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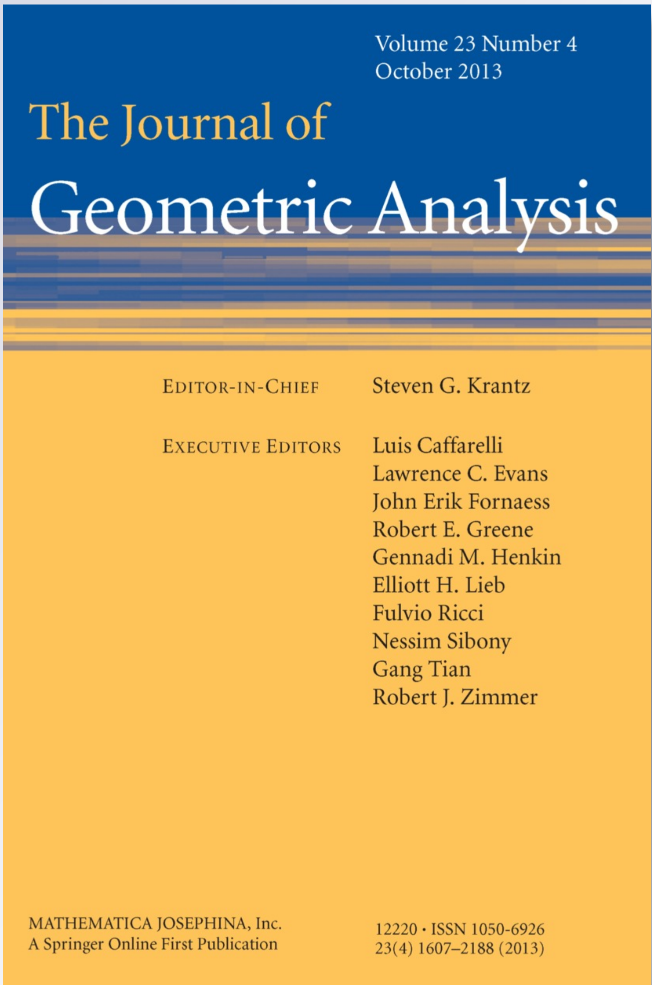

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Abstract In this note we obtain results regarding the preservation of homogeneity properties along the whole orbit of a given iterated function system (IFS). We have essentially two types of results. The first class of them contains negative results: it is possible for a classical IFS to have a complete non-homogeneous sequence of spaces along the orbit, starting from very classical homogeneous spaces such as those defined by Muckenhoupt weights. The second class contains positive results which can be summarized here by saying that the sequence of spaces defined by the orbit of contractive similitudes starting at a normal space in the sense of Ahlfors, Macías, and Segovia, preserves doubling. As a consequence of these results we conclude boundedness properties of the Hardy–Littlewood maximal operator along the orbits.

Mathematics Subject Classification Primary 28A80 · 42B25 · Secondary 60B10

Keywords Iterated function systems · Spaces of homogeneous nature · Hutchinson orbits · Maximal functions

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1 Introduction

Fourier and harmonic analysis, after the introduction of the methods of real analysis in the first years of the second half of the 20th century, relies strongly on the boundedness properties of some crucial operators. Among them, the role of the Hardy–Littlewood maximal operator M is central because of at least two reasons. The first one is that in one sense or another M controls all the singular integral operators of Fourier and harmonic analysis. The second is that it involves the subtle relationships between the two underlying structures of the space on which functions are defined: measure and metric. If (Y, d) is a metric space and μ is a positive Borel measure such that the d-balls $B = B(y, r) = \{z \in Y : d(z, y) < r\}$ have finite and positive measure, the Hardy–Littlewood maximal operator is defined, for a locally integrable function f, as

$$Mf(y) = \sup_{B \ni y} \frac{1}{\mu(B)} \int_B \left| f(z) \right| d\mu(z),$$

where the supremum is taken over the family of the d-balls B containing y.

Singular integrals and Hardy–Littlewood type maximal operators are generally not integral operators in the sense that they do not improve regularity. This fact is reflected by the boundedness properties of singular integrals; they preserve the Lebesgue L^p spaces for $1 < p < \infty$. For the Hardy–Littlewood maximal operator a natural setting for its L^p boundedness and for its weak type $(1, 1)$ is provided by the structure of space of homogeneous type: a quasi-metric space with a doubling measure. In other words, if μ is doubling then $\|Mf\|_{L^p(d\mu)} \leq C_p \|f\|_{L^p(d\mu)}$ and $\mu(\{Mf > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1(d\mu)}$. Moreover, the theory developed by Muckenhoupt in [12] provides necessary and sufficient conditions on a weight (positive density) w in order to obtain weighted estimates of the type $\|Mf\|_{L^p(w\,d\mu)} \leq C_{p,w} \|f\|_{L^p(w\,d\mu)}$ for the maximal operator M. These functions w are known as A_p-Muckenhoupt weights, and we shall recall them in the next section.

In [11] Mosco shows that the classical fractals obtained as fixed points of iterated function systems have some precise homogeneity property. Hence, on those fractals the Hardy–Littlewood maximal operator is a well-behaved operator in Lebesgue spaces. Since, on the other hand, the uniqueness of the Banach fixed point leads us to that limit space no matter what is the initial space, we ask whether or not having good behavior of the Hardy–Littlewood maximal operator in the initial space and, of course, in the limit space (the attractor) would guarantee good behavior of the Hardy–Littlewood maximal operator on each one of the spaces determined by the orbit of the iterated function system. Loosely speaking, we would expect some kind of interpolation of the behavior of the Hardy–Littlewood maximal operator, providing boundedness properties in the middle spaces from the same properties in the extreme spaces. We prove that the answer to this general question is negative, and we obtain some positive results under more restrictive doubling conditions, such as those known as Ahlfors conditions.

The geometric results are contained in Theorem 2.1, and some of their analytical consequences in Theorem 2.2.

2 Statement of Results

Let us start by describing our general framework. Let (X, d) be a metric space. A mapping $\phi : X \to X$ is called *contraction map* if there exists a constant $a > 1$ such that

$$d\big(\phi(x), \phi(y)\big) \leq \frac{1}{a} d(x, y),$$

for every $x, y \in X$. The constant $\frac{1}{a}$ is called the contractivity coefficient. We will call a finite set of contraction maps $\{\phi_1, \dots, \phi_M\}$ on X an *iterated function system* (IFS). We will use the term ISS as an acronym of *iterated system of similitudes* for designating an iterated function system Φ consisting of

(a) a compact metric space (X, d) with finite metric dimension and d-diameter equal to one;

(b) M contractive similitudes $\phi_i : X \to X$ with the same contractivity coefficient $1/a$, with $a > 1$. Precisely, each ϕ_i satisfies

$$d\big(\phi_i(x), \phi_i(y)\big) = \frac{1}{a} d(x, y)$$

for every $x, y \in X$;

(c) a non-empty open set U such that

$$\bigcup_{i=1}^{M} \phi_i(U) \subseteq U,$$

and $\phi_i(U) \cap \phi_j(U) = \emptyset$ if $i \neq j$.

A metric space (X, d) is of *finite metric dimension* (also known as finite Assouad dimension) if there exists a constant $N \in \mathbb{N}$ such that for every $x \in X$, every $r > 0$, and every r-disperse subset E of X, we have that $\operatorname{card}(E \cap B(x, 2r)) \leq N$. A set E is said to be r-disperse if $d(x, y) \geq r$ for every $x, y \in E$, $x \neq y$. If (X, d) has finite metric dimension, then every r-disperse subset of X has at most N^m points in each ball of radius $2^m r$, for all $m \in \mathbb{N}$ (see [4] and [3]). Property (c) is known as the *open set condition* (OSC) for Φ, and U is called an *open set for the OSC* for Φ (see, for example, [5, 8], and [10]).

For the sake of simplicity in further reference, we shall say that Φ as above is an iterated system of similitudes, briefly $\Phi \in \mathrm{ISS}(M, a)$ or simply $\Phi \in \mathrm{ISS}$.

Notice that a compact metric space (X, d), a positive integer M, a constant $a > 1$, M similitudes ϕ_i, and an open set U are associated with each given $\Phi \in \mathrm{ISS}$.

Of course the basic examples are the systems generating the most classical and best-known fractal sets, such as the ternary Cantor set and the von Koch snowflake. For example, in the case of the ternary Cantor set the system Φ consists of $X = [0, 1]$ equipped with the Euclidean distance $d(x, y) = |x - y|$, $M = 2$, $a = 3$, $\phi_1(x) = x/3$, $\phi_2(x) = x/3 + 2/3$, and $U = (0, 1)$.

Let $\mathcal{K} = \{K \subseteq X : K \neq \emptyset,\ \ K \text{ closed}\}$. With $[A]_\varepsilon$ we shall denote the ε-enlargement of the set $A \subset X$, i.e., $[A]_\varepsilon = \bigcup_{x \in A} B(x, \varepsilon) = \{y \in X : d(y, A) < \varepsilon\}$.

Here $d(x, A) = \inf\{d(x, y) : y \in A\}$. Given two sets in $\mathcal{K}$ A and B, the Hausdorff distance from A to B is given by

$$d_H(A, B) = \inf\bigl\{\varepsilon > 0 : A \subseteq [B]_\varepsilon \text{ and } B \subseteq [A]_\varepsilon\bigr\}.$$

It is well known that $(\mathcal{K}, d_H)$ is a complete metric space (see [6]).

