Computer Physics Communications 138 (2001) 29–43

Computer Physics Communications

www.elsevier.com/locate/cpc

Numerical evaluation of Appell's F_1 hypergeometric function

F.D. Colavecchia [a,*], G. Gasaneo [b,1], J.E. Miraglia [c]

[a] Centro Atómico Bariloche and Consejo Nacional de Investigaciones Científicas y Técnicas, 8400 S.C. de Bariloche, Argentina

[b] Departmento de Física, Universidad Nacional del Sur, Av. Alem 1453, 8000 Bahía Blanca, Argentina

[c] Instituto de Física y Astronomía del Espacio and Consejo Nacional de Investigaciones Científicas y Técnicas, 1428 Buenos Aires, Argentina

Received 13 December 2000; received in revised form 28 January 2001; accepted 3 March 2001

Abstract

In this work we present a numerical scheme to compute the two-variable hypergeometric function $F_1(\alpha, \beta, \beta', \gamma; x, y)$ of Appell for complex parameters α, β, β' and γ, and real values of the variables x and y. We implement a set of analytic continuations that allow us to obtain the F_1 function outside the region of convergence of the series definition. These continuations can be written in terms of the Horn's G_2 function, Appell's F_2 function related, and the F_1 hypergeometric itself. The computation of the function inside the region of convergence is achieved by two complementary methods. The first one involves a single-index series expansion of the F_1 function, while the second one makes use of a numerical integration of a third order ordinary differential equation that represents the system of partial differential equations of the F_1 function. We briefly sketch the program and show some examples of the numerical computation. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 02.60.Gf; 02.30.Gp; 34.50.Fa

Keywords: Numerical methods; Special functions; Hypergeometric functions; Appell functions; Gauss function

1. Introduction

The numerical evaluation of special functions has been an active field of research in physics since the introduction of the computer in science. There exists a variety of numerical approaches for each special function of mathematics and nowadays this generally does not represent a particular problem [1]. However, there are some special functions that still present difficulties. The most important of them is the Gauss hypergeometric function commonly denoted as ${}_2F_1(\alpha, \beta, \gamma, z)$. This function is mathematically convergent for all values of $|z| < 1$ and can be expanded as a series [2]:

$$ {}_2F_1(\alpha, \beta, \gamma, z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{z^m}{m!} \tag{1} $$

* Corresponding author. Present address: Theoretical Division (T-12, MS B268), Los Alamos National Laboratory, Los Alamos, NM 87545, USA.

E-mail address: flavioc@t12.lanl.gov (F.D. Colavecchia).

1 Present address: Department of Physics and Astronomy, University of Tennessee, Knoxville, TN 37996-1501, USA.

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PII: S0010-4655(01)00186-2

for $\gamma \neq 0, -1, -2, \ldots$ and as usual $(a)_m = \Gamma(a+m)/\Gamma(a)$ represents the Pochhammer symbol and $\Gamma(z)$ is the Gamma function. The Gauss function presents a cut in the complex plane for real values of $z > 1$. As quoted in Ref. [1], "numerical computation of Gauss hypergeometric function for all values of the variables and parameters is practically impossible". From the numerical point of view, the series (1) can be summed with confidence for $|z| \lesssim r_{\max}$ with $r_{\max} \sim 0.5$ in double precision arithmetic. There are several strategies that enable the calculation of the Gauss function for $|z| > 1$. The first one is to solve parametrically the differential equation of the ${}_2F_1$ function [1]:

$$z(1-z)\frac{\partial^2 F}{\partial z^2} + \left[\gamma - (\alpha+\beta+1)z\right]\frac{\partial F}{\partial z} - \alpha\beta F = 0 \tag{2}$$

with some numerical methods, such as Runge–Kutta, using the series expansion (1) as a starting point of the calculation for a z_0 with $|z_0| \ll r_{\max}$. This gives good results for values not very close to the cut. In such cases a different path of integration should be considered. On the other hand, the function ${}_2F_1$ has several analytic continuations that are useful to compute the hypergeometric function in 'problematic' regions [2]. For example, the continuation

$$\begin{aligned}{}_2F_1(\alpha,\beta,\gamma,z) = {} & \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,{}_2F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-z) \\ & + \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z)\end{aligned} \tag{3}$$

allow the computation of the function in the vicinity of $z = 1$. This approach has been taken by Forrey to obtain the Gauss function for real values of the variables [3].

There are a lot of problems where the computation of the Gauss function is necessary. On the one hand, many special functions can be obtained as particular cases of the hypergeometric ${}_2F_1$, such as the Jacobi, Legendre polynomials, etc. On the other hand, the outcome of many integrals arising in physics can be expressed in terms of ${}_2F_1$ functions. This is the case for three-dimensional integrals resulting in the computation of transition matrices in atomic and molecular physics, such as the transitions that involve Coulombic continuum states given by the Kummer ${}_1F_1$ function [4–6]. These integrals are commonly known as Nordsieck integrals and include a product of two Kummer functions:

$$J_1 = \int \frac{\mathrm{d}\mathbf{r}}{r}\,\mathrm{e}^{-zr+\mathrm{i}\mathbf{q}.\mathbf{r}}\mathcal{F}_1\mathcal{F}_2 \tag{4}$$

with $\mathcal{F}_j = {}_1F_1(\mathrm{i}a_j, 1, \mathrm{i}p_j r + \mathrm{i}\mathbf{p}_j.\mathbf{r})$ and z, $\mathbf{q}$, a_j and $\mathbf{p}_j$ are real parameters. Recently we have found analytic solutions of generalized Nordsieck integrals where the second parameter of $\mathcal{F}_j$ is no longer one, but any complex value. This result can be represented by a multivariable hypergeometric function [7]. When only one of these parameters is one, the function reduces to the Appell's F_1 function [8]. This simplification has been useful to compute ionization cross sections in ion–atom collisions [9].

The evaluation of two-variable hypergeometric functions is more difficult than the ${}_2F_1$ due to the increasing number of parameters and variables. Besides, the mathematical regions where the functions are defined by series expansion can be very rare. For example, the Appell's $F_4(\alpha,\beta,\gamma,\gamma',x,y)$ function is convergent in the region:

$$\sqrt{|x|} + \sqrt{|y|} < 1. \tag{5}$$

Moreover, analytic continuations of these functions are well known; however, as we will see in the case of Appell's F_1, they are not as simple as their one-variable partners.

