Annals of Physics 354 (2015) 570–589

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Entanglement and the process of measuring the position of a quantum particle

V.M. Apel [a], S. Curilef [a], A.R. Plastino [b,*]

[a] *Departamento de Física, Universidad Católica del Norte, Av. Angamos 0610, Antofagasta, Chile*

[b] *Universidad Nacional Noroeste-Buenos Aires, UNNOBA and Conicet, Roque Saenz Peña 456, Junin, Argentina*

HIGHLIGHTS

- We explore entanglement features of a quantum position measurement.
- We consider instantaneous and finite-duration measurements.
- We evaluate the entanglement of exact time-dependent particle–pointer states.

ARTICLE INFO

Article history:
Received 26 August 2014
Accepted 21 January 2015
Available online 2 February 2015

Keywords:
Quantum measurement
Quantum dynamics
Entanglement

ABSTRACT

We explore the entanglement-related features exhibited by the dynamics of a composite quantum system consisting of a particle and an apparatus (here referred to as the "pointer") that measures the position of the particle. We consider measurements of finite duration, and also the limit case of instantaneous measurements. We investigate the time evolution of the quantum entanglement between the particle and the pointer, with special emphasis on the final entanglement associated with the limit case of an impulsive interaction. We consider entanglement indicators based on the expectation values of an appropriate family of observables, and also an entanglement measure computed on particular exact analytical solutions of the particle–pointer Schrödinger equation. The general behavior exhibited by the entanglement indicators is consistent with that shown by the entanglement measure evaluated on particular analytical solutions of the Schrödinger equation. In the limit of instantaneous measurements the system's entanglement dynamics corresponds to that of an ideal quantum measurement process. On the contrary, we show that the entanglement evolu-

* Corresponding author.
E-mail address: arplastino@unnoba.edu.ar (A.R. Plastino).

http://dx.doi.org/10.1016/j.aop.2015.01.019

tion corresponding to measurements of finite duration departs in important ways from the behavior associated with ideal measurements. In particular, highly localized initial states of the particle lead to highly entangled final states of the particle–pointer system. This indicates that the above mentioned initial states, in spite of having an arbitrarily small position uncertainty, are not left unchanged by a finite-duration position measurement process.

0. Introduction

Quantum entanglement [1,2] and the quantum measurement process [3,4] are two closely related and fundamentally non-classical aspects of quantum physics. If initially the system being measured is described by a pure state (which, consequently, is factorized from the initial, standard state of the measuring apparatus), the measurement process in general generates entanglement between the system and the apparatus [5,6]. Therefore, after the measurement takes place (but before the result of the measurement is "read") the system and the apparatus are, in general, in an entangled state (except in the case of an ideal measurement with the system starting in an eigenstate of the observable being measured). Within the standard quantum formalism one can consider the measurement of general physical observables described by appropriate hermitian operators acting on the relevant Hilbert space. However, it is generally acknowledged that the measurement of the position of quantum particles plays a particularly fundamental role among the set of all possible physical measurements. In fact, most, if not all, physical measurements can be reduced to the measurement of the position of some particle (for instance, a pointer in the measuring apparatus) [7,8]. This is one of the main reasons why position observables play a central role in many approaches to the quantum measurement problem and related aspects of the foundations of quantum mechanics. Among the interesting position-centered contributions to these fundamental issues we can mention the de Broglie–Bohm pilot wave approach to quantum mechanics [9–11], the Ghirardi–Rimini–Weber model of wave-function collapse [12], non-linear modifications of Schrödinger equation describing the continuous measurement of a particle's position [13,14], the Fisher information-based derivation of the fundamental Lagrangians leading to relativistic wave equations [15], and the entropic-dynamics approach to quantum evolution [16].

A central point concerning the quantum measurement problem is whether one regards the measurement process as arising from a physical interaction between the system and the measuring apparatus describable by the standard, linear Schrödinger equation. A useful tool for analyzing the conceptual issues associated with this point of view is given by the celebrated von Neumann model for quantum measurements [17–19]. In this model the measuring apparatus is characterized by one, single relevant coordinate Q (the "pointer" coordinate). If the system being measured is described by a coordinate R, then the von Neumann model assumes that the system–apparatus Hamiltonian has an interaction term of the form,

$$G\delta(t)F\left(R, \frac{\hbar}{i}\frac{\partial}{\partial R}\right)\frac{\hbar}{i}\frac{\partial}{\partial Q}, \tag{1}$$

where $F\left(R, \frac{\hbar}{i}\frac{\delta}{\delta R}\right)$ corresponds to the observable being measured (which is here expressed as a function of the position and the momentum observables) and G is a coupling constant. Notice that this interaction term is time-dependent, and describes an impulsive interaction that is switched on at the instant $t = 0$. This means that the interaction between the system and the apparatus has a very short duration and is very strong. Therefore, under the impulsive assumption, the contribution to the evolution of the system–apparatus composite due to the "free" Hamiltonians associated with the two parts (i.e., the system and the apparatus) can be neglected during the measurement process.

As already mentioned, after an "unread" measurement the measured system and the measuring apparatus are, in general, in an entangled state. In an ideal measurement, however, this creation of entanglement does not occur in the important case in which the initial state of the system is an eigenstate $|\psi_\alpha\rangle$ of the observable F being measured. Indeed, in this case the (unitary) quantum evolution describing an ideal measurement acts as follows,

$$|\Sigma_0\rangle|\psi_\alpha\rangle \longrightarrow |\Sigma_{\psi_\alpha}\rangle|\psi_\alpha\rangle, \tag{2}$$

where $|\Sigma_0\rangle$ and $|\Sigma_{\psi_\alpha}\rangle$ are respectively the initial and final states of the measuring apparatus. Note that, while the standard, initial state of the apparatus $|\Sigma_0\rangle$ is always the same, the final state $|\Sigma_{\psi_\alpha}\rangle$ depends on the initial state of the system. That is, information concerning the system's initial state is transferred into the state of the apparatus. In fact, the result of the measurement is encoded in the final state $|\Sigma_{\psi_\alpha}\rangle$ of the apparatus. Eq. (2) is at the heart of the basic property of repeatability of quantum measurements: if one performs, in close temporal succession, repeated measurements of the same observable F, one always gets the same result. It has been recently shown by Zurek [20,21] that the implicit assumption behind Eq. (2) of the existence of a set of system's states left unperturbed by the measurement process has itself the following important consequence: If two states of the system $|\psi_\alpha\rangle$ and $|\psi_{\alpha'}\rangle$ are both left undisturbed by the measurement process and yield different measurement results, then these states must be orthogonal. This constitutes a generalization of the celebrated no-cloning principle [22–25]. It means that the orthogonality of the aforementioned undisturbed states does not need to be assumed as a postulate [20,26], as is usually done in text-book presentations of quantum mechanics.

An instantaneous measurement described by the Hamiltonian (1) generates the transformation

$$\Phi_0(Q)\psi_\alpha(R) \longrightarrow \Phi_0(Q - Gf_\alpha)\psi_\alpha(R), \tag{3}$$

where $\psi_\alpha(R)$ is an eigenfunction of the observable F with eigenvalue f_α, and $\Phi_0(Q)$ is the wave function describing the initial, standard state of the measuring apparatus. The transformation (3) constitutes a particular instance, corresponding to the Hamiltonian (1), of the general transformation (2). The effect of the measurement process corresponding to an arbitrary initial wave function $\psi(R)$ of the system being measured is obtained expanding $\psi(R)$ in the eigenbasis of the observable F, $\psi(R) = \sum_\alpha c_\alpha \psi_\alpha(R)$, leading to,

$$\Phi_0(Q)\sum_\alpha c_\alpha\psi_\alpha(R) \longrightarrow \sum_\alpha c_\alpha \Phi_0(Q - Gf_\alpha)\psi_\alpha(R). \tag{4}$$

In the present work we shall consider the entanglement generated during a position-measurement process between the measuring apparatus and the particle whose position is measured. The particle (of mass m) interacts with another particle of mass M (the "pointer") that represents the measuring apparatus. The particle–pointer composite system is governed by the Hamiltonian [18]

$$H = \frac{p^2}{2m} + V(x) + \frac{P^2}{2M} + \frac{1}{T}f(t)xP, \tag{5}$$

where x (Q) represents the position, p (P) the momentum of the *particle* (*pointer*), and the particle evolves under the effect of the potential function $V(x)$. The last term of the above Hamiltonian, $\frac{1}{T}f(t)xP$, describes the interaction between the pointer and the particle during the quantum measuring process, where T is the length of the time interval within which this interaction takes place and $f(t)$ is a dimensionless time-dependent coupling function. We assume that $f(t)$ adopts a constant value G within the interval $[0, T]$ and vanishes for time values outside this interval. Consequently, the interaction can be characterized by an effective, time-integrated coupling constant given by

$$G = \frac{1}{T}\int_0^T f(t)dt. \tag{6}$$

The Hamiltonian (5) allows for the study of non-instantaneous measurement processes of a finite duration T. In the limit $T \to 0$ one recovers the von Neumann, instantaneous measurement described by an impulsive interaction. For the particle's potential function $V(x)$ we consider two cases,

$$V(x) = \begin{cases} 0 & \text{free particle,} \\ \frac{1}{2}m\omega^2 x^2 & \text{harmonic oscillator.} \end{cases} \tag{7}$$

As already mentioned, we are working with a time dependent Hamiltonian. The limit of an instantaneous measurement corresponds to an instantaneous change in the function $f(t)$ characterizing the Hamiltonian's time dependence. This limit does not refer to any approximation in the *solution* $|\Psi\rangle$ of the (time dependent) Schrödinger equation. That is, for a given form of $f(t)$, we consider the exact dynamics determined by the concomitant Schrödinger equation. We either consider explicit exact time dependent solutions $|\Psi\rangle$ of this equation or, alternatively, the exact time evolution of the expectation values of an appropriate family of observables. Since we are considering the system's exact evolution, our results are fully consistent with any quantum "speed limit" derived from the Schrödinger equation [27]. Finally, it is worth to mention that our present (non-relativistic) analysis concerns only local measurements and thus, it is not affected by relativistic constraints on the measurement of non-local observables, such as the celebrated Landau–Peierls relation [28].