Let us now introduce the Kantorovich distance d_K on the set

$$\mathcal{P}(X) = \bigl\{\mu : \mu \text{ is a non-negative Borel probability measure on } X \text{ and } \mu(X) = 1\bigr\}.$$

Given two measures μ and ν in $\mathcal{P}$, $d_K(\mu, \nu) = \sup\{|\int f\, d\mu - \int f\, d\nu| : f \in \mathrm{Lip}_1\}$, where Lip_1 denotes the space of all Lipschitz continuous functions defined on X with Lipschitz constant less than or equal to one, i.e., $f \in \mathrm{Lip}_1$ if and only if $|f(x) - f(y)| \le d(x, y)$ for every x and $y \in X$. Since (X, d) is compact, d_K gives a distance on $\mathcal{P}(X)$ such that the d_K-convergence of a sequence is equivalent to its weak star convergence to the same limit (see [5] or [2] for the metric case).

The family Φ gives rise to dynamical systems by iteration of some basic operations. We are interested in two of these dynamical systems. The first one is defined on $\mathcal{K}$ and is given by

$$T_1 Y = \bigcup_{i=1}^{M} \phi_i(Y)$$

for $Y \in \mathcal{K}$.

The second, T_2, is defined on $\mathcal{P}$. The operation T_2 on $\mu \in \mathcal{P}$, which we denote $T_2\mu$, is given by

$$T_2\mu(B) = \frac{1}{M} \sum_{i=1}^{M} \mu\bigl(\phi_i^{-1}(B)\bigr),$$

where B is any Borel measurable subset of X.

The original result of Hutchinson and some further developments like those obtained by Mosco in [11] refer to the pair of limit objects: the invariant measure and the fractal set supporting it.

Then we consider, instead of two independent dynamical systems induced by T_1 and T_2, the system generated by a single operation T defined by

$$T(Y, \mu) = (T_1 Y, T_2\mu)$$

on the set

$$\mathcal{E} = \bigl\{(Y, \mu) \in \mathcal{K} \times \mathcal{P} : \mathrm{supp}(\mu) \subseteq Y\bigr\}$$

equipped with the inherited distance $\delta((Y_1, \mu_1), (Y_2, \mu_2)) = d_H(Y_1, Y_2) + d_K(\mu_1, \mu_2)$ on $\mathcal{K} \times \mathcal{P}$. Since $\mathcal{E}$ is closed in $\mathcal{K} \times \mathcal{P}$, we have that $(\mathcal{E}, \delta)$ is also a complete metric space (see [2]).

From the Banach fixed point theorem we have that T has a unique fixed point which we shall denote, from now on, by (Y_∞, μ_∞). Of course Y_∞ is the attractor set and μ_∞ is its invariant measure. Moreover, (Y_∞, μ_∞) can be achieved as the limit, in the distance δ, of the iterations of T starting at any initial space $(Y_0, \mu_0) \in \mathcal{E}$.

For a given $(Y_0, \mu_0) \in \mathcal{E}$, we shall denote by $\mathcal{O}(Y_0, \mu_0)$ the *orbit* of T with initial point (Y_0, μ_0). In other words,

$$\mathcal{O}(Y_0, \mu_0) = \left\{T^n(Y_0, \mu_0) : n \in \mathbb{N}_0\right\} \cup \left\{(Y_\infty, \mu_\infty)\right\},$$

where $T^0(Y_0, \mu_0) = (Y_0, \mu_0)$ and $T^{n+1}(Y_0, \mu_0) = T(T^n(Y_0, \mu_0))$, $n \geq 0$. Let us write (Y_n, μ_n) to denote $T^n(Y_0, \mu_0)$.

We shall now illustrate the above definitions in the case of the ternary Cantor set C. If we take $Y_0 = [0, 1]$ and $\mu_0 =$ Lebesgue measure on Y, we have that $T_1(Y_0) = [0, 1/3] \cup [2/3, 1]$, $T_1^2(Y_0) = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, and in general, $T_1^n(Y_0)$ is the union of 2^n disjoint intervals of the n-th step in the usual construction of the Cantor set. Denoting this union by C_n, we have that $T_2^n(\mu_0)$ coincides with the uniform measure on C_n normalized to a probability. But if we now take $Y_0 = \{0\}$ and $\mu_0 =$ Dirac delta concentrated at 0, then $T_1^n(Y_0)$ is the collection L_n of all the left endpoints of each interval in C_n, and $T_2^n(\mu_0)$ is the counting measure on L_n divided by 2^n. We point out, in accordance with the results of this work, no matter what the starting space is, the limit (Y_∞, μ_∞) is the Cantor set C with the Hausdorff s-dimensional measure on C, where $s = \log 2/\log 3$ (see [5, 8]).

For $(Y, \mu) \in \mathcal{E}$ we define the Hardy–Littlewood maximal operator by

$$M_Y f(y) = \sup_{B \ni y} \frac{1}{\mu(B)} \int_B |f(z)|\, d\mu(z),$$

for functions $f \in L^1(Y, \mu)$.

So that, once an ISS Φ is given, our universal space shall be $(\mathcal{E}, \delta)$. A basic question regarding the above considerations is the preservation of some subclasses of $\mathcal{E}$, in particular those that could guarantee the preservation of the boundedness properties of the Hardy–Littlewood operator along the orbits. From the results in [11], subclasses of homogeneity, doubling, or normality are of particular interest.

Let us proceed to define, and briefly illustrate in the interval [0, 1], the subclasses $\mathcal{D}_A$, $\mathcal{D}$, $\mathcal{N}_{\beta,C}$, $\mathcal{N}_\beta$, and $\mathcal{N}$ of $\mathcal{E}$ that we are going to consider.

- ***Doubling***. For a given $A \geq 1$, we say that a pair $(Y, \mu) \in \mathcal{E}$ belongs to $\mathcal{D}_A$ if (Y, d, μ) is a *space of homogeneous type* or a *doubling space* with doubling constant A, and we write $(Y, \mu) \in \mathcal{D}_A$. In other words, $\mathcal{D}_A$ is the collection of all $(Y, \mu) \in \mathcal{E}$ such that

$$0 < \mu\big(B(y, 2r)\big) \leq A\mu\big(B(y, r)\big) \tag{1}$$

holds for every $y \in Y$ and every $r > 0$. The constant A in 1 is called the *doubling constant* for μ. Since $\operatorname{supp}(\mu) \subseteq Y$, we have that $\mu(B(y, s)) = \mu(B(y, s) \cap Y)$. The family $\mathcal{D} = \bigcup_{A \geq 1} \mathcal{D}_A$ is the class of all doubling spaces in $\mathcal{E}$. We say that a subset $\mathcal{F}$ of $\mathcal{E}$ is **uniformly doubling** if there exists a constant $A \geq 1$ such that $\mathcal{F} \subseteq \mathcal{D}_A$.

For example, in $X=[0,1]$ with the usual distance, the space (Y,μ) with $Y=X$ and μ the Lebesgue length, belongs to $\mathcal{D}_2$. But the subspace $Y=[0,\frac{1}{2}]\cup[\frac{3}{4},1]$ with μ being normalized length does not belong to $\mathcal{D}_2$, since, for example, $\mu(B(\frac{1}{2},\frac{1}{4})\cap Y)=\frac{1}{3}$ but $\mu(B(\frac{1}{2},\frac{1}{2})\cap Y)=1$. However, $(Y,\mu)\in\mathcal{D}_3$. The Dirac delta at the origin in $[0,1]$, in other words the pair $([0,1],\delta_0)$, provides an example of an element of $\mathcal{E}$ which is not in $\mathcal{D}$.

Let us point out that if $(Y,\mu)\in\mathcal{D}_A$ then $\operatorname{supp}(\mu)=Y$. In fact for $y\in Y\setminus\operatorname{supp}(\mu)$ there exists an open set G containing y with $\mu(G)=0$. So that for some ball B in Y we should have $\mu(B)=0$, which is impossible.

Another special class of metric measure spaces that we shall consider is that of normal spaces. The concept of normality of a space of homogeneous type was introduced by Macías and Segovia in [9]. Actually, they define the concept that in our terminology is called 1-dimensional normality.

- ***Normality***. For $\beta>0$ and $c\geq 1$, we shall write $\mathcal{N}_{\beta,c}$ to denote the set of all pairs $(Y,\mu)\in\mathcal{E}$ such that

$$c^{-1}r^{\beta}\leq\mu\big(B(y,r)\big)\leq cr^{\beta}$$

for every $y\in Y$ and $0<r<1$. In other words, the measure of each ball of radius r is comparable to r^{β}. The family $\mathcal{N}_{\beta}=\bigcup_{c\geq 1}\mathcal{N}_{\beta,c}$ denotes the class of all *β-normal* (also named *Ahlfors β-regular* or *β-dimensional*) spaces in $\mathcal{E}$, and $\mathcal{N}=\bigcup_{\beta>0}\mathcal{N}_{\beta}$ is the class of all *normal spaces* in $\mathcal{E}$. For a given $\beta>0$, we say that a subset $\mathcal{F}$ of $\mathcal{E}$ is **uniformly $\boldsymbol{\beta}$-normal** if there exists a constant $c\geq 1$ such that $\mathcal{F}\subset\mathcal{N}_{\beta,c}$. We say that $\mathcal{F}$ is **uniformly normal** if $\mathcal{F}$ is uniformly β-normal for some $\beta>0$.