In this work we develop a numerical scheme to compute the function $F_1 = F_1(\alpha,\beta_1,\beta_2,\gamma,x,y)$ defined as a double series [8,10]:

$$F_1(\alpha,\beta_1,\beta_2,\gamma,x,y) = \sum_{m,n}\frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}m!n!}x^m y^n. \tag{6}$$

This series is mathematically convergent when:

$$|x| < 1 \quad \text{and} \quad |y| < 1. \tag{7}$$

The plan of the paper is the following. In Section 2 we outline the main properties of Appell's F_1 function. In Section 3 we present the numerical procedure to compute the function, pointing out the different regions of the variables where the function can be computed. In Section 4 we summarize the computer code and present some results. Finally we draw some conclusions and consider the generalization of this procedure for complex variables.

2. The Appell's F_1 function

In this section we review some important properties of the two-variable hypergeometric function of Appell. Appell's F_1 function is a solution of the system of partial differential equations:

$$\begin{aligned} &x(1-x)\frac{\partial^2 F}{\partial x^2} + y(1-x)\frac{\partial^2 F}{\partial x \partial y} + \left[\gamma - (\alpha + \beta_1 + 1)x\right]\frac{\partial F}{\partial x} - \beta_1 y \frac{\partial F}{\partial y} - \alpha\beta_1 F = 0, \\ &y(1-y)\frac{\partial^2 F}{\partial y^2} + x(1-y)\frac{\partial^2 F}{\partial x \partial y} + \left[\gamma - (\alpha + \beta_2 + 1)y\right]\frac{\partial F}{\partial y} - \beta_2 x \frac{\partial F}{\partial x} - \alpha\beta_2 F = 0. \end{aligned} \tag{8}$$

The series expansion (6) is the solution of the equation in the vicinity of the singular point $(x, y) = (0, 0)$. Further solutions of the equations can be obtained by path integration in the complex plane. Le Vavasseur studied these solutions and obtained a table of sixty integrals of the above system [8,10]. This table contains all the solutions of the system of equations that are expressible in terms of F_1 functions. However, Erdélyi pointed out that there exist other solutions of the equation that can not be associated with F_1 functions [11]. These extra solutions can be defined in terms of the two variable G_2 function:

$$G_2(\alpha_1, \alpha_2, \beta_1, \beta_2, x, y) = \sum (\alpha_1)_m (\alpha_2)_n (\beta_1)_{n-m} (\beta_2)_{m-n} \frac{x^m y^n}{m!n!}. \tag{9}$$

As well as the F_1 function, G_2 belongs to the thirty four hypergeometric functions of two-variables of order two given by Horn [12]. The whole set of functions given by F_1 and G_2 functions can be used to obtain solutions of the system (8) near any singular point and is equivalent to the set of twenty four one-variable hypergeometric functions given by Kummer that enable the computation of the Gauss function near every singular point of Eq. (2).

The simple inspection of the set of Eqs. (8) shows that there exists a variety of singular points and directions. They are given by the (x, y) pairs:

$$\begin{array}{ccc} (0, y) & (1, y) & (\infty, y) \\ (x, 0) & (x, 1) & (x, \infty) \\ & (x, x = y) & \end{array} \tag{10}$$

We can obtain analytic continuations of the solutions in the vicinity of any of these singular manifolds. However, some particular points will deserve a special treatment since they can be intersections of two or three of these manifolds. For example, the point $(0, 1)$ is the intersection of the $(0, y)$ and $(x, 1)$. The situation is worse for the $(1, 1)$ point, since it is a three manifold intersection. Olsson performed a detailed study of the analytic continuations in these points [13]. In the next sections we make use of these analytic continuations to compute the function in the whole $\{x, y\}$ real plane.

3. Numerical scheme

The numerical approach to compute Appell's F_1 function is built upon both the set of analytic continuations and the numerical solution of the system of partial differential equations. We make use of the first ones to reduce the computation of the function to the vicinity of the point $(0,0)$. There we use different strategies to obtain the functions.

3.1. Transformations of the F_1 function

There are a variety of transformations of the F_1 function, ranging from the simplest cases where one of the variables or parameters is zero, to the analytic continuations obtained by Le Vavasseur and Olsson. We divide them into simple transformations, that involve only one single-variable hypergeometric, and analytic continuations, that in general include also the function G_2.

3.1.1. Simple transformations

The simplest transformations of the Appell's F_1 function are quoted in advanced textbooks on special functions. If one of the parameters β_1 or β_2 or only one of the variables become zero, the function reduces to a Gauss hypergeometric [14]:

$$F_1(\alpha,0,\beta_2,\gamma,x,y) = {}_2F_1(\alpha,\beta_2,\gamma,y), \tag{11}$$

$$F_1(\alpha,\beta_1,0,\gamma,x,y) = {}_2F_1(\alpha,\beta_1,\gamma,x), \tag{12}$$

$$F_1(\alpha,\beta_1,\beta_2,\gamma,0,y) = {}_2F_1(\alpha,\beta_2,\gamma,y), \tag{13}$$

$$F_1(\alpha,\beta_1,\beta_2,\gamma,x,0) = {}_2F_1(\alpha,\beta_1,\gamma,x). \tag{14}$$

There are also some simple transformations similar to the Gauss function:

$$F_1 = (1-x)^{-\beta_1}(1-y)^{-\beta_2} F_1\left(\gamma-\alpha,\beta_1,\beta_2,\gamma,\frac{x}{x-1},\frac{y}{y-1}\right) \tag{15}$$

$$= (1-x)^{-\alpha} F_1\left(\alpha,\gamma-\beta_1-\beta_2,\beta_2,\gamma,\frac{x}{x-1},\frac{x-y}{x-1}\right) \tag{16}$$

$$= (1-y)^{-\alpha} F_1\left(\alpha,\beta_1,\gamma-\beta_1-\beta_2,\gamma,\frac{y-x}{y-1},\frac{y}{y-1}\right). \tag{17}$$

Furthermore, there exists for this function a set of Kummer theorems:

$$F_1(\alpha,\beta_2,\beta_2,\gamma,x,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_2)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_2)}\,{}_2F_1(\alpha,\beta_1,\gamma-\beta_2,x), \tag{18}$$

$$F_1(\alpha,\beta_1,\beta_2,\gamma,1,y) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_1)}\,{}_2F_1(\alpha,\beta_2,\gamma-\beta_1,y), \tag{19}$$

$$F_1(\alpha,\beta_1,\beta_2,\gamma,1,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1-\beta_2)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_1-\beta_2)}. \tag{20}$$

