Robust Estimates in Balanced Norms for Singularly Perturbed Reaction Diffusion Equations Using Graded Meshes

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Robust estimates in balanced norms for singularly perturbed reaction diffusion equations using graded meshes

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Abstract The goal of this paper is to provide almost robust approximations of singularly perturbed reaction-diffusion equations in two dimensions by using finite elements on graded meshes. When the mesh grading parameter is appropriately chosen, we obtain quasioptimal error estimations in a balanced norm for piecewise bilinear elements, by using a weighted variational formulation of the problem introduced by N. Madden and M. Stynes, Calcolo 58(2) 2021. We also prove a supercloseness result, namely, that the difference between the finite element solution and the Lagrange interpolation of the exact solution, in the weighted balanced norm, is of higher order than the error itself. We finish the work with numerical examples which show the good performance of our approach.

Keywords reaction diffusion problems · singularly perturbed problems · balanced norms · graded meshes · supercloseness

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1 Introduction

The reaction-diffusion equations arise in many applications, indeed, these equations appear naturally in systems consisting of many interacting components and

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are used to describe pattern-formation phenomena of biological, chemical and physical systems (see, for example, [8, 18, 19]).

It is well known that, when the singular perturbation parameter is very small, the solution of the problem presents boundary layers which downgrade the approximability of the solution when uniform or quasi-uniform meshes are used. The approximation by finite element methods of these singularly perturbed problems have been extensively studied (see, for instance, [21, 13, 10, 14] and its references) where uniform error estimates where analyzed for different norms, including the energy and L^∞ ones.

It turns out that the natural energy norm associated to the problem is not balanced, i.e, when the singular perturbation parameter tends to zero, the energy norm of the layer contribution vanishes while the energy norm of the smooth part of the solution does not. Balanced norms were introduced to reflect the behavior of layers more accurately in the finite element method for singularly perturbed reaction–diffusion problems. This is extensively discussed in [12] where a new bilinear form and a finite element method were designed to facilitates the analysis for a new balanced norm. Subsequently new analysis were performed in several articles, in particular [1, 2, 7, 17].

Therefore, the problem requires especially designed schemes for its effective numerical solution. In a recent work, N. Madden and M. Stynes [16] introduced a weighted balanced norm (whose H^1 component is scaled to the correct size) and obtained an robust almost first-order error bound for piecewise bilinears on the unit square by using Shishkin meshes.

In this paper we consider the bilinear formulation and the weighted balanced norm introduced in [16], and obtain a robust approximation of singularly perturbed reaction-diffusion equation, with homogeneous Dirichlet boundary conditions, in two dimensions by using piecewise bilinear elements on graded meshes. We present quasi-optimal error estimates when appropriate graded meshes are used, in addition we also obtain a supercloseness result for the balanced norm, i.e., we prove that, under suitable hypothesis, the difference between the approximate solution and the Lagrange interpolation of the exact solution is of higher order than the error itself. In particular, to obtain the supercloseness result we need to prove some properties over the weight function which characterize the discrete formulation and also we need to prove some estimations over the derivatives of the solution.

In [4] graded meshes were also used, with bilinear finite elements, to obtain robust and almost optimal error estimates in the energy norm for a reaction diffusion problem similar to the one we consider here. In that work the grading parameter (and therefore the meshes) could be taken independently of the singular perturbation parameter of the equation. Adjusting the grading parameter, but still being independent of the singular perturbation, supercloseness results in the energy norm were obtained in [6]. In the present paper, to obtain almost uniform results in the balanced norm, we use meshes of the same type to those introduced in [4] but with a grading depending on the singular perturbation parameter (see Section 3).

Although the numerical results obtained with Shishkin meshes and graded meshes are similar, graded meshes satisfy some desirable properties. In fact, when one is approximating a singularly perturbed problem with an a priori adapted mesh, it is natural to expect that a mesh designed for some value of the perturbation parameter works well also for larger values of it (we include a numerical test of this performance). This is the case for graded meshes as it is mentioned in [5,6]. This fact could be an important property in problems where the diffusion parameter is not constant or, also, to treat systems of equations in which different equations have singular perturbations of different orders.

The paper is organized as follows. In Section 2 we present the reaction diffusion problem, the weighted formulation and the weighted balanced norm under consideration. In Section 3 we introduce the graded meshes and we present some interpolation properties in standard Sobolev norms and in Section 4 we obtain interpolation error estimates on the weighted balanced norm. Section 5 is devoted to the supercloseness results. In Section 6 we present some numerical examples which show the good performance of our method. We finish the paper with an Appendix which includes a technical Lemma used along the paper.

Throughout the paper, the letter C will denote a generic positive constant, not necessarily the same at each occurrence, which is independent of the singular perturbation parameter ε and the mesh size.

2 Problem Statement

Let Ω be a bounded domain on $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We consider the following reaction-diffusion problem

$$\begin{aligned} -\varepsilon^2 \Delta u + b(x,y)u = f(x,y) \qquad (x,y) \in \Omega \\ u = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $0 < \varepsilon < 1$ and $b \in L^\infty(\Omega)$, with $0 < b_0^2 < b(x,y) < b_1^2$ for almost all $(x,y) \in \Omega$.

In a recent paper, Madden and Stynes [16] propose a new variational formulation of this problem as follows. Let

$$\beta(x,y) = 1 + \frac{1}{\varepsilon} e^{-\frac{\gamma d(x,y)}{\varepsilon}}$$

be a weighting function, with γ a fixed positive parameter and $d(x,y)$ the distance to the boundary $\partial\Omega$. It is appropriate to mention that, also this weight function is basically the same used in Adler et al. [2], but there the authors rewrite the reaction-diffusion problem as a system of equations. The property (see [16])

$$|\nabla\beta(x,y)| \leq \frac{C}{\varepsilon}\beta(x,y)$$

almost everywhere in $(x,y) \in \Omega$ will be used along the manuscript.

We consider the weighted norm

$$|||v|||_\beta = \left(\varepsilon^2 \|\nabla v\|_\beta^2 + \|v\|_\beta^2\right)^{\frac{1}{2}}$$

where $\|v\|_\beta = (\beta v, v)^{\frac{1}{2}}$. We use the notation $|||\cdot|||_{\beta,D}, \|\cdot\|_{\beta,D}$ to denote the β-weighted norms on the subdomain D. The domain subscript is dropped for the case $D = \Omega$.

Defining the weighted bilinear form $B_\beta : H_0^1(\Omega)^2 \to \mathbb{R}$ by

$$B_\beta(v,w) = \varepsilon^2 \int_\Omega \nabla v \cdot \nabla(\beta w)\, dx\, dy + \int_\Omega b(x) v\, (\beta w)\, dx\, dy.$$

Then, the variational formulation of problem (1) is given by: find $u \in H_0^1(\Omega)$ such that

$$B_\beta(u,v) = \int_\Omega f(x)(\beta v)\, dx\, dy \qquad \forall v \in H_0^1(\Omega).$$

Remark 1 The β-norm $|||\cdot|||_\beta$ is balanced, indeed, its components $\varepsilon^2\|\nabla u\|_\beta$ and $\|u\|_\beta^2$ are both $O(1)$ for a typical solution u of (1) (see [16] for more details).

If $V_h \subseteq H_0^1(\Omega)$ is a finite element space, we define the finite element formulation: find $u_h \in V_h$ such that

$$B_\beta(u_h,v) = \int_\Omega f(x)(\beta v)\, dx\, dy \qquad \forall v \in V_h. \tag{2}$$

Following [16] we assume $0 < \gamma \leq b_0$. In this case, the bilinear form $B_\beta(\cdot,\cdot)$ is coercive and continuous, and by using Lax-Milgram Theorem and Céa Lemma, the following approximation error estimate holds (see [16, Section 3]):

$$|||u - u_h|||_\beta \leq C \inf_{w_h \in V_h} |||u - w_h|||_\beta. \tag{3}$$

It follows that in order to estimate the error in the balanced norm $|||\cdot|||_\beta$ is enough to compare u with some interpolant Πu of u.

3 Graded meshes and preliminary results

Let $\Omega = (0,1)^2$. Let us introduce a family of meshes in the following way. We consider two parameters, $h > 0$ which is related with the mesh size (see Remark 2), and the grading parameter α given by

$$\alpha := 1 - \frac{1}{2\log\frac{1}{\varepsilon}}.$$

Let $x_0, x_1, \ldots, x_{mid}$ the grid points on the interval $[0, \frac{1}{2}]$ given by

$$\begin{cases} x_0 = 0, \\ x_1 = h^s, \qquad \text{with } s := \dfrac{1}{1-\alpha} \\ x_{i+1} = x_i + h x_i^\alpha, \quad i = 2, \ldots, mid-1, \\ x_{mid} = \dfrac{1}{2}. \end{cases} \tag{4}$$

This partition is extended to a grid $\{x_0, x_1, \ldots, x_{mid}, \ldots, x_M\}$ with $M = 2mid$ of $[0,1]$ by setting $x_i = 1 - x_{M-i}$ for $i = mid+1, \ldots, M$. We consider a 2-dimensional mesh $\mathcal{T}_h = \{R\}$ of tensor product type of $\Omega = (0,1)^2$, composed by rectangles $R = R_{ij}$ defined by

$$R_{ij} = (x_{i-1}, x_i) \times (x_{j-1}, x_j).$$

Set $h_k = x_k - x_{k-1}$. Then the lengths of the sides of R_{ij} are h_i and h_j. We will use repeatedly along this paper the following property for the meshes $\mathcal{T}_h$: For $R_{ij} \in \mathcal{T}_h$ with $1 < i < M$ we have

$$h_i \le h \min\{x, 1-x\}^\alpha \qquad \forall (x,y) \in R_{ij}.$$

Similarly, for $R_{ij} \in \mathcal{T}_h$ with $1 < j < M$ we have

$$h_j \le h \min\{y, 1-y\}^\alpha \qquad \forall (x,y) \in R_{ij}.$$

Remark 2 The number $M+1$ of grid points along the x and y axis is related with the parameter h which define the mesh $\mathcal{T}_h$ by

$$h \le C \frac{1}{M} \log \frac{1}{\varepsilon} \log M.$$

(see [4, proof of Corollary 4.5]). Hence, we see that h is bounded almost uniformly with respect to ε and similarly to the case of quasi-uniform meshes except for the logarithmic factor $\log M$. In what follows, for simplicity, we write the error estimates in terms of h, but they can be traduced in terms of the number of degrees of freedom using this relationship.