The two examples of doubling given before are also normal and 1-dimensional in the above sense. However, a measure can be doubling but not normal. For example, if we take $Y=[0,\frac{1}{2}]\cup\{\frac{3}{4}\}$ with $\mu(E)=\lambda(E\cap[0,\frac{1}{2}])+\frac{1}{2}\operatorname{card}(E\cap\{\frac{3}{4}\})$, where λ denotes the Lebesgue length, we have that (Y,μ) is doubling but not normal. Another example can even be obtained in the interval $[0,1]$ for measures that are absolutely continuous with respect to Lebesgue measure. In fact, Lebesgue measure is 1-normal on the interval $[0,1]$ and $d\mu(x)=w(x)dx$ with $w(x)=x^{-1/2}$ is a doubling measure, but μ is not β-normal for any $\beta>0$. This is a consequence of the fact that for $\varepsilon<1/2$ we have $\int_{1-\varepsilon}^{1}w\,dx\simeq\varepsilon$ while $\int_0^{\varepsilon}w\,dx\simeq\sqrt{\varepsilon}$.

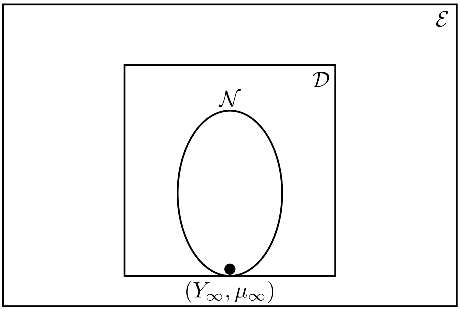
We note that for every $(Y,\mu)\in\mathcal{N}_{\beta}$, the Hausdorff dimension of Y with respect to d is exactly β. Moreover, every open set in Y has dimension β. Let us observe that if $\beta_1,\beta_2>0$ and $\beta_1\neq\beta_2$, then $\mathcal{N}_{\beta_1}\cap\mathcal{N}_{\beta_2}=\emptyset$. Note also that the attractor (Y_∞,μ_∞) is normal; moreover, it is shown in [8] and [11] that $(Y_\infty,\mu_\infty)\in\mathcal{N}_s$, where $s=\log_a M$ is completely determined by Φ.

We have the obvious inclusions $\mathcal{N}\subset\mathcal{D}\subset\mathcal{E}$, which we depict schematically with different shapes in Figure 1 with the goal of making self-explanatory the diagrams in Figure 2.

For $1<p<\infty$ the Muckenhoupt class $A_p(Y,\mu)$ is defined as the set of all non-negative, measurable, and locally integrable functions w defined on Y for which there exists a constant C such that the inequality

$$\left(\int_B w\,d\mu\right)\left(\int_B w^{-\frac{1}{p-1}}\,d\mu\right)^{p-1}\leq C\big(\mu(B)\big)^p$$

Fig. 1 Subclasses of $\mathcal{E}$



holds for every d-ball B in Y. We say that w is a *Muckenhoupt weight* on (Y, μ) if $w \in A_p(Y, \mu)$ for some $1 < p < \infty$. It is well known (see [1]) that, if μ is doubling on Y, then $w \in A_p(Y, \mu)$ suffices for the $L^p(w\, d\mu)$ boundedness of M_Y. Also, it is a classical result in the theory of Muckenhoupt weights that if $w \in A_p(Y, \mu)$ and $(Y, \mu) \in \mathcal{D}$, then $w d\mu$ is doubling on Y.

In the Euclidean space $(\mathbb{R}^n, \lambda)$, where λ denotes the Lebesgue measure, one of the most classical examples of weights in $A_p(\mathbb{R}^n, \lambda)$ is given by $w(x) = |x|^\alpha$ for $-n < \alpha < n(p-1)$, with $1 < p < \infty$ (see [7, 12]).

The geometric results in this note can be summarized in the following theorem.

Theorem 2.1

(I) *There exists $\Phi \in ISS$ such that for some $(Y_0, \mu_0) \in \mathcal{D}$ we have that*

$$\mathcal{O}(Y_0, \mu_0) \cap \mathcal{D} = \big\{(Y_0, \mu_0), (Y_\infty, \mu_\infty)\big\}.$$

(II) *There exists $\Phi \in ISS$ such that for some $(Y_0, \mu_0) \in \mathcal{D}$ we have that $\mathcal{O}(Y_0, \mu_0) \subset \mathcal{D}$ and $\mathcal{O}(Y_0, \mu_0) \nsubseteq \mathcal{D}_A$ for every $A \geq 1$.*

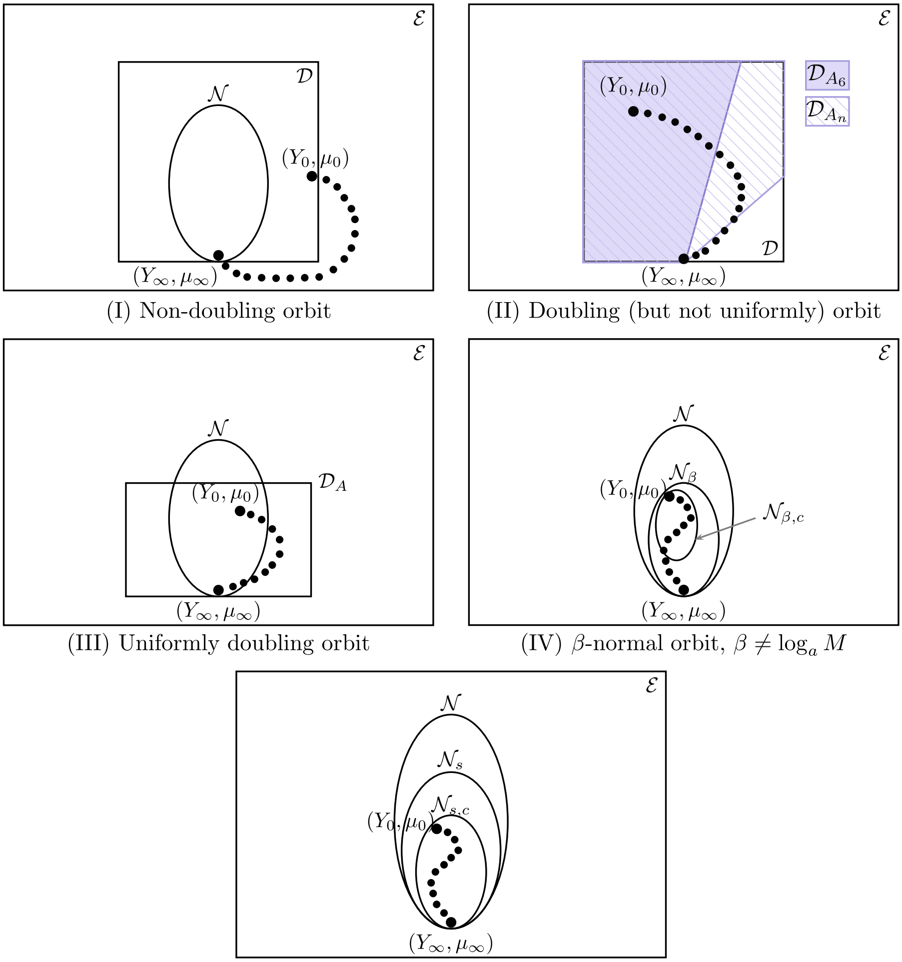
(III) *For every $\Phi \in ISS$ and every $(Y_0, \mu_0) \in \mathcal{N}$ with $Y_0 \cap U \neq \emptyset$, we have that $\mathcal{O}(Y_0, \mu_0) \subset \mathcal{D}_A$ for some doubling constant $A \geq 1$, where U is the open set for the OSC in the definition of Φ.*

(IV) *For every $\Phi \in ISS$, every $\beta > 0$, and every $(Y_0, \mu_0) \in \mathcal{N}_\beta$ with $Y_0 \cap U \neq \emptyset$, we have that $\mathcal{O}(Y_0, \mu_0) \subset \mathcal{N}_\beta$, where U is the open set for the OSC in the definition of Φ. If $\beta \neq \log_a M$ then there is no $c \geq 1$ such that $\mathcal{O}(Y_0, \mu_0) \subset \mathcal{N}_{\beta,c}$.*

(V) *For every $\Phi \in ISS$ and every $(Y_0, \mu_0) \in \mathcal{N}_s$ with $Y_0 \cap U \neq \emptyset$, we have that $\mathcal{O}(Y_0, \mu_0) \subset \mathcal{N}_{s,c}$ for some constant $c \geq 1$, where U is the open set for the OSC in the definition of Φ.*

A schematic picture of the results in Theorem 2.1 is contained in Figure 2, and a loose description of these situations is as follows.

(I) There are orbits that start at doubling spaces but do not remain doubling.
(II) There are orbits that start at doubling spaces and remain doubling, but are not uniformly doubling.
(III) All orbits starting at normal spaces are uniformly doubling.
(IV) All orbits starting at β-normal spaces remain β-normal, but if $\beta \neq \log_a M$ then the orbit is not uniformly normal.



(I) Non-doubling orbit

(II) Doubling (but not uniformly) orbit

(III) Uniformly doubling orbit

(IV) β-normal orbit, $\beta \neq \log_a M$

(V) Uniformly s-normal orbit, $s = \log_a M$

Fig. 2 Behavior of orbits

(V) All orbits starting at s-normal spaces, where $s = \log_a M$, are uniformly s-normal.