3.1.2. Analytic continuations

We make use of some of the analytic continuations found by Olsson that we quote here for completeness [13]. In Table 1 we summarize the different regions related to each singular manifold as well as the equations involved. Near the singular point $(1,1)$ we have:

$$\begin{aligned} F_1 = {}& \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1-\beta_2)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_1-\beta_2)} F_1(\alpha,\beta_1,\beta_2,1+\alpha+\beta_1+\beta_2-\gamma,1-x,1-y) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta_2-\gamma)}{\Gamma(\alpha)\Gamma(\beta_2)}(1-x)^{-\beta_1}(1-y)^{\gamma-\alpha-\beta_2} \end{aligned}$$

Table 1
Analytic continuations of the $F_1(\alpha, \beta_1, \beta_2, \gamma, x, y)$ function

Region	Singular points (x, y)	Transformation variables (u, w)	Transformation equation
1	$(1, 1)$	$(1-x, 1-y)$	(21), (22)
2	$(0, \infty)$	$(x/y, 1/y)$	(23)
3	$(\infty, 0)$	$(1/x, y/x)$	(24)
4	$(1, \infty)$	$(1-x, 1/y)$	(25)
5	$(\infty, 1)$	$(1/x, 1-y)$	(26)
6	(∞, ∞)	$(1/x, 1/y)$	(27), (28)
7a	$(\infty, \infty), x \sim y$	$(\frac{x-y}{y(x-1)}, \frac{1}{y})$ if $\|x-y\| < \|1-x\|$	(29)
7b	$(\infty, \infty), x \sim y$	$(\frac{1}{x}, \frac{x-y}{x(y-1)})$ if $\|x-y\| < \|1-y\|$	(30)

$$\times F_1\left(\gamma-\alpha, \beta_1, \gamma-\beta_1-\beta_2, \gamma-\alpha-\beta_2+1, \frac{1-y}{1-x}, 1-y\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_2)\Gamma(\alpha+\beta_1+\beta_2-\gamma)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\gamma-\alpha)}(1-x)^{\gamma-\alpha-\beta_1-\beta_2}$$
$$\times G_2\left(\gamma-\beta_1-\beta_2, \beta_2, \alpha+\beta_1+\beta_2-\gamma, \gamma-\alpha-\beta_2, x-1, \frac{1-y}{x-1}\right). \tag{21}$$

A similar equation can be obtained by interchanging the roles of x with y and β_1 with β_2:

$$F_1 = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1-\beta_2)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_1-\beta_2)}F_1(\alpha, \beta_1, \beta_2, 1+\alpha+\beta_1+\beta_2-\gamma, 1-x, 1-y)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta_1-\gamma)}{\Gamma(\alpha)\Gamma(\beta_1)}(1-y)^{-\beta_2}(1-x)^{\gamma-\alpha-\beta_1}$$
$$\times F_1\left(\gamma-\alpha, \gamma-\beta_2-\beta_1, \beta_2, \gamma-\alpha-\beta_1+1, 1-x, \frac{1-x}{1-y}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1)\Gamma(\alpha+\beta_1+\beta_2-\gamma)}{\Gamma(\alpha)\Gamma(\beta_2)\Gamma(\gamma-\alpha)}(1-y)^{\gamma-\alpha-\beta_1-\beta_2}$$
$$\times G_2\left(\beta_1, \gamma-\beta_1-\beta_2, \gamma-\alpha-\beta_1, \alpha+\beta_1+\beta_2-\gamma, \frac{1-x}{y-1}, y-1\right). \tag{22}$$

To get into the convergence region, we use (21) when $|1-y| < |1-x|$ and (22) otherwise. The analytic continuation near the singular point $(0, \infty)$ is :

$$F_1 = \frac{\Gamma(\gamma)\Gamma(\beta_2-\alpha)}{\Gamma(\beta_2)\Gamma(\gamma-\alpha)}(-y)^{-\alpha}F_1\left(\alpha, \beta_1, 1+\alpha-\gamma, \alpha-\beta_2+1, \frac{x}{y}, \frac{1}{y}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta_2)}{\Gamma(\alpha)\Gamma(\gamma-\beta_2)}(-y)^{-\beta_2}G_2\left(\beta_1, \beta_2, 1+\beta_2-\gamma, \alpha-\beta_2, -x, -\frac{1}{y}\right). \tag{23}$$

Analytical continuation near $(\infty, 0)$:

$$\begin{aligned} F_1 =\ & \frac{\Gamma(\gamma)\Gamma(\beta_1-\alpha)}{\Gamma(\beta_1)\Gamma(\gamma-\alpha)}(-x)^{-\alpha} F_1\left(\alpha, 1+\alpha-\gamma, \beta_2, \alpha-\beta_1+1, \frac{1}{x}, \frac{y}{x}\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_1)}{\Gamma(\alpha)\Gamma(\gamma-\beta_1)}(-x)^{-\beta_1} G_2\left(\beta_1, \beta_2, \alpha-\beta_1, 1+\beta_1-\gamma, -\frac{1}{x}, -y\right). \end{aligned} \tag{24}$$

Analytic continuation in the vicinity of $(1, \infty)$:

$$\begin{aligned} F_1 =\ & \frac{\Gamma(\gamma)\Gamma(\beta_2-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta_2)}(1-y)^{-\alpha} F_1\left(\alpha, \beta_1, \gamma-\beta_1-\beta_2, 1+\alpha-\beta_2, \frac{1-x}{1-y}, \frac{1}{1-y}\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta_1-\gamma)}{\Gamma(\alpha)\Gamma(\beta_1)}(1-x)^{\gamma-\alpha-\beta_1}(1-y)^{-\beta_2} \\ & \times F_1\left(\gamma-\alpha, \beta_2, \gamma-\beta_2-\beta_1, \gamma-\alpha-\beta_1+1, \frac{1-x}{1-y}, 1-x\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_2)\Gamma(\gamma-\alpha-\beta_1)}{\Gamma(\alpha)\Gamma(\gamma-\beta_1-\beta_2)\Gamma(\gamma-\alpha)}(1-y)^{-\beta_2} \\ & \times G_2\left(\beta_1, \beta_2, \gamma-\alpha-\beta_1, \alpha-\beta_2, x-1, \frac{1}{y-1}\right). \end{aligned} \tag{25}$$