Given the (time-dependent) joint pure state $|\Psi\rangle$ describing the particle–pointer-system, a useful quantitative indicator of the amount of entanglement existing between the particle and the pointer is given by the linear entropy of the pointer's marginal density matrix ρ_P, given by,

$$\mathcal{E} = 1 - Tr\left(\rho_P^2\right), \tag{8}$$

where the marginal density matrix ρ_P is obtained by computing the partial trace of the global density matrix $\rho = |\Psi\rangle\langle\Psi|$ over the degrees of freedom corresponding to the particle, $\rho_P = Tr_{\text{part.}}[\rho]$. Linear entropy-based measures of quantum entanglement like (8) have been used for the study of the entanglement properties of several quantum systems [29–31]. The entanglement measure (8) constitutes a quantitative measure of the degree of mixedness of the marginal density matrix describing the states of either the particle or the pointer at a given time. These two quantities are equal, that is, $1 - Tr\left(\rho_P^2\right) = 1 - Tr\left(\rho_p^2\right)$, where $\rho_p = Tr_P(|\Psi\rangle\langle\Psi|)$ is the marginal density matrix associated with the particle. For pure states of bipartite systems the concurrence, another important entanglement measure, is closely related to the linear entropy (8). Indeed, for pure states the concurrence is given by $C = \sqrt{2(1 - Tr\left[\rho_P^2\right])}$ [32]. In general, quantitative measures of entanglement for *pure states* of bipartite quantum systems are measures of the degree of mixedness of the marginal states describing each of the parts constituting the system (or are closely related to such measures). For instance, the entropy of entanglement is given by the von Neumann entropies of the marginal entropy matrices of the two subsystems (which, for pure states, are equal to each other) [2]. When the composite system under consideration is in a mixed state, its entanglement cannot be assessed directly by entropic functionals characterizing the degree of mixedness of its parts (although entropic entanglement indicators based upon the violation of classical entropic inequalities are still useful [32]). To assess quantitatively the entanglement of composite systems in mixed states one needs other measures, such as positivity (see [33] and references therein). In the present work we are going to focus our considerations in pure states of the particle–pointer systems. Consequently, when working with explicit solutions of the concomitant Schrödinger equation, we are going to consider the entanglement measure (8).

In the present work we are going to explore entanglement-related effects during the quantum evolution associated with the position measurement process governed by the Hamiltonian (5). First we shall focus our attention on the (exact) time evolution of the expectation values of a family $\mathcal{F}$ of relevant observables. The investigation of the entanglement created during the measurement will allow us to assess to what extent the ideal condition (2) is satisfied when one considers initial states of the system increasingly localized, approaching therefore the eigenstates of the position observable. In particular, we shall consider how the duration T of the measurement affects the validity of (2). We are also going to obtain a family of exact solutions for the particle–pointer system, and then explore the evolution of the particle–pointer entanglement during the position-measurement process. The study of the entanglement features of the solutions of the Schrödinger equation associated with the Hamiltonian (5) has connections to various lines of research that have been pursued in recent years. On the one hand, entanglement-related aspects and correlation properties of other quadratic Hamiltonians have attracted the interest of researchers (see, for instance [31,34–37]), although most of these studies have focused upon the entanglement of the Hamiltonian's eigenstates, and not upon the entanglement of time-dependent solutions of the associated Schrödinger equation. On the other hand, the interaction between the particle and the pointer can be regarded as a scattering process, and the entanglement properties of these process have been the focus of some recent research activity [38].

1. Dynamical evolution of relevant expectation values

In this section we shall obtain the exact time evolution of the expectation values of a set $\mathcal{F}$ of relevant observables of the particle–pointer system. It is possible to choose the set $\mathcal{F}$ in such a way that the commutator of the Hamiltonian H with any member of the set $\mathcal{F}$ is equal to a linear combination of the members of $\mathcal{F}$. Then, the time derivative of the expectation value of an observable $A \in \mathcal{F}$, given by $\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle [H, A]\rangle$, it equal to a linear combination of the mean values of the members of $\mathcal{F}$, meaning that the evolution of the expectation values of the observables in $\mathcal{F}$ is governed by a set of closed, ordinary differential equations that can be solved without explicitly solving the full Schrödinger equation (this kind of treatment, based on a family of observables that is closed under the commutation operation, is at the basis of powerful techniques for studying the dynamics of quantum systems. See for example [39] for an interesting recent application). We choose the set of observables

$$\mathcal{F} = \left\{x, p, x^2, p^2, xp + px, Q, P, Q^2, P^2, QP + PQ, xQ, xP, pQ, pP\right\}. \tag{9}$$

In what follows we are going to investigate separately two instances of a position measurement. On the one hand, we shall consider the measurement of the position of a "free particle", that is, of a particle that interacts with the pointer but is not under the effect of any other force. On the other hand, we shall consider a particle that, besides interacting with the pointer, is under the effect of a harmonic force.

When the particle whose position is measured is under he effect of a harmonic potential, the equations of motion for the relevant expectation values are the following,

$$\begin{aligned}
\frac{d}{dt}\langle x\rangle &= \frac{\langle p\rangle}{m},\\
\frac{d}{dt}\langle x^2\rangle &= \frac{\langle xp + px\rangle}{m},\\
\frac{d}{dt}\langle Q\rangle &= \frac{\langle P\rangle}{M} + \frac{f(t)}{T}\langle x\rangle,\\
\frac{d}{dt}\langle Q^2\rangle &= \frac{\langle QP + PQ\rangle}{M} + 2\frac{f(t)}{T}\langle xQ\rangle,\\
\frac{d}{dt}\langle p\rangle &= -\frac{f(t)}{T}\langle P\rangle - m\omega^2\langle x\rangle,\\
\frac{d}{dt}\langle p^2\rangle &= -2\frac{f(t)}{T}\langle pP\rangle - m\omega^2\langle xp + px\rangle,\\
\frac{d}{dt}\langle P\rangle &= 0,\\
\frac{d}{dt}\langle P^2\rangle &= 0,\\
\frac{d}{dt}\langle xp + px\rangle &= \frac{2}{m}\langle p^2\rangle - 2\frac{f(t)}{T}\langle xP\rangle - 2m\omega^2\langle x^2\rangle,\\
\frac{d}{dt}\langle QP + PQ\rangle &= \frac{2}{M}\langle P^2\rangle + 2\frac{f(t)}{T}\langle xP\rangle,\\
\frac{d}{dt}\langle xQ\rangle &= \frac{1}{m}\langle pQ\rangle + \frac{1}{M}\langle xP\rangle + \frac{f(t)}{T}\langle x^2\rangle,
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\langle xP\rangle &= \frac{1}{m}\langle pP\rangle,\\
\frac{d}{dt}\langle pQ\rangle &= \frac{1}{m}\langle pP\rangle + \frac{f(t)}{T}\langle px - PQ\rangle - m\omega^2\langle xQ\rangle,\\
\frac{d}{dt}\langle pP\rangle &= -\frac{f(t)}{T}\langle P^2\rangle - m\omega^2\langle xP\rangle,\\
\frac{d}{dt}\langle px - PQ\rangle &= \frac{1}{m}\langle p^2\rangle - \frac{1}{M}\langle P^2\rangle - 2\frac{f(t)}{T}\langle xP\rangle - m\omega^2\langle x^2\rangle.
\end{aligned}
\tag{10}
$$

If one now considers a constant coupling function $f(t) = G$, the above equations admit an exact analytical solution, which is given in the Appendix A. The equations of motion of the relevant observables corresponding to the free particle case are obtained by setting $\omega = 0$ in (10). The concomitant solution is recovered by putting $\omega = 0$ in the solution given in Appendix A.

1.1. Free particle

We first consider the measurement of the position of a free particle. That is, the particle interacts only with the apparatus (pointer) and is not under the effect of any external potential. The Hamiltonian of the particle–pointer system comprises the kinetic energy terms corresponding to the particle and to the pointer, and the interaction term. We assume that before the measurement the pointer is described by a symmetric wave packet centered at the origin, with $\langle Q\rangle_0 = 0$ and $\langle P\rangle_0 = 0$. Let $\langle A\rangle(T)$ represent the expectation value of the observable A at the end of the measurement process (that has a duration T). It is interesting to see that some of the main features of the measurement process can be appreciated from the analysis of the evolution of the mean values of the relevant observables. For instance, let as consider the total displacement of the mean value of the pointer position. From our previous results one obtains,

$$
\langle Q\rangle(T) - \langle Q\rangle_0 = G\frac{\langle x\rangle(T) + \langle x\rangle_0}{2}. \tag{11}
$$

We see that the total displacement $\langle Q\rangle(T) - \langle Q\rangle_0$ of the mean pointer's position is proportional to the average of the initial and final mean positions of the particle. This means that information concerning the position of the particle has been "transferred" from the state of the particle to the state of the pointer, as is to be expected to occur after a position measurement. The result summarized in Eq. (11), connecting the displacement of the pointer's location after a position measurement of *finite time duration* with the average initial and final positions of the particle, has been discussed by Roig [18] using an approach based on path integrals, which is different from the one followed in the present work, based on the evolution of the mean values of selected observables.