The analytic results are contained in the following result, which is essentially a consequence of Theorem 2.1.

Theorem 2.2

(A) *There exists* $\Phi \in ISS$ *such that for some* $(Y_0, \mu_0) \in \mathcal{D}$, *some* $1 < p < \infty$, *and some* $w_0 \in A_p(Y_0, \mu_0)$ *we have that*

(a) M_0 *is bounded on* $L^p(Y_0, w_0 d\mu_0)$,

(b) M_∞ *is bounded on* $L^p(Y_\infty, d\mu_\infty)$,

(c) *M_n is an unbounded operator on $L^q(Y_n, w_n d\mu_n)$ for every $n = 1, 2, \dots$ and every $1 < q < \infty$.*

Here M_k denotes the Hardy–Littlewood maximal operator on (Y_k, μ_k).

(B) *For all $\Phi \in ISS$ and every $(Y_0, \mu_0) \in \mathcal{N}$ with $Y_0 \cap U \neq \emptyset$, where U is the open set for the OSC in the definition of Φ, the sequence of maximal operators M_n is uniformly bounded on $L^p(Y_n, \mu_n)$.*

This paper is organized in four sections. Section 3 is devoted to providing three examples showing that statements (I) and (II) are true. In Sect. 4 we prove three lemmas from which statements (III), (IV), and (V) shall be corollaries. In Sect. 5 we prove Theorem 2.2.

3 Proofs of Statements (I) and (II)

We shall show first that it may happen that the only point in the orbit satisfying the doubling property is (Y_0, μ_0) and of course the limit space (Y_∞, μ_∞) but no other $T^n(Y_0, \mu_0)$, $n \in \mathbb{N}$, is a space of homogeneous type. In these constructions we shall use Muckenhoupt weights.

We shall construct our first example in the interval $I = [0, 1]$. The iterated system considered is $\Phi = \{\phi_1, \phi_2\}$ with $\phi_1(x) = x/2$ and $\phi_2(x) = x/2 + 1/2$, defined on $X = I$ with the usual distance $d(x, y) = |x - y|$. Here we have $M = 2$ and $a = 2$. Notice that the attractor (Y_∞, μ_∞) is the interval I equipped with Lebesgue measure.

Now we shall describe the initial space (Y_0, μ_0). Take $Y_0 = I$ and μ_0 the absolutely continuous probability on I induced by the density $w(x) = x^{-1/2}$. In other words, $d\mu_0 = \frac{1}{2} w(x)dx$. Since $w \in A_2(Y_0, dx)$ we have that $(Y_0, \mu_0) \in \mathcal{D}$.

Next we shall prove that $(Y_1, \mu_1) := T(Y_0, \mu_0)$ is not a space of homogeneous type. In order to show the above statement, take $0 < \varepsilon < 1/4$, $E_\varepsilon = (\frac{1}{2} - \varepsilon, \frac{1}{2})$, and $F_\varepsilon = (\frac{1}{2}, \frac{1}{2} + \varepsilon)$. Notice that $Y_1 = Y_0 = [0, 1]$ and that $d\mu_1 = \frac{\sqrt{2}}{4} v(x)dx$ with

$$v(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x \leq 1/2, \\ (x - \frac{1}{2})^{-1/2} & \text{if } 1/2 < x < 1. \end{cases}$$

(see Figure 3). Hence $\mu_1(E_\varepsilon) = \frac{1}{2}(1 - \sqrt{1 - 2\varepsilon})$ and $\mu_1(F_\varepsilon) = \sqrt{2\varepsilon}/2$. So that $\mu_1(F_\varepsilon)/\mu_1(E_\varepsilon)$ tends to ∞ when ε tends to zero.

If μ_1 were doubling, for every $r > 0$ and every $x_1, x_2 \in Y_1$ satisfying $d(x_1, x_2) \leq 2r$ we have that

$$\mu_1\big(B(x_1, r)\big) \leq \mu_1\big(B(x_2, 3r)\big) \leq A^2 \mu_1\big(B(x_2, r)\big),$$

where A denotes the doubling constant for μ_1. Since E_ε and F_ε are balls with the same radius $\frac{\varepsilon}{2}$ and with centers $x_1 = \frac{1}{2} - \frac{\varepsilon}{2}$ and $x_2 = \frac{1}{2} + \frac{\varepsilon}{2}$ respectively, by taking $\varepsilon \to 0$ we realize the impossibility of the doubling for μ_1. For $(Y_n, \mu_n) = T^n([0, 1], \frac{1}{2} w dx) = ([0, 1], w_n\, dx)$, the same situation appears at each point of the form $j/2^n$, $j = 1, 2, \dots, 2^n - 1$. Hence no (Y_n, μ_n) is a space of homogeneous type for $n \in \mathbb{N}$.

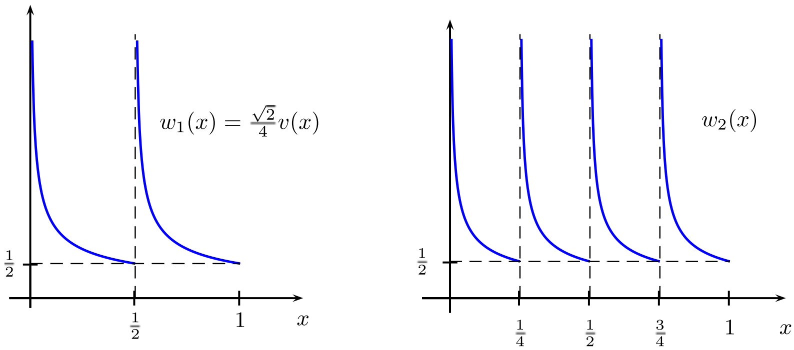


Fig. 3 $T^n([0,1], \frac{1}{2}w dx) = ([0,1], w_n dx)$

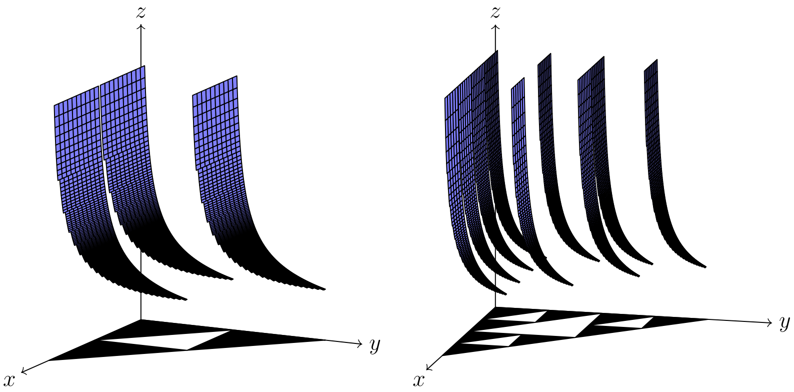
Notice that in the above example we have that $d(\phi_1([0,1], \phi_2([0,1]) = 0$, and also the attractor is the initial set $[0,1]$.

The second example is given by a similar construction associated with the classical Sierpinski contraction T_s. In this case, let X be the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$, and $(0,1)$, and take $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ as the distance on X. The ISS is given by $\phi_1(x, y) = (x/2, y/2)$, $\phi_2(x, y) = \phi_1(x, y) + (0, 1/2)$ and $\phi_3(x, y) = \phi_1(x, y) + (1/2, 0)$.

Let us define a weight function $\widetilde{w}(x, y)$ on the basic triangle X, given by $\widetilde{w}(x, y) = \frac{1}{2}w(y)$, where w is the weight function defined on $[0,1]$ by $w(y) = y^{-1/2}$. It is not difficult to see directly or to deduce from Muckenhoupt theory that $\widetilde{w}$ is a doubling weight on the metric space (X, d). So that, in particular, $(X, d, \widetilde{w}\, dxdy)$ is a space of homogeneous type. Notice that $d(\phi_i(X), \phi_j(X)) = 0$ for every $i, j = 1, 2, 3$, and taking $Y_0 = X$ again $T_s(Y_0, \widetilde{w}\, dxdy)$ is not a space of homogeneous type since precisely at each contact point of $\phi_i(X)$ and $\phi_j(X)$ for $i \neq j$, we have a singularity of $\widetilde{w}$ in one of these sets and boundedness on the other (see Figure 4). The limit space (Y_∞, μ_∞) is the Sierpinski triangle with μ_∞ being the restriction of the Hausdorff measure of dimension $\log 3/\log 2$, which is doubling.

Either one of the above two examples proves statement (I).