When the values of (x, y) approach to the point $(\infty, 1)$ we make use of:

$$\begin{aligned} F_1 =\ & \frac{\Gamma(\gamma)\Gamma(\beta_1-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta_1)}(1-x)^{-\alpha} F_1\left(\alpha, \gamma-\beta_1-\beta_2, \beta_2, 1+\alpha-\beta_1, \frac{1}{1-x}, \frac{1-y}{1-x}\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta_2-\gamma)}{\Gamma(\alpha)\Gamma(\beta_2)}(1-y)^{\gamma-\alpha-\beta_2}(1-x)^{-\beta_1} \\ & \times F_1\left(\gamma-\alpha, \beta_1, \gamma-\beta_1-\beta_2, \gamma-\alpha-\beta_2+1, \frac{1-y}{1-x}, 1-y\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_1)\Gamma(\gamma-\alpha-\beta_2)}{\Gamma(\alpha)\Gamma(\gamma-\beta_1-\beta_2)\Gamma(\gamma-\alpha)}(1-x)^{-\beta_1} \\ & \times G_2\left(\beta_1, \beta_2, \alpha-\beta_1, \gamma-\alpha-\beta_2, \frac{1}{x-1}, y-1\right). \end{aligned} \tag{26}$$

There are several continuations near (∞, ∞). The first ones

$$\begin{aligned} F_1 =\ & \frac{\Gamma(\gamma)\Gamma(\beta_1-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta_1)}(-x)^{-\alpha} F_1\left(\alpha, 1+\alpha-\gamma, \beta_2, 1+\alpha-\beta_1, \frac{1}{x}, \frac{y}{x}\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_1-\beta_2)}{\Gamma(\alpha)\Gamma(\gamma-\beta_1-\beta_2)}(-x)^{-\beta_1}(-y)^{-\beta_2} \\ & \times F_1\left(1+\beta_1+\beta_2-\gamma, \beta_1, \beta_2, 1+\beta_1+\beta_2-\alpha, \frac{1}{x}, \frac{1}{y}\right) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_1)\Gamma(\beta_1+\beta_2-\alpha)}{\Gamma(\alpha)\Gamma(\beta_2)\Gamma(\gamma-\alpha)}(-x)^{-\beta_1}(-y)^{\beta_1-\alpha} \\ & \times G_2\left(\beta_1, 1+\alpha-\gamma, \alpha-\beta_1, \beta_1+\beta_2-\alpha, -\frac{y}{x}, -\frac{1}{y}\right) \end{aligned} \tag{27}$$

$$
\begin{aligned}
F_1 = {} & \frac{\Gamma(\gamma)\Gamma(\beta_2-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta_2)}(-y)^{-\alpha} \\
& \times F_1\left(\alpha,\beta_1,1+\alpha-\gamma,1+\alpha-\beta_2,\frac{x}{y},\frac{1}{y}\right) \\
& + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_1-\beta_2)}{\Gamma(\alpha)\Gamma(\gamma-\beta_1-\beta_2)}(-x)^{-\beta_1}(-y)^{-\beta_2} \\
& \times F_1\left(1+\beta_1+\beta_2-\gamma,\beta_1,\beta_2,1+\beta_1+\beta_2-\alpha,\frac{1}{x},\frac{1}{y}\right) \\
& + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_2)\Gamma(\beta_1+\beta_2-\alpha)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\gamma-\alpha)}(-x)^{\beta_2-\alpha}(-y)^{-\beta_2} \\
& \times G_2\left(1+\alpha-\gamma,\beta_2,\beta_1+\beta_2-\alpha,\alpha-\beta_2,-\frac{1}{x},-\frac{x}{y}\right)
\end{aligned} \tag{28}
$$

are useful for $x \nsim y$. As in the $(1,1)$ case, we use (27) when $x < y$ and (28) in the other case. If $x \sim y$, we can make use of:

$$
\begin{aligned}
F_1 = {} & \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_1-\beta_2)}{\Gamma(\alpha)\Gamma(\gamma-\beta_1-\beta_2)}(-y)^{\alpha-\gamma}(1-x)^{-\beta_1}(1-y)^{\gamma-\alpha-\beta_2} \\
& \times F_1\left(\gamma-\alpha,\beta_1,1-\alpha,1+\beta_1+\beta_2-\alpha,\frac{x-y}{y(x-1)},\frac{1}{y}\right) \\
& + \frac{\Gamma(\gamma)\Gamma(\beta_1+\beta_2-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta_1+\beta_2)}(-y)^{\beta_1+\beta_2-\gamma}(1-x)^{-\beta_1}(1-y)^{\gamma-\alpha-\beta_2} \\
& \times G_2\left(\beta_1,\gamma-\beta_1-\beta_2,1-\beta_1-\beta_2,\beta_1+\beta_2-\alpha,\frac{x-y}{1-x},-\frac{1}{y}\right)
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
F_1 = {} & \frac{\Gamma(\gamma)\Gamma(\alpha-\beta_1-\beta_2)}{\Gamma(\alpha)\Gamma(\gamma-\beta_1-\beta_2)}(-x)^{\alpha-\gamma}(1-y)^{-\beta_2}(1-x)^{\gamma-\alpha-\beta_1} \\
& \times F_1\left(\gamma-\alpha,1-\alpha,\beta_2,1+\beta_1+\beta_2-\alpha,\frac{1}{x},\frac{y-x}{x(y-1)}\right) \\
& + \frac{\Gamma(\gamma)\Gamma(\beta_1+\beta_2-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta_1+\beta_2)}(-x)^{\beta_1+\beta_2-\gamma}(1-y)^{-\beta_2}(1-x)^{\gamma-\alpha-\beta_1} \\
& \times G_2\left(\gamma-\beta_1-\beta_2,\beta_2,\beta_1+\beta_2-\alpha,1-\beta_1-\beta_2,-\frac{1}{x},\frac{y-x}{1-y}\right).
\end{aligned} \tag{30}
$$

We found that this set of analytic continuations enables us to map the points outside the convergence region into it. There are other analytic continuations for other singular manifolds (see Le Vavasseur, Ref. [10]). However, we have found that the set presented here is complete enough to compute the Appell's F_1 outside the convergence region. A larger set of analytic continuations could be useful when considering complex variables.