Given a generic rectangle R with edges of lengths h_x and h_y, let $\mathcal{Q}_1 : H^2(R) \to H^1(R)$ be the classical interpolation operator on R. We know the error estimates (see [3, Th. 2.7])

$$\|v - \mathcal{Q}_1 v\|_{0,R} \le C\{h_x^2 \|\partial_x^2 v\|_{0,R} + h_y^2 \|\partial_y^2 v\|_{0,R}\} \tag{5}$$

and

$$\begin{aligned} \|\partial_x(v - \mathcal{Q}_1 v)\|_{0,R} &\le C\{h_x \|\partial_x^2 v\|_{0,R} + h_y \|\partial_x \partial_y v\|_{0,R}\}, \\ \|\partial_y(v - \mathcal{Q}_1 v)\|_{0,R} &\le C\{h_x \|\partial_x \partial_y v\|_{0,R} + h_y \|\partial_y^2 v\|_{0,R}\}. \end{aligned} \tag{6}$$

We also have the following results that will be useful later on.

Lemma 1 *Let $R = (a,b) \times (c,d)$ be a rectangle with sides of lengths $h_x = b - a$ and $h_y = d - c$. Then we have*

$$\|\nabla(\mathcal{Q}_1 f)\|_{\infty,R} \le 2\sqrt{2} \|\nabla f\|_{\infty,R}$$

for all $f \in \mathcal{C}^1(\overline{R})$.

Proof Let $A = (a,c)$, $B = (b,c)$, $C = (b,d)$ and $D = (a,d)$, and the Lagrange bilinear bases functions

$$\lambda_A(x,y) = \frac{(x-b)(y-d)}{h_x h_y}, \qquad \lambda_B(x,y) = -\frac{(x-a)(y-d)}{h_x h_y},$$
$$\lambda_C(x,y) = \frac{(x-a)(y-c)}{h_x h_y}, \qquad \lambda_D(x,y) = -\frac{(x-b)(y-c)}{h_x h_y}.$$

Then

$$\mathcal{Q}_1 f = f(A)\lambda_A + f(B)\lambda_B + f(C)\lambda_C + f(D)\lambda_D$$

and

$$\partial_x(\mathcal{Q}_1 f)(x,y) = \frac{f(A) - f(B)}{h_x} \frac{y-d}{h_y} + \frac{f(C) - f(D)}{h_x} \frac{y - y_1}{h_y}.$$

Then, by the Mean Value Theorem we have that there exist $x_{m_1}, x_{m_2} \in (a,b)$ such that

$$\partial_x (\mathcal{Q}_1 f)(x,y) = \partial_x f(x_{m_1}, c)\frac{y-d}{h_y} + \partial_x f(x_{m_2}, d)\frac{y-c}{h_y}.$$

Since, for $(x,y) \in R$ it holds $|y-c|, |y-d| \leq h_y$ then it results

$$|\partial_x (\mathcal{Q}_1 f)(x,y)| \leq |\partial_x f(x_{m_1}, c)| + |\partial_x f(x_{m_2}, d)| \leq 2\|\nabla f\|_{\infty,R}.$$

A similar estimate hold for $|\partial_y (\mathcal{Q}_1 f)(x,y)|$ and then the proof concludes. □

Lemma 2 *Let $R = (a,b) \times (c,d)$ be a rectangle with sides of lengths $h_x = b-a$ and $h_y = d-c$. If $0 < \alpha \leq 1$ then, for any $v \in H^2(R)$, we have*

$$\|\partial_x(v - \mathcal{Q}_1 v)\|_{0,R} \leq C\{h_x^{1-\alpha}\|(x-a)^\alpha \partial_x^2 v\|_{0,R} + h_y\|\partial_x\partial_y v\|_{0,R}\}, \qquad (7)$$

$$\|\partial_y(v - \mathcal{Q}_1 v)\|_{0,R} \leq C\{h_x\|\partial_x\partial_y v\|_{0,R} + h_y^{1-\alpha}\|(y-c)^\alpha \partial_y^2 v\|_{0,R}\} \qquad (8)$$

Proof Let $\hat{R} = (0,1)^2$ and $\hat{\mathcal{Q}}_1 : H^2(\hat{R}) \to H^1(\hat{R})$ be the bilinear interpolation operator. For a function $v \in H^2(\hat{R})$ define

$$\Pi v(x,y) = v(0,y)(1-x) + xv(1,y), \qquad (x,y) \in \hat{R}.$$

Note that $\Pi v(\cdot,y)$ is the linear interpolation of $v(\cdot,y)$ for each $y \in [0,1]$. Then we know that for smooth functions v we have (see [15, Corollary 1.2.3])

$$\|\partial_x[v(\cdot,y) - \Pi v(\cdot,y)]\|_{0,(0,1)} \leq C\|x^\alpha \partial_x^2 v(\cdot,y)\|_{0,(0,1)} \qquad \forall y \in (0,1)$$

and therefore

$$\begin{aligned}\|\partial_x(v - \Pi v)\|_{0,\hat{R}}^2 &= \int_0^1 \|\partial_x[v(\cdot,y) - \Pi v(\cdot,y)]\|_{0,(0,1)}^2\, dy \\ &\leq C\int_0^1 \|x^\alpha \partial_x^2 v(\cdot,y)\|_{0,(0,1)}^2\, dy = C\|x^\alpha \partial_x^2 v\|_{0,\hat{R}}^2.\end{aligned}$$

Now,

$$\begin{aligned}v - \hat{\mathcal{Q}}_1 v &= (v - \Pi v) + (\Pi v - \hat{\mathcal{Q}}_1 v) \\ &= (v - \Pi v) + [\Pi v - \hat{\mathcal{Q}}_1(\Pi v)]\end{aligned}$$

since $\hat{\mathcal{Q}}_1 v = \hat{\mathcal{Q}}_1(\Pi v)$. Then

$$\begin{aligned}\|\partial_x(v - \hat{\mathcal{Q}}_1 v)\|_{0,\hat{R}} &\leq C\|x^\alpha \partial_x^2 v\|_{0,\hat{R}} + \|\partial_x[\Pi v - \hat{\mathcal{Q}}_1(\Pi v)]\|_{0,\hat{R}} \\ &\leq C\{\|x^\alpha \partial_x^2 v\|_{0,\hat{R}} + \|\partial_x^2 \Pi v\|_{0,\hat{R}} + \|\partial_x\partial_y \Pi v\|_{0,\hat{R}}\}\end{aligned} \qquad (9)$$

where we used the estimate (6) for $\hat{\mathcal{Q}}_1$. But

$$\partial_x^2 \Pi v = 0$$

and, since

$$\partial_x \Pi v(x,y) = v(1,y) - v(0,y)$$

it follows, for smooth functions v, that

$$\partial_y \partial_x \Pi v(x,y) = \partial_y (v(1,y) - v(0,y)) = \int_0^1 \partial_y \partial_x v(t,y)\, dt.$$

Then

$$\begin{aligned}
\|\partial_x \partial_y \Pi v\|_{0,\hat{R}}^2 &= \int_0^1 \int_0^1 \left| \int_0^1 \partial_x \partial_y v(t,y)\, dt \right|^2 dy\, dx \\
&\leq \int_0^1 \int_0^1 \int_0^1 |\partial_x \partial_y v(t,y)|^2 \, dt\, dy\, dx \\
&= \|\partial_x \partial_y v\|_{0,\hat{R}}^2.
\end{aligned}$$

From (9) we obtain

$$\|\partial_x (v - \hat{\mathcal{Q}}_1 v)\|_{0,\hat{R}} \leq C\{\|x^\alpha \partial_x^2 v\|_{0,\hat{R}} + \|\partial_x \partial_y v\|_{0,\hat{R}}\}.$$

By a density argument, the previous inequality holds for all $v \in H^2(\hat{R})$. Then, the inequality (7) is obtained by a simple rescaling argument. Inequality (8) follows analogously, and then the proof concludes. □

We will denote the global continuous piecewise bilinear interpolation operator $H^2(\Omega) \to H^1(\Omega)$ also by $\mathcal{Q}_1$.

4 Estimates on graded meshes

In this Section we obtain interpolation error estimates in the β-norm with graded meshes. We will assume the compatibility conditions (see [10] and the references therein)

$$f(0,0) = f(1,0) = f(1,1) = f(0,1) = 0$$

which ensure that the solution u belong to $\mathcal{C}^4(\Omega) \cap \mathcal{C}^2(\overline{\Omega})$. Such compatibility conditions are necessary for the following pointwise estimates, which for $k \leq 2$ are proved in [10, Lemmata 3.1, 3.3 and 3.5] and for $k = 3, 4$ were stated in [11, Lemma 4.1].

Lemma 3 *We have that, for $(x,y) \in \Omega$ and $0 \leq k \leq 4$,*

$$\begin{aligned}
\left|\partial_x^k u(x,y)\right| &\leq C\{1 + \frac{1}{\varepsilon^k} e^{-b_0 \frac{x}{\varepsilon}} + \frac{1}{\varepsilon^k} e^{-b_0 \frac{1-x}{\varepsilon}}\}, \\
\left|\partial_y^k u(x,y)\right| &\leq C\{1 + \frac{1}{\varepsilon^k} e^{-b_0 \frac{y}{\varepsilon}} + \frac{1}{\varepsilon^k} e^{-b_0 \frac{1-y}{\varepsilon}}\}.
\end{aligned}$$

Note that, with $k = 0$ we obtain that the solution u is uniformly bounded on the domain Ω.

In our analysis we make the following reasonable Assumption.

Assumption 1 *Assume that $h < e^{-\frac{3}{2}}$ and $\varepsilon < h$, as otherwise the subsequent analysis can be carried out using standard techniques.*

First, we consider the L^2-part of the β-norm of the interpolation error.

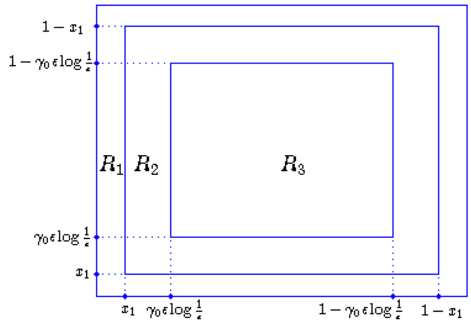

Fig. 1 Decomposition of Ω for the proof of Proposition 1

Proposition 1 *Let u be the solution of (1) and $\mathcal{Q}_1 u$ be the piecewise bilinear interpolation of u on the mesh $\mathcal{T}_h$. Then, under Assumption 1, we have that there exists a constant C such that*

$$\|u-\mathcal{Q}_1 u\|_\beta \le Ch^2\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}.$$

Proof Let us define

$$R_1 = \{[(0,x_1)\cup(1-x_1,1)]\times(0,1)\}\cup\{(0,1)\times[(0,x_1)\cup(1-x_1,1)]\},$$

$$R_2 = \left\{\left[(x_1,\gamma_0\varepsilon\log\frac{1}{\varepsilon})\cup(1-\gamma_0\varepsilon\log\frac{1}{\varepsilon},1-x_1)\right]\times(x_1,1-x_1)\right\}$$
$$\bigcup\left\{(x_1,1-x_1)\times\left[(x_1,\gamma_0\varepsilon\log\frac{1}{\varepsilon})\cup(1-\gamma_0\varepsilon\log\frac{1}{\varepsilon},1-x_1)\right]\right\},$$

$$R_3 = (\gamma_0\varepsilon\log\frac{1}{\varepsilon},1-\gamma_0\varepsilon\log\frac{1}{\varepsilon})^2,$$

where γ_0 is taken greater than or equal to $\max\left\{\frac{2}{b_0},\frac{1}{\gamma}\right\}$ and such that $\gamma_0\varepsilon\log\frac{1}{\varepsilon}$ and $1-\gamma_0\varepsilon\log\frac{1}{\varepsilon}$ are grid points. Then, $\Omega = R_1\cup R_2\cup R_3$ (see Figure 1).