In order to prove (II) we give our third example, which shows that with some separation of the sets $\{\phi_i(X) : i = 1, \ldots, M\}$, no uniform doubling property for the whole orbit can be expected. In fact, let us consider now $(X, d) = ([0,1], |\cdot|)$ and $\Phi = \{\phi_1, \phi_2\}$ with $\phi_1(x) = 2x/5$ and $\phi_2(x) = 2x/5 + 3/5$. Take again $w(x) = x^{-1/2}$ and $(Y_0, \mu_0) = ([0,1], \frac{1}{2}w(x)dx)$ as the starting space. We already know that the space $(Y_0, \mu_0) \in \mathcal{D}$. It is not difficult to show that $\mathcal{O}(Y_0, \mu_0) \subset \mathcal{D}$. In fact, let us write (Y_n, μ_n) to denote $T^n(Y_0, \mu_0)$. Each Y_n is a finite union of 2^n intervals which are at a distance at least 5^{-n} from each other. Since μ_n is doubling on each one of these intervals, we readily see that μ_n is doubling on Y_n. But we claim that if $T^n(Y_0, \mu_0)$ belongs to $\mathcal{D}_{A_n}$ then $A_n \geq 2^{n/2}$, for each non-negative integer n. To see this, for a fixed n take $y_1 = (\frac{2}{5})^n$, $y_2 = \frac{3}{2}(\frac{2}{5})^n$, and $r = d(y_1, y_2) = \frac{1}{2}(\frac{2}{5})^n$. Notice that y_1 and y_2 belong to Y_n and they are the endpoints of the first "gap" of Y_n. It is easy to see that $\mu_n(B(y_1, r)) = C5^{-n/2}$ and $\mu_n(B(y_2, r)) = C(\frac{2}{5})^{n/2}$, where C is a constant which



(A) $\widetilde{w}_1(x,y)$ defined on Y_1 (B) $\widetilde{w}_2(x,y)$ defined on Y_2

Fig. 4 $T_s^n(Y_0, \widetilde{w}\,dxdy) =: (Y_n, \widetilde{w}_n dxdy)$

does not depend on n. Then

$$\frac{\mu_n(B(y_1, 2r))}{\mu_n(B(y_1, r))} \geq \frac{\mu_n(B(y_2, r))}{\mu_n(B(y_1, r))} = 2^{n/2}.$$

4 Proofs of Statements (III), (IV), and (V)

In this section, we prove the positive results contained in statements (III), (IV), and (V). The basic tools will be the next two lemmas. The first one shows the scaling properties of the approximating measure for different sizes of balls.

Lemma 4.1 *Let $\Phi \in ISS(M, a)$, let U be an open set for the OSC for Φ, and Y_0 a closed subset of X such that $Y_0 \cap U \neq \emptyset$.*

(i) *If (Y_0, μ_0) is β-normal and $0 < r \leq a^{-n}$, then the μ_n-measure of the ball $B(y, r)$ scales like r^β, with constant depending on n. In other words, if $(Y_0, \mu_0) \in \mathcal{N}_{\beta,c}$ for some $\beta > 0$ and $c \geq 1$, then there exists a constant $\widehat{C} = \widehat{C}(\beta, c, Y_0) \geq 1$ such that the inequalities*

$$\frac{1}{\widehat{C}} \frac{a^{\beta n}}{M^n} r^\beta \leq \mu_n\big(B(y, r)\big) \leq \widehat{C} \frac{a^{\beta n}}{M^n} r^\beta$$

hold for every $y \in Y_n$, every r such that $0 < r \leq a^{-n}$, and every $n \in \mathbb{N}$;

(ii) *If $a^{-n} < r \leq 1$, then for all $(Y_0, \mu_0) \in \mathcal{E}$ the μ_n-measure of the ball $B(y, r)$ scales like r^s, where $s = \log_a M$. In other words, for every $(Y_0, \mu_0) \in \mathcal{E}$ there exists a constant $\widetilde{C} = \widetilde{C}(a, M, Y_0) \geq 1$ such that the inequalities*

$$\widetilde{C}^{-1} r^s \leq \mu_n\big(B(y, r)\big) \leq \widetilde{C} r^s$$

hold for every $y \in Y_n$, every r such that $a^{-n} < r \leq 1$, and every $n \in \mathbb{N}$.

The second lemma states that if a sequence of spaces is uniformly β-normal and has a limit, then this limit remains β-normal. More precisely, we have the following result.

Lemma 4.2 *If $\{(Z_n,\nu_n): n\in\mathbb{N}\}$ is a sequence in $\mathcal{N}_{\beta,c}$ for some β and some c, such that $Z_n \xrightarrow{d_H} Z$ and $\nu_n \xrightarrow{d_K} \nu$, then $(Z,\nu)\in\mathcal{N}_{\beta,2^\beta c}$.*

Let us start by proving (III), (IV), and (V) assuming that Lemmas 4.2 and 4.1 hold.

Proof of (III) Take $(Y_0,\mu_0)\in\mathcal{N}$; then there exist $\beta>0$ and $c\geq 1$ such that $(Y_0,\mu_0)\in\mathcal{N}_{\beta,c}$. Let $\widehat{C}$ and $\widetilde{C}$ be as in Lemma 4.1, and let $C=\max\{\widehat{C},\widetilde{C}\}$. For a fixed n, take $y\in Y_n$ and $r>0$. We shall consider two cases:

(a) Assume first that $2r\leq a^{-n}$. Then from (i) in Lemma 4.1 we have

$$\mu_n\big(B(y,2r)\big)\leq C2^\beta\frac{a^{\beta n}}{M^n}r^\beta\leq C^2 2^\beta\mu_n\big(B(y,r)\big).$$

(b) For the case $2r>a^{-n}$ we shall consider separately two possibilities for r:

(b.1) $r>a^{-n}$. Using (ii) in Lemma 4.1, we have

$$\mu_n\big(B(y,2r)\big)\leq C2^s r^s\leq C^2 2^s\mu_n\big(B(y,r)\big)$$

when $2r\leq 1$, and if $2r>1$ we have

$$\mu_n\big(B(y,r)\big)\geq\frac{1}{C}r^s\geq\frac{1}{C2^s}=\frac{1}{C2^s}\mu_n\big(B(y,2r)\big),$$

where we have used that the d-diameter of Y_n is less than or equal to 1, so that $\mu_n(B(y,2r))=\mu_n(Y_n)=1$.

(b.2) $r\leq a^{-n}$. If $2r>1$, since $\mu_n(B(y,2r))=1$, we only need to prove that $\mu_n(B(y,r))$ is bounded below by a constant. In fact, from (i) in Lemma 4.1 and since $n\leq\log_a 2$, we have

$$\mu_n\big(B(y,r)\big)\geq\mu_n\big(B(y,1/2)\big)\geq\frac{1}{C}\frac{a^{n\beta}}{M^n}\frac{1}{2^\beta}\geq\frac{1}{C2^\beta M^{\log_a 2}}.$$

We only have to deal with the case $r\leq a^{-n}<2r\leq 1$. In this case, we have that $\frac{r^{s-\beta}}{2^{|s-\beta|}}\leq a^{-n(s-\beta)}$, where $s=\log_a M$, so that

$$\begin{aligned}\mu_n\big(B(y,2r)\big)&\leq C2^s r^s\\&=C2^s r^{s-\beta}r^\beta\\&\leq C^2 2^s r^{s-\beta}\frac{M^n}{a^{\beta n}}\mu_n\big(B(y,r)\big)\\&\leq C^2 2^{s+|s-\beta|}\frac{M^n}{a^{sn}}\mu_n\big(B(y,r)\big)\end{aligned}$$

$$= C^2 2^{s+|s-\beta|} \mu_n\big(B(y,r)\big),$$

where we have first used (ii) and then (i) in Lemma 4.1.

We have proved that there exists a constant $A = A(c, \beta, a, M, Y_0)$ such that $\{(Y_n, \mu_n)\} \subseteq \mathcal{D}_A$. □

Proof of (IV) Take $(Y_0, \mu_0) \in \mathcal{N}_\beta$; then there exists $c \geq 1$ such that $(Y_0, \mu_0) \in \mathcal{N}_{\beta,c}$. Let $\widehat{C}$ and $\widetilde{C}$ be as in Lemma 4.1, and let $C = \max\{\widehat{C}, \widetilde{C}\}$. The first statement in (IV) is an immediate consequence of Lemma 4.1 and the fact that for $r \in (a^{-n}, 1]$, the function r^s is comparable to the function r^β; moreover, $a^{-n|s-\beta|} r^\beta \leq r^s \leq a^{n|s-\beta|} r^\beta$. Thus, for each $n \in \mathbb{N}$, when $a^{-n} < r \leq 1$ and $y \in Y_n$, from (ii) in Lemma 4.1 we get that

$$C^{-1} a^{-n|s-\beta|} r^\beta \leq \mu_n\big(B(y,r)\big) \leq C a^{n|s-\beta|} r^\beta.$$

On the other hand, for each $n \in \mathbb{N}$ and each $y \in Y_n$, when $0 < r \leq a^{-n}$ we apply (i) in Lemma 4.1 to obtain

$$C^{-1} m^{-n} r^\beta \leq \mu_n\big(B(y,r)\big) \leq C m^n r^\beta,$$

where $m = \max\{\frac{a^\beta}{M}, \frac{M}{a^\beta}\}$. Hence, we get that $(Y_n, \mu_n) \in \mathcal{N}_{\beta, c_n}$ for every $n \in \mathbb{N}$, with $c_n = C \max\{m^n, a^{n|s-\beta|}\}$.