3.1.3. The convergence region

Once the point $\mathbf{P} = (x, y)$ has been mapped into the convergence region, we define a key parameter t_0 to decide whether to use a series expansion or to solve the differential equation. For practical purposes, a value of $t_0 = 0.5$ works well since the series can be evaluated confidently for absolute values of the variables smaller than this t_0. The double series (6) is hard to be summed up, so we make use of a single index series expansion found by Burchnall and Chaundy [16,17]:

$$F_1(\alpha,\beta_1,\beta_2,\gamma,x,y) = \sum_r \frac{(\alpha)_r(\beta_1)_r(\beta_2)_r(\gamma-\alpha)_r}{(\gamma+r-1)_r(\gamma)_{2r}r!}(xy)^r \times {}_2F_1(\alpha+r,\beta_1+r,\gamma+2r,x) \times {}_2F_1(\alpha+r,\beta_2+r,\gamma+2r,y). \tag{31}$$

From a numerical point of view this series seems to be more difficult to evaluate since each term would require the evaluation of two Gauss functions. However, we found that this series is strongly convergent and suitable for numerical calculations. In the case

$$t_0 < |\mathbf{P}-\mathbf{0}| < 1 \tag{32}$$

we solve numerically a third-order ordinary differential equation that represents the whole system of partial differential equations (8) [11]. This equation has been found by Burchnall for the function $z(xt,yt)$ in terms of the parametric variable t [15]. The equation is:

$$z\delta(-2+\gamma+\delta)(-1+\gamma+\delta) - z(\alpha+\delta)(-1+\gamma+\delta)t\big(x(\beta_1+\delta)+y(\delta+\beta_2)\big) + t^2xy\,z(\alpha+\delta)(1+\alpha+\delta)(\beta_1+\delta+\beta_2) = 0, \tag{33}$$

where $\delta = t\frac{\partial}{\partial t}$. The Appell's F_1 function verifies this equation. This equation can be numerically integrated from a proper initial condition to retrieve the F_1 function, in the same way as that proposed by Press and co-workers in Ref. [1]. The selection $z(0,0)=0$ and a numerical path integration following a straight line to the point $\mathbf{P}$ gives the desired result, that is, the function in the region (32). Furthermore, this enables us to check the result from the sum of the series. This strategy cannot be used straightforwardly for a point outside the convergence regions, since in that case the path would cross the lines $x=1$ or $y=1$ which are singular points of Eq. (33).

On the other hand, the use of analytic continuations results in a further problem, that is, the computation of the G_2 function. This function can be related to Appell's F_2 function by:

$$G_2(\beta_1,\beta_2,\gamma_1,\gamma_2,x,y) = (1+x)^{-\beta_1}(1+y)^{-\beta_2} \times F_2\left(1-\gamma_1-\gamma_2,\beta_1,\beta_2,1-\gamma_1,1-\gamma_2,\frac{x}{x+1},\frac{y}{y+1}\right) \tag{34}$$

and

$$F_2(\alpha,\beta_1,\beta_2,\gamma_1,\gamma_2,x,y) = \sum \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma_1)_m(\gamma_1)_n m!n!}x^m y^n. \tag{35}$$

There exists a single index series for Appell's F_2 function:

$$F_2(a,\beta_1,\beta_2,\gamma_1,\gamma_2,x,y) = \sum_r \frac{(a)_r(\beta_1)_r(\beta_2)_r}{(\gamma_1)_r(\gamma_2)_r r!}(xy)^r \times {}_2F_1(a+r,\beta_1+r,\gamma_1+2r,x) \times {}_2F_1(a+r,\beta_2+r,\gamma_2+2r,y) \tag{36}$$

which enables us to compute the G_2 function in its region of convergence. With these methods we can efficiently compute the function in the whole real plane.

4. The structure of the program

In this section we make a more detailed description of the code. We have two versions of these routines. We have first developed the general structure of the program in Mathematica, using their built-in Gauss and Gamma functions. We then translated the code into a Fortran program. The structure of the program is the same, although in the latter case we should rely on numerical functions to compute the core parts of the program. There are a variety of routines that compute Gamma and Gauss functions, and, for brevity, we will not comment here on the evaluation of them. For example, the functions described in Ref. [1] are well suited to perform this task. Also, these functions have been nicely programmed by Forrey [3]. The FORTRAN code uses double precision arithmetic.

4.1. Programming details

The computation of the Appell's hypergeometric function is based on a careful determination of the most convenient analytic continuation for each given pair of variables (x, y). These procedures are encapsulated in the function `f1`, that gets all the parameters and variables as arguments, returning the complex value of the F_1 function. First, the program checks the trivial cases: whether or not some of the variables are zero, see Eqs. (11)–(14) and when they fulfill some particular relations (Eqs. (15)–(17)). These simple expressions imply the evaluation of only a Gauss hypergeometric function. We found that the comparisons of double precision variables with the mathematical zero can be performed against a numerical zero value defined as 10^{-12} for all practical purposes. The next step is to determine the best analytic continuation to be used. Whenever the absolute values of the variables are less than one, we call the routine `f1bnl`. This routine manages two possible behaviours: if the absolute values of both variables are less than $t_0 = 0.5$ we make use of the series expansion in terms of the hypergeometric ${}_2F_1$, Eq. (31), which provides relative and absolute error criteria. Otherwise, the routine switches to the numerical solution of the third-order differential equation, Eq. (33). This numerical integration is performed with the routine `odeint` from Ref. [1], slightly modified to manage a set of complex coefficients in the differential equation. The relative error in the integration is set to 10^{-10} through the variable `eps`.

If both variables in absolute value are greater than unity, we should choose an analytic continuation. We evaluate the effective distance in each possible region as:

$$t_{\max} = \sqrt{w^2 + u^2},$$

where w and u are given in Table 1 for each region. To obtain the minimum of all distances, we make a sequential searching, storing in the integer variable `flag` the best region obtained. When the region is selected, we call the corresponding analytic continuation. This in general would imply a computation of an Appell's F_1 function with $|x| < 1$ and $|y| < 1$ through the routine `f1bnl` and an evaluation of the G_2 function.

We must note that, in a similar way to the Gauss function ${}_2F_1$, there are many particular cases of the set of parameters $\{\alpha, \beta_1, \beta_2, \gamma\}$ that would lead to simplifications in the computation of the function. Some of them are associated with the occurrence of divergences in the analytic continuations. For example, if the routine is called with the fourth parameter $\gamma = -n$ with n a natural number or zero; the program first checks whether this function can be evaluated using one of the simple transformations. If not, the program stops. To handle this kind of divergences, a whole new routine should be written for these irregular solutions of Eq. (8). For practical purposes, we did not include them in the present version of the program. The functions frequently called such as the Gamma functions provide the corresponding error checking. To avoid some problems with the evaluation of the Gamma function near negative integers, we make use of a `cgammar` function that returns $1/\Gamma(z)$. If the variable is close enough to a pole, the functions returns zero.

