theory becomes finite just supposing that the physical observables are regular (namely, just using as observables the real physical apparatuses in our laboratory which give us* **finite measurements***) or the functions $\rho(x_1)\rho(x_2)...\rho(x_N)$ are regular (***which is the usual supposition***) and adopting the* **functional** *approach based in the ideas of the authors of papers* [2], [16], [17], and [18]. In this way the substraction method is justified. Instead if we use the naive usual formalism where all the characters belong to Hilbert spaces and are multiplied using the ordinary *inner product* $Z[\rho]$ will be singular and the theory must be renormalized.

VII. CONCLUSION.

Sometimes renormalization is considered as a *miracle* ([5], page 243, [13], page 172). In fact: there is an infinite bare mass m_0 (which being infinite it can hardly be considered as "bare"), and an infinite counterterm, that plus the bare mass gives the finite physical "dressed" mass m (which being finite is less dressed than the bare one); there is an infinite bare coupling constant and a counterterm such that,...etc. The substraction of all these infinities give (bingo!) the right answer. This is a *pure miracle*[34]!

Now let us consider the same phenomenon according to the ideas of this paper: We have chosen the simplest Lorentz invariant lagrangian L, constructed using a scalar filed ϕ, to base our theory. It is too much to assume that L would give us the right answers both for long and short distances. In fact, it works remarkably well for long distances but it behaves badly for short ones, since it produces short distance singularities in the relevant N-points functions. So let us eliminate these singularities and we will obtain the correct both short and long distance behavior. This is the best we can do with lagrangian L and the best we have until more refined lagrangian will be invented (using perhaps superstrings, membranes, etc.). Moreover, the singular structure is point-like and a pure mathematical artifact, and therefore undetectable by the measurement apparatuses, so it must be eliminated, in some way or other. So there is no miracle in the finite nature of the theory and there is a logical explanation of what really is going on. All these facts are embodied in the rigorous mathematical structure of section VI.

Only a *minor miracle* remains. The numerical constant of some (renormalizable) models are determined by a finite number of measurements, while others (unrenormalizable) need an infinite number. Really it is a very small miracle compared with the former one. We are used to deal with systems that can be defined with a finite number parameters (e. g. mechanical systems) while others have an infinite number (e.g.: the initial conditions of classical electromagnetic fields or mechanical systems with an infinite number of parameters like fluid with variable density or viscosity). Then what really remains is a very big practical problem, how to work and solve quantum field systems similar to the latter kind[35]. We do not propose a solution but we believe that we have enlightened the real nature of the problem.

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[34] The author himself confesses that it was really difficult to understand and to teach this miracle.

[35] Maybe superstrings or membranes are book keeping devices that allow us to tame an infinite number of data as function $y = f(x)$ encompassing an infinite number of data: the infinite relations between each value of variable x with each value of variable y.

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