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NEW COMPLEXITY RESULTS ON ROMAN {2}-DOMINATION

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Abstract. The study of a variant of Roman domination was initiated by Chellali *et al.* [*Discrete Appl. Math.* **204** (2016) 22–28]. Given a graph G with vertex set V, a Roman {2}-dominating function $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex u adjacent to v with $f(u) = 2$, or at least two vertices x, y adjacent to v with $f(x) = f(y) = 1$. The weight of a Roman {2}-dominating function is the value $f(V) = \sum_{v\in V} f(v)$. The minimum weight of a Roman {2}-dominating function is called the Roman {2}-domination number and is denoted by $\gamma_{\{R2\}}(G)$. In this work we find several NP-complete instances of the Roman {2}-domination problem: chordal graphs, bipartite planar graphs, chordal bipartite graphs, bipartite with maximum degree 3 graphs, among others. A result by Chellali *et al.* [*Discrete Appl. Math.* **204** (2016) 22–28] shows that $\gamma_{\{R2\}}(G)$ and the 2-rainbow domination number of G coincide when G is a tree, and thus, the linear time algorithm for k-rainbow domination due to Brešar *et al.* [*Taiwan J. Math.* **12** (2008) 213–225] can be followed to compute $\gamma_{\{R2\}}(G)$. In this work we develop an efficient algorithm that is independent of k-rainbow domination and computes the Roman {2}-domination number on a subclass of trees called caterpillars.

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1. Definitions and preliminaries

The notion of Roman {2}-domination was defined just a few years ago and is nowadays being widely studied. Roman {2}-domination (also called Italian domination) was introduced by Chellali *et al.* as a variant of Roman domination [7].

All graphs in this paper are undirected and simple. Let G be a graph, and let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Whenever it is clear from the context, we simply write V and E. For basic definitions not included here, we refer the reader to [5].

For a graph G, two vertices of V are *adjacent* in G if there is an edge of E between them. For $v \in V$, $N_G(v)$ denotes the set of all the vertices adjacent to v in G, and $N_G[v]$ denotes the *closed neighborhood* of v, *i.e.* $N_G(v)$ together with v. The *degree* of $v \in V$ is $d(v) = |N_G(v)|$. For $J \subseteq V$, with $N_G(J)$ we denote $\bigcup_{v\in J} N_G(v)$.

A *pendant* vertex is a vertex of degree one.

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Given a graph G and $S \subseteq V$, $G \setminus S$ denotes the *subgraph of G induced by $V \setminus S$*, *i.e.* the graph with vertex set $V \setminus S$ and such that two vertices of $V \setminus S$ are adjacent in $G \setminus S$ if and only if they are adjacent in G. In other words, with $G \setminus S$ we mean the *deletion* from G of the vertices in S.

Given two graphs G and H, the *union* of G and H is denoted by $G \cup H$ and refers to the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

The *1-clique-sum* of graphs G and H, $G \oplus H$, is formed from their disjoint union by identifying a vertex from G with a vertex from H.

A *path* is a connected graph whose vertices have all degree at most two. A path with n vertices is denoted by P_n.

A graph G is a *bipartite* graph if its vertex set can be partitioned into two sets B_1, B_2 of pairwise nonadjacent vertices.

A graph G is *chordal* if for every cycle of length at least four there is a chord, *i.e.* an edge not in the cycle whose endpoints lie in the cycle.

A bipartite graph G is *chordal bipartite* if for every cycle of length at least six there is a chord. Clearly, chordal bipartite graphs may not be chordal.

A *star* is a connected graph in which at most one vertex has degree greater than one. An n-star is a star with $n+1$ vertices.

A *tree* is a connected acyclic graph.

A graph G is a *caterpillar* if G is a tree in which the deletion of all the pendant vertices (the *leaves*) results in a path (the *spine* or *central path*).

Given a graph G, a *Roman dominating function* $f : V \to \{0, 1, 2\}$ has the property that every vertex $v \in V$ with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$ [8].