Let $S_1 = (0,x_1)\times(0,1)$. Note that, since $1-\alpha = \frac{1}{-2\log\varepsilon}$ and $\varepsilon = h^{\frac{\log\varepsilon}{\log h}}$, we have

$$\frac{x_1}{\varepsilon} = \frac{h^{-2\log\varepsilon}}{\varepsilon} = h^{\frac{\log\varepsilon}{\log h}[2(-\log h)-1]}. \tag{10}$$

then, by Assumption 1, since $h<e^{-\frac{3}{2}}$ and $\varepsilon<h$, it follows

$$\frac{\log\varepsilon}{\log h}[2(-\log h)-1] > 2(-\log h)-1 > 2$$

and therefore

$$\frac{x_1}{\varepsilon} \leq h^2. \tag{11}$$

Then, since u is uniformly bounded, we get

$$\begin{aligned}\|u - \mathcal{Q}_1 u\|^2_{\beta,S_1} &= \int_0^{x_1}\int_0^1 \beta(u-\mathcal{Q}_1 u)^2\,dy\,dx \\ &\leq Cx_1\varepsilon^{-1} \leq h^2.\end{aligned}$$

Then clearly, by symmetry arguments, we obtain

$$\|u - \mathcal{Q}_1 u\|^2_{\beta,R_1} \leq Ch^2. \tag{12}$$

Let now $S_2 = (x_1, \gamma_0\varepsilon\log\frac{1}{\varepsilon}) \times (x_1, 1-x_1)$. Using anisotropic interpolation error estimate [3, Th. 2.7] and taking into account that $\beta \leq C\varepsilon^{-1}$, we have

$$\begin{aligned}\|u - \mathcal{Q}_1 u\|^2_{\beta,S_2} &\leq C\varepsilon^{-1}\|u - \mathcal{Q}_1 u\|^2_{0,S_2} \\ &\leq C\varepsilon^{-1}\sum_{R_{ij}\subset S_2}\left(h_i^4\|\partial_x^2 u\|^2_{0,R_{ij}} + h_j^4\|\partial_y^2 u\|^2_{0,R_{ij}}\right).\end{aligned}$$

For $R_{ij} \subset S_2$ we have $h_i \leq Chx^\alpha$, $h_j \leq Ch\min(y, 1-y)^\alpha$ for all $(x,y) \in R_{ij}$. Using also that $h_i, h_j \leq h$ and the a priori estimates of Lemma 3 we obtain

$$\begin{aligned}&\|u - \mathcal{Q}_1 u\|^2_{\beta,S_2} \leq Ch^4\varepsilon^{-1}\times \\ &\int_{S_2}\left(1 + x^{4\alpha}\varepsilon^{-4}e^{-2b_0\frac{x}{\varepsilon}} + y^{4\alpha}\varepsilon^{-4}e^{-2b_0\frac{y}{\varepsilon}} + (1-y)^4\varepsilon^{-4}e^{-2b_0\frac{1-y}{\varepsilon}}\right)dx\,dy.\end{aligned} \tag{13}$$

Now, taking into account that $|S_2| \leq \gamma_0\varepsilon\log\frac{1}{\varepsilon}$ we get

$$Ch^4\varepsilon^{-1}\int_{S_2} dx\,dy \leq Ch^4\log\frac{1}{\varepsilon}.$$

On the other hand, we have

$$\begin{aligned}&Ch^4\varepsilon^{-1}\int_{S_2}\left(x^{4\alpha}\varepsilon^{-4}e^{-2b_0\frac{x}{\varepsilon}} + y^{4\alpha}\varepsilon^{-4}e^{-2b_0\frac{y}{\varepsilon}}\right)dx\,dy = \\ &\quad Ch^4(1-2x_1)\varepsilon^{4(\alpha-1)}\int_{x_1}^{\gamma_0\varepsilon\log\frac{1}{\varepsilon}}(x\varepsilon^{-1})^{4\alpha}e^{-2b_0x/\varepsilon}\frac{dx}{\varepsilon} \\ &\qquad + Ch^4(\gamma_0\varepsilon\log\frac{1}{\varepsilon} - x_1)\varepsilon^{4(\alpha-1)}\int_{x_1}^{1-x_1}(y\varepsilon^{-1})^{4\alpha}e^{-2b_0y/\varepsilon}\frac{dy}{\varepsilon} \\ &\quad \leq Ch^4.\end{aligned}$$

where we used that $\varepsilon\log\frac{1}{\varepsilon} \leq C$, $\varepsilon^{4(\alpha-1)} = e^2$ and that for $\delta \in [0,4]$ the integrals $\int_0^\infty x^\delta e^{-2b_0x}\,dx$ are uniformly bounded. Similarly, we have

$$Ch^4\varepsilon^{-1}\int_{S_2}(1-y)^4\varepsilon^{-4}e^{-2b_0\frac{1-y}{\varepsilon}}\,dx\,dy \leq Ch^4.$$

Then from (13) we get

$$\|u - \mathcal{Q}_1 u\|_{\beta,S_2} \le Ch^2 \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}}.$$

Now, with similar arguments we obtain

$$\|u - \mathcal{Q}_1 u\|_{\beta,R_2} \le Ch^2 \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}}. \tag{14}$$

Finally, since $\gamma_0 \ge \frac{2}{b_0}$, it follows from Lemma 3 that

$$|\partial_x^2 u| + |\partial_y^2 u| \le C \qquad \text{on } R_3.$$

Similarly, $\beta \le C$ on R_3 since $\gamma_0 > \frac{1}{\gamma}$. Then, using again the anisotropic interpolation error estimates for the operator $\mathcal{Q}_1$ and that $h_i, h_j \le h$ for all i, j, we easily obtain

$$\|u - \mathcal{Q}_1 u\|_{\beta,R_3} \le Ch^2. \tag{15}$$

Since $\Omega = R_1 \cup R_2 \cup R_3$, from (12), (14) and (15) we get the desired result. □

Also, we can prove the following result involving the H^1-seminorm.

Proposition 2 *Let u be the solution of (1) and $\mathcal{Q}_1 u$ be the piecewise bilinear interpolation of u on the mesh $\mathcal{T}_h$. Then, under Assumption 1, we have*

$$\|\nabla(u - \mathcal{Q}_1 u)\|_0 \le C\varepsilon^{-\frac{1}{2}} h.$$

Proof Let us estimate $\|\nabla(u-\mathcal{Q}_1 u)\|_{0,\Omega_s}$ where $\Omega_s = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$. Then estimate on the rest of the domain follows by symmetry. Let us introduce the notation

$$\Omega_i = \cup_{j=1}^{mid} R_{ij}, \qquad \Omega^j = \cup_{i=1}^{mid} R_{ij}.$$

Using inequalities (6) and (7) on each element R_{ij} we have

$$\begin{aligned}
\|\partial_x(u-\mathcal{Q}_1 u)\|_{0,\Omega_s}^2 &\le h_1^{2-2\alpha}\|x^\alpha \partial_x^2 u\|_{0,\Omega_1}^2 + \sum_{i=2}^{mid} h_i^2 \|\partial_x^2 u\|_{0,\Omega_i}^2 + \sum_{j=1}^{mid} h_j^2 \|\partial_x \partial_y u\|_{0,\Omega^j}^2 \\
&\le h_1^{2-2\alpha}\|x^\alpha \partial_x^2 u\|_{0,\Omega_1}^2 + \sum_{i=2}^{mid} h^2 \|x^\alpha \partial_x^2 u\|_{0,\Omega_i}^2 + \sum_{j=1}^{mid} h_j^2 \|\partial_x \partial_y u\|_{0,\Omega^j}^2
\end{aligned}$$

where, for the second line, we used that

$$h_i \le hx^\alpha \qquad \forall (x,y) \in R_{ij}, 2 \le i.$$

Since $h_1 = h^s$ with $s = \frac{1}{1-\alpha}$, we have that $h_1^{2-2\alpha} = h^2$. On the other hand we have

$$h_j \le hy^\alpha \qquad \forall (x,y) \in R_{ij},\ 2 \le j.$$

Then we have

$$\|\partial_x(u - \mathcal{Q}_1 u)\|_{0,\Omega_s}^2 \le h^2\|x^\alpha \partial_x^2 u\|_{0,\Omega_s}^2 + h^2\|y^\alpha \partial_x \partial_y u\|_{0,\Omega_s \setminus \Omega^1}^2 + h_1^2 \|\partial_x \partial_y u\|_{0,\Omega^1}^2. \tag{16}$$

We need to bound each term in the last inequality. By integration by parts twice, Lemma 3 and using that

$$\partial_y u = 0 \quad \text{on } x = 0 \text{ and } x = 1, \qquad \partial_x^2 u = 0 \quad \text{on } y = 0$$

we have

$$\begin{aligned}
\|\partial_x \partial_y u\|_{0,\Omega^1}^2 &\le \|\partial_x \partial_y u\|_{0,[0,1]\times[0,x_1]}^2 \\
&= \int_0^{x_1} \partial_x \partial_y u \, \partial_y u|_0^1 \, dy - \int_0^1 \int_0^{x_1} \partial_y u \, \partial_y \partial_x^2 u \, dy \, dx \\
&= \int_0^1 \int_0^{x_1} \partial_y^2 u \, \partial_x^2 u \, dy \, dx - \int_0^1 \partial_y u \, \partial_x^2 u|_0^{x_1} \, dx \\
&\le \frac{x_1}{\varepsilon^2} \int_0^1 \left(1 + \frac{1}{\varepsilon^2} e^{-b_0 \frac{x}{\varepsilon}}\right) dx + \frac{1}{\varepsilon} \int_0^1 \left(1 + \frac{1}{\varepsilon^2} e^{-b_0 \frac{x}{\varepsilon}}\right) dx \\
&\le C\varepsilon^{-3}.
\end{aligned}$$

By Assumption 1 we have, in particular, that $\varepsilon, h < e^{-1}$, thus

$$h_1^2 = h^{2s} = h^{4 \log \frac{1}{\varepsilon}} = h^{2 \log \frac{1}{\varepsilon}} h^{2 \log \frac{1}{\varepsilon}} \le h^2 \varepsilon^{2 \log \frac{1}{h}} = h^2 \varepsilon^2,$$

therefore

$$h_1^2 \|\partial_x \partial_y u\|_{0,\Omega^1}^2 \le C h^2 \varepsilon^{-1}. \tag{17}$$