The second statement in (IV) follows from Lemma 4.2 and the facts that the attractor $(Y_\infty, \mu_\infty) \in \mathcal{N}_s$ and that $\mathcal{N}_\beta \cap \mathcal{N}_s = \emptyset$ for $\beta \neq s$. □

Proof of (V) Take $(Y_0, \mu_0) \in \mathcal{N}_s$; then there exists $c \geq 1$ such that $(Y_0, \mu_0) \in \mathcal{N}_{s,c}$. Let $\widehat{C}$ and $\widetilde{C}$ be as in Lemma 4.1, and let $C = \max\{\widehat{C}, \widetilde{C}\}$. Since $a^s/M = 1$, from (i) and (ii) in Lemma 4.1 we get $(Y_n, \mu_n) \in \mathcal{N}_{s,C}$ for every n. □

Proof of Lemma 4.2 Fix $z \in Z$ and $0 < r \leq 1$. Since $Z_n \xrightarrow{d_H} Z$, there exists a sequence $\{z_n\}$ with $z_n \in Z_n$ and $d(z_n, z) \to 0$ as $n \to \infty$. In order to estimate the ν-measure of a d-ball $B(x,t)$ with $x \in X$ and $t > 0$, taking into account that the d_K-convergence is the weak star convergence, we approximate the indicator function of $B(x,t)$ by a Lipschitz function. Let ψ be the continuous function defined on $\mathbb{R}_0^+$ which takes values 1 in $[0,1]$, 0 out of $[0,2]$, and is linear on the interval $[1,2]$. Let $\psi_{x,t}$ denote the function defined on X as $\psi_{x,t}(y) = \psi(\frac{d(y,x)}{t})$. Since $(Z_n, \nu_n) \in \mathcal{N}_{\beta,c}$ for each n, we have

$$c^{-1} t^\beta \leq \nu_n\big(B(z_n,t)\big) \leq \int_X \psi_{z_n,t}(y)\, d\nu_n(y) \leq \nu_n\big(B(z_n,2t)\big) \leq c 2^\beta t^\beta,$$

for every n. We also have that

$$\int_X \psi_{z,t}(y)\, d\nu_n(y) \xrightarrow[n\to\infty]{} \int_X \psi_{z,t}(y)\, d\nu(y).$$

On the other hand, since the convergence of $\psi_{z_n,t}(y)$ to $\psi_{z,t}(y)$ is uniform in y for t fixed, from the fact that each ν_n is a probability measure on X, we get

$$\int_X \big[\psi_{z_n,t}(y) - \psi_{z,t}(y)\big]\, d\nu_n(y) \underset{n\to\infty}{\longrightarrow} 0.$$

Now

$$\int_X \psi_{z,t}(y)\, d\nu_n(y) = \int_X \big[\psi_{z,t}(y) - \psi_{z_n,t}(y)\big]\, d\nu_n(y) + \int_X \psi_{z_n,t}(y)\, d\nu_n(y).$$

So that by taking the limit as n tends to ∞, we obtain

$$c^{-1}t^{\beta} \le \int_X \psi_{z,t}(y)\, d\nu(y) \le c2^{\beta}t^{\beta},$$

for every $t > 0$. Hence, applying the lower estimate with $t = r/2$ and the upper with $t = r$, we have

$$c^{-1}2^{-\beta}r^{\beta} \le \int_X \psi_{z,r/2}(y)\, d\nu(y) \le \nu\big(B(z,r)\big) \le \int_X \psi_{z,r}(y)\, d\nu(y) \le c2^{\beta}r^{\beta},$$

equivalently $(Z,\nu) \in \mathcal{N}_{\beta,2^{\beta}c}$, as desired. □

To prove Lemma 4.1 we shall state, and for the sake of completeness, prove some basic results. Given $\boldsymbol{i} = (i_1, i_2, \dots, i_k) \in \{1,2,\dots,M\}^k$, we denote by $\boldsymbol{\phi_i}$ the composition $\phi_{i_k} \circ \phi_{i_{k-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}$. Also, if $i_0 \in \{1,2,\dots,M\}$, we write $\boldsymbol{i}' = (i_0, \boldsymbol{i})$ to denote the $(k+1)$-tuple $(i_0, i_1, i_2, \dots, i_k)$.

Lemma 4.3 *With U an open set for the OSC for Φ, we have*

(a) *if $\boldsymbol{i}, \boldsymbol{j} \in \{1,2,\dots,M\}^k$ and $\boldsymbol{i} \ne \boldsymbol{j}$, then $\boldsymbol{\phi_i}(U) \cap \boldsymbol{\phi_j}(U) = \emptyset$;*
(b) *if $\boldsymbol{i} = (i, \boldsymbol{i}')$ with $\boldsymbol{i}' \in \{1,2,\dots,M\}^k$ and $i \in \{1,2,\dots,M\}$, then $\boldsymbol{\phi_i}(U) \subseteq \boldsymbol{\phi_{i'}}(U)$;*
(c) *if $\boldsymbol{i}'$ and $\boldsymbol{j}'$ are two different elements in $\{1,2,\dots,M\}^k$ and $\boldsymbol{i} = (i, \boldsymbol{i}')$ where $i \in \{1,2,\dots,M\}$, then $\boldsymbol{\phi_i}(U) \cap \boldsymbol{\phi_{j'}}(U) = \emptyset$;*
(d) *for any fixed $u \in U$ and each positive integer n, if we define*

$$\Delta_n = \big\{\boldsymbol{\phi_j}(u) : \boldsymbol{j} \in \{1,2,\dots,M\}^n\big\},$$

then we have that

$$\operatorname{card}\big(\boldsymbol{\phi_\ell}(U) \cap \Delta_n\big) = M^{n-k}$$

for every $k \le n$ and every $\boldsymbol{\ell} \in \{1,2,\dots,M\}^k$.

Proof In order to prove (a), fix $\boldsymbol{i} = (i_1, i_2, \dots, i_k)$ and $\boldsymbol{j} = (j_1, j_2, \dots, j_k)$ such that $\boldsymbol{i} \ne \boldsymbol{j}$. Let us first assume that $i_k \ne j_k$. Since $\boldsymbol{\phi_i}(U) \subseteq \phi_{i_k}(U)$, $\boldsymbol{\phi_j}(U) \subseteq \phi_{j_k}(U)$, and $\phi_{i_k}(U) \cap \phi_{j_k}(U) = \emptyset$, in this case we have that $\boldsymbol{\phi_i}(U)$ and $\boldsymbol{\phi_j}(U)$ are disjoint. Assume now that $\boldsymbol{i} \ne \boldsymbol{j}$ but $i_k = j_k$. Let ℓ be the largest index satisfying $j_\ell \ne i_\ell$. So that $j_m = i_m$ for every $m > \ell$, and then

$$\boldsymbol{\phi_i}(U) = (\varphi \circ \phi_{i_\ell} \circ \cdots \circ \phi_{i_1})(U),$$

$$\boldsymbol{\phi}_{\boldsymbol{j}}(U) = (\varphi \circ \phi_{j_\ell} \circ \cdots \circ \phi_{j_1})(U),$$

where $\varphi = \phi_{i_k} \circ \cdots \circ \phi_{i_{\ell+1}} = \phi_{j_k} \circ \cdots \circ \phi_{j_{\ell+1}}$. From the OSC we have

$$(\phi_{i_{\ell-1}} \circ \cdots \circ \phi_{i_1})(U) \subseteq U \quad \text{and} \quad (\phi_{j_{\ell-1}} \circ \cdots \circ \phi_{j_1})(U) \subseteq U.$$

Hence,

$$\boldsymbol{\phi}_{\boldsymbol{i}}(U) \subseteq \varphi\big(\phi_{i_\ell}(U)\big),$$
$$\boldsymbol{\phi}_{\boldsymbol{j}}(U) \subseteq \varphi\big(\phi_{j_\ell}(U)\big).$$

Since $\phi_{i_\ell}(U) \cap \phi_{j_\ell}(U) = \emptyset$ and φ is one-to-one, we have $\varphi(\phi_{i_\ell}(U)) \cap \varphi(\phi_{j_\ell}(U)) = \emptyset$, which implies (a).

To prove (b), let $\boldsymbol{i}' = (i_1, i_2, \ldots, i_k)$ and $\boldsymbol{i} = (i, \boldsymbol{i}')$. Since $\phi_i(U) \subseteq U$, we have that

$$\boldsymbol{\phi}_{\boldsymbol{i}}(U) = (\phi_{i_k} \circ \cdots \circ \phi_{i_1})\big(\phi_i(U)\big) \subseteq (\phi_{i_k} \circ \cdots \circ \phi_{i_1})(U) = \boldsymbol{\phi}_{\boldsymbol{i}'}(U).$$

To see (c), let $\boldsymbol{i}' = (i_1, \ldots, i_k)$, $\boldsymbol{i} = (i, \boldsymbol{i}')$, and $\boldsymbol{j}' \in \{1, \ldots, M\}^k$ such that $\boldsymbol{j}' \neq \boldsymbol{i}'$. Since $\boldsymbol{\phi}_{\boldsymbol{i}}(U) \subseteq \boldsymbol{\phi}_{\boldsymbol{i}'}(U)$, and from (a) we have that

$$\boldsymbol{\phi}_{\boldsymbol{i}'}(U) \cap \boldsymbol{\phi}_{\boldsymbol{j}'}(U) = \emptyset,$$

we also have $\boldsymbol{\phi}_{\boldsymbol{i}}(U) \cap \boldsymbol{\phi}_{\boldsymbol{j}'}(U) = \emptyset$.