Of special interest is the behavior of quantities of the form $\langle AB\rangle - \langle A\rangle\langle B\rangle$, where A is an observable involving only the particle and B is an observable referred only to the pointer. These quantities vanish for non-entangled, pure states of the particle–pointer system. Consequently, for pure states, non-zero values of these quantities indicate the presence of entanglement. In other words, these quantities constitute entanglement indicators. For the system investigated here we have,

$$
\begin{aligned}
(\langle xQ\rangle - \langle x\rangle\langle Q\rangle)(T) &= \frac{G}{6m^2}\left[\frac{G^2}{2} - \frac{2m+M}{M}\right]\langle P^2\rangle_0 T^2 + \frac{G}{2m^2}\Delta_{p0}^2 T^2\\
&\quad + \frac{G}{m}\left[\langle xp + px\rangle_0 - \frac{3}{2}\langle x\rangle_0\langle p\rangle_0\right] T + G\Delta_{x0}^2\\
(\langle pQ\rangle - \langle p\rangle\langle Q\rangle)(T) &= \frac{G}{2m}\left[\frac{G^2}{3} - \frac{M+m}{M}\right]\langle P^2\rangle_0 T + \frac{G}{2m}\Delta_{p0}^2 T\\
&\quad + \frac{G}{2}\left[\langle xp + px\rangle_0 - 2\langle x\rangle_0\langle p\rangle_0\right]
\end{aligned}
$$

$$(\langle xP\rangle - \langle x\rangle\langle P\rangle)(T) = -\frac{1}{2}G\frac{\langle P^2\rangle_0}{m}T$$

$$(\langle pP\rangle - \langle p\rangle\langle P\rangle)(T) = -G\langle P^2\rangle_0, \quad (12)$$

where $\Delta_A^2 = \langle A^2\rangle - \langle A\rangle^2$. Defining now $C_{AB} = |\langle AB\rangle - \langle A\rangle\langle B\rangle|$, we have in the limit $T \to 0$ of instantaneous measurement,

$$C_{xQ}(0) = G\Delta_{x0}^2$$

$$C_{pP}(0) = G\langle P^2\rangle_0$$

$$C_{xP}(0) = 0$$

$$C_{pQ}(0) = \frac{G}{2}\left[\langle xp + px\rangle_0 - 2\langle x\rangle_0\langle p\rangle_0\right]. \quad (13)$$

Note that by $\langle A\rangle_0$ and $\langle A\rangle(0)$ we designate different things. By $\langle A\rangle_0$ we denote the initial expectation value of the observable A immediately before the measurement, while $\langle A\rangle(0)$ denotes the expectation value of A immediately after an impulsive measurement (of duration $T = 0$). Similarly, $C_{AB}(0)$ designates the value adopted by the indicator C_{AB} immediately after an instantaneous measurement. We see from (13) that $C_{xQ}(0)$, which corresponds to the correlation in configuration space exhibited by an (entangled) pure state of the particle–pointer system, is proportional to the initial dispersion Δ_{x0} of the particle's position. This result is consistent with Eq. (2) describing an ideal measurement when the initial state of the measured system (in our case, the particle) is an eigenstate of the observable being measured. As indicated by Eq. (2), in this case we expect zero entanglement of the particle–pointer system after the measurement is performed. In the model here discussed we see that as Δ_{x0} decreases (which means that the initial particle's state approaches an eigenstate of the position observable) the quantity $C_{xQ}(0)$ goes to zero. The particle–pointer correlation in momentum space due to an instantaneous measurement, given by $C_{pP}(0)$, is equal to $G\Delta_{p0}$. That is, $C_{pP}(0)$ is proportional to the initial dispersion of the pointer's momentum (taking into account that $\langle P\rangle_0 = 0$). The final correlation between the particle's momentum and the pointer's position depends only on features of the initial state of the particle, as indicated by $C_{pQ}(0) = \frac{G}{2}\left[\langle xp + px\rangle_0 - 2\langle x\rangle_0\langle p\rangle_0\right]$. Finally, the instantaneous measurement does not generate correlations between the particle's position and the pointer's momentum ($C_{xP}(0) = 0$).

In the case of measurements of finite duration, the particle–pointer correlation in momentum space, given by C_{pP}, is independent of the initial state of the particle and is also independent of the duration T of the measurement. On the other hand, the correlation in configuration space, C_{xQ}, grows quadratically with T for measurements of large enough duration. In the case of $m < M$, and for measurements of short duration, this configuration-space correlation can actually decrease with the duration T. Indeed, for small values of the pointer's mass M the terms in T^2 in the right hand sides of the first two equations in (12) become negative. When this happens there is a special value T^* of the measurement's duration such that for $T \leq T^*$ the indicators C_{xQ} and C_{pQ} are decreasing functions of T (each of these two indicators has a different critical value T^*). For $T > T^*$ the indicators increase with the duration of the measurement process. This non-monotonic behavior of the indicators C_{xQ} and C_{pQ} has a clear correlate with the behavior of the amount of entanglement corresponding to the exact time-dependent solution of the particle–pointer Schrödinger equation studied in Section 3. In fact, in Fig. 3 one can see that for small M there is an interval of T-values for which entanglement decreases with T.

It is also interesting to consider, for measurements of finite duration T, the dependence of the spatial correlation indicator C_{xQ} on the initial dispersion Δ_{x0} of the particle's position. For ideal position measurements one expects that $C_{xQ} \to 0$ when $\Delta_{x0} \to 0$, since in this limit the initial state of the particle (as it becomes more localized) approached an eigenstate of the position observable. However, we see from the first equation in (12) that, for $T \neq 0$ and Δ_{x0} small enough, C_{xQ} increases as Δ_{x0} decreases. This effect is due to the term $\frac{G}{2m^2}\Delta_{p0}^2T^2$ appearing in the expression for C_{xQ} (that is, the modulus of the right hand side of the first equation in (12)). From the position–momentum uncertainty relation one gets,

$$\frac{G}{2m^2}\Delta_{p0}^2 T^2 \geq \frac{G}{2m^2}\frac{\hbar^2}{\Delta_{x0}^2}T^2, \tag{14}$$

implying that C_{xQ} actually diverges in the limit $\Delta_{x0} \to 0$. This means that, with regards to this fundamental point a finite-time measurement, no matter how small its duration T is (as long as it is finite) differs drastically from an instantaneous measurement. In the limit of an instantaneous measurement the kinetic energy terms in the particle–pointer Hamiltonian become negligible compared with the interaction term. The same occurs in the limit $M, m \to \infty$. In fact, in this last case we recover a behavior similar to the one associated with an instantaneous measurement, with $C_{xQ} \to 0$ when $\Delta_{x_0} \to 0$.

1.2. Harmonic oscillator

One can determine the final displacement of the mean pointer's position,

$$\langle Q\rangle(T) - \langle Q\rangle_0 = G\frac{\tan(\omega T/2)}{\omega T/2}\frac{\langle x\rangle(T) + \langle x\rangle_0}{2}, \tag{15}$$

which again indicates that the displacement of the pointer's location is proportional to the average between the initial and final mean positions of the particle.

As occurs in the free particle case, the entanglement of the particle–pointer (pure) state leads to correlations in position and in momentum that can be assessed with the indicators,

$$\begin{aligned}
C_{xQ}(T) = \Bigg| & 2\left(\frac{G}{T}\right)^3 \frac{\langle P^2\rangle_0}{m^2\omega^5}\sin^2(\omega T/2)(\omega T - \sin(\omega T)) \\
& + 2\,\frac{G}{T}\frac{\Delta_{p0}^2}{m^2\omega^3}\sin(\omega T)\sin^2(\omega T/2) + \frac{1}{2}\frac{G}{T}\frac{\Delta_{x0}^2}{\omega}\sin(2\omega T) \\
& + \frac{1}{2}\frac{G}{T}\frac{\langle P^2\rangle_0}{m^2M\omega^3}\left((m-M)\sin(\omega T) + (m+M)\cos(\omega T)\omega T\right) \\
& + \frac{G}{T}\frac{1}{m\omega^2}(1 + 2\cos(\omega T))\sin^2\left(\frac{\omega T}{2}\right)\left[\langle xp + px\rangle_0 - 2\langle x\rangle_0\langle p\rangle_0\right] - G\frac{\langle P^2\rangle_0}{mM\omega^2}\Bigg| \\
C_{pP}(T) = & \left|\frac{G}{T}\frac{\langle P^2\rangle_0}{\omega}\sin(\omega T)\right| \\
C_{xP}(T) = & \left|2\frac{G}{T}\frac{\langle P^2\rangle_0}{m\omega^2}\sin^2(\omega T/2)\right|,
\end{aligned} \tag{16}$$

In the impulsive limit we have,

$$\begin{aligned}
& C_{xQ}(0) = G\Delta_{x0}^2, \\
& C_{pP}(0) = G\langle P^2\rangle_0, \\
& C_{xP}(0) = 0.
\end{aligned} \tag{17}$$

We see that in the limit of an instantaneous measurement the final values of the entanglement indicators after the measurement are the same as in the case of an instantaneous measurement performed on a free particle. This, of course, is consistent with the fact that in this limit the interaction term of the Hamiltonian becomes dominant and determines completely the evolution of the particle–pointer system during the measurement process.

2. Exact time dependent Gaussian wave packet solutions of the Schrödinger equation

In the previous sections we have studied the behavior of the particle–pointer system by recourse to the analysis of the evolution of the expectations values of an appropriate family of observables. That analysis was general, in the sense of yielding results valid for arbitrary initial conditions of the particle–pointer system. Now we shall consider an ansatz for the particle–pointer wavefunction leading to particular exact solutions of the concomitant Schrödinger equation, on which we can explicitly evaluate an entanglement measure in order to determine its time evolution.