On the other hand, using again the estimates of Lemma 3 we have

$$\begin{aligned}
\|x^\alpha \partial_x^2 u\|_{0,\Omega_s}^2 &\le C \int_0^{\frac{1}{2}} x^{2\alpha} \left(1 + \frac{1}{\varepsilon^4} e^{-2b_0 \frac{x}{\varepsilon}}\right) dx \\
&\le C + C\varepsilon^{2\alpha - 3} \int_0^{\frac{1}{2}} \left(\frac{x}{\varepsilon}\right)^{2\alpha} e^{-2b_0 \frac{x}{\varepsilon}} \frac{dx}{\varepsilon} \\
&\le C\varepsilon^{-1},
\end{aligned}$$

and so

$$h^2 \|x^\alpha \partial_x^2 u\|_{0,\Omega}^2 \le C h^2 \varepsilon^{-1}. \tag{18}$$

Finally, with the same arguments, see [4, ineq. (4.31)] for a similar computation, we have

$$
\begin{aligned}
\|y^\alpha \partial_x\partial_y u\|^2_{0,\Omega_s\setminus\Omega^1} &\le \|y^\alpha \partial_x\partial_y u\|^2_{0,[0,1]\times[0,\frac{1}{2}]} \\
&= -\left(\frac{1}{2}\right)^{2\alpha}\int_0^1 \partial_y u\left(x,\frac{1}{2}\right)\partial_x^2 u\left(x,\frac{1}{2}\right)\,dx \\
&\quad + \int_0^1\int_0^{\frac{1}{2}} 2\alpha y^{2\alpha-1}\partial_y u\,\partial_x^2 u\,dy\,dx \\
&\quad + \int_0^1\int_0^{\frac{1}{2}} y^{2\alpha}\partial_y^2 u\partial_x^2 u\,dy\,dx \\
&\le C\int_0^1\left(1+\frac{1}{\varepsilon^2}e^{-b_0\frac{x}{\varepsilon}}\right)dx \\
&\quad + C\int_0^1 y^{2\alpha-1}\left(1+\frac{1}{\varepsilon}e^{-b_0\frac{y}{\varepsilon}}\right)dy\int_0^1\left(1+\frac{1}{\varepsilon^2}e^{-b_0\frac{x}{\varepsilon}}\right)dx \\
&\quad + C\int_0^1 y^{2\alpha}\left(1+\frac{1}{\varepsilon^2}e^{-b_0\frac{y}{\varepsilon}}\right)dy\int_0^1\left(1+\frac{1}{\varepsilon^2}e^{-b_0\frac{x}{\varepsilon}}\right)dx \\
&\le C\varepsilon^{-1},
\end{aligned}
$$

that is

$$h^2\|y^\alpha \partial_x\partial_y u\|^2_{0,\Omega_s\setminus\Omega^1} \le Ch^2\varepsilon^{-1}. \tag{19}$$

Now inserting (17)–(19) in (16) we obtain

$$\|\partial_x(u-\mathcal{Q}_1 u)\|_{0,\Omega_s} \le Ch\varepsilon^{-\frac{1}{2}},$$

and by symmetry it follows

$$\|\partial_x(u-\mathcal{Q}_1 u)\|_{0,\Omega} \le Ch\varepsilon^{-\frac{1}{2}}.$$

Clearly, the estimate

$$\|\partial_y(u-\mathcal{Q}_1 u)\|_{0,\Omega} \le Ch\varepsilon^{-\frac{1}{2}}$$

can be proved in a similar way concluding the proof. □

We remark that, from the definition of the β-norm $\|\cdot\|_\beta$ and Proposition 2, since $\beta \le C\frac{1}{\varepsilon}$, we get

$$\varepsilon\|\nabla(u-\mathcal{Q}_1 u)\|_{\beta,\Omega} \le C\varepsilon^{\frac{1}{2}}\|\nabla(u-\mathcal{Q}_1 u)\|_{0,\Omega} \le Ch, \tag{20}$$

which together with Proposition 1 allows us to obtain the main result of this Section.

Theorem 1 *Let u be the solution of* (1) *and $\mathcal{Q}_1 u$ be the piecewise bilinear interpolation of u on the mesh $\mathcal{T}_h$. Then, under Assumption 1, we have*

$$|||u-\mathcal{Q}_1 u|||_\beta \le Ch\left(1+h\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}\right).$$

Clearly as a consequence of Céa Lemma, equation (3) and this Theorem we have the corresponding error estimate for the finite element approximation u_h.

5 Supercloseness

In this section we prove that the $\beta-$norm of the difference between the interpolation of the exact solution u and the finite element approximation u_h is of higher order than the β-norm of the error $u - u_h$.

Let us denote by β_{min} and β_{max} the piecewise constant functions such that on each element $R \in \mathcal{T}_h$ hold

$$\beta_{min}|_R = \min_{(x,y)\in R} \beta(x,y), \qquad \beta_{max}|_R = \max_{(x,y)\in R} \beta(x,y).$$

Clearly β_{min} and β_{max} depend on the mesh $\mathcal{T}_h$ but this dependence is omitted for the sake of simplicity of the notation. The following Lemma presents an estimation of the relation of β_{min} and β_{max} inside the elements which is fundamental for our estimations.

Lemma 4 *There exists a positive constant η, independent of h and ε, such that on graded meshes $\mathcal{T}_h$, assuming $h < e^{-1}$, we get*

$$\frac{\beta_{max}}{\beta_{min}} \le C\varepsilon^{-\eta h} \qquad \text{on } \Omega.$$

Proof Due to the symmetry of the problem it is enough to estimate β_{max}/β_{min} for elements contained in $\Omega_s = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$. On elements R_{1j} or R_{i1} we have

$$\beta_{min} = 1 + \frac{1}{\varepsilon} e^{-\frac{\gamma}{\varepsilon} d_{max}} = 1 + \frac{1}{\varepsilon} e^{-\frac{\gamma}{\varepsilon} h^s},$$
$$\beta_{max} = 1 + \frac{1}{\varepsilon} e^{-\frac{\gamma}{\varepsilon} d_{min}} = 1 + \frac{1}{\varepsilon},$$

where naturally d_{max} and d_{min} represent the maximum and the minimum of the distance to the boundary. But

$$h^s = h^{2\log\frac{1}{\varepsilon}} = \varepsilon^{2\log\frac{1}{h}}$$

and so, since $h < \frac{1}{e}$,

$$-\frac{\gamma}{\varepsilon} h^s = -\gamma \varepsilon^{2\log\frac{1}{h} - 1} > -\gamma\varepsilon$$

and therefore

$$\beta_{min} > 1 + \frac{1}{\varepsilon} e^{-\gamma\varepsilon}.$$

Then we can conclude that

$$\frac{\beta_{max}}{\beta_{min}} < \frac{1 + \frac{1}{\varepsilon}}{1 + \frac{1}{\varepsilon} e^{-\gamma\varepsilon}} < C.$$

Now, we consider a rectangle R_{ij} with

$$i > 1 \qquad \text{and} \qquad x_j \le \gamma_0 \varepsilon \log\frac{1}{\varepsilon}$$

or

$$j > 1 \qquad \text{and} \qquad x_i \le \gamma_0 \varepsilon \log\frac{1}{\varepsilon}.$$

It can be checked that it is enough to consider a case as in the Figure 2, and we will use the notation of that Figure. We have

$$d_{min} = y_1, \qquad d_{max} = y_2,$$

and then $\beta_{min} = 1 + \frac{1}{\varepsilon} e^{-\gamma \frac{y_2}{\varepsilon}}$ and $\beta_{max} = 1 + \frac{1}{\varepsilon} e^{-\gamma \frac{y_1}{\varepsilon}}$. Then

$$\begin{aligned}
\frac{\beta_{max}}{\beta_{min}} &= \frac{1 + \frac{1}{\varepsilon} e^{-\gamma \frac{d_{min}}{\varepsilon}}}{1 + \frac{1}{\varepsilon} e^{-\gamma \frac{d_{max}}{\varepsilon}}} = 1 + \frac{\frac{1}{\varepsilon} e^{-\gamma \frac{d_{max}}{\varepsilon}}}{1 + \frac{1}{\varepsilon} e^{-\gamma \frac{d_{max}}{\varepsilon}}} \left(e^{-\frac{\gamma}{\varepsilon}(d_{min} - d_{max})} - 1 \right) \\
&= 1 + \frac{\frac{1}{\varepsilon} e^{-\gamma \frac{y_2}{\varepsilon}}}{1 + \frac{1}{\varepsilon} e^{-\gamma \frac{y_2}{\varepsilon}}} \left(e^{-\frac{\gamma}{\varepsilon}(y_1 - y_2)} - 1 \right) \le 1 + e^{\frac{\gamma}{\varepsilon}(y_2 - y_1)} - 1 \\
&= e^{\frac{\gamma}{\varepsilon} h y_1^\alpha}.
\end{aligned}$$

But, since $\varepsilon^{\frac{1}{\log \varepsilon}} = e$, we obtain

$$\begin{aligned}
h y_1^\alpha &\le h \left(\gamma_0 \varepsilon \log \frac{1}{\varepsilon} \right)^{1 - \frac{1}{2 \log \frac{1}{\varepsilon}}} = h \gamma_0 \varepsilon \log \frac{1}{\varepsilon} \left(\gamma_0 \varepsilon \log \frac{1}{\varepsilon} \right)^{\frac{1}{2 \log \varepsilon}} \\
&= h \varepsilon e^{\frac{1}{2}} \gamma_0^{1 + \frac{1}{2 \log \varepsilon}} \log \frac{1}{\varepsilon} \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2 \log \varepsilon}} \le C h \varepsilon \log \frac{1}{\varepsilon}
\end{aligned}$$

where we used that $\gamma_0^{1 + \frac{1}{2 \log \varepsilon}} \le C$ and $(\log \frac{1}{\varepsilon})^{\frac{1}{2 \log \frac{1}{\varepsilon}}} \le C$. Then we have

$$e^{\frac{\gamma}{\varepsilon} h y_1^\alpha} \le e^{C h \gamma \log \frac{1}{\varepsilon}} = \varepsilon^{-\gamma C h}.$$

Finally, it is clear that on rectangles R_{ij} with x_i or x_j greater than $\gamma_0 \varepsilon \log \frac{1}{\varepsilon}$ we have

$$\frac{\beta_{max}}{\beta_{min}} \sim 1$$

and this concludes the proof. □

For the meshes $\mathcal{T}_h$ we also introduce the piecewise constant function h_{min} which on each rectangle $R \in \mathcal{T}_h$ take the minimum of the lengths of the sides of R. Taking into account that the graph of the distance function d is a square pyramid with its apex on the point $(\frac{1}{2}, \frac{1}{2})$, it can be checked that, given an element $R \in \mathcal{T}_h$ and $(x, y) \in R$ there exists $(x_{int}, y_{int}) \in R$ such that

$$|\beta(x, y) - \beta_{min}| \le C h_{min} |\nabla \beta(x_{int}, y_{int})|. \tag{21}$$