Finally, to prove (d), let us fix two positive integers n and k with $k \leq n$, and let $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots, \ell_k) \in \{1, 2, \ldots, M\}^k$. If $x \in \boldsymbol{\phi}_{\boldsymbol{\ell}}(U) \cap \Delta_n$, (c) implies that $x = \boldsymbol{\phi}_{\boldsymbol{i}}(u)$ for some $\boldsymbol{i} = (i_1, i_2, \ldots, i_{n-k}, \boldsymbol{\ell})$. Then $\operatorname{card}(\boldsymbol{\phi}_{\boldsymbol{\ell}}(U) \cap \Delta_n) \leq M^{n-k}$. On the other hand, from (b) we have that if $\boldsymbol{j}$ is any n-tuple of the type $(j_1, j_2, \ldots, j_{n-k}, \boldsymbol{\ell})$, then $\boldsymbol{\phi}_{\boldsymbol{j}}(u) \in \boldsymbol{\phi}_{\boldsymbol{\ell}}(U) \cap \Delta_n$. We also have that $\boldsymbol{\phi}_{\boldsymbol{i}}(u) \neq \boldsymbol{\phi}_{\boldsymbol{j}}(u)$ for every $\boldsymbol{i} = (\boldsymbol{i}', \boldsymbol{\ell})$, $\boldsymbol{j} = (\boldsymbol{j}', \boldsymbol{\ell})$, with $\boldsymbol{i}', \boldsymbol{j}' \in \{1, 2, \ldots, M\}^{n-k}$, $\boldsymbol{i}' \neq \boldsymbol{j}'$. Then $\operatorname{card}(\boldsymbol{\phi}_{\boldsymbol{\ell}}(U) \cap \Delta_n) \geq M^{n-k}$. □

Proof of Lemma 4.1 To prove (i), let $(Y_0, \mu_0) \in \mathcal{N}_{\beta,c}$ such that $Y_0 \cap U \neq \emptyset$. Fix $n \in \mathbb{N}$, $y \in Y_n$, and $r > 0$. Set $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$ to denote an element of $\{1, 2, \ldots, M\}^n$ such that $y \in Y_n^{\boldsymbol{i}} := \boldsymbol{\phi}_{\boldsymbol{i}}(Y_0)$. Since $0 < r \leq a^{-n}$ we have

$$\begin{aligned}
\mu_n\big(B(y,r)\big) &= \frac{1}{M^n} \sum_{\boldsymbol{j} \in \{1,\ldots,M\}^n} \mu_0\big(\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}\big(B(y,r)\big)\big) \\
&\geq \frac{1}{M^n} \mu_0\big(\boldsymbol{\phi}_{\boldsymbol{i}}^{-1}\big(B(y,r)\big)\big) \\
&= \frac{1}{M^n} \mu_0\big(B\big(\boldsymbol{\phi}_{\boldsymbol{i}}^{-1}(y), a^n r\big)\big) \\
&\geq c^{-1} \frac{a^{\beta n}}{M^n} r^{\beta}.
\end{aligned}$$

For the upper bound, let us start by noting that when $\boldsymbol{j}$ is such that $B(y,r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) = \emptyset$, then $\mu_0(\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}(B(y,r))) = 0$. Also, we claim that if

$$\mathcal{J}(n,y,r) = \big\{\boldsymbol{j} \in \{1, 2, \ldots, M\}^n : B(y,r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) \neq \emptyset\big\},$$

then $\operatorname{card}(\mathcal{J}(n,y,r)) \leq \Lambda$ for some constant Λ which does not depend on y, r, and n. In fact, fix $u \in U \cap Y_0$ and set $\Delta_n = \{\boldsymbol{\phi}_{\boldsymbol{j}}(u) : \boldsymbol{j} \in \{1,\dots,M\}^n\}$. The OSC implies that Δ_n is a δa^{-n}-disperse set, with $\delta = \operatorname{dist}(u, \partial U)$. To see this, take $\boldsymbol{j}, \boldsymbol{i} \in \{1,\dots,M\}^n$ with $\boldsymbol{j} \neq \boldsymbol{i}$, and set $x_n^{\boldsymbol{j}} = \boldsymbol{\phi}_{\boldsymbol{j}}(u)$ and $x_n^{\boldsymbol{i}} = \boldsymbol{\phi}_{\boldsymbol{i}}(u)$. Since U is an open set, we have that $B(u,\delta) \subseteq U$. Then

$$B\big(x_n^{\boldsymbol{j}}, \delta a^{-n}\big) = \boldsymbol{\phi}_{\boldsymbol{j}}\big(B(u,\delta)\big) \subseteq \boldsymbol{\phi}_{\boldsymbol{j}}(U),$$
$$B\big(x_n^{\boldsymbol{i}}, \delta a^{-n}\big) = \boldsymbol{\phi}_{\boldsymbol{i}}\big(B(u,\delta)\big) \subseteq \boldsymbol{\phi}_{\boldsymbol{i}}(U),$$

and since $\boldsymbol{\phi}_{\boldsymbol{j}}(U)$ and $\boldsymbol{\phi}_{\boldsymbol{i}}(U)$ are disjoint, we have $B(x_n^{\boldsymbol{j}}, \delta a^{-n}) \cap B(x_n^{\boldsymbol{i}}, \delta a^{-n}) = \emptyset$. This implies that $d(x_n^{\boldsymbol{j}}, x_n^{\boldsymbol{i}}) \geq \delta a^{-n}$.

On the other hand, if $v_{\boldsymbol{j}} \in B(y,r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0)$ for some $\boldsymbol{j} \in \{1,\dots,M\}^n$, then

$$d\big(\boldsymbol{\phi}_{\boldsymbol{j}}(u), y\big) \leq d\big(\boldsymbol{\phi}_{\boldsymbol{j}}(u), v_{\boldsymbol{j}}\big) + d(v_{\boldsymbol{j}}, y) < a^{-n} + r \leq 2a^{-n},$$

because $\operatorname{diam}(\boldsymbol{\phi}_{\boldsymbol{j}}(X)) = a^{-n}$ and $r < a^{-n}$. Then

$$\begin{aligned}\operatorname{card}\big(\mathcal{J}(n,y,r)\big) &\leq \operatorname{card}\big\{\boldsymbol{j} \in \{1,\dots,M\}^n : \boldsymbol{\phi}_{\boldsymbol{j}}(u) \in B\big(y, 2a^{-n}\big)\big\} \\ &= \operatorname{card}\big(\Delta_n \cap B\big(y, 2a^{-n}\big)\big) \\ &\leq N^{m_0},\end{aligned}$$

where N denotes the constant in the definition of the finite metric dimension of (X,d) and m_0 is any integer satisfying $2^{m_0} \geq 2/\delta$ (see p. 1834).

Furthermore, if there exists $v_{\boldsymbol{j}} \in B(y,r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0)$, then $B(y,r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) \subseteq B(v_{\boldsymbol{j}}, 2r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0)$. Hence, $\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}(B(y,r)) \subseteq B(\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}(v_{\boldsymbol{j}}), 2ra^n)$. Finally,

$$\mu_n\big(B(y,r)\big) \leq \frac{1}{M^n} \sum_{\boldsymbol{j} \in \mathcal{J}(n,y,r)} \mu_0\big(B\big(\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}(v_{\boldsymbol{j}}), 2ra^n\big)\big) \leq cN^{m_0} 2^{\beta} \frac{a^{\beta n}}{M^n} r^{\beta},$$

and (i) is proved.

In order to prove (ii), let $(Y_0, \mu_0) \in \mathcal{E}$ such that $Y_0 \cap U \neq \emptyset$. For $u \in Y_0 \cap U$ fixed, let, as before, $\Delta_n = \{\boldsymbol{\phi}_{\boldsymbol{j}}(u) : \boldsymbol{j} \in \{1,\dots,M\}^n\}$ and $\delta = \operatorname{dist}(u, \partial U)$. Fix n, $y \in Y_n$ and r with $a^{-n} < r \leq 1$. Let us also fix $1 \leq k \leq n$ such that $a^{-k} < r \leq a^{-k+1}$. Since

$$\begin{aligned}\mu_n\big(B(y,r)\big) &= \frac{1}{M^n} \sum_{\boldsymbol{j} \in \{1,\dots,M\}^n} \mu_0\big(\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}\big(B(y,r)\big)\big) \\ &= \frac{1}{M^n} \sum_{\boldsymbol{j} \in \mathcal{J}(n,y,r)} \mu_0\big(\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}\big(B(y,r)\big)\big),\end{aligned}$$

where, as before,

$$\mathcal{J}(n,y,r) = \big\{\boldsymbol{j} \in \{1,2,\dots,M\}^n : B(y,r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) \neq \emptyset\big\},$$

we have that

$$\mu_n\big(B(y,r)\big) \leq \frac{1}{M^n}\operatorname{card}\big(\mathcal{J}(n,y,r)\big).$$

In this case, if $v_{\boldsymbol{j}} \in B(y,r) \cap \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0)$ for some $\boldsymbol{j} \in \{1,\dots,M\}^n$, then

$$d\big(y, \boldsymbol{\phi}_{\boldsymbol{j}}(u)\big) \leq d(y, v_{\boldsymbol{j}}) + d\big(v_{\boldsymbol{j}}, \boldsymbol{\phi}_{\boldsymbol{j}}(u)\big) < r + a^{-n} < 2r.$$

In other words, if $\boldsymbol{j} \in \mathcal{J}(n,y,r)$, then $\boldsymbol{\phi}_{\boldsymbol{j}}(u) \in B(y,2r)$. Hence,

$$\mu_n\big(B(y,r)\big) \leq \frac{1}{M^n}\operatorname{card}\big(\Delta_n \cap B(y,2r)\big).$$

For the fixed k, we define

$$\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}(y,r) = \big\{\boldsymbol{\ell} \in \{1,2,\dots,M\}^k : B(y,2r) \cap \boldsymbol{\phi}_{\boldsymbol{\ell}}(U) \neq \emptyset\big\}.$$