Given a graph G, a *Roman $\{2\}$-dominating function* $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex $u \in N_G(v)$ with $f(u) = 2$, or at least two vertices $x, y \in N_G(v)$ with $f(x) = f(y) = 1$ [7]. The *weight* of a Roman $\{2\}$-dominating function is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman $\{2\}$-dominating function is called the *Roman $\{2\}$-domination number* and is denoted by $\gamma_{\{R2\}}(G)$ (also $\gamma_I(G)$). Roman $\{2\}$-dominating functions and the Roman $\{2\}$-domination number are also called Italian functions and Italian domination number respectively. Since 2004, several papers have been published on this topic where some new variations were introduced: weak Roman domination [9], maximal Roman domination [1], mixed Roman domination [2], double Roman domination [3], among others.

A Roman $\{2\}$-dominating function f can be represented by a triple (V_0, V_1, V_2), where V_i is the subset of vertices v of G such that $f(v) = i$. Thus, we use the notation $f = (V_0, V_1, V_2)$.

Given a non-connected graph G, it is clear that a Roman $\{2\}$-dominating function of G is the union of Roman $\{2\}$-dominating functions of its connected components and even more, that the Roman $\{2\}$-domination number of G is the sum of the Roman $\{2\}$-domination numbers of its connected components.

In this work we will say that f is a $\gamma_{\{R2\}}(G)$-function when f is a Roman $\{2\}$-dominating function of G with minimum weight.

The decision problem associated with Roman $\{2\}$-domination, the ROMAN $\{2\}$-DOMINATION PROBLEM (R2D), can be stated as follows:

Instance: A graph G, $j \in \mathbb{N}$.
Question: Is there a Roman $\{2\}$-dominating function with weight at most j?.

The first NP-complete result for R2D is presented in [7], proving that R2D is NP-complete even for bipartite graphs by reducing the Exact-3-Cover problem. Other NP-complete results for R2D are shown in [12] (for star convex bipartite graphs, comb convex bipartite graphs and bisplit graphs) also by reducing the Exact-3-Cover problem, and in [13] for planar graphs by reducing the 3-Satisfiability problem. Linear algorithms for computing $\gamma_{\{R2\}}(G)$ are presented in [13] for chain graphs, threshold graphs and unicyclic graphs.

A celebrated result by Courcelle *et al.* states that each graph property that is expressible in MSOL_1 (resp. MSOL_2) can be solved in polynomial time for graphs with bounded treewidth (resp. cliquewidth) [8]. Note that this result is mainly of theoretical interest and does not lead to practical algorithms. Since the problem of finding a minimum Roman {2}-dominating function can be expressed in MSOL_1 [12], this motivates our search of efficient algorithms for classes of graphs with this property, in particular for trees.

The Roman {2}-domination number on trees is studied in [6] and [10], but not from an algorithmic point of view as our aim is. On the one hand, in [6] it is proved that $\gamma_{\{R2\}}(T) = \gamma_{r2}(T)$ for a tree T, where $\gamma_{r2}(T)$ denotes the 2-rainbow domination number of T, *i.e.* the minimum weight between all 2-rainbow dominating functions. For a positive integer k, a *k-rainbow dominating function* of G is a function f from $V(G)$ to the set of all subsets of $\{1, 2, \ldots, k\}$ such that for any vertex v with $f(v) = \emptyset$ we have $\bigcup_{N_G(v)} f(u) = \{1, 2, \ldots, k\}$. There is a linear time algorithm that finds the k-rainbow number of a given tree [6]. On the other hand and regarding bounds on trees, the following one is proved in [7] for any tree T: $\gamma_R(T) \leq \frac{4}{3}\gamma_{\{R2\}}(T)$, where $\gamma_R(T)$ denotes the Roman domination number of T.

This work is organized as follows. We start by showing in Section 2, a reduction of the classical domination problem to R2D. In this way we derive many new NP-complete graph classes for R2D. In Section 3, we show an efficient algorithm for a very sparse class of graphs, a subclass of trees called caterpillars. We conclude the paper with some final remarks in Section 4.