4.2. Numerical tests

To illustrate the functionality of the code, we show in this section some examples of the computation of the Appell's F_1 function. The routines have been carefully tested for a variety of situations and have been shown to be very fast. For example, a typical evaluation in the region 7a takes less than 0.01 s of CPU time in a 400 MHz Pentium II single processor computer, involving the evaluation of ten Gauss functions. We note that also the computing time strongly depends on the value of the parameters of the F_1. There are several checks of the Appell's functions that involve the reduction to a Gauss function. First we study the ability of the methods proposed in the

Table 2
Test of the F_1 function, Eq. (37), in the convergence region with $\alpha = \gamma = 1$, $\beta_1 = 2 + \mathrm{i}$ and $\beta_2 = \frac{3}{2} - \frac{\mathrm{i}}{2}$

x	y	$\lvert F_1\rvert$ Eq. (33)	$(1-x)^{-\beta_1}(1-y)^{-\beta_2}$	ε
−0.95	−0.95	0.09657815	0.09657815	6.3[−12]
−0.95	−0.57	0.13368457	0.13368457	2.7[−12]
−0.95	−0.19	0.20258642	0.20258642	2.8[−12]
−0.95	0.19	0.36074743	0.36074743	7.4[−13]
−0.95	0.57	0.93267019	0.93267019	9.1[−12]
−0.95	0.95	23.52208260	23.52208260	4.4[−12]
−0.53	−0.95	0.15646981	0.15646981	3.8[−11]
−0.53	−0.57	0.21658730	0.21658730	8.2[−13]
−0.53	−0.19	0.32821773	0.32821773	3.2[−13]
−0.53	0.19	0.58446021	0.58446021	1.3[−12]
−0.53	0.57	1.51105333	1.51105333	1.1[−12]
−0.53	0.95	38.10899256	38.10899256	2.0[−11]
−0.11	−0.95	0.29592231	0.29592231	2.1[−15]
−0.11	−0.57	0.40961906	0.40961906	1.1[−12]
−0.11	−0.19	0.62073923	0.62073923	5.5[−13]
−0.11	0.19	1.10535582	1.10535582	1.7[−12]
−0.11	0.57	2.85776787	2.85776787	1.4[−12]
−0.11	0.95	72.07333626	72.07333626	1.9[−11]
0.30	−0.95	0.75810548	0.75810548	2.5[−12]
0.30	−0.57	1.04937829	1.04937829	7.3[−13]
0.30	−0.19	1.59023429	1.59023429	1.1[−11]
0.30	0.19	2.83174424	2.83174424	3.0[−14]
0.30	0.57	7.32114269	7.32114269	1.2[−12]
0.30	0.95	184.64032172	184.64032172	8.5[−12]
0.72	−0.95	4.75180494	4.75180494	3.1[−11]
0.72	−0.57	6.57750282	6.57750282	9.9[−12]
0.72	−0.19	9.96758802	9.96758802	2.2[−13]
0.72	0.19	17.74937206	17.74937206	1.2[−11]
0.72	0.57	45.88892014	45.88892014	6.0[−12]
0.72	0.95	1157.32548026	1157.32548030	3.0[−11]

[a] The numbers between brackets denote multiplicative powers of 10. The relative error ε is defined in the text.

preceding sections to deal with variables in the convergence region. When $\alpha = \gamma$ the function F_1 reduces to a simple expression:

$$F_1(\alpha, \beta_1, \beta_2, \alpha, x, y) = (1-x)^{-\beta_1}(1-y)^{-\beta_2}. \tag{37}$$

In Table 2 we show the results of the computation of the F_1 in this case. The computation is restricted to the convergence region and we make use only of the routine that numerically integrates the differential equation (33).

The relative error ε is defined with respect to the exact value of the function given by the RHS of (37). We observe that in all cases the relative error is smaller than 10^{-10}, the specified relative error in the routine.

A similar comparison can be done with the computation of the series (31). However, when $\alpha = \gamma$ all terms but the first of this relation are equal to zero, because $(\gamma - \alpha)_r = (0)_r = 0$ for $r \neq 0$. Then in this case the series reduces to the product of two Gauss functions that can be further simplified to get exactly the result (37).

Another useful expression with which make a test can be found when $\gamma = \beta_1 + \beta_2$:

$$F_1(\alpha, \beta_1, \beta_2, \beta_1 + \beta_2, x, y) = (1 - y)^{-\alpha}\, {}_2F_1\big(\alpha, \beta_1, \beta_1 + \beta_2, (y - x)/(y - 1)\big). \tag{38}$$

In Table 3 we show the results of both the series summation F_1^{ser} and the numerical integration F_1^{int} in the convergence region, compared with the result given by the Gauss function. Both relative errors ε_a and ε_b are computed against the 'exact' ${}_2F_1$. Again, both methods give a good accuracy in the whole convergence region. The 0.0 value in the relative error of the numerical integration F_1^{int} when $x = y$ indicates that the numerical integration is not possible since this line is also a pole of the differential equations even when $|x, y| < 1$ as in this example. This does not represent a difficulty because there exists an exact result that has been included in the code:

$$F_1(\alpha, \beta_1, \beta_2, \gamma, x, x) = {}_2F_1(\alpha, \beta_1 + \beta_2, \gamma, x). \tag{39}$$

Also, we note that when the variables approach the point $(1, 1)$ the numerical integration achieves better results than the series expansion. Again, the last method also deteriorates when the parameters become large.

Finally, we show another simplification of the previous example. When $\alpha = -\frac{1}{2}$, $\beta_1 = 2$, $\beta_2 = 1$ and $\gamma = \beta_1 + \beta_2$ we have:

$${}_2F_1\big(-\tfrac{1}{2}, 2, 3, z\big) = \tfrac{4}{15}(1 - y)^{-\alpha}\big[2 - (2 + 3z)(1 - z)^{3/2}\big]. \tag{40}$$

In Table 4 we show the results of the F_1 program together with a result from a Gauss function and the exact result (40). In this example we scan a whole set of variables, $|x, y| < \frac{7}{2}$ that travel through all the possible regions defined in Table 1. The computation has been performed with a relative error of 10^{-6}. The obtained error ε_{F_1} shows an excellent agreement of the computed F_1 function relative to the exact values. Surprisingly, in a few cases the relative error is smaller than the obtained with the ${}_2F_1$ function. The routine manages all the presented pair of variables and moves into the best regions according to the criteria presented before.

5. Conclusion and outlook

In this work we have presented a numerical scheme implemented to compute Appell's F_1 hypergeometric function of two variables. We make use of a mixed strategy, combining series computation and numerical integration of differential equations. Even when the F_1 function is a solution of a system of partial differential equations and is usually defined as a double series, there exists a single index series representation of the function. Also, the system of PDEs can be expressed as an ordinary third-order differential equation. These properties are highly convenient for numerical purposes. Due to the convergence behaviour of the series definition and the distribution of singularities, these methods are suitable inside the zone (7). However, there are several well known analytical continuations that enable the transformation of the function from outside the convergence region into it. The price to pay is that we need also to compute a G_2 function, an Appell's F_1 relative. The numerical code has been thoroughly tested and some of these runs have been presented as examples. We would like to remark that the code is able to compute some simplified cases, when the variables and/or the parameters fulfill some particular conditions, but do not manage all of them. For example, the introduction of simplifications that involve negative integer values of the parameters would imply a high numerical cost in terms of computing time.