2.1. Gaussian particle–pointer wave packet

We introduce the following Gaussian wave packet ansatz for the particle–pointer wave function,

$$\Psi(x;Q) = \mathcal{N}\exp(-\lambda_1 x - \lambda_2 x^2 - \lambda_3 Q - \lambda_4 Q^2 - \lambda_5 xQ) \tag{18}$$

where the normalization constant $\mathcal{N}$ is given by,

$$\mathcal{N} = \left[\frac{\sqrt{4l_4l_2 - l_5^2}}{2\pi}\exp\left(-\frac{l_1^2 l_4 + l_3^2 l_2 - l_3 l_1 l_5}{l_5^2 - 4l_4 l_2}\right)\right]^{1/2}, \tag{19}$$

with $\lambda_j = l_j/2 + \imath\lambda_j^I$, $l_j = 2\mathrm{Re}(\lambda_j)$, and $\lambda_j^I = \mathrm{Im}(\lambda_j)$, $(j = 0, 1, \ldots, 5)$. After some algebra the real and the imaginary parts of the coefficients λ_i can be expressed in terms of the set of selected expectation values. For the l_j's we obtain,

$$\begin{aligned}
l_1 &= \frac{\langle xQ\rangle\langle Q\rangle - \langle x\rangle\langle Q^2\rangle}{\mathcal{D}_{xQ}^2}\\
l_2 &= \frac{\Delta_Q^2}{2\mathcal{D}_{xQ}^2}\\
l_3 &= \frac{\langle x\rangle\langle xQ\rangle - \langle x^2\rangle\langle Q\rangle}{\mathcal{D}_{xQ}^2}\\
l_4 &= \frac{\Delta_x^2}{2\mathcal{D}_{xQ}^2}\\
l_5 &= \frac{\langle x\rangle\langle Q\rangle - \langle xQ\rangle}{\mathcal{D}_{xQ}^2},
\end{aligned} \tag{20}$$

where (see Appendix B),

$$\begin{aligned}
\mathcal{D}_{xQ}^2 &= -\left(\langle x^2\rangle\langle Q\rangle^2 - \langle x^2\rangle\langle Q^2\rangle + \langle x\rangle^2\langle Q^2\rangle + \langle xQ\rangle^2 - 2\langle x\rangle\langle xQ\rangle\langle Q\rangle\right)\\
&= \Delta_Q^2\Delta_x^2 - (\langle x\rangle\langle Q\rangle - \langle xQ\rangle)^2 \geq 0.
\end{aligned} \tag{21}$$

The imaginary parts λ_j^I, of the λ_i are,

$$\begin{aligned}
\lambda_1^I &= \frac{1}{\hbar}\left(\left(\langle x\rangle\langle p\rangle - \frac{1}{2}\langle xp + px\rangle\right)l_1 + (\langle Q\rangle\langle p\rangle - \langle pQ\rangle)l_3 - \langle p\rangle\right)\\
\lambda_2^I &= -\frac{1}{2\hbar}(l_1\langle p\rangle + l_2\langle xp + px\rangle + l_5\langle pQ\rangle)\\
\lambda_3^I &= \frac{1}{2\hbar\Delta_Q^2}(\langle Q\rangle\langle QP + PQ\rangle - 2\langle Q^2\rangle\langle P\rangle - \mathcal{D}_{xQ}^2(2l_3\langle p\rangle + 4l_4\langle pQ\rangle + l_5\langle xp + px\rangle)l_1)
\end{aligned}$$

$$\lambda_4^I = -\frac{1}{4\hbar\Delta_Q^2}(\langle QP+PQ\rangle - 2\langle Q\rangle\langle P\rangle + \mathcal{D}_{xQ}^2(2l_3\langle p\rangle + 4l_4\langle pQ\rangle + l_5\langle xp+px\rangle)l_5)$$

$$\lambda_5^I = -\frac{1}{\hbar}\left(l_3\langle p\rangle + 2l_4\langle pQ\rangle + \frac{1}{2}l_5\langle xp+px\rangle\right). \tag{22}$$

From Eqs. (20) and (22) we can evaluate the λ_i's at any time t if we know at that time the expectation values of the relevant observables.

2.2. Initial conditions

We assume that before the beginning of the measurement process the state of the particle–pointer system is represented by a factorized Gaussian wave packet. That is, at $t = 0$ we have $\lambda_5 = 0$. The characterization of this initial Gaussian state requires three initial parameters for the particle, $\langle x\rangle_0 = x_0$, $\langle p\rangle_0 = p_0$ and $\Delta_x = \Delta_{x0}$. Similarly, the initial state of the pointer is determined by $\langle Q\rangle_0 = Q_0$, $\langle P\rangle_0 = P_0$ y $\Delta_Q = \Delta_{Q0}$.

$$\Psi(x;Q) = \frac{1}{\sqrt{2\pi\Delta_{x0}\Delta_{Q0}}}\exp\left(i\frac{p_0}{\hbar}(x-x_0) - \frac{(x-x_0)^2}{4\Delta_{x0}^2}\right) \times \exp\left(i\frac{P_0}{\hbar}(Q-Q_0) - \frac{(Q-Q_0)^2}{4\Delta_{Q0}^2}\right). \tag{23}$$

2.3. The impulsive limit: $T \to 0$

In the limit of an instantaneous measurement the final state of the system when we measure the position of a free particle is the same as the one obtained when the particle is under the effect of a harmonic potential. If the pointer is initially represented by a factorized Gaussian wave packet, with $Q_0 = \langle Q\rangle_0 = 0$ and arbitrary widths Δ_x and Δ_Q, we have,

$$\Delta_x^2(0) = \Delta_{x0}^2$$
$$\Delta_Q^2(0) = G^2\Delta_{x0}^2 + \Delta_{Q0}^2$$
$$\mathcal{D}_{xQ}(0) = \Delta_{x0}^2\Delta_{Q0}^2. \tag{24}$$

Note that $\Delta_A(0)$ and Δ_{A0} denote different things. The quantity $\Delta_A(0)$ is the value adopted by Δ_A immediately after an instantaneous measurement (corresponding to the limit $T \to 0$) while Δ_{A0} is the initial value of Δ_A (that is, the value at $t = 0$). The values of the λ-parameters characterizing our ansatz after an instantaneous measurement are,

$$\lambda_1(0) = -\frac{x_0}{2\Delta_x^2} - i\frac{\langle p\rangle_0}{\hbar}$$
$$\lambda_2(0) = G^2\frac{1}{4\Delta_Q^2} + \frac{1}{4\Delta_x^2}$$
$$\lambda_3(0) = i\frac{1}{2\hbar(\Delta_x^2+\Delta_Q^2)}\left(2G + \frac{\Delta_Q^2}{G\Delta_x^2+\Delta_Q^2}\right)\langle x\rangle_0\langle p\rangle_0$$
$$\lambda_4(0) = \frac{1}{4\Delta_Q^2}$$
$$\lambda_5(0) = -G\frac{1}{2\Delta_Q^2}. \tag{25}$$

In the above equations, and in the remaining equations of this Section we drop the "0" subindex from Δx and ΔQ with the convention that when these quantities appear with no subindices they refer to the initial values.

We have assumed that just before measuring, the system state is separable

$$\Psi(x;Q) = \phi_0(x)\Phi_0(Q), \tag{26}$$

as described in Eq. (23), which corresponds to Eq. (18) with $\lambda_5 = 0$. Due to the particle–pointer interaction the system evolves to a final state with $\lambda_5 \neq 0$, corresponding to a non-separable Gaussian wave packet. It is instructive to compute the final marginal probability density for the positions of the particle and the pointer. It is possible to verify after some algebra that for the final, non-separable (that is, not of the form (26)) Gaussian wave packet the marginal probability density for the particle is,

$$\begin{aligned} P(x) &= \int_{-\infty}^{\infty} |\Psi(x,Q)|^2 dQ \\ &= |\phi_0(x)|^2. \end{aligned} \tag{27}$$

The last equality in the above equation is due to the fact that the value of Δ_x after an instantaneous measurement is equal to its initial value. The marginal probability density for the pointer's position is,

$$\begin{aligned} \tilde{P}(Q) &= \int_{-\infty}^{\infty} |\Psi(x,Q)|^2 dx \\ &= \frac{1}{2\sqrt{\pi}\sigma} \exp\left(-\frac{(Q-Gx_0)^2}{4\sigma^2}\right), \end{aligned} \tag{28}$$

where

$$\sigma^2 = \frac{G^2\Delta_x^2 + \Delta_Q^2}{2}. \tag{29}$$

We see that the probability density for the particle's position is the same before and after the measurement. This might suggest that the quantum state of the particle is unchanged by the measurement. This, however, is not in general the case, because the act of measurement generates entanglement between the particle and the pointer.

3. Particle–pointer entanglement

In order to investigate the time evolution of the particle–pointer entanglement we need to consider the global density matrix $\rho = |\Psi\rangle\langle\Psi|$ describing the joint state of the particle–pointer system. From this matrix we obtain the marginal density matrix ρ_P describing the pointer. The matrix elements of ρ_P are $\rho_P(Q,Q') = \int dx \Psi(x,Q)\Psi^*(x,Q')$. The entanglement measure (8) is then given by the linear entropy of ρ_P, yielding

$$\mathcal{E} = 1 - (4l_2 l_4 - l_5^2)|\lambda_4|\lambda_5^2 \frac{1}{\sqrt{B}} \exp\left(2\frac{R_1}{R_2}\right), \tag{30}$$

where,

$$\begin{aligned} R_1 &= \lambda_3^2\lambda_4(\lambda_5^2+|\lambda_5|^2)^2(l_1^2 l_4 + l_2 l_3^2 - l_1 l_3 l_5) - \lambda_3^2\lambda_5^2(\lambda_4^2+|\lambda_4|^2)(l_1^2 l_5^2 - 2l_2 l_3)^2 \\ &\quad + \lambda_4\lambda_5\left(\lambda_3(|\lambda_3|^2\lambda_5^2 + |\lambda_5^2\lambda_3^2| + \lambda_3^2\lambda_5^2 + \lambda_3^2|\lambda_5|^2)l_1 - \lambda_5(\lambda_3^2+|\lambda_3|^2)^2 l_2\right)(-4l_2 l_4 + l_5^2) \\ R_2 &= (4l_4 l_2 - l_5^2)(4\lambda_5^2(\lambda_4^2+|\lambda_4|^2)l_2 - \lambda_4(\lambda_5^2+|\lambda_5|^2)^2)\lambda_3^2 \\ B &= 16 l_2^2\lambda_5^4(\lambda_4^2+|\lambda_4|^2)^2 + \lambda_4^2(\lambda_5^4 - |\lambda_5|^4)^2 \\ &\quad - 8l_2\lambda_4\lambda_5^2(|\lambda_5^4\lambda_4^2| + \lambda_5^2|\lambda_4|^2 + \lambda_4^2(\lambda_5^4+|\lambda_5|^4)). \end{aligned} \tag{31}$$

We now evaluated the post-measurement entanglement $\mathcal{E}$ for the impulsive limit ($T \to 0$) using Eq. (25) to obtain a compact closed form for expression (30), yielding,

$$\mathcal{E} = 1 - \frac{\Delta_Q}{\sqrt{\Delta_Q^2 + \Delta_x^2}}. \tag{32}$$

Since in this impulsive limit the evolution of the particle–pointer system is determined exclusively by the interaction term in the Hamiltonian, the final entanglement depends neither on the mass of the particle nor on the mass of the pointer (in fact, it can be verified that expression (32) for the post-measurement entanglement also holds for finite-duration measurements in the limit $m, M \to \infty$ where, again, only the interaction term in the Hamiltonian survives). Note that the system is entangled more strongly with decreasing Δ_Q/Δ_x. In other words, if the initial position of the pointer is defined with accuracy much greater than that of the particle, the system ends up in a highly entangled state. In the limit of $\Delta_Q/\Delta_x \to 0$, we have $\mathcal{E} \to 1$. We see that when $\Delta_x \to 0$ (for a given Δ_Q) the final particle–pointer entanglement tends to zero. This is consistent with an ideal position measurement, since when Δ_x decreases the particle becomes more localized and its state approaches an eigenstate of the position observable.