From the coerciveness and the Galerkin orthogonality of the bilinear form $B_\beta(\cdot, \cdot)$ we get

$$\begin{aligned}
C |||u_h - \mathcal{Q}_1 u|||_\beta^2 &\le B_\beta(u_h - \mathcal{Q}_1 u, u_h - \mathcal{Q}_1 u) \\
&= B_\beta(u - \mathcal{Q}_1 u, u_h - \mathcal{Q}_1 u)
\end{aligned} \tag{22}$$

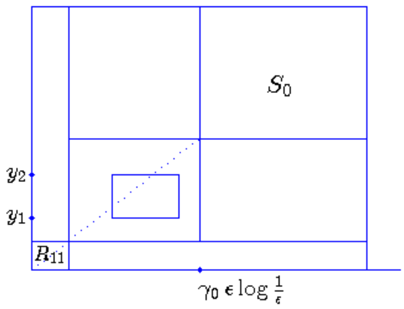

Fig. 2 Notation on Ω_s for the proof of Lemma 4

Now, for any $w \in V_h$ we have

$$\begin{aligned}
B_\beta(u-\mathcal{Q}_1 u, w) &= \int_\Omega \varepsilon^2 \nabla(u-\mathcal{Q}_1 u)\cdot\nabla(\beta w)\,dx + \int_\Omega b(x)(u-\mathcal{Q}_1 u)\,(\beta w)\,dx \\
&= \int_\Omega \varepsilon^2 \nabla(u-\mathcal{Q}_1 u)\beta\cdot\nabla(w)\,dx + \int_\Omega \varepsilon^2\nabla(u-\mathcal{Q}_1 u)\cdot\nabla(\beta)w\,dx + \\
&\qquad \int_\Omega b(x)(u-\mathcal{Q}_1 u)\,(\beta w)\,dx \\
&= \int_\Omega \varepsilon^2\nabla(u-\mathcal{Q}_1 u)(\beta-\beta_{min})\cdot\nabla w + \int_\Omega \varepsilon^2\beta_{min}\nabla(u-\mathcal{Q}_1 u)\cdot\nabla w + \\
&\qquad \int_\Omega \varepsilon^2\nabla(u-\mathcal{Q}_1 u)\cdot\nabla(\beta)w + \int_\Omega b(x)(u-\mathcal{Q}_1 u)\beta w \\
&=: I + II + III + IV
\end{aligned} \tag{23}$$

In the next subsections, we will prove the following estimates for I, II, III and IV assuming that $\varepsilon \le ch^3$, for some fixed constant $c \ge 1$,

$$|I|, |III| \le Ch^2\varepsilon^{-\eta h}\log\frac{1}{\varepsilon}|||w|||_\beta \quad \text{and} \quad |II|, |IV| \le Ch^2|||w|||_\beta.$$

Therefore,

$$|B_\beta(u-\mathcal{Q}_1 u, u_h-\mathcal{Q}_1 u)| \le Ch^2\varepsilon^{-\eta h}\log\frac{1}{\varepsilon}|||w|||_\beta,$$

which together with (22) will allow us to conclude the following supercloseness result.

Theorem 2 *There exist positive constants C and η, independent of ε and h, such that on graded meshes $\mathcal{T}_h$, assuming that $h < e^{-\frac{3}{2}}$ and $\varepsilon < ch^3$, we get*

$$|||u_h-\mathcal{Q}_1 u|||_\beta \le C\varepsilon^{-\eta h}\log(1/\varepsilon)^{\frac{1}{2}}h^2.$$

5.1 Estimation of term I

Let us estimate

$$I = \int_{\Omega} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u)(\beta - \beta_{min}) \cdot \nabla w.$$

Using property (21) and Lemma 4 we have

$$\begin{aligned}|\beta(x,y) - \beta_{min}| &\leq Ch_{min}\,|\nabla\beta(x_{int}, y_{int})| \leq Ch_{min}\varepsilon^{-1}|\beta(x_{int}, y_{int})| \\ &\leq Ch_{min}\varepsilon^{-1}\varepsilon^{-\eta h}|\beta(x,y)|.\end{aligned}$$

Let

$$S_0 = \left\{(x,y) \in \Omega : \min(x, y, 1-x, 1-y) \leq \gamma_0 \varepsilon \log \frac{1}{\varepsilon}\right\}.$$

For elements $R \subset S_0$ we have $h_{\min,R} \leq \gamma_0\, h\varepsilon \log \frac{1}{\varepsilon}$, and therefore

$$|\beta(x,y) - \beta_{min}| \leq Ch\varepsilon^{-\eta h}\beta(x,y) \log \frac{1}{\varepsilon}.$$

Then

$$\begin{aligned}&\left|\int_{S_0} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u)(\beta - \beta_{min}) \cdot \nabla w\right| \\ &\quad\leq Ch\varepsilon^{-\eta h} \log \frac{1}{\varepsilon} \left[\varepsilon \|\beta^{\frac{1}{2}} \nabla(u - \mathcal{Q}_1 u)\|_{0,S_0}\right] \left[\varepsilon \|\beta^{\frac{1}{2}} \nabla w\|_{0,S_0}\right] \\ &\quad\leq Ch^2 \varepsilon^{-\eta h} \log\left(\frac{1}{\varepsilon}\right) |||w|||_{\beta,S_0}\end{aligned}$$

where in the last inequality we used estimate (20).

Now, let $S_1 = \Omega \setminus S_0$. Since $\gamma_0 \geq \frac{2}{b_0}$ we have

$$\beta(x,y), |\nabla\beta(x,y)| \leq C \qquad \forall (x,y) \in S_1$$

and therefore it is easy to check that

$$|\beta(x,y) - \beta_{min}| \leq Ch \qquad (x,y) \in R, \quad R \subset S_1.$$

Also, since $\beta \geq 1$ we have, on $\|\cdot\|_{\beta,S_1} \sim \|\cdot\|_{0,S_1}$. Then, using again (20) we get

$$\begin{aligned}\left|\int_{S_1} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u)(\beta - \beta_{min}) \cdot \nabla w\right| &\leq Ch\varepsilon \|\nabla(u - \mathcal{Q}_1 u)\|_{0,S_1} \varepsilon \|\nabla w\|_{0,S_1} \\ &\leq Ch\varepsilon \|\nabla(u - \mathcal{Q}_1 u)\|_{\beta,S_1} \varepsilon \|\nabla w\|_{\beta,S_1} \\ &\leq Ch^2 |||w|||_{\beta,S_1}.\end{aligned}$$

Then finally we obtain

$$|I| \leq Ch^2 \varepsilon^{-\eta h} \log\left(\frac{1}{\varepsilon}\right) |||w|||_{\beta}. \tag{24}$$

5.2 Estimate of II

Now we consider
$$II = \int_{\Omega} \beta_{min} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot \nabla w.$$

Since β_{min} is piecewise constant we can use an argument due to Zlamal [23], as in [6, Lema 4.5], to obtain that for each element R_{ij} we have

$$\begin{aligned}
&\left| \varepsilon^2 \int_{R_{ij}} \beta_{min} \partial_x (u - \mathcal{Q}_1 u) \partial_x w \right| \\
&\leq C \varepsilon \beta_{min}^{\frac{1}{2}} \left\{ h_i^2 \|\partial_{xxx} u\|_{0,R_{ij}} + h_i h_j \|\partial_{xxy} u\|_{0,R_{ij}} + h_j^2 \|\partial_{xyy} u\|_{0,R_{ij}} \right\} \\
&\qquad \times \varepsilon \beta_{min}^{\frac{1}{2}} \|\partial_x w\|_{0,R_{ij}} \\
&\leq C \varepsilon \beta_{min}^{\frac{1}{2}} \left\{ h_i^2 \|\partial_{xxx} u\|_{0,R_{ij}} + h_i h_j \|\partial_{xxy} u\|_{0,R_{ij}} + h_j^2 \|\partial_{xyy} u\|_{0,R_{ij}} \right\} \\
&\qquad \times |||w|||_{\beta,R_{ij}}
\end{aligned}$$

In Lemma 6, in the Appendix, we prove that

$$\varepsilon \left[\sum_{i,j} \beta_{min} \left(h_i^2 \|\partial_x^3 u\|_{0,R_{ij}} + h_i h_j \|\partial_x^2 \partial_y u\|_{0,R_{ij}} + h_j^2 \|\partial_x \partial_y^2 u\|_{0,R_{ij}} \right)^2 \right]^{\frac{1}{2}} \leq C \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2$$

from which we can conclude that
$$II \leq C \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2 |||w|||_{\beta}. \tag{25}$$

5.3 Estimate of III

Now we deal with the estimate for III. Let $\Omega_s = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$. Then it is clear that due to symmetry arguments it is enough to estimate III_s which is defined as III but with the integral over Ω_s. We have

$$\begin{aligned}
III_s &= \int_{\Omega_s} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla \beta)\, w \\
&= \int_{D_1} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla \beta)\, w + \int_{D_2} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla \beta)\, w \\
&= III_1 + III_2
\end{aligned}$$

with
$$D_1 = \bigcup \{R : \nabla d \text{ is discontinuous on } R\}, \quad D_2 = \Omega \setminus D_1.$$

Let $S_A = \left[0, \gamma_0 \varepsilon \log \frac{1}{\varepsilon}\right]^2$ and $S_B = \left[\gamma_0 \varepsilon \log \frac{1}{\varepsilon}, \frac{1}{2}\right]^2$. Then
$$D_1 = (D_1 \cap S_A) \cup (D_1 \cap S_B)$$

and we can put

$$III_{1,A} := \int_{D_1 \cap S_A} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta)\, w,$$
$$III_{1,B} := \int_{D_1 \cap S_B} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta)\, w.$$

We will use that

$$|\nabla(u - \mathcal{Q}_1 u)| \le C\|\nabla u\|_\infty \le C\varepsilon^{-1}$$

which follows from Lemma 1 and from the a priori estimates of Lemma 3. Then using that $|\nabla\beta| \le C\beta/\varepsilon$ we have

$$\begin{aligned}
|III_{1,A}| &= \left| \int_{D_1 \cap S_A} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta)\, w \right| \le C \int_{D_1 \cap S_A} \beta |w| \\
&\le C \left(\int_{D_1 \cap S_A} \beta \right)^{\frac{1}{2}} \|\beta^{\frac{1}{2}} w\|_{0,D_1 \cap S_A} \le C \left(\int_{D_1 \cap S_A} \beta \right)^{\frac{1}{2}} |||w|||_\beta.
\end{aligned}$$

Since rectangles in $D_1 \cap S_A$ have sides of lengths $O\left(h\left(\varepsilon \log \frac{1}{\varepsilon}\right)^\alpha\right)$ it is not difficult to see that

$$\begin{aligned}
\int_{D_1 \cap S_A} \beta &\le Ch \left(\varepsilon \log \frac{1}{\varepsilon} \right)^\alpha \int_0^{\gamma_0 \varepsilon \log \frac{1}{\varepsilon}} \frac{1}{\varepsilon} e^{-\gamma x/\varepsilon}\, dx + |D_1 \cap S_A| \\
&\le Ch \left(\varepsilon \log \frac{1}{\varepsilon} \right)^\alpha \\
&\le Ch\varepsilon \log \frac{1}{\varepsilon}
\end{aligned}$$

where for the last inequality we recall that $\alpha = 1 - \frac{1}{2\log\frac{1}{\varepsilon}}$. It follows that

$$|III_{1,A}| \le Ch^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} |||w|||_\beta \le Ch^2 \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} |||w|||_\beta,$$

since $\varepsilon \le ch^3$.