2. NP-complete results

We already know from [7, 12, 13] that R2D is NP-complete. The reductions in [7] (for bipartite graphs) and [12] (for star convex bipartite graphs, comb convex bipartite graphs and bisplit graphs) come in both cases from the Exact-3-Cover problem. In [13] the reduction comes from the 3-Satisfiability problem on planar graphs. In this section we present a simple proof that just reduces the classical domination problem, that not only allows us to give a unified alternative and simpler proof, but also an NP-complete proof of R2D for chordal graphs and chordal bipartite graphs. As a by-product, from the large list of NP-complete graph classes for the domination problem, we derive many NP-complete graph classes for R2D.

Theorem 2.1. *The Roman {2}-domination problem is NP-complete for general graphs.*

Proof. We will reduce the domination problem to the Roman {2}-domination problem. Given a graph G on n vertices, $V(G) = \{v_1, \ldots, v_n\}$, consider the graph G' with vertex set $V(G') = V(G) \cup \{w_1, \ldots, w_n\}$ and edge set $E(G') = E(G) \cup \{v_i w_i : i \in \{1, \ldots, n\}\}$. Namely, we add n leaves to G. We claim that G has a dominating set of cardinality at most s if and only if G' has a Roman {2}-dominating function of weight at most $s + n$.

Suppose G has a dominating set D of cardinality at most s. Consider the function f from $V(G')$ to $\{0, 1, 2\}$ defined by $f(u) = 1$ if $u \in D$, $f(u) = 0$ if $u \in V(G) \setminus D$, and $f(w_i) = 1$, for $i \in \{1, \ldots, n\}$.

Take $u \in V(G')$ with $f(u) = 0$. By the definition of f, $u \in V(G) \setminus D$ and thus $u = v_i$ for some $i \in \{1, \ldots, n\}$ and moreover, u has a neighbor $v \in D$ (since D is a dominating set in G). Since $f(v) = 1$ and $f(w_i) = 1$, we have $f(N_{G'}(u)) = 2$. Therefore, f is a Roman {2}-dominating function of G' with weight $|D| + n \leq s + n$.

On the other hand, suppose G' has a Roman {2}-dominating function f of weight at most $s + n$. For each $v_i \in V(G)$, we may assume that $|f(w_i)| = 1$ (if $f(w_i) = 2$, we turn $f(w_i)$ to 1 and add 1 to $f(v_i)$; if $f(w_i) = 0$, we turn $f(w_i)$ to 1 and subtract 1 from $f(v_i)$) to obtain a a Roman {2}-dominating function of weight at most $s + n$. Now, consider the set $D = \{v \in V(G) : f(v) \neq 0\}$.

For any vertex $v_i \in V(G) \setminus D$, we have $f(v_i) = 0$ and $f(N_{G'}(v_i)) = 2$. Since $|f(w_i)| = 1$, we have $f(u) \neq 0$ for some $u \in N_G(v)$ which implies $u \in D$. Therefore, D is a dominating set of G. It is straightforward from our assumption that the cardinality of D is at most the weight of f minus n, *i.e.* $s + n - n = s$. □

Corollary 2.2. *R2D is NP-complete on every graph class that is closed under adding pendant vertices and for which the dominating set problem is NP-complete. In particular, on chordal graphs, bipartite planar graphs, chordal bipartite graphs and bipartite with maximum degree 3 graphs.*

3. Roman {2}-domination on caterpillars

As trees have bounded treewidth and, as mentioned in the introduction, the result by Courcelle *et al.* is mainly of theoretical interest and does not lead to practical algorithms, in this section our aim is to find an efficient algorithm for a specific subclass of trees, namely caterpillars.

We will show that for caterpillars, Roman {2}-dominating sets are very particular, and give an efficient algorithm to compute the Roman {2}-domination number on them.

Recall that *caterpillar* is a tree where there is a path, called the *central path*, such that every vertex that is not in the path is adjacent to a vertex of the path. Notice that a caterpillar is connected.

It is clear that an induced subgraph of a caterpillar may be non-connected. Each of the connected components of a caterpillar can be a caterpillar or a path.

For a caterpillar G, a *father* is a vertex with at least 3 neighbors. Clearly, any father has two neighbors in the central path and at least one pendant neighbor (a leaf). The *children* of a father is the set of leaves it is adjacent to. Besides, we call F_1^G, F_2^G and $F_{>2}^G$ the subsets of the father set with exactly one child, exactly two and more than two children in G, respectively.