It follows from (32) and from the uncertainty relation connecting Δ_Q and Δ_P, that,

$$\mathcal{E} \leq 1 - \frac{1}{\sqrt{1 + \frac{\Delta_x^2 \Delta_P^2}{\hbar^2}}}. \tag{33}$$

We see that $\mathcal{E} \to 0$ when $\Delta_P \to 0$. This occurs because the interaction term in the Hamiltonian (5) admits an alternative interpretation as describing a measurement of the momentum P of the pointer. From this point of view the particle (with coordinate x) plays the role of the measuring apparatus, while the pointer (with coordinate Q) is the system being measured.

The amount of entanglement $\mathcal{E}$ generated by an instantaneous ($T \to 0$) measurement process (with the initial state of the particle–pointer system given by the Gaussian wave packet ansatz (18)) is depicted in Fig. 1 as a function of (a) Δ_x with fixed values of $\Delta_Q = 0.1, 1, 10$, and (b) Δ_Q with $\Delta_x = 0.1, 1, 10$. It transpires from Fig. 1, that for given values of Δ_Q, $\mathcal{E}$ increases with Δ_x from zero to the maximum value given by 1. On the other hand, for given Δ_x, the final entanglement decreases with Δ_Q. We see that for small values of Δ_Q the final entanglement increases quickly with Δ_x while, for large values of Δ_Q the entanglement increases slowly with the initial dispersion of the particle's position. Similarly, for given values of Δ_x, the smaller the value of Δ_x the faster is the decrease of the final entanglement with Δ_Q. The amount of entanglement $\mathcal{E}$ resulting from a *non-instantaneous* measurement with a finite duration $T = 1$ is plotted in Fig. 2 as a function of Δ_Q and Δ_x, with the main parameters adopting the values $M = 100$, $m = 1$ and $G = 1$. The results observed for $M > 1$ are almost independent of the pointer's mass M. The surface obtained for $M = 10$ practically coincides with the one corresponding to $M = 100$. Any difference with the surface obtained for $M = 1$ is also negligible. It is interesting that, contrary to what one expects for an ideal position measurement, for small values of Δ_x the entanglement of the particle–pointer system increases as Δ_x decreases, approaching its maximum value $\mathcal{E} = 1$ as $\Delta_x \to 0$. This is consistent with the behavior of the indicator C_{xQ} which, as was shown in Section 1.1, for measurements of finite duration diverges as $\Delta_x \to 0$.

In Fig. 3 the final particle–pointer entanglement resulting after measuring the position of a free particle is depicted as a function of the measurement's duration T. We assume $G = 1$ and initial conditions $x_0 = 1$, $\Delta_x = 2$, $p_0 = 1$, $Q_0 = 0$, and $P_0 = 0$, with (a) $\Delta_Q = 3$ and (b) $\Delta_Q = 0.1$. It transpires from Fig. 3 that, for a given value of the measurement duration T (and a given particle's mass m) the amount of entanglement generated by the measurement tends to increase with the mass M of the measuring apparatus (pointer). It can also be appreciated in Fig. 3 that the finite amount of entanglement generated by an instantaneous measurement ($T \to 0$) does not depend on M, as is to be expected, since in the impulsive limit the contribution of the particle's and pointer's free Hamiltonians to the system's evolution is negligible (in fact, in the limit $T \to 0$ the post-measurement entanglement does not depend on m either).

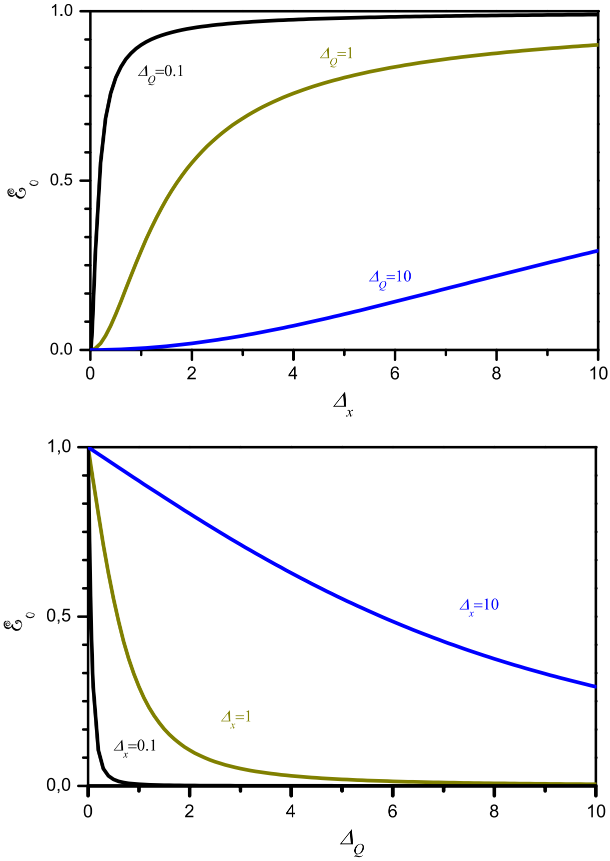

Fig. 1. The entanglement $\mathcal{E}$, for $T \to 0$, is depicted as a function of: (a) Δ_x for several values of Δ_Q; (b) Δ_Q for several values of Δ_x.

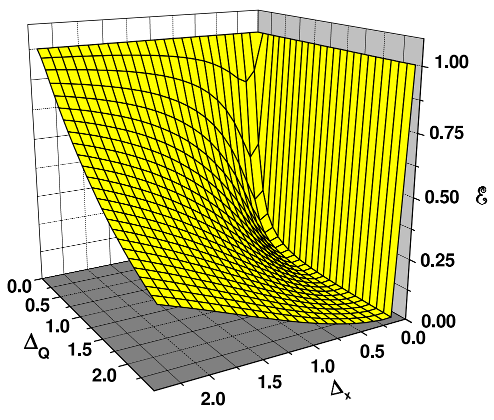

Fig. 2. A surface is plotted to represent the entanglement $\mathcal{E}$ as a function of Δ_Q and Δ_x where main parameters are $M = 100$, $m = 1$ and $T = 1$, and $G = 1$. It is observed that for $M > 1$ the results are almost independent of mass M. The surface obtained for $M = 10$ practically coincides with the one got for $M = 100$. Any difference with the surface obtained for $M = 1$ is also negligible.

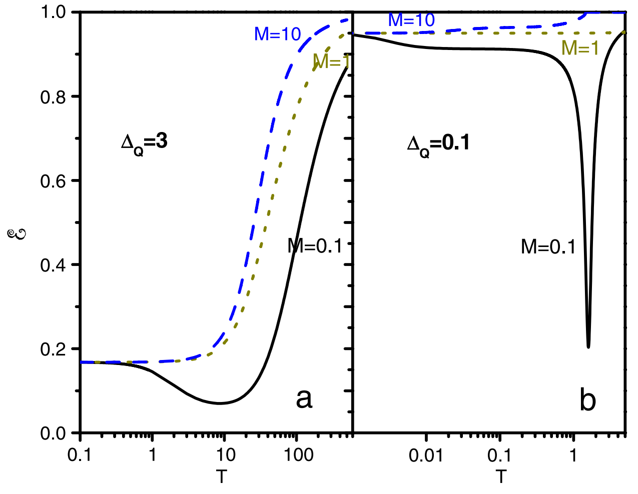

Fig. 3. The particle–pointer entanglement is depicted as a function of the measurement's duration T for different values of the pointer's mass. We have the following parameters $\hbar = 1$, $G = 1$, $\Delta_x = 2$, and $m = 1$, with (a) $\Delta_Q = 3$ and (b) $\Delta_Q = 0.1$. We considered three values of the pointer's mass, $M = 0.1, 1, 10$. The particle is not affected by any external potential, interacting only with the pointer.

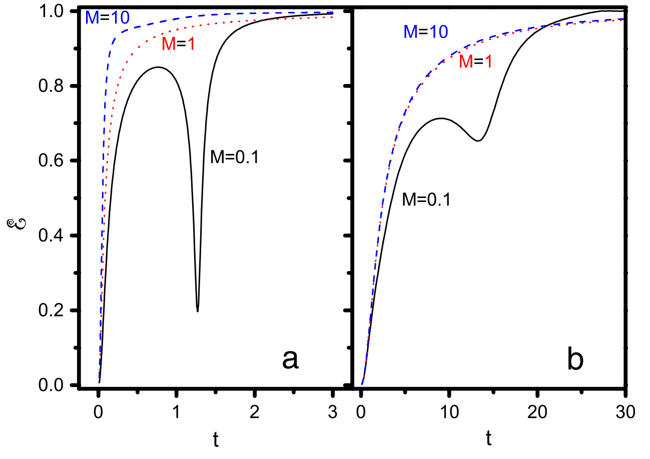

Fig. 4. The particle–pointer entanglement is depicted as a function of time for different values of the pointer's mass. We have the following parameters $\hbar = 1$, $G/T = 1$, $\Delta_x = 2$, $m = 1$, with (a) $\Delta_Q = 0.1$ and (b) $\Delta_Q = 3$. Three values for the mass of the pointer are considered, $M = 0.1, 1, 10$. The particle is not under the effect of any external potential, and interacts solely with the pointer.