On the other hand on S_B we have $|\nabla\beta|, |\nabla u|, |\nabla \mathcal{Q}_1 u|$ bounded independently of ε. We also have $|D_1 \cap S_B| \le Ch$. Then

$$\begin{aligned}
|III_{1,B}| &= \left| \int_{D_1 \cap S_B} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta)\, w \right| \\
&\le C\varepsilon^2 |D_1 \cap S_B|^{\frac{1}{2}} \|w\|_{0,D_1 \cap S_B} \\
&\le C\varepsilon^2 h^{\frac{1}{2}} |||w|||_\beta \le Ch^2 |||w|||_\beta,
\end{aligned}$$

by using that $\varepsilon < h$.

Thus, we obtain

$$III_1 \le Ch^2 \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} |||w|||_\beta. \tag{26}$$

Now we consider

$$III_2 = \int_{D_2} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta)\, w.$$

On each element R let $(\nabla\beta)_{min,R}$ be the componentwise minimum of $\nabla\beta$ on R. We write

$$\begin{aligned} III_2 = &\int_{D_2} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta - (\nabla\beta)_{min})\, w + \\ &\int_{D_2} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta)_{min}\, w \\ = &\, III_{21} + III_{22}. \end{aligned}$$

Let $D^1_{2A} = D_2 \cap \{(x,y) : x \le y \le 1-x, x \le \gamma_0 \varepsilon \log \frac{1}{\varepsilon}\}$. Notice that $d(x,y) = x$ on D^1_{2A}. It follows that

$$\nabla\beta(x,y) = \left(-\frac{\gamma}{\varepsilon^2} e^{-\gamma\frac{x}{\varepsilon}}, 0\right) \qquad \text{on } D^1_{2A}.$$

By the Mean Value Theorem, since $\nabla\beta$ depends only on x, and that for rectangular elements on D^1_{2A} the horizontal sides have the minimum length, we have

$$\nabla\beta(x,y) - (\nabla\beta)_{min,R} = \left(\frac{\gamma}{\varepsilon^3} e^{-\gamma\frac{x_{int}}{\varepsilon}} (x - x_{min,R}), 0\right)$$

with $x_{int} \in R$ and $x_{min,R}$ being the minimum value of x on R. Now, since $\frac{1}{\varepsilon} e^{-\gamma\frac{x_{int}}{\varepsilon}} \le \beta(x_{int}) \le \beta_{max}$ taking into account Lemma 4 we have

$$|\nabla\beta(x,y) - (\nabla\beta)_{min,R}| \le C\varepsilon^{-\eta h} \varepsilon^{-2} \beta_{min,R} h_{min,R}.$$

But $h_{min,R} \le Ch(\varepsilon \log \frac{1}{\varepsilon})^\alpha \le Ch\varepsilon \log \frac{1}{\varepsilon}$ (on elements touching $x = 0$ we also have $h_{min} \le Ch\varepsilon$). So

$$|\nabla\beta(x,y) - (\nabla\beta)_{min,R}| \le C\varepsilon^{-\eta h} \varepsilon^{-1} \beta_{min,R} h \log \frac{1}{\varepsilon}.$$

Then

$$\begin{aligned} &\left| \int_{D^1_{2A}} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta - (\nabla\beta)_{min})\, w \right| \\ &\qquad \le C\varepsilon^{-\eta h} h \log\left(\frac{1}{\varepsilon}\right) \int_{D^1_{2A}} \left[\varepsilon \beta^{\frac{1}{2}} |\nabla(u - \mathcal{Q}_1 u)|\right] \left[\beta^{\frac{1}{2}} |w|\right] \\ &\qquad\qquad \le C\varepsilon^{-\eta h} h^2 \log \frac{1}{\varepsilon} |||w|||_\beta \end{aligned}$$

where we have used (20).

On $D^2_{2A} = D_2 \cap \{(x,y) : x \le y \le 1-x, x > \gamma_0 \varepsilon \log \frac{1}{\varepsilon}\}$ we also have that β, $|D^\delta(\beta)|$, $0 \le |\delta| \le 2$, are uniformly bounded respect of ε and we also have $\beta \ge 1$, so a simple computation leaves

$$\begin{aligned} &\left| \int_{D^2_{2A}} \varepsilon^2 \nabla(u - \mathcal{Q}_1 u) \cdot (\nabla\beta - (\nabla\beta)_{min})\, w \right| \\ &\qquad \le Ch\varepsilon \|\varepsilon \beta^{\frac{1}{2}} \nabla(u - \mathcal{Q}_1 u)\|_{0,\Omega} \|\beta^{\frac{1}{2}} w\|_{0,\Omega} \\ &\qquad\qquad \le Ch^2 \varepsilon |||w|||_\beta. \end{aligned}$$

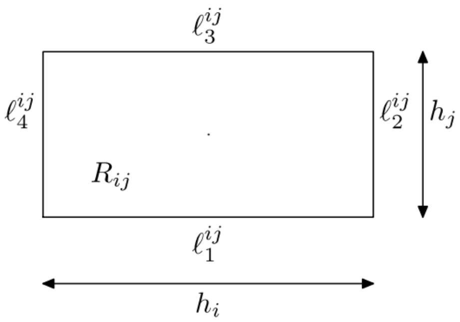

Fig. 3 Notation

Clearly, similar arguments can be used on $D_2 \setminus (D_{2A}^1 \cup D_{2A}^2)$ to obtain

$$|III_{21}| \leq C\varepsilon^{-\eta h} h^2 \log\left(\frac{1}{\varepsilon}\right) |||w|||_\beta. \tag{27}$$

Now we have to estimate III_{22}. Let call $(\nabla\beta)_{min,R_{ij}} = q_{ij} = (q_{ij}^1, q_{ij}^2)$. Then we will estimate

$$\sum_{R_{ij}\subset D_2} \int_{R_{ij}} \varepsilon^2 q_{ij}^1 \partial_x(u - \mathcal{Q}_1 u)w.$$

We will follow a technique used in [22,5]. Take into account Figure 3 for the notation of the sides of an element and its lengths. Let

$$K_{ij}(u,w) = \int_{R_{ij}} \partial_x(u - \mathcal{Q}_1 u)w - \frac{h_i^2}{12}\left(\int_{\ell_2^{ij}} (\partial_{xx} u)w\, dy - \int_{\ell_4^{ij}} (\partial_{xx} u)w\, dy\right).$$

Then we can write

$$\sum_{R_{ij}\subset D_2} \int_{R_{ij}} \varepsilon^2 q_{ij}^1 \partial_x(u - \mathcal{Q}_1 u)w =$$
$$\sum_{R_{ij}\subset D_2} \varepsilon^2 q_{ij}^1 K_{ij}(u,w) + \sum_{R_{ij}\subset D_2} \varepsilon^2 q_{ij}^1 \frac{h_i^2}{12}\left(\int_{\ell_2^{ij}} (\partial_{xx} u)w\, dy - \int_{\ell_4^{ij}} (\partial_{xx} u)w\, dy\right).$$

From [5, eq. (3.14)] we know that

$$|K_{ij}(u,w)| \leq C\left(h_i^2 \|\partial_x^3 u\|_{0,R_{ij}} + h_i h_j \|\partial_x^2 \partial_y u\|_{0,R_{ij}} + h_j^2 \|\partial_x \partial_y^2\|_{0,R_{ij}}\right) \|w\|_{0,R_{ij}}$$

then, since

$$|q_{ij}| \leq |\nabla\beta| \leq C\frac{\beta}{\varepsilon} \leq C\varepsilon^{-\eta h}\frac{\beta_{min}}{\varepsilon}$$

using Lemma 6 it follows that

$$\sum_{R_{ij}\subset D_2} \varepsilon^2 q_{ij}^1 K_{ij}(u,w) \le C\varepsilon^{-\eta h} \times$$
$$\sum_{ij} \varepsilon \beta_{min}^{\frac{1}{2}} \left(h_i^2 \|\partial_x^3 u\|_{0,R_{ij}} + h_i h_j \|\partial_x^2 \partial_y u\|_{0,R_{ij}} + h_j^2 \|\partial_x \partial_y^2\|_{0,R_{ij}} \right) \|\beta^{\frac{1}{2}} w\|_{0,R_{ij}}$$
$$\le C\varepsilon^{-\eta h} h^2 \log\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} \|\beta^{\frac{1}{2}} w\|_{0,\Omega}. \quad (28)$$

It remains to deal with

$$\Xi := \sum_{R_{ij}\subset D_2} \varepsilon^2 q_{ij}^1 \frac{h_i^2}{12} \left(\int_{\ell_2^{ij}} (\partial_{xx} u) w \, dy - \int_{\ell_4^{ij}} (\partial_{xx} u) w \, dy \right)$$

which can be written as

$$\begin{aligned}
\Xi &= -\sum_{R_{ij}\subset D_2} \varepsilon^2 q_{ij}^1 \frac{h_i^2}{12} \int_{R_{ij}} \partial_x \left[(\partial_x^2 u) w \right] \\
&= -\sum_{R_{ij}\subset D_2} \varepsilon^2 q_{ij}^1 \frac{h_i^2}{12} \int_{R_{ij}} (\partial_x^3 u) w - \sum_{R_{ij}\subset D_2} \varepsilon^2 q_{ij}^1 \frac{h_i^2}{12} \int_{R_{ij}} \partial_x^2 u \partial_x w \\
&=: \Xi_1 + \Xi_2.
\end{aligned}$$

Now we take into account that

$$|q_{ij}^1| \le |\nabla \beta| \le C\frac{\beta}{\varepsilon} \le C\varepsilon^{-\eta h} \frac{\beta_{min}}{\varepsilon}.$$

Then

$$|\Xi_1| \le C\varepsilon^{-\eta h} \sum_{R_{ij}\subset D_2} \left(\varepsilon \beta_{min}^{\frac{1}{2}} h_i^2 \|\partial^3 u\|_{0,R_{ij}} \right) \left(\beta_{min}^{\frac{1}{2}} \|w\|_{0,R_{ij}} \right).$$

Therefore, after applying Cauchy–Schwarz inequality and by using Lemma 6 it follows that

$$|\Xi_1| \le C\varepsilon^{-\eta h} \log\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} h^2 |||w|||_\beta.$$