In the sequel for a caterpillar G, its central path has at least three vertices, then G has at least four vertices.

We start by proving a simple characterization of those caterpillars with Roman {2}-domination number equal to two.

Lemma 3.1. *Let G be a caterpillar. Then $\gamma_{\{R2\}}(G) = 2$ if and only if G is a star.*

Proof. Clearly, if G is a star then $\gamma_{\{R2\}}(G) = 2$.

Now let G be a caterpillar with $\gamma_{\{R2\}}(G) = 2$ and let u, v two distinct vertices of G. Then, there exist at most two different Roman {2}-dominating functions, let's say $f = (V \setminus \{u\}, \emptyset, \{u\})$ and $g = (V \setminus \{u, v\}, \{u, v\}, \emptyset)$. We will see that in fact g cannot exist. Since f is a Roman {2}-dominating function of G and $V \setminus \{u\}$ is a nonempty set, every vertex is adjacent to u in G. Then since G is a tree, thus triangle-free, no pair of vertices in $V \setminus \{u\}$ are pairwise adjacent. Thus G is a star.

In the second case, for g to be a Roman {2}-dominating function of G, it must happen that every vertex in $V \setminus \{u, v\}$ is adjacent to both u and v. But in this case G would be itself a P_3 or, otherwise, would have a 4-vertex cycle. Both situations lead to a contradiction. □

The following reduction is not difficult to prove:

Proposition 3.2. *There exists a linear time transformation that reduces R2D on a general caterpillar, to R2D on a caterpillar without fathers with more than two children.*

Proof. Let G be a caterpillar with $F_{>2}^G \neq \emptyset$ and H be the induced subgraph of G obtained by deleting all but two children of each vertex in $F_{>2}^G$.

Let $f = (V_0, V_1, V_2)$ be a $\gamma_{\{R2\}}(G)$-function. If $F_{>2}^G \subseteq V_2$ and thus all the children of vertices in $F_{>2}^G$ are in V_0, it turns out that the restriction of f to $V(H)$ is a Roman {2}-dominating function of H of the same weight. Otherwise, if there exists $x \in F_{>2}^G$ that doesn't belong to V_2 and thus all its children are in V_1, it turns out that the restriction of f to $V(H)$ is a Roman {2}-dominating function of H with weight not greater than the weight of f.

Now let $g = (V_0, V_1, V_2)$ be a $\gamma_{\{R2\}}(H)$-function and y be a vertex of $V(H)$ that belongs also to $F_{>2}^G$. Notice that y has only two children in H. If $y \in V_2$, then its two children in H are in V_0. By assigning 0 to the children of y in G that were deleted from G, we obtain a Roman {2}-dominating function of G of the same weight. Otherwise, if $y \notin V_2$, and then its two children in H are in V_1, by assigning 0 to every children of y in G, and 2 to y, we obtain a Roman {2}-dominating function of G with weight not greater than the weight of g.

□

Proposition 3.2 reduces our study to caterpillars G with $F_{>2}^G = \emptyset$. First, we have:

Lemma 3.3. *Let G be a caterpillar with $F_2^G \neq \emptyset$ and $F_{>2}^G = \emptyset$. Then there exists a $\gamma_{\{R2\}}(G)$-function (V_0, V_1, V_2) such that $F_2^G \subseteq V_2$ and $N_G(F_2^G) \setminus F_2^G \subseteq V_0$.*

Proof. Choose a $\gamma_{\{R2\}}(G)$-function $g = (V_0, V_1, V_2)$. If $F_2^G \cap (V_0 \cup V_1) \neq \emptyset$, for a father x in $F_2^G \cap (V_0 \cup V_1)$ it is clear from the definition of g that its two children belong to V_1. We can then turn to 2 the weight of x, to zero the weights of its two children, and eventually to zero the weight of a vertex $w \in \left(N_G(x) \setminus F_2^G\right) \cap (V_1 \cup V_2)$ if $\left(N_G(F_2^G) \setminus F_2^G\right) \cap (V_1 \cup V_2) \neq \emptyset$ and add at the same time the weight of w to its other neighbor in the central path. In this way we build a Roman {2}-dominating function with weight at most the weight of g, thus minimum.