In Fig. 4 the final particle–pointer entanglement generated after the position's measurement of a free particle is depicted as a function of time for different values of the pointer mass. Note that in this Figure we are not showing the dependence of entanglement on the total duration T of the measurement process, but its dependence on the time t along the course of the measurement process. In Fig. 4 we have the following parameters $\hbar = 1$, $\Delta_x = 2$, $m = 1$. The difference with the case represented by Fig. 3 is that G is not constant, we consider $G/T = 1$. In (a) $\Delta_Q = 0.1$ and (b) $\Delta_Q = 3$. In every case, the mass of the pointer is fixed, as $M = 0.1, 1, 10$. The particle does not interact with other potential except with the pointer. In Fig. 5 the dependence of entanglement on Δ_Q is shown for $G = 1$, $T = 1$, $m = 1$ and $\Delta_x = 1$ and two particular values of M ($M = 1{,}100$). The entanglement generated by the interaction Hamiltonian alone is also shown by the dot-dashed orange line. In Fig. 6

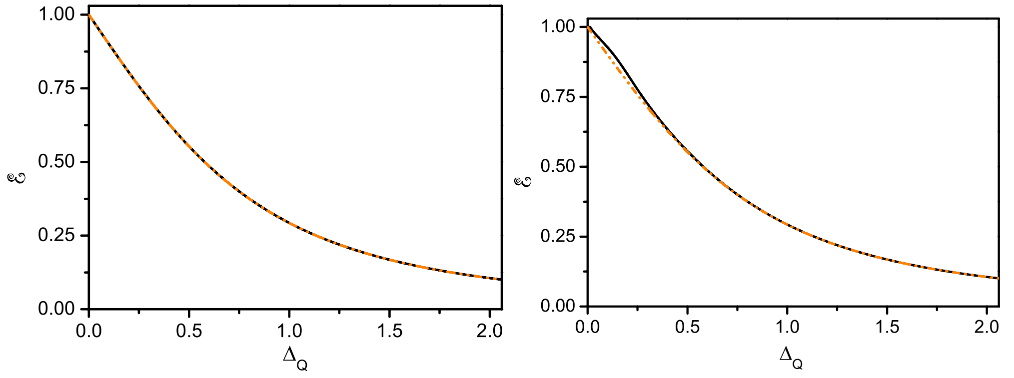

Fig. 5. The entanglement $\mathcal{E}$ (full black line) is represented as a function of Δ_Q for $\Delta_x = 1$. The parameters of the system are $m = 1$, $G = 1$ and $T = 1$. The part of the entanglement produced by the interaction term alone (dash–dot–dot orange line) is also represented. For the pointer's mass we consider in (a) $M = 100$ and in (b) $M = 1$. Besides, it is plotted the entanglement $\mathcal{E}$ (dash–dot–dot orange line) produced only by the interaction term. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

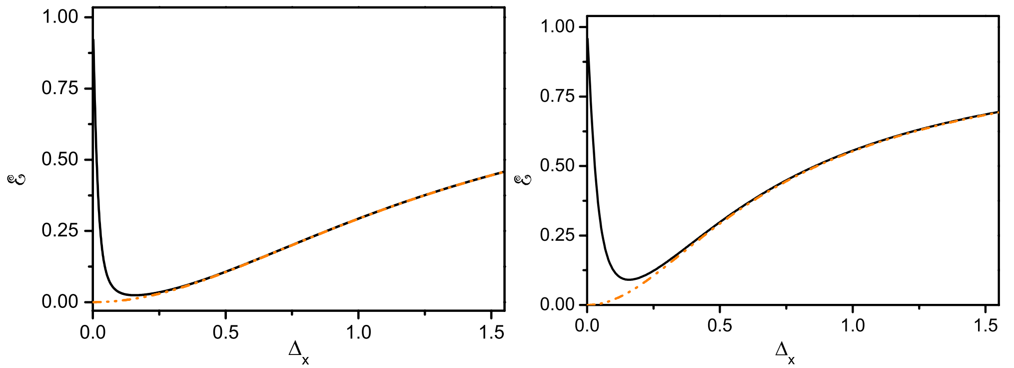

Fig. 6. The entanglement $\mathcal{E}$ (solid black line) is represented as a function of Δ_x with $M = 100$, $m = 1$, $G = 1$ and $T = 1$ for (a) $\Delta_Q = 1$, and (b) $\Delta_Q = 0.5$ having a minimum close to zero. In addition, the entanglement (dash–dot–dot orange line) produced by the interaction term alone is depicted. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the dependence of entanglement on Δ_x is depicted for $G = 1$, $T = 1$, $m = 1$, $M = 100$, and the pair of particular values $\Delta_Q = 1, 0.5$. The entanglement (dash–dot–dot orange line) produced by the interaction term alone is also depicted. In this last case, which is similar to an instantaneous measurement where the effect of the free Hamiltonians of the particle and the pointer are negligible, we see that the amount of entanglement generated by the measurement tends to zero when $\Delta_x \to 0$. On the other hand, we see that when the kinetic energy terms make a finite contribution to the evolution, the entanglement tends to its maximum value when $\Delta_x \to 0$, illustrating again this already mentioned important aspect in which a finite-duration position measurement differs from an ideal one. In general, the effect of an instantaneous measurement process on the initial particle–pointer wave function $\psi(x, Q, 0) = \phi(x)\Phi(Q)$, is given by

$$\phi(x)\Phi(Q) \longrightarrow \phi(x)\Phi(Q - Gx). \tag{34}$$

This transformation corresponds to the particular instance of the general transformation (4) obtained when the observable being measured is the position of a particle. Note that, in general, the final particle–pointer state is entangled. The final state of the particle is described by a marginal density matrix

ρ_p with matrix elements given by,

$$\rho_p(x, x') = \phi(x)\phi^*(x') \int \Phi(Q - Gx)\Phi^*(Q - Gx')dQ. \quad (35)$$

In order to determine the entanglement of the final particle–pointer state we first compute $Tr[\rho_p^2]$. We have,

$$Tr[\rho_p^2] = \int |\phi(x)|^2|\phi(x')|^2\gamma(x - x')dxdx', \quad (36)$$

where $\gamma(x - x') = \left|\int \Phi(u)\Phi^*(u + G(x - x'))du\right|^2$. Note that the function $\gamma(x - x')$ does not depend on the initial state $\phi(x)$ of the particle whose position is to be measured. The function $\gamma(x - x')$ is determined by the initial "setting" of the measuring apparatus, given by the apparatus initial wave function $\Phi(Q)$. Applying the Schwartz inequality to (36) one gets $Tr[\rho_p^2] \leq \Gamma \int |\phi(x)|^4dx$, with $\Gamma = \left|\int \gamma^2(x - x')dxdx'\right|^{1/2}$. From this we obtain a lower bound for the entanglement (as measured by the linear entropy of ρ_p) of the final state of the particle–pointer system,

$$\mathcal{E} \geq 1 - \Gamma \int |\phi(x)|^4dx. \quad (37)$$

This lower bound is given in terms of the initial wave function of the particle and of the constant Γ (which depends solely on the initial setting of the measuring apparatus). We see that for very delocalized initial states of the particle the integral appearing in (37) adopts small values, leading to a final entanglement between the particle and the pointer approaching the maximum value $\mathcal{E} = 1$.

4. Conclusions

We have investigated some entanglement features of a model system describing the measurement of the position of a quantum particle. We considered two complementary approaches to this problem. On the one hand, we analyzed the evolution of the expectation values of an appropriate family of quantum observables that obey a closed system of equations of motion. We obtained exact analytical solutions to these equations and derived from them the behavior of some entanglement indicators. This approach is general, in the sense of corresponding to arbitrary initial conditions of the particle–pointer system and, consequently, to arbitrary solutions of the system's Schrödinger equation. As a second approach we obtained a particular exact solution to the system's time-dependent Schrödinger equation. This solution is given by a parameterized two-dimensional anisotropic Gaussian wave packet. We computed exactly an entanglement measure on this solution and investigated its behavior.

From the time evolution of the expectation values of the aforementioned family of observables we proved that for a finite-duration measurement the final shift in the mean position of the pointer is proportional to the average between the initial and the final mean positions of the particle. This constitutes a new, alternative proof of the shift-property for finite-duration measurements of the position of a particle obtained in [18] on the bases of Feynman's path integrals formalism. We also used the time-depending expectation values to determined the evolution of some entanglement indicators associated with the correlations between the particle and the pointer in configuration space (C_{xQ}), in momentum space (C_{pP}), and "mixed" position–momentum correlations (C_{xP} and C_{pQ}). Since we are solely considering pure global states of the particle–pointer system, these indicators adopt non-vanishing values only for entangled states of the system. We found that, when measuring the position x of a free particle these indicators behave as follows. The indicator C_{xQ} is given by a polynomial of second degree on the duration T of the measurement process. On the other hand, the indicators C_{xP} and C_{pQ} are linear functions of T, the former being a linear homogeneous function of T while the latter involving an inhomogeneous term equal to $\frac{G}{2}[\langle xp + px\rangle_0 - 2\langle x\rangle_0\langle x\rangle_0]$. The indicator C_{pP}, giving the correlation between the particle's and the pointer's momenta, adopts a constant value independent of the duration of the measurement. In the limit $T \to 0$, corresponding to an instantaneous measurement, C_{xQ} tends to the limit value $G\Delta_{x0}^2$, proportional to the squared initial uncertainty of the particle's position (this limit value is also proportional to the coupling constant G characterizing the intensity of the particle–pointer interaction). This limit value of C_{xQ} occurs because in the instantaneous limit the behavior of the model here analyzed corresponds to an ideal position measurement, implying that in the limit $\Delta_{x0} \to 0$ the measurement does not generate entanglement between the particle and the pointer. On the other hand, for finite non-vanishing values of the duration T one observes that, in this important respect, the measuring process' behavior exhibits drastic departures from that of an ideal measurement. In fact, for very small values of Δ_{x0} the indicator C_{xQ} actually increases when Δ_{x0} decreases, showing that for highly localized initial states of the particle (corresponding to a well-defined initial position of the particle) a large amount of entanglement between the particle and the pointer is generated during the measurement. Similar results are obtained when measuring the position of a particle that moves in a harmonic potential, although in this case the entanglement indicators exhibit a more complicated dependence on the measurement duration. In particular, the limit $T \to 0$ yields the same results for the free particle and for the harmonic oscillator, because in this limit the behavior of the system is fully determined by the particle–pointer interaction term, and the contributions of the free Hamiltonians of the particle and the pointer become negligible.