Analogously we have

$$|\Xi_2| \le C\varepsilon^{-\eta h} \sum_{R_{ij}\subset D_2} \left(\beta_{min}^{\frac{1}{2}} h_i^2 \|\partial_x^2 u\|_{0,R_{ij}} \right) \left(\varepsilon \beta_{min}^{\frac{1}{2}} \|\partial_x w\|_{0,R_{ij}} \right).$$

With similar arguments we easily obtain

$$|\Xi_2| \le C\varepsilon^{-\eta h} h^2 |||w|||_\beta.$$

Then we arrived at

$$|\Xi| \le C\varepsilon^{-\eta h} \log\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} h^2 |||w|||_\beta.$$

This inequality together with (28) give

$$\left| \sum_{R_{ij} \subset D_2} \int_{R_{ij}} \varepsilon^2 q^1_{ij} \partial_x (u - \mathcal{Q}_1 u) w \right| \leq C \varepsilon^{-\eta h} \log \left(\frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2 |||w|||_\beta .$$

Clearly a similar argument allow us to conclude that

$$|III_{22}| \leq C \varepsilon^{-\eta h} \log \left(\frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2 |||w|||_\beta . \tag{29}$$

With inequalities (26), (27) and (29) we arrive at

$$|III| \leq C \varepsilon^{-\eta h} \log \left(\frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2 |||w|||_\beta . \tag{30}$$

5.4 Acotación de IV

From Proposition 1 we have

$$\begin{aligned} |IV| &\leq C \|\beta^{\frac{1}{2}} (u - \mathcal{Q}_1 u)\|_{0,\Omega} \|\beta^{\frac{1}{2}} w\|_{0,\frac{1}{2}} \\ &\leq C h^2 \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} |||w|||_\beta . \end{aligned} \tag{31}$$

5.5 Proof of Theorem 2

From (22), the splitting (23) with $w = u_h - \mathcal{Q}_1 u \in V_h$ and the estimates (24), (25), (30) and (31) we obtain

$$|||u_h - \mathcal{Q}_1 u|||^2_\beta \leq C \varepsilon^{-\eta h} \log \left(\frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2 |||u_h - \mathcal{Q}_1 u|||_\beta$$

from where the poof concludes.

6 Numerical experiments

We consider the problem

$$-\varepsilon^2 \Delta u + u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{32}$$

on $\Omega = [0,1]^2$ with two different choices for the function f. The first one is taken from [6] and the second one was introduced by Kopteva [9] and is widely used in the literature (see, for example, [1,16]). In both cases we take $\varepsilon = 1e-6$ and $\varepsilon = 1e-8$. All the numerical results were computed using Firedrake [20]. In Tables 1-5 we report the estimated order of convergence (eoc) of distinct quantities with respect to M, the number of grid points along x and y axis. We recall that the number of degrees of freedom is $\sim M^2$.

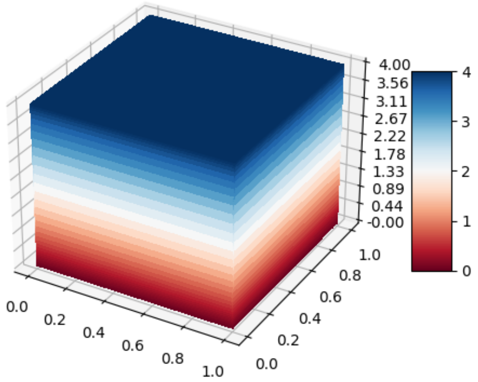

Fig. 4 Solution of Example 1 with $\varepsilon = 10^{-6}$.

h	M	$\|u - u_h\|_0$	eoc	$\|\|\|u - u_h\|\|\|_\beta$	eoc	$\|\|\|u_I - u_h\|\|\|_\beta$	eoc
0.2	245	5.5510e-5	-	1.2455e-1	-	2.2199e-2	-
0.1	521	1.6126e-5	1.6384	6.6549e-2	0.8307	6.3379e-3	1.6614
0.05	1055	4.3530e-6	1.8561	3.4499e-2	0.9312	1.7021e-3	1.8633
0.03	1758	1.6160e-6	1.9406	2.1014e-2	0.9708	6.3126e-4	1.9425

Table 1 Report of errors for the numerical experiment of Example 1 with $\varepsilon = 10^{-6}$.

Example 1 Take f given by

$$f(x, y) = -2\frac{1 - e^{-\frac{1}{\sqrt{2\varepsilon}}}}{1 - e^{-\frac{\sqrt{2}}{\varepsilon}}} \left(e^{-\frac{x}{\sqrt{2\varepsilon}}} + e^{-\frac{1-x}{\sqrt{2\varepsilon}}} + e^{-\frac{y}{\sqrt{2\varepsilon}}} + e^{-\frac{1-y}{\sqrt{2\varepsilon}}}\right) + 4.$$

By setting

$$u_0(t) = -2\frac{1 - e^{-\frac{1}{\sqrt{2\varepsilon}}}}{1 - e^{-\frac{\sqrt{2}}{\varepsilon}}} \left(e^{-\frac{t}{\sqrt{2\varepsilon}}} + e^{-\frac{1-t}{\sqrt{2\varepsilon}}}\right) + 2$$

it follows that the exact solution u is

$$u(x, y) = u_0(x)u_0(y).$$

We report in Table 1 (resp. Table 2) the errors and convergence orders obtained using the discretization (2) with V_h being the space of piecewise bilinear functions on the graded meshes introduced in Section 3 with $\varepsilon = 1e - 6$ (resp. $\varepsilon = 1e - 8$).

Example 2 Now, f is chosen such that

$$u(x, y) = \left[\cos\left(\frac{\pi}{2}x\right) - \frac{e^{-\frac{x}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}\right] \left(1 - y - \frac{e^{-\frac{y}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}\right)$$

is the solution of (32). This solution exhibits boundary layers only along the sides $x = 0$ and $y = 0$. In Table 3 (resp. Table 4) we report the convergence results

h	M	$\|u-u_h\|_0$	eoc	$\|\|\|u-u_h\|\|\|_\beta$	eoc	$\|\|\|u_I-u_h\|\|\|_\beta$	eoc
0.2	333	5.6529e-6	-	1.2443e-01	-	2.2354e-2	-
0.1	707	1.6423e-6	1.6418	6.6517e-02	0.8318	6.3922e-3	1.6628
0.05	1431	4.4411e-7	1.8547	3.4493e-02	0.9314	1.7183e-3	1.8632
0.03	2384	1.6498e-7	1.9401	2.1013e-02	0.9710	6.3753e-4	1.9425

Table 2 Report of errors for the numerical experiment of Example 1 with $\varepsilon = 10^{-8}$.

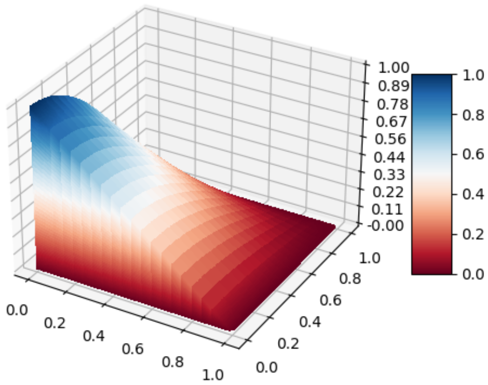

Fig. 5 Solution of Example 2 for $\varepsilon = 10^{-6}$.

h	M	$\|u-u_h\|_0$	eoc	$\|\|\|u-u_h\|\|\|_\beta$	eoc	$\|\|\|u_I-u_h\|\|\|_\beta$	eoc
0.2	245	1.3125e-3	-	2.0775e-2	-	6.6800e-3	-
0.1	521	3.3423e-4	1.8129	1.1058e-2	0.8358	1.8082e-3	1.7320
0.05	1055	8.4654e-5	1.9464	5.7287e-3	0.9322	4.7234e-4	1.9026
0.03	1758	2.9457e-5	2.0673	3.4892e-3	0.9710	1.6900e-4	2.0128

Table 3 Report of errors for Example 2 using graded meshes towards the entire boundary of Ω with $\varepsilon = 10^{-6}$.

obtained by using meshes graded towards the entire boundary $\partial\Omega$ for $\varepsilon = 1e-6$ (resp. $\varepsilon = 1e-8$), and we note that the expected orders of convergence are observed. On the other hand, in Table 5 (resp. Table 6) we report the results obtained by grading the mesh only close to the boundary layers of the solution. In this case, we observe the correct order of convergence in $|||\cdot|||_\beta$, but the ones for the L^2-norm and the supercloseness are suboptimal. This curious behavior will be in the future subject of further investigation.

Remark 3 As we mentioned in the Introduction, it is desirable that graded meshes designed for a small value of ε work well for reaction–diffusion problems with larger values of the diffusion parameter. Although this fact is not included in our analysis, we show computationally that behaviour. As an example, Figure 6 exhibits the solution of Example 2 with $\varepsilon = 1e-3$ obtained using the graded mesh designed for $h = 0.1$ and $\varepsilon = 1e-8$. Using the same fixed graded mesh, Table 7 shows the

h	M	$\|u-u_h\|_0$	eoc	$\|\|\|u-u_h\|\|\|_\beta$	eoc	$\|\|\|u_I-u_h\|\|\|_\beta$	eoc
0.2	333	1.2517e-3	-	2.0709e-2	-	6.4672e-3	-
0.1	707	3.2785e-4	1.7794	1.1032e-2	0.8365	1.7876e-3	1.7079
0.05	1431	8.3918e-5	1.9327	5.7167e-3	0.9324	4.7048e-4	1.8932
0.03	2384	2.9208e-5	2.0678	3.4823e-3	0.9712	1.6842e-4	2.0126

Table 4 Report of errors for Example 2 using graded meshes towards the entire boundary of Ω with $\varepsilon = 10^{-8}$.

h	M	$\|u-u_h\|_0$	eoc	$\|\|\|u-u_h\|\|\|_\beta$	eoc	$\|\|\|u_I-u_h\|\|\|_\beta$	eoc
0.2	125	4.6034e-2	-	7.9234e-2	-	7.6137e-2	-
0.1	265	1.9384e-2	1.1511	3.3988e-2	1.1264	3.2117e-2	1.1487
0.05	537	6.9052e-3	1.4614	1.2799e-2	1.3828	1.1448e-2	1.4606
0.03	895	3.2052e-3	1.5024	6.3562e-3	1.3703	5.3146e-3	1.5024

Table 5 Report of errors for Example 2 using graded meshes towards $x = 0$ and $y = 0$ with $\varepsilon = 10^{-6}$.

h	M	$\|u-u_h\|_0$	eoc	$\|\|\|u-u_h\|\|\|_\beta$	eoc	$\|\|\|u_I-u_h\|\|\|_\beta$	eoc
0.2	169	4.5459e-2	-	7.8290e-2	-	7.5187e-2	-
0.1	358	1.9286e-2	1.1423	3.3825e-2	1.1180	3.1953e-2	1.1400
0.05	725	6.8893e-3	1.4588	1.2771e-2	1.3804	1.1423e-2	1.4578
0.03	1208	3.1980e-3	1.5032	6.3424e-3	1.3708	5.3026e-3	1.5031

Table 6 Report of errors for Example 2 using graded meshes towards $x = 0$ and $y = 0$ with $\varepsilon = 10^{-8}$.