The scheme presented can be also applied to other Appell's functions. We have found that it is possible to obtain analytical continuations of Appell's functions, but they present an important drawback: most of these continuations

Table 3
Numerical test of the F_1 function, Eq. (38) in the convergence region with $\alpha = 1$, $\beta_1 = 3 + \mathrm{i}$, $\beta_2 = 2 - \frac{\mathrm{i}}{2}$ and $\gamma = \beta_1 + \beta_2$

x	y	F_1^{ser} (Eq. (31))	F_1^{int} (Eq. (33))	F_1 (Eq. (38))	ε_{ser} [a]	ε_{int} [a]
−0.95	−0.95	0.51282051	0.51282051	0.51282051	0.0	0.0
−0.95	−0.57	0.55535479	0.55535479	0.55535479	7.2[−09]	2.0[−10]
−0.95	−0.19	0.60791691	0.60791691	0.60791691	4.1[−09]	2.2[−10]
−0.95	0.19	0.67545324	0.67545323	0.67545324	6.8[−09]	1.1[−08]
−0.95	0.57	0.76755519	0.76755518	0.76755365	2.0[−06]	2.0[−06]
−0.95	0.95	0.90428982	0.90428984	0.90428984	1.9[−08]	3.5[−09]
−0.57	−0.95	0.58342152	0.58342152	0.58342152	2.7[−09]	8.9[−11]
−0.57	−0.57	0.63694268	0.63694268	0.63694268	0.0	0.0
−0.57	−0.19	0.70425421	0.70425421	0.70425421	8.8[−10]	4.8[−10]
−0.57	0.19	0.79279973	0.79279973	0.79279974	4.0[−09]	3.5[−09]
−0.57	0.57	0.91786342	0.91786341	0.91786346	4.0[−08]	5.1[−08]
−0.57	0.95	1.11675223	1.11675226	1.11675226	2.2[−08]	1.1[−09]
−0.19	−0.95	0.67982327	0.67982327	0.67982327	8.9[−10]	1.6[−09]
−0.19	−0.57	0.75004065	0.75004065	0.75004065	3.9[−09]	2.5[−10]
−0.19	−0.19	0.84033613	0.84033613	0.84033613	0.0	0.0
−0.19	0.19	0.96273876	0.96273876	0.96273876	5.1[−10]	4.7[−10]
−0.19	0.57	1.14366891	1.14366890	1.14366892	6.1[−09]	1.7[−08]
−0.19	0.95	1.45953732	1.45953734	1.45953733	1.2[−08]	7.4[−09]
0.19	−0.95	0.82196313	0.82196313	0.82196312	6.1[−09]	8.5[−09]
0.19	−0.57	0.92026724	0.92026724	0.92026724	5.7[−09]	2.1[−09]
0.19	−0.19	1.05054372	1.05054372	1.05054372	6.2[−10]	6.9[−10]
0.19	0.19	1.23456790	1.23456790	1.23456790	0.0	0.0
0.19	0.57	1.52444538	1.52444536	1.52444537	9.1[−09]	1.9[−09]
0.19	0.95	2.10562432	2.10562436	2.10562427	2.5[−08]	4.3[−08]
0.57	−0.95	1.06293284	1.06293284	1.06293284	3.5[−10]	5.8[−09]
0.57	−0.57	1.21816169	1.21816169	1.21816168	1.1[−08]	1.1[−08]
0.57	−0.19	1.43365812	1.43365813	1.43365812	4.1[−09]	8.3[−09]
0.57	0.19	1.75843150	1.75843151	1.75843150	1.9[−09]	2.5[−09]
0.57	0.57	2.32558140	2.32558140	2.32558140	0.0	0.0
0.57	0.95	3.77963829	3.77963837	3.77963715	3.0[−07]	3.2[−07]
0.95	−0.95	1.66982378	1.66982335	1.66982033	2.1[−06]	1.8[−06]
0.95	−0.57	2.02668624	2.02668669	2.02668322	1.5[−06]	1.7[−06]
0.95	−0.19	2.58134890	2.58135132	2.58134727	6.3[−07]	1.6[−06]
0.95	0.19	3.56763283	3.56764083	3.56763606	9.1[−07]	1.3[−06]
0.95	0.57	5.85733701	5.85737543	5.85737585	6.6[−06]	7.1[−08]
0.95	0.95	20.00000000	20.00000000	20.00000000	0.0	0.0

[a] See note on Table 2.