Finally, we obtained a particular exact solution for the system's Schrödinger equation and investigated the evolution of the associated particle–pointer entanglement. The main features of the entanglement dynamics are observed to be consistent with the ones indicated by the evolution of the expectation values of the family (9) of observables. In particular, one can see in Fig. 3 that, for small values of M, there is a range of T-values for which the final particle–pointer entanglement decreases with the duration of the measurement process. We also see in Fig. 2 that there is an important aspect in which finite-duration measurements differ from instantaneous ones. For small values of the initial uncertainty Δ_x in the particle's position the final joint particle–pointer state is highly entangled. Indeed, its amount of entanglement (as measured by the linear entropy of the particle's marginal density matrix) actually approaches its maximum value as $\Delta_x \to 0$. This means that, for finite-time position measurements, highly localized initial states of the particle are not left unaffected by the measurement process, since the final marginal state of the particle is highly mixed. This implies that finite-duration position measurements differ in essential ways from ideal quantum position measurements. In contrast to what occurs in the finite-duration case, for ideal position measurements initial particle's states of arbitrary small position uncertainty should tend to be unaffected by the position measurement process. Our present results suggest that the finite duration T of a quantum measurement might impose universal limitations on the extent to which the measurement under consideration is an ideal measurement. Given the privileged role played by position measurements in Physics, these limitations may also affect the measurement of other observables besides position. We plan to address some of these issues in a future communication.

Acknowledgment

We would like to thank partial financial support by Conicyt-UCN, Grant PSD-065.

Appendix A

If the particle whose position is measured is under the effect of an harmonic potential, the solution to the equations of motion (10) for the relevant expectation values is,

$$\langle x\rangle(t) = \frac{\langle p\rangle_0}{m\omega}\sin(\omega t) - 2\frac{G}{T}\frac{\langle P\rangle_0}{m\omega^2}\sin^2\left(\frac{\omega t}{2}\right) + \langle x\rangle_0\cos(\omega t)$$

$$\begin{aligned}\langle x^2\rangle(t) \;=\; & \frac{1}{2}\left(\frac{\langle xp+px\rangle_0}{m\omega}\right)\sin(2\omega t) + \left(\frac{\langle p^2\rangle_0}{m^2\omega^2} - \left(\frac{G}{T}\right)^2\frac{\langle P^2\rangle_0}{m^2\omega^4}\right)\sin^2(\omega t)\\ & - 4\frac{G}{T}\left(\frac{\langle pP\rangle_0}{m^2\omega^3}\sin(\omega t) + \frac{\langle xP\rangle_0}{m\omega^2}\cos(\omega t) - \frac{G}{T}\frac{\langle P^2\rangle_0}{m^2\omega^4}\right)\sin^2\left(\frac{\omega t}{2}\right) + \langle x^2\rangle_0\cos^2(\omega t)\end{aligned}$$

$$\langle Q\rangle(t) = \frac{G}{T}\frac{\langle x\rangle_0}{\omega}\sin(\omega t) + \frac{\langle P\rangle_0}{M}t + 2\frac{G}{T}\frac{\langle p\rangle_0}{m\omega^2}\sin^2\left(\frac{\omega t}{2}\right) + \left(\frac{G}{T}\right)^2\frac{\langle P\rangle_0}{m\omega^2}\left(\frac{\sin(\omega t)}{\omega t} - 1\right)t + \langle Q\rangle_0$$

$$\begin{aligned}\langle Q^2\rangle(t) = {} & \left(\frac{G}{T}\right)^4\frac{1}{m^2\omega^5}\left[\omega t^2 - 2t\sin(\omega t) - \frac{\cos(2\omega t)}{2\omega}\right]\langle P^2\rangle_0 \\ & - \left(\frac{G}{T}\right)^2\frac{1}{Mm\omega^2}\left[\frac{4}{\omega^2}\cos(\omega t) - \frac{(m+M)t}{m\omega}\sin(\omega t) - \frac{2}{\omega^2} + 2t^2\right]\langle P^2\rangle_0 \\ & + \frac{t^2}{M^2}\langle P^2\rangle_0 + \left(\frac{G}{T}\right)^3\frac{2}{m\omega^3}\left[\frac{\sin^2(\omega t)}{\omega} - t\sin(\omega t)\right]\langle xP\rangle_0 \\ & - \frac{G}{T}\frac{1}{M}\left[\frac{4}{\omega^2}\cos(\omega t) + \frac{(m+M)t\sin(\omega t)}{m\omega} - \frac{2}{\omega^2}\right]\langle xP\rangle_0 \\ & + \left(\frac{G}{T}\right)^3\frac{2}{m^2\omega^5}\left[\sin(\omega t) + \omega t\cos(\omega t) - \frac{\sin(\omega t)}{2} - \omega t\right]\langle pP\rangle_0 \\ & + \frac{G}{T}\frac{1}{m^2M\omega^3}\left[(M-m)\sin(\omega t) - (M+m)\omega t\cos(\omega t) + 2m\omega t\right]\langle pP\rangle_0 \\ & + \frac{2}{m^2\omega^4}\left(\frac{G}{T}\right)^2\left[\frac{1}{4}\cos(2\omega t) - \cos(\omega t)\right]\langle p^2\rangle_0 - \frac{1}{2\omega^2}\left(\frac{G}{T}\right)^2\cos(2\omega t)\langle x^2\rangle_0 \\ & + \frac{2}{m\omega^3}\left(\frac{G}{T}\right)^2\left[\sin(\omega t) - \frac{1}{4}\sin(2\omega t) - \frac{\omega t}{2}\right]\langle xp + px\rangle_0 \\ & + \frac{2}{m\omega^3}\left(\frac{G}{T}\right)^2\left[\omega t - \sin(\omega t)\right]\langle px + PQ\rangle_0 \\ & - \frac{2}{m\omega^2}\frac{G}{T}\cos(\omega t)\langle pQ\rangle_0 + \frac{2}{\omega}\frac{G}{T}\sin(\omega t)\langle xQ\rangle_0 + \frac{t}{M}\langle QP + PQ\rangle_0 + \langle Q^2\rangle_0\end{aligned}$$

$$\langle p\rangle(t) = -\left(m\omega\langle x\rangle_0 + \frac{G}{T}\frac{\langle P\rangle_0}{\omega}\right)\sin(\omega t) + \langle p\rangle_0\cos(\omega t)$$

$$\begin{aligned}\langle p^2\rangle(t) = {} & \left((m\omega)^2\langle x^2\rangle_0 + 2\frac{G}{T}m\langle xP\rangle_0 + \left(\frac{G}{T}\right)^2\frac{\langle P^2\rangle_0}{\omega^2}\right)\sin^2(\omega t) \\ & - \left(\frac{m\omega}{2}\langle xp + px\rangle_0 + \frac{G}{T}\frac{\langle pP\rangle_0}{\omega}\right)\sin(2\omega t) + \langle p^2\rangle_0\cos^2(\omega t)\end{aligned}$$

$$\begin{aligned}\langle xp + px\rangle(t) = {} & \left(\frac{\langle p^2\rangle_0}{m\omega} - m\omega\langle x^2\rangle_0 - 2\frac{\langle xP\rangle_0}{\omega}\frac{G}{T} - \left(\frac{G}{T}\right)^2\frac{\langle P^2\rangle_0}{m\omega^3}\right)\sin(2\omega t) \\ & - 2\frac{G}{T}\frac{\langle pP\rangle_0}{m\omega^2}\cos(\omega t) + 2\left(\frac{G}{T}\frac{\langle xP\rangle_0}{\omega} + \left(\frac{G}{T}\right)^2\frac{\langle P^2\rangle_0}{m\omega^3}\right)\sin(\omega t) \\ & + \left(2\frac{G}{T}\frac{\langle pP\rangle_0}{m\omega^2} + \langle xp + px\rangle_0\right)\cos(2\omega t)\end{aligned}$$

$$\begin{aligned}\langle QP + PQ\rangle(t) = {} & 2\frac{G}{T}\left(\frac{\langle xP\rangle_0}{\omega}\sin(\omega t) + 2\frac{\langle pP\rangle_0}{m\omega^2}\sin^2\left(\frac{\omega t}{2}\right) + \frac{G}{T}\frac{\langle P^2\rangle_0}{m\omega^3}(\sin(\omega t) - \omega t)\right) \\ & + 2\frac{\langle P^2\rangle_0}{M}t + \langle QP + PQ\rangle_0\end{aligned}$$