If $F_2^G \cap (V_0 \cup V_1) = \emptyset$ but $\left(N_G(F_2^G) \setminus F_2^G\right) \cap (V_1 \cup V_2) \neq \emptyset$, take $w \in \left(N_G(x) \setminus F_2^G\right) \cap (V_1 \cup V_2)$ for some $x \in F_2^G$. Since g is minimum, it is clear that both x's children are in V_0. We can then add the weight of w to its other neighbor in the central path and turn to 0 the weight of w, building in this way another Roman {2}-dominating function with weight at most the weight of g, thus minimum. ☐

From Lemma 3.3 we can prove:

Proposition 3.4. *Let G be a caterpillar with $F_2^G \neq \emptyset$ and $F_{>2}^G = \emptyset$. If $G' := G \setminus \bigcup_{x \in F_2^G} N_G[x]$ then*

$$\gamma_{\{R2\}}(G) = \gamma_{\{R2\}}(G') + 2\left|F_2^G\right|,$$

Proof. We will proceed by induction on $\left|F_2^G\right|$.

- If $F_2^G = \{x\}$, then following Lemma 3.3 we can choose a $\gamma_{\{R2\}}(G)$-function $f = (V_0, V_1, V_2)$ such that $x \in V_2$ and $N_G(x) \subseteq V_0$. Let us denote $G' = G \setminus N_G[x]$. It is not difficult to see that the restriction of f to G' is a Roman {2}-dominating function of G'. Thus, $\gamma_{\{R2\}}(G') \leq \gamma_{\{R2\}}(G) - 2$. To prove the opposite inequality, consider a $\gamma_{\{R2\}}(G')$-function and extend it to $V(G)$ by assigning weight 2 to x and 0 to its four neighbors. It turns out that the function built in this way is a Roman {2}-dominating function of G with weight $\gamma_{\{R2\}}(G') + 2$, implying that $\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(G') + 2$.
- If $\left|F_2^G\right| \geq 2$, then choose $x \in F_2^G$. Again, let us denote $G' = G \setminus N_G[x]$.
  - If both neighbors of x in the central path do not belong to F_2^G, notice that $F_2^{G'} = F_2^G \setminus \{x\}$. The induction hypothesis holds for G', *i.e.* $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G'') + 2(|F_2^G| - 1)$, where $G'' := G' \setminus \bigcup_{y \in F_2^{G'}} N_{G'}[y] = G \setminus \bigcup_{y \in F_2^G} N_G[y]$.
    Take a $\gamma_{\{R2\}}(G')$-function (V_0, V_1, V_2). Then the function $f = (V_0 \cup N_G(x), V_1, V_2 \cup \{x\})$ is a Roman {2}-dominating function of G with weight $\gamma_{\{R2\}}(G') + 2$. Thus $\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(G') + 2$. The induction hypothesis implies $\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(G'') + 2\left(\left|F_2^G\right| - 1\right) + 2 = \gamma_{\{R2\}}\left(G \setminus \bigcup_{y \in F_2^G} N_G[y]\right) + 2|F_2^G|$.
  - If exactly one of the two neighbors of x in the central path, let's say w, belongs to F_2^G, notice that $F_2^{G'} = F_2^G \setminus \{x, w\}$ and that G' has two isolated vertices (the children w_1 and w_2 of w in G). The induction hypothesis holds for G', *i.e.* $\gamma_{\{R2\}}(G') = \gamma_{\{R2\}}(G'') + 2(|F_2^G| - 2)$, where $G'' := G' \setminus \bigcup_{y \in F_2^{G'}} N_{G'}[y] = (G \setminus \bigcup_{y \in F_2^G} N_G[y]) \cup 2K_1$ and $2K_1$ is the graph with no edges and two vertices (w_1 and w_2 in this case). Take a $\gamma_{\{R2\}}(G')$-function (V_0, V_1, V_2). Since w_1 and w_2 are isolated vertices in G', then $\{w_1, w_2\} \subseteq V_1$. Thus the function $f = (V_0 \cup N_G(x), V_1, V_2 \cup \{x\})$ is a Roman {2}-dominating function of G with weight $\gamma_{\{R2\}}(G') + 2$. Thus $\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(G') + 2$. In this case, the induction hypothesis valid for G' implies $\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(G'') + 2\left(\left|F_2^G\right| - 2\right) + 2 \leq \gamma_{\{R2\}}\left(\left(G \setminus \bigcup_{y \in F_2^G} N_G[y]\right) \cup 2K_1\right) + 2\left|F_2^G\right| - 2 = \gamma_{\{R2\}}\left(G \setminus \bigcup_{y \in F_2^G} N_G[y]\right) + 2 + 2\left|F_2^G\right| - 2$, and the desired inequality holds.
  - We omit the analysis for the case in which both neighbors of x in the central path belong to F_2^G since it follows a similar reasoning.