Table 4
Numerical test of the F_1 function given by Eq. (40) for $x < 0$

x	y	Region	F_1	${}_2F_1$ (Eq. (38))	Exact (Eq. (40))	ε_{F_1}	$\varepsilon_{{}_2F_1}$
−3.50	−3.50	[b]	2.12132034	2.12132034	2.12132034	0.0	0.0
−3.50	−2.50	7b	2.04040982	2.04040981	2.04040992	5.2[−08]	5.2[−08]
−3.50	−1.50	7b	1.95401180	1.95401180	1.95401190	5.0[−08]	5.0[−08]
−3.50	−0.50	3	1.86035462	1.86035566	1.86035569	5.8[−07]	1.7[−08]
−3.50	0.50	3	1.75598179	1.75598179	1.75598193	7.9[−08]	7.8[−08]
−3.50	1.50	5	1.62917838	1.62917838	1.62917847	5.9[−08]	5.9[−08]
−3.50	2.50	6	1.48548529	1.48548531	1.48548541	8.1[−08]	6.3[−08]
−3.50	3.50	6	1.35499406	1.35499406	1.35499419	9.5[−08]	9.7[−08]
−2.50	−3.50	7a	1.95697834	1.95697834	1.95697845	5.2[−08]	5.2[−08]
−2.50	−2.50	[b]	1.87082869	1.87082869	1.87082869	0.0	0.0
−2.50	−1.50	7b	1.77824921	1.77824921	1.77824930	5.1[−08]	5.1[−08]
−2.50	−0.50	3	1.67700137	1.67700356	1.67700363	1.3[−06]	4.4[−08]
−2.50	0.50	3	1.56257050	1.56257050	1.56257065	1.0[−07]	9.8[−08]
−2.50	1.50	5	1.41872399	1.41872399	1.41872407	5.9[−08]	5.9[−08]
−2.50	2.50	6	1.25857059	1.25857059	1.25857068	7.1[−08]	7.2[−08]
−2.50	3.50	6	1.12513362	1.12513362	1.12513380	1.6[−07]	1.6[−07]
−1.50	−3.50	7a	1.77471786	1.77471786	1.77471794	4.6[−08]	4.6[−08]
−1.50	−2.50	7a	1.68182193	1.68182193	1.68182202	5.2[−08]	5.2[−08]
−1.50	−1.50	[b]	1.58113883	1.58113883	1.58113883	0.0	0.0
−1.50	−0.50	[c]	1.46969378	1.46969383	1.46969392	9.4[−08]	6.2[−08]
−1.50	0.50	3	1.34118576	1.34118577	1.34118599	1.7[−07]	1.6[−07]
−1.50	1.50	5	1.17126079	1.17126079	1.17126086	5.9[−08]	5.9[−08]
−1.50	2.50	6	0.99247161	0.99247160	0.99247171	1.1[−07]	1.1[−07]
−1.50	3.50	6	0.86922669	0.86922668	0.86922703	4.0[−07]	4.1[−07]
−0.50	−3.50	2	1.56577391	1.56578858	1.56578860	9.4[−06]	1.4[−08]
−0.50	−2.50	2	1.46350101	1.46351856	1.46351861	1.2[−05]	3.1[−08]
−0.50	−1.50	[c]	1.35127910	1.35127918	1.35127925	1.1[−07]	4.9[−08]
−0.50	−0.50	[b]	1.22474487	1.22474487	1.22474487	0.0	0.0
−0.50	0.50	[d]	1.07407681	1.07407684	1.07407686	4.3[−08]	1.7[−08]
−0.50	1.50	2	0.85764535	0.85764535	0.85764540	6.3[−08]	6.3[−08]
−0.50	2.50	2	0.67330010	0.67330009	0.67330036	4.0[−07]	4.1[−07]
−0.50	3.50	2	0.61598960	0.61598957	0.61599065	1.7[−06]	1.8[−06]

[a] See note in Table 2.
[b] In these cases $x = y$ and F_1 always reduces to a Gauss function.
[c] Simple continuations, Eqs. (15)–(17).
[d] Convergence region.

Table 5
Same as Table 4 for $x > 0$

x	y	Region	F_1	${}_2F_1$ (Eq. (38))	Exact (Eq. (40))	ε_{F_1} [a]	$\varepsilon_{{}_2F_1}$ [a]
0.50	−3.50	2	1.30813932	1.30814809	1.30814761	6.3[−06]	3.6[−07]
0.50	−2.50	2	1.19046476	1.19047288	1.19047262	6.6[−06]	2.2[−07]
0.50	−1.50	2	1.05833564	1.05834335	1.05834326	7.2[−06]	8.3[−08]
0.50	−0.50	[d]	0.90400843	0.90400846	0.90400847	4.4[−08]	1.4[−08]
0.50	0.50	[b]	0.70710678	0.70710678	0.70710678	0.0	0.0
0.50	1.50	1	0.38872999	0.38872999	0.38873015	4.0[−07]	4.1[−07]
0.50	2.50	2	0.42426224	0.42426219	0.42426409	4.4[−06]	4.5[−06]
0.50	3.50	2	0.60368908	0.60368899	0.60369237	5.5[−06]	5.6[−06]
1.50	−3.50	4	0.92086392	0.92086378	0.92086920	5.7[−06]	5.9[−06]
1.50	−2.50	4	0.77207798	0.77207787	0.77208236	5.7[−06]	5.8[−06]
1.50	−1.50	4	0.60368908	0.60368899	0.60369237	5.5[−06]	5.6[−06]
1.50	−0.50	3	0.42426219	0.42426219	0.42426409	4.5[−06]	4.5[−06]
1.50	0.50	1	0.38872999	0.38872999	0.38873015	4.0[−07]	4.1[−07]
1.50	1.50	[b]	0.70710678	0.70710678	0.70710678	0.0	0.0
1.50	2.50	4	0.90399999	0.90400846	0.90400847	9.4[−06]	1.4[−08]
1.50	3.50	4	1.05833564	1.05834335	1.05834326	7.2[−06]	8.3[−08]
2.50	−3.50	6	0.73484376	0.73484366	0.73484696	4.4[−06]	4.5[−06]
2.50	−2.50	6	0.65183291	0.65183285	0.65183505	3.3[−06]	3.4[−06]
2.50	−1.50	6	0.61598958	0.61598957	0.61599065	1.7[−06]	1.8[−06]
2.50	−0.50	3	0.67330009	0.67330009	0.67330036	4.1[−07]	4.1[−07]
2.50	0.50	3	0.85764534	0.85764535	0.85764540	6.5[−08]	6.3[−08]
2.50	1.50	5	1.07407624	1.07407684	1.07407686	5.7[−07]	1.7[−08]
2.50	2.50	[b]	1.22474487	1.22474487	1.22474487	0.0	0.0
2.50	3.50	7a	1.35127918	1.35127918	1.35127925	4.9[−08]	4.9[−08]
3.50	−3.50	6	0.79693132	0.79693127	0.79693294	2.0[−06]	2.1[−06]
3.50	−2.50	6	0.80691370	0.80691371	0.80691460	1.1[−06]	1.1[−06]
3.50	−1.50	6	0.86922668	0.86922668	0.86922703	4.0[−07]	4.1[−07]
3.50	−0.50	3	0.99247160	0.99247160	0.99247171	1.1[−07]	1.1[−07]
3.50	0.50	3	1.17126079	1.17126079	1.17126086	6.0[−08]	5.9[−08]
3.50	1.50	5	1.34118577	1.34118577	1.34118599	1.6[−07]	1.6[−07]
3.50	2.50	7b	1.46969385	1.46969383	1.46969392	5.0[−08]	6.2[−08]
3.50	3.50	[b]	1.58113883	1.58113883	1.58113883	0.0	0.0

[a–d] See notes in Table 4.

would require the computation of higher order single variable hypergeometric functions, such as ${}_3F_2$ or ${}_4F_3$, which are more complicated to implement numerically.

Finally, we would like to note that this method can also be used for complex variables, but some modifications would be needed. Basically, the set of analytic continuations presented is not sufficiently complete to cover the whole two-variable complex plane, and should be extended to include the regions not represented by this scheme.

Acknowledgement

We would like to acknowledge Dr. C.R. Garibotti and Dr. W. Cravero for helpful discussions and for testing some of the routines used here. This work was partially supported by ANCyT grant PICT98 03-04021.

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