$$
\begin{aligned}
\langle xQ\rangle(t) \;=\; & 2\left(\frac{G}{T}\right)^3 \frac{\langle P^2\rangle_0}{m^2\omega^5}\sin^2(\omega t/2)(\omega t-\sin(\omega t)) \\
& +\left(\frac{G}{T}\right)^2 \frac{\langle xP\rangle_0}{m\omega^3}(\sin(\omega t)(2\cos(\omega t)-1)-\omega t\cos(\omega t)) \\
& +\left(\frac{G}{T}\right)^2 \frac{\langle pP\rangle_0}{m^2\omega^4}(-\omega t\sin(\omega t)+4\cos(\omega t)\sin^2(\omega t/2)) \\
& +2\frac{G}{T}\frac{\langle p^2\rangle_0}{m^2\omega^3}\sin(\omega t)\sin^2(\omega t/2) \\
& +\frac{1}{2}\frac{G}{T}\frac{\langle x^2\rangle_0}{\omega}\sin(2\omega t)+\frac{1}{2}\left(\frac{G}{T}\frac{\langle P^2\rangle_0}{m^2M\omega^3}+\frac{\langle xP\rangle_0}{mM\omega}\right)((m-M)\sin(\omega t) \\
& +(m+M)\cos(\omega t)\omega t)+2\frac{G}{T}\left(\frac{\langle xp+px\rangle_0}{m\omega^2}\cos(\omega t)+\frac{\langle px-PQ\rangle_0}{m\omega^2}\right) \\
& \times\sin^2(\omega t/2)+\frac{1}{2}\frac{\langle pP\rangle_0}{m^2M\omega}(M+m)t\sin(\omega t) \\
& -\frac{G}{T}\frac{\langle P^2\rangle_0}{mM\omega^2}t+\frac{\langle pQ\rangle_0}{m\omega}\sin(\omega t)+\langle xQ\rangle_0\cos(\omega t)
\end{aligned}
$$

$$
\langle xP\rangle(t)=\frac{\langle pP\rangle_0}{m\omega}\sin(\omega t)-2\frac{G}{T}\frac{\langle P^2\rangle_0}{m\omega^2}\sin^2(\omega t/2)+\langle xP\rangle_0\cos(\omega t)
$$

$$
\begin{aligned}
\langle pQ\rangle(t) \;=\; & -1\left(\frac{G}{T}\right)^3\frac{\langle P^2\rangle_0}{m\omega^4}(\sin(\omega t)-\omega t)\sin(\omega t)+\left(\frac{G}{T}\right)^2 \\
& \times\left(\frac{\langle pP\rangle_0}{m\omega^3}\cos(\omega t)-\frac{\langle xP\rangle_0}{\omega^2}\sin(\omega t)\right)(2\sin(\omega t)-\omega t) \\
& -\left(\frac{G}{T}\right)^2\frac{\langle pP\rangle_0}{m\omega^3}\sin(\omega t)+\frac{G}{T}\left(\frac{\langle px-PQ\rangle_0}{\omega}\right. \\
& \left.-m\langle x^2\rangle_0\sin(\omega t)-\frac{1}{2}\frac{M+m}{mM}\frac{\langle P^2\rangle_0}{\omega}t\right)\sin(\omega t) \\
& +2\frac{G}{T}\left(\frac{\langle p^2\rangle_0}{m\omega^2}\cos(\omega t)-\frac{\langle xp+px\rangle_0}{\omega}\sin(\omega t)\right)\sin^2(\omega t/2) \\
& -\frac{1}{2}\frac{m+M}{M}\langle xP\rangle_0\sin(\omega t)\omega t\frac{1}{2}\frac{\langle pP\rangle_0}{mM\omega}((M+m)\omega t\cos(\omega t) \\
& +(M-m)\sin(\omega t))-\langle xQ\rangle_0 m\omega\sin(\omega t)+\langle pQ\rangle_0\cos(\omega t)
\end{aligned}
$$

$$
\langle pP\rangle(t)=-m\omega\langle xP\rangle_0\sin(\omega t)-\frac{G}{T}\frac{\langle P^2\rangle_0}{\omega}\sin(\omega t)+\langle pP\rangle_0\cos(\omega t)
$$

$$
\begin{aligned}
\langle px-PQ\rangle(t) \;=\; & -\frac{\langle P^2\rangle_0}{M}t+\left(\frac{G}{T}\right)^2\frac{\langle P^2\rangle}{m\omega^2}t-\left(\langle xp+px\rangle_0+2\frac{G}{T}\frac{\langle pP\rangle_0}{m\omega^2}\right)\sin^2(\omega t) \\
& +\left(\frac{1}{2}\frac{\langle p^2\rangle_0}{m\omega}-1\frac{G}{T}\frac{\langle xP\rangle_0}{\omega}-\frac{1}{2}\left(\frac{G}{T}\right)^2\frac{\langle P^2\rangle}{m\omega^3}\right. \\
& \left.-\frac{1}{2}m\omega\langle x^2\rangle_0\right)\sin(2\omega t)+\langle px-PQ\rangle_0.
\end{aligned}
\tag{38}
$$

Putting $T=t$ yields the relevant expectation values at the end of the measurement. Setting $\omega=0$ in (38) one obtains the time dependent expectation values corresponding to the measurement of the position of a free particle.

Appendix B

The non-negativity of the quantity defined in Eq. (21) follows from the Cauchy–Schwarz inequality. We have,

$$\begin{aligned}\Big(\langle xQ\rangle - \langle x\rangle\langle Q\rangle\Big)^2 &= \left[\int (x-\langle x\rangle)(Q-\langle Q\rangle)|\Psi(x,Q)|^2 dxdQ\right]^2 \\ &\le \left[\int (x-\langle x\rangle)^2|\Psi(x,Q)|^2 dxdQ\right]\left[\int (Q-\langle Q\rangle)^2|\Psi(x,Q)|^2 dxdQ\right] \\ &= \Delta_x^2\Delta_Q^2, \end{aligned} \tag{39}$$

from which follows that $\mathcal{D}_{xQ}$ in Eq. (21) is non-negative.

References

[1] I. Bengtsson, K. Zyczkowski, Geometry of Quantum States an Introduction to Quantum Entanglement, Cambridge University Press, 2008.
[2] G. Jaeger, Entanglement, Information, and the Interpretation of Quantum Mechanics, Springer, 2007.
[3] J.A. Wheeler, W.H. Zurek (Eds.), Quantum Theory and Measurement, Princeton University Press, 1983.
[4] V.B. Braginsky, F.Y. Khalili, Quantum Measurement, Cambridge University Press, 1995.
[5] W.H. Zurek, Phys. Today 44 (1991) 36.
[6] G.B. Roston, M. Casas, A. Plastino, A.R. Plastino, European J. Phys. 26 (2005) 657.
[7] M. Jammer, Concepts of Mass in Contemporary Physics and Philosophy, Princeton University Press, 2000.
[8] J.S. Bell, Speakable and Unspeakable in Quantum Mechanics, second ed., Cambridge University Press, 2004.
[9] P.R. Holland, The Quantum Theory of Motion: An Account of the de Broglie–Bohm Causal Interpretation of Quantum Mechanics, Cambridge University Press, 1995.
[10] A. Valentini, H. Westman, Proc. R. Soc. Lond. Ser. A 461 (2005) 253.
[11] G. Naaman-Marom, N. Erez, L. Vaidman, Ann. Phys. 327 (2012) 2522.
[12] G.C. Ghirardi, A. Rimini, T. Weber, Phys. Rev. D 34 (1986) 470.
[13] A.B. Nassar, Ann. Phys. 331 (2013) 317.
[14] A.B. Nassar, S. Miret-Artés, Phys. Rev. Lett. 111 (2013) 150401.
[15] B.R. Frieden, B.H. Soffer, Phys. Rev. E 52 (1995) 2274.
[16] A. Caticha, J. Phys. A 44 (2011) 225303.
[17] J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton University Press, Princeton, 1955.
[18] F.S. Roig, Phys. Rev. A 73 (2006) 042106.
[19] C.M. Caves, G.J. Milburn, Phys. Rev. A 36 (1987) 5543.
[20] W.H. Zurek, Phys. Rev. A 76 (2007) 052110.
[21] W.H. Zurek, Phys. Rev. A 87 (2013) 052111.
[22] W.K. Wootters, W.H. Zurek, Nature 299 (1982) 802.
[23] G.C. Ghirardi, Entanglement, Nonlocality, Superluminal Signaling, and Cloning, 2013, arXiv:1305.2305.
[24] It is now recognized that the no-cloning theorem was first proved by G.C. Ghirardi in 1981. Interesting historical remarcks concerning this theorem, and the relevant references, can be found in [23].
[25] S. Luo, Phys. Lett. A 374 (2010) 1350.
[26] C. Zander, A.R. Plastino, Europhys. Lett. 86 (2009) 18004.
[27] M.M. Taddei, B.M. Escher, L. Davidovich, R.L. de Matos Filho, Phys. Rev. Lett. 110 (2013) 050402.
[28] L. Landau, R. Peierls, Z. Phys. 69 (1931) 56.
[29] J. Naudts, T. Verhulst, Phys. Rev. A 75 (2007) 062104.
[30] C. Zander, A.R. Plastino, A. Plastino, M. Casas, J. Phys. A 40 (2007) 2861.
[31] N.L. Harshman, W.F. Flynn, Quantum Inf. Comput. 11 (2011) 278.
[32] A. Borras, A.P. Majtey, A.R. Plastino, M. Casas, A. Plastino, Phys. Rev. A 79 (2009) 022112.
[33] A. Borras, A.P. Majtey, A.R. Plastino, M. Casas, A. Plastino, Phys. Rev. A 79 (2009) 022108.
[34] H.G. Laguna, R.P. Sagar, Phys. Rev. A 84 (2011) 012502.
[35] R.G. Nazmitdinov, A.V. Chizhov, Opt. Spectra 112 (2012) 319.
[36] P. Koscik, A. Okopinska, Few-Body Syst. 54 (2013) 1637–1640.
[37] M. Srednicki, Phys. Rev. Lett. 71 (1993) 666.
[38] R. Weder, Phys. Rev. A 84 (2011) 062320.
[39] M. Kuna, J. Naudts, Rep. Math. Phys. 65 (2010) 77–108.