To prove the opposite inequality, we follow the reasoning of the base case: due to Lemma 3.3, we can choose a $\gamma_{\{R2\}}(G)$-function $f = (V_0, V_1, V_2)$ such that $F_2^G \subseteq V_2$ and $N_G(F_2^G) \setminus F_2^G \subseteq V_0$. The restriction of f to the subgraph $G \setminus \bigcup_{x \in F_2^G} N_G[x]$ is a Roman $\{2\}$-dominating function of $G \setminus \bigcup_{x \in F_2^G} N_G[x]$. Thus, $\gamma_{\{R2\}}\left(G \setminus \bigcup_{x \in F_2^G} N_G[x]\right) \leq \gamma_{\{R2\}}(G) - 2|F_2^G|$.

□

Now, Proposition 3.4 reduces even more our study. Proposition 3.7 below refers to special caterpillars G with $F_2^G = \emptyset$. In order to prove Proposition 3.7, we need to prove a simple fact valid for any graph.

Lemma 3.5. *Let G be a graph, $v \in V$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{\{R2\}}(G)$-function with $v \in V_0$. Then $\gamma_{\{R2\}}(G) \geq \sum \gamma_{\{R2\}}(G_k)$, where each G_k is a connected component of $G \setminus \{v\}$.*

Proof. For $u \in N_G(v) \cap V_0$ it happens that $f(N_G(u)) = f(N_G(u) \setminus \{v\}) = f(N_{G\setminus\{v\}}(u)) \geq 2$. Thus $(V_0 \setminus \{v\}, V_1, V_2)$ is a Roman $\{2\}$-dominating function of $G \setminus \{v\}$ with same weight as f. Thus, $\gamma_{\{R2\}}(G \setminus \{v\}) \leq \gamma_{\{R2\}}(G)$, and since $\gamma_{\{R2\}}(G \setminus \{v\}) = \sum \gamma_{\{R2\}}(G_k)$, the inequality follows. □

Remark 3.6. We need to remark the following facts concerning Roman $\{2\}$-domination in paths:

– For a path P_n with $n \geq 1$, it is known that $\gamma_{\{R2\}}(P_n) = \lceil \frac{n+1}{2} \rceil$ [7]. Thus it is clear that

$$\gamma_{\{R2\}}(P_{n+1}) = \begin{cases} \gamma_{\{R2\}}(P_n) + 1 & \text{if } n \text{ is odd} \\ \gamma_{\{R2\}}(P_n) & \text{if } n \text{ is even.} \end{cases}$$

– Denote by $P_n = u_1, u_2, \ldots, u_n$, for a path P_n with $n \geq 1$.
  - When n is even, then there exists a $\gamma_{\{R2\}}(P_n)$-function $f = (V_0, V_1, V_2)$ such that either $u_{n-1} \in V_1$ (and thus $u_n \in V_1$) or $u_{n-1} \in V_2$ (and thus $u_n \in V_0$).
  - When $n \geq 5$ is odd, then a $\gamma_{\{R2\}}(P_n)$-function is unique and satisfies $V_2 = \emptyset$ and $\{u_1, u_n\} \subset V_1$.