Since $\{\boldsymbol{\phi}_{\boldsymbol{\ell}}(U) : \boldsymbol{\ell} \in \{1,2,\dots,M\}^k\}$ is a covering of Δ_n, we have that

$$\Delta_n \cap B(y,2r) \subseteq \Delta_n \cap \bigcup_{\boldsymbol{\ell} \in \widetilde{\mathcal{J}}} \boldsymbol{\phi}_{\boldsymbol{\ell}}(U).$$

Hence, by (d) in Lemma 4.3, we have

$$\begin{aligned}\operatorname{card}\big(\Delta_n \cap B(y,2r)\big) &\leq \sum_{\boldsymbol{\ell} \in \widetilde{\mathcal{J}}} \operatorname{card}\big(\Delta_n \cap \boldsymbol{\phi}_{\boldsymbol{\ell}}(U)\big)\\ &= \operatorname{card}(\widetilde{\mathcal{J}})M^{n-k}\\ &= \operatorname{card}(\widetilde{\mathcal{J}})M^n a^{-ks}\\ &\leq \operatorname{card}(\widetilde{\mathcal{J}})M^n r^s.\end{aligned}$$

To obtain the upper bound, we only have to show that $\operatorname{card}(\widetilde{\mathcal{J}})$ is bounded by a constant which does not depend on y, n, or $r \in (a^{-n}, 1]$. In order to prove it, since $\boldsymbol{\phi}_{\boldsymbol{\ell}}(U)$ are pairwise disjoint, let us identify each $\boldsymbol{\ell} \in \widetilde{\mathcal{J}}$ with the point $\boldsymbol{\phi}_{\boldsymbol{\ell}}(u) \in \boldsymbol{\phi}_{\boldsymbol{\ell}}(U)$, and let us define the set $\mathcal{A} = \{\boldsymbol{\phi}_{\boldsymbol{\ell}}(u) : \boldsymbol{\ell} \in \widetilde{\mathcal{J}}\}$. Then $\operatorname{card}(\widetilde{\mathcal{J}}) = \operatorname{card}(\mathcal{A})$. Notice that $\mathcal{A} \subseteq B(y,3r)$. In fact, if $\boldsymbol{\ell} \in \widetilde{\mathcal{J}}$, then there exists $z \in B(y,2r) \cap \boldsymbol{\phi}_{\boldsymbol{\ell}}(U)$, and

$$d\big(\boldsymbol{\phi}_{\boldsymbol{\ell}}(u), y\big) \leq d\big(\boldsymbol{\phi}_{\boldsymbol{\ell}}(u), z\big) + d(z,y) < a^{-k} + 2r \leq 3r.$$

Take $m_1 \in \mathbb{N}$ such that $2^{m_1} \geq 3a/\delta$. Since, being a subset of Δ_k, the set $\mathcal{A}$ is δa^{-k}-disperse, we have that

$$\operatorname{card}(\mathcal{A}) = \operatorname{card}\big(B(y,3r) \cap \mathcal{A}\big) \leq \operatorname{card}\big(B\big(y, 3a^{-k+1}\big) \cap \mathcal{A}\big) \leq N^{m_1}.$$

For the lower bound, notice that

$$\mu_n\big(B(y,r)\big) = \frac{1}{M^n}\sum_{\boldsymbol{j} \in \{1,\dots,M\}^n} \mu_0\big(\boldsymbol{\phi}_{\boldsymbol{j}}^{-1}\big(B(y,r)\big)\big)$$

$$\geq \frac{1}{M^n} \mathrm{card}\big(\{\boldsymbol{j} \in \{1, \dots, M\}^n : \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) \subseteq B(y, r)\}\big).$$

If we show that

$$\mathrm{card}\big(\{\boldsymbol{j} \in \{1, \dots, M\}^n : \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) \subseteq B(y, r)\}\big) \geq M^{n-k}, \tag{2}$$

then we have

$$\mu_n\big(B(y, r)\big) \geq \frac{1}{M^n} M^{n-k} = M^{-k} = a^{-ks} \geq a^{-s} r^s,$$

and the lemma is proved. In order to show (2), set $y = \phi_{\boldsymbol{i}}(y_0)$ for some $y_0 \in Y_0$ and some $\boldsymbol{i} \in \{1, \dots, M\}^n$. Since it is easy to see that $\boldsymbol{\phi}_{\boldsymbol{i}}(Y_0) \subseteq B(y, r)$, we obtain (2) if $k = n$. Finally, if $1 \leq k < n$, let us write $\boldsymbol{i} = (i_1, i_2, \dots, i_n) = (\boldsymbol{i}'', \boldsymbol{i}')$, where $\boldsymbol{i}' = (i_{n-k+1}, i_{n-k+2}, \dots, i_n)$ and $\boldsymbol{i}'' = (i_1, i_2, \dots, i_{n-k})$. If $\boldsymbol{j} = (\boldsymbol{j}', \boldsymbol{i}')$ for some $\boldsymbol{j}' \in \{1, 2, \dots, M\}^{n-k}$, then $\boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) \subseteq B(y, r)$. In fact, if $z \in \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0)$ there exists $v \in Y_0$ with $z = \boldsymbol{\phi}_{\boldsymbol{j}}(v)$, and

$$d(y, z) = d\big(\boldsymbol{\phi}_{\boldsymbol{i}}(y_0), \boldsymbol{\phi}_{\boldsymbol{j}}(v)\big) = a^{-k} d\big(\boldsymbol{\phi}_{\boldsymbol{i}''}(y_0), \boldsymbol{\phi}_{\boldsymbol{j}'}(v)\big) \leq a^{-k} < r.$$

Hence,

$$\big\{(\boldsymbol{j}', \boldsymbol{i}') : \boldsymbol{j}' \in \{1, 2, \dots, M\}^{n-k}\big\} \subseteq \big\{\boldsymbol{j} \in \{1, \dots, M\}^n : \boldsymbol{\phi}_{\boldsymbol{j}}(Y_0) \subseteq B(y, r)\big\},$$

as desired. □

5 Proof of Theorem 2.2

Proof of (A) Let us consider the same Φ as in the proof of statement (I) of Theorem 2.1. Since $w(x) = x^{-1/2}$ on $[0, 1]$ is an A_2 Muckenhoupt weight with respect to the Lebesgue measure, we have that M, the standard Hardy–Littlewood maximal operator on $[0, 1]$, is bounded on $L^2(w dx)$. This proves (a). Statement (b) is clear since $Y_\infty = [0, 1]$ and μ_∞ is the Lebesgue measure on $[0, 1]$. In order to prove (c), we have only to recall that $w_n \in A_q$ is equivalent to the $L^q(w_n dx)$ boundedness of $M_n = M_0 = M_\infty$, and that the doubling property of $w_n d\mu_n = w_n dx$ is necessary for $w_n \in A_q$. Since we have already proved that $w_n dx$ is not doubling, $w_n \notin A_q$, hence the Hardy–Littlewood Maximal operator is unbounded on $L^q(Y_n, w_n d\mu_n) = L^q([0, 1], w_n dx)$. □

Proof of (B) It follows from statement (III) in Theorem 2.1. In fact, since each (Y_n, d, μ_n) is a space of homogeneous type, we have that M_n is bounded on $L^p(Y_n, \mu_n)$. Moreover, since the constant C_n in the inequality

$$\|M_n f\|_{L^p(Y_n, \mu_n)} \leq C_n \|f\|_{L^p(Y_n, \mu_n)}$$

depends only on the doubling constant for μ_n, and statement (III) in Theorem 2.1 guarantees a uniform bound for C_n, we get the desired result. □

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References

1. Aimar, H., Macías, R.A.: Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type. Proc. Am. Math. Soc. **91**(2), 213–216 (1984)
2. Aimar, H.A., Carena, M., Iaffei, B.: Discrete approximation of spaces of homogeneous type. J. Geom. Anal. **19**(1), 1–18 (2009)
3. Assouad, P.: Étude d'une dimension métrique liée à la possibilité de plongements dans $\mathbf{R}^n$. C. R. Acad. Sci. Paris Sér. A–B **288**(15), A731–A734 (1979)
4. Coifman, R.R., Weiss, G.: Analyse Harmonique Non-commutative sur Certains Espaces Homogènes. Lecture Notes in Mathematics, vol. 242. Springer, Berlin (1971). Étude de certaines intégrales singulières
5. Falconer, K.: Techniques in Fractal Geometry. Wiley, Chichester (1997)
6. Federer, H.: Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften, vol. 153. Springer, New York (1969)
7. García-Cuerva, J., Rubio de Francia, J.L.: Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies, vol. 116. North-Holland, Amsterdam (1985). Notas de Matemática [Mathematical Notes], 104
8. Hutchinson, J.E.: Fractals and self-similarity. Indiana Univ. Math. J. **30**(5), 713–747 (1981)
9. Macías, R. A., Segovia, C.: Lipschitz functions on spaces of homogeneous type. Adv. Math. **33**(3), 257–270 (1979)
10. Moran, P.A.P.: Additive functions of intervals and Hausdorff measure. Proc. Camb. Philos. Soc. **42**, 15–23 (1946)
11. Mosco, U.: Variational fractals. Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **25**(3–4), 683–712 (1998). Dedicated to Ennio De Giorgi
12. Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. **165**, 207–226 (1972)