– The Roman $\{2\}$-domination number of the 1-clique sum of paths P_n and P_m with $n, m \geq 1$ is equal to $\lceil \frac{n+m}{2} \rceil$.

Now we can state and prove the following fact concerning caterpillars with a unique child. We consider the number 0 as odd and denote indistinctly by P_0, the empty graph or the path without vertices. In this case, we define $\gamma_{\{R2\}}(P_0) := 0$.

Proposition 3.7. *Let G be a caterpillar with $F_2^G = \emptyset$, $x \in F_1^G$ such that $G' := G \setminus N_G[x]$ is the union of two paths P_n and P_m, for non negative integers n and m. Then,*

(1) $\gamma_{\{R2\}}(G) = \gamma_{\{R2\}}(P_n) + \gamma_{\{R2\}}(P_m) + 1$ *for even n and m,*
(2) $\gamma_{\{R2\}}(G) = \gamma_{\{R2\}}(P_n) + \gamma_{\{R2\}}(P_m) + 2$ *otherwise.*

Proof. Let $P_n := u_1, u_2, \ldots, u_n$ and $P_m := v_m, v_{m-1}, \ldots, v_1$, where u_n and v_m are both at distance two from x in the central path. Also, let $u_{n+1} \in N_G(u_n) \cap N_G(x)$, $v_{m+1} \in N_G(v_m) \cap N_G(x)$ and y be the only child of x, *i.e.* $N_G(x) = \{y, u_{n+1}, v_{n+1}\}$.

Take a $\gamma_{\{R2\}}(P_n \cup P_m)$-function (V_0, V_1, V_2). Clearly, $(V_0 \cup N_G(x), V_1, V_2 \cup \{x\})$ is a Roman $\{2\}$-dominating function of G, implying

$$\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(P_n) + \gamma_{\{R2\}}(P_m) + 2.$$

In particular, when n and m are both even, from Remark 3.6 we can assume that $\{u_{n-1}, v_{m-1}\} \subseteq V_1$, and thus $\{u_n, v_m\} \subseteq V_1$. Then $(V_0 \cup \{x, u_n, v_m\}, (V_1 \setminus \{u_n, v_m\}) \cup N_G(x), \emptyset)$ is a Roman $\{2\}$-dominating function of G, implying

$$\gamma_{\{R2\}}(G) \leq \gamma_{\{R2\}}(P_n) + \gamma_{\{R2\}}(P_m) + 1.$$

To see the reverse inequalities, let $g = (V_0, V_1, V_2)$ be a $\gamma_{\{R2\}}(G)$-function and consider all the possible cases for x.

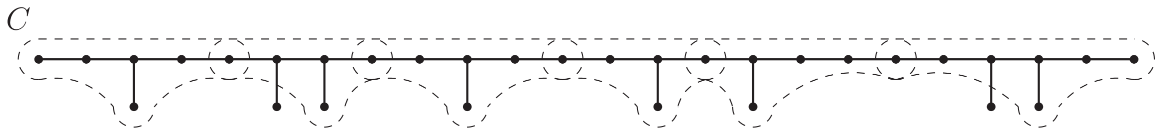

FIGURE 1. A decomposition for a caterpillar C with $F_2^C = F_{>2}^C = \emptyset$.

– If $x \in V_2$, since g is minimum we can assume w.l.o.g. that $N_G(x) \subset V_0$. Thus, $(V_0 \setminus N_G(x), V_1, V_2 \setminus \{x\})$ is a Roman {2}-dominating function of $P_n \cup P_m$. Therefore, $\gamma_{\{R2\}}(P_n) + \gamma_{\{R2\}}(P_m)$ is at most $\gamma_{\{R2\}}(G) - 2$.
– If $x \in V_1$ then $y \in V_1$. We can then move the weight from y to x to obtain another Roman {2}-dominating function of G with weight $\gamma_{\{R2\}}(G)$ and follow the reasoning of the previous case.
– If $x \in V_0$, then $y \in V_1$ and from Lemma 3.5, $\gamma_{\{R2\}}(G) \geq \gamma_{\{R2\}}(P_{n+1}) + \gamma_{\{R2\}}(P_{m+1}) + 1$.
Following Remark 3.6 we have

$$\gamma_{\{R2\}}(P_n) + \gamma_{\{R2\}}(P_m) \leq \begin{cases} \gamma_{\{R2\}}(G) - 1 & \text{if } n \text{ and } m \text{are even} \\ \gamma_{\{R2\}}(G) - 2 & \text{if } n - m \text{ is odd} \\ \gamma_{\{R2\}}(G) - 3 & \text{if } n \text{ and } m \text{are odd,} \end{cases}$$

implying

$$\gamma_{\{R2\}}(P_n) + \gamma_{\{R2\}}(P_m) \leq \begin{cases} \gamma_{\{R2\}}(G) - 1 & \text{if } n \text{ and } m \text{ are even} \\ \gamma_{\{R2\}}(G) - 2 & \text{in any other case.} \end{cases}$$

The result follows. □

Corollary 3.8. *Let G and H be two caterpillars with $F_2^G = F_2^H = \emptyset$, $x \in F_1^G$ such that $G' := G \setminus N_G[x]$ is the union of two paths P_n and P_m, $y \in F_1^H$ such that $H' := H \setminus N_H[y]$ is the union of two paths P_r and P_s, for non negative integers n, m, r and s. Then, for the 1-clique of G and H obtained by identifying the last vertex of P_m with the first vertex of P_r we have:*

(1) $\gamma_{\{R2\}}(G \oplus H) = \gamma_{\{R2\}}(G) + \gamma_{\{R2\}}(H) - 2$ *when both, n and s are even, and m and r have distinct parity,*
(2) $\gamma_{\{R2\}}(G \oplus H) = \gamma_{\{R2\}}(G) + \gamma_{\{R2\}}(H) - 1$, *otherwise.*

In all, for a given general caterpillar, from the results in this section we can restrict its Roman {2}-domination study to a caterpillar subgraph C with $F_2^C = F_{>2}^C = \emptyset$. Clearly, C is the 1-clique sum of a certain number of caterpillars as those in Proposition 3.7, and some isolated vertices. Consider such a decomposition with minimum number of isolated vertices (see Fig. 1). Now Proposition 3.7, Corollary 3.8 and Lemma 3.5 derive into an efficient algorithm that computes the Roman {2}-domination number of the given caterpillar. Thus we can state:

Theorem 3.9. *For any caterpillar, the Roman {2}-domination number can be obtained efficiently.*

For the graph C in Figure 1, $\gamma_{\{R2\}}(C) = 20$.

4. Final remarks

A future line of work is to continue studying Roman {2}-domination on subclasses of trees, for instance in lobsters which generalize caterpillars.

On the other hand, the following result appears in [7] (Prop. 8). For every graph G, there exists a $\gamma_{\{R2\}}(G)$-function $f = (V_0, V_1, V_2)$ such that either $V_2 = \emptyset$ or every vertex of V_2 has at least three private neighbors in V_0 with respect to the set $V_1 \cup V_2$. A vertex u is said to be a *private neighbor* of v with respect to D if $v \notin D$ and $N_G(u) \cap D = \{v\}$.

We notice that there is a mistake in the mentioned result, as the following counterexample shows: Consider a graph on 5 vertices not a P_5 consisting in a P_4 together with a pendant vertex. The Roman {2}-domination number for this graph is 3, but the thesis of Proposition 8 in [7] does not hold for this graph. In fact, the only minimum Roman $\{2\}$-dominating function for it assigns the value 2 to the vertex of degree three, 0 to its three neighbors and 1 to the remaining pendant vertex. The vertex of degree 3 has then only 2 private neighbors with respect to $V_1 \cup V_2$.

We think that a correct restatement of Proposition 8 in [7] is the following: For every graph G, there exists a $\gamma_{\{R2\}}(G)$-function $f = (V_0, V_1, V_2)$ such that either $V_2 = \emptyset$ or every vertex of V_2 has at least three private neighbors in V_0 with respect to the set V_2. We hope that this result would help in making a breakthrough in the study of Roman $\{2\}$-domination on lobsters and also on other subclasses of trees, or in trees in general.

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