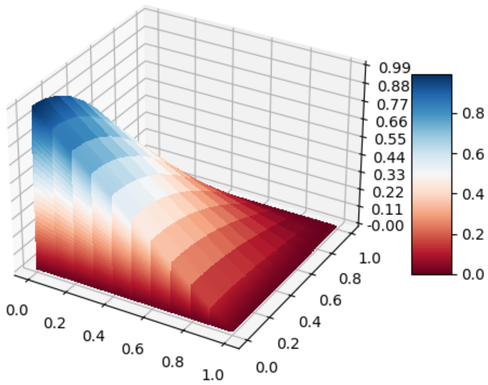

Fig. 6 Solution of Example 2, for $\varepsilon = 10^{-3}$, obtained using the mesh designed with $h = 0.1$ and $\varepsilon = 10^{-8}$.

errors obtained for ε varying between $1e-8$ and $1e-3$. We observe that for all the values of diffusion parameter the errors are almost the same.

ε	$\|u-u_h\|_0$	$\|\|\|u-u_h\|\|\|_\beta$	$\|\|\|u_I-u_h\|\|\|_\beta$
1.0e-3	3.2820e-04	8.4978e-03	1.5048e-03
1.0e-4	3.2793e-04	8.7397e-03	1.5486e-03
1.0e-5	3.2786e-04	9.2645e-03	1.5964e-03
1.0e-6	3.2785e-04	9.8217e-03	1.6513e-03
1.0e-7	3.2785e-04	1.0410e-02	1.7146e-03
1.0e-8	3.2785e-04	1.1032e-02	1.7876e-03

Table 7 Report of errors for the numerical experiment of Example 2 for distinct values of ε with a fixed mesh designed with $h = 0.1$ and $\varepsilon = 10^{-8}$.

Declarations

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Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Appendix

In this section we present some technical results which have been used along the paper.

The following Lemma is a consequence of [14, Lemmata 1.1 and 1.2]. In addition to the compatibility conditions of Section 4 we assume here that the fourth order derivatives of f and b are Hölder continuous up to the boundary. It is also assumed that $b(x,y) \geq 2b_0^2$.

Lemma 5 *Let u be the solution of (1). Then for all $x \in (0, \frac{3}{4}) \times (0, \frac{3}{4})$ and $k \leq 2$, it holds*

$$\left|\partial_x \partial_y^k u(x,y)\right| \leq C\left(1+\varepsilon^{1-k}\right) + \varepsilon^{-1} e^{-b_0 \frac{x}{\varepsilon}} + \varepsilon^{-k} e^{-b_0 \frac{y}{\varepsilon}} + \varepsilon^{-1-k} e^{-b_0 \frac{x+y}{\varepsilon}},$$
$$\left|\partial_y \partial_x^k u(x,y)\right| \leq C\left(1+\varepsilon^{1-k}\right) + \varepsilon^{-k} e^{-b_0 \frac{x}{\varepsilon}} + \varepsilon^{-1} e^{-b_0 \frac{y}{\varepsilon}} + \varepsilon^{-1-k} e^{-b_0 \frac{x+y}{\varepsilon}}.$$

Similar estimates are valid on the subdomains $(0, \frac{3}{4}) \times (\frac{1}{4}, 1)$ (replace y by $1-y$), $(\frac{1}{4}, 1) \times (0, \frac{3}{4})$ (replace x by $(1-x)$) and $(\frac{1}{4}, 1) \times (\frac{1}{4}, 1)$ replace (x by $1-x$ and y by $1-y$).

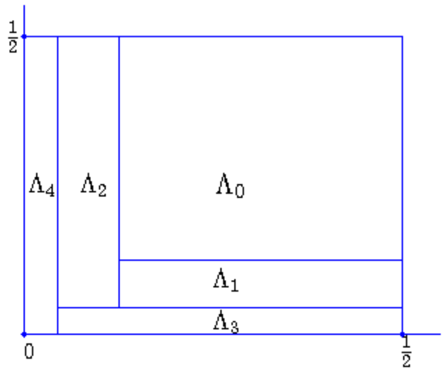

Fig. 7 Split of $\Omega_s = [0, \frac{1}{2}]^2$ used in the proof of Lemma 6

This Lemma allows us to obtain the next result.

Lemma 6 *Let u be the solution of (1). Then, under Assumption 1, we have that there exists a constant C such that*

$$\varepsilon \left[\sum_{i,j} \beta_{min} \left(h_i^2 \|\partial_x^3 u\|_{0,R_{ij}} + h_i h_j \|\partial_x^2 \partial_y u\|_{0,R_{ij}} + h_j^2 \|\partial_x \partial_y^2 u\|_{0,R_{ij}} \right)^2 \right]^{\frac{1}{2}} \leq C \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2. \quad (33)$$

Proof It is clear that by symmetry arguments it is enough to obtain (33) when the sum on the right hand side is restricted to the indices i, j with $R_{ij} \subseteq \Omega_s := [0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Let us split Ω_s as indicated in Figure 7. More precisely we set

$$\Lambda_0 = \left(x_{\bar{m}}, \tfrac{1}{2}\right) \times \left(x_{\bar{m}}, \tfrac{1}{2}\right), \qquad \Lambda_1 = \left(x_{\bar{m}}, \tfrac{1}{2}\right) \times (x_1, x_{\bar{m}}),$$

$$\Lambda_2 = (x_1, x_{\bar{m}}) \times \left(x_1, \tfrac{1}{2}\right), \qquad \Lambda_3 = \left(x_1, \tfrac{1}{2}\right) \times (0, x_1),$$

$$\Lambda_4 = (0, x_1) \times \left(0, \tfrac{1}{2}\right),$$

where $x_{\bar{m}}$ is a grid point with $x_{\bar{m}} = \gamma_0 \varepsilon \log \frac{1}{\varepsilon}$. We use the notation

$$A(\Lambda_k) := \varepsilon \left[\sum_{i,j: R_{ij} \subset \Lambda_k} \beta_{min} \left(h_i^2 \|\partial_x^3 u\|_{0,R_{ij}} + h_i h_j \|\partial_x^2 \partial_y u\|_{0,R_{ij}} + h_j^2 \|\partial_x \partial_y^2 u\|_{0,R_{ij}} \right)^2 \right]^{\frac{1}{2}}.$$

We will estimate separately $A(\Lambda_k)$ for $k = 0, \ldots, 4$.

0. Since $\gamma_0 \geq \frac{2}{b_0}$ we have from Lemmata 3 and 5 that $|D^3 u(x, y)| \leq C\varepsilon^{-1}$ and being $\gamma_0 \geq \frac{1}{\gamma}$ we also have $\beta_{min} \leq |\beta(x, y)| \leq C$ for all $(x, y) \in \Lambda_0$. Since $h_i \leq h$ for all i easily arrive at

$$A(\Lambda_0) \leq Ch^2.$$

1. On Λ_1 we also have $\beta \leq C/\varepsilon$. Taking into account that the length of Λ_1 in the y-direction is $\leq C\varepsilon \log \frac{1}{\varepsilon}$, $h_i \leq hx^\alpha$ for $(x,y) \in R_{ij} \subseteq \Lambda_1$, and using Lemma 3 we have

$$\sum_{R_{ij}\subseteq \Lambda_1} \beta_{min} \left(h_i^2 \|\partial_x^3 u\|_{0,R_{ij}} \right)^2 \leq C\varepsilon^{-1} h^4 \log \frac{1}{\varepsilon}. \tag{34}$$

Now we again have into account the estimate

$$\left|\partial_x^2 \partial_y u(x,y)\right| \leq C \left(1+\varepsilon^{-1}\right) + \varepsilon^{-2} e^{-b_0 \frac{x}{\varepsilon}} + \varepsilon^{-1} e^{-b_0 \frac{y}{\varepsilon}} + \varepsilon^{-3} e^{-b_0 \frac{x+y}{\varepsilon}}. \tag{35}$$

With the previous arguments, and in addition using that $\gamma_0 \geq \frac{2}{b_0}$, we have $\varepsilon^{-2} e^{-b_0 \frac{x}{\varepsilon}} \leq C$ on Λ_1, $h_i, h_j \leq h$, $h_j \leq Chy^\alpha$ for $(x,y) \in R_{ij} \subseteq \Lambda_1$. Thus we obtain

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_i h_j \|(1+\varepsilon^{-1})\|_{0,R_{ij}} \right)^2 \leq Ch^4 \varepsilon^{-2} \log \frac{1}{\varepsilon},$$

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_i h_j \|\varepsilon^{-2} e^{-b_0 \frac{x}{\varepsilon}}\|_{0,R_{ij}} \right)^2 \leq Ch^4 \log \frac{1}{\varepsilon},$$

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_i h_j \|\varepsilon^{-1} e^{-b_0 \frac{y}{\varepsilon}}\|_{0,R_{ij}} \right)^2 \leq Ch^4,$$

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_i h_j \|\varepsilon^{-3} e^{-b_0 \frac{(x+y)}{\varepsilon}}\|_{0,R_{ij}} \right)^2 \leq Ch^4.$$

Then, together with (35) we arrive at

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_i h_j \|\partial_x^2 \partial_y u\|_{0,R_{ij}} \right)^2 \leq Ch^4 \varepsilon^{-2} \log \frac{1}{\varepsilon}. \tag{36}$$

Now, from Lemma 5 we further have

$$\left|\partial_x \partial_y^2 u(x,y)\right| \leq C \left(1+\varepsilon^{-1}\right) + \varepsilon^{-1} e^{-b_0 \frac{x}{\varepsilon}} + \varepsilon^{-2} e^{-b_0 \frac{y}{\varepsilon}} + \varepsilon^{-3} e^{-b_0 \frac{x+y}{\varepsilon}}. \tag{37}$$

Now we use that $h_j \leq h$, $|S_4| \leq C\varepsilon \log \frac{1}{\varepsilon}$, $\varepsilon^{-2} e^{-b_0 \frac{x}{\varepsilon}} \leq C$ on Λ_1 and $h_j \leq hy^\alpha$ for $(x,y) \in R_{ij} \subseteq \Lambda_1$ to obtain

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_j^2 \|(1+\varepsilon^{-1}) + \varepsilon^{-1} e^{-b_0 \frac{x}{\varepsilon}}\|_{0,R_{ij}} \right)^2 \leq Ch^4 \varepsilon^{-2} \log \frac{1}{\varepsilon},$$

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_j^2 \|\varepsilon^{-2} e^{-b_0 \frac{y}{\varepsilon}}\|_{0,R_{ij}} \right)^2 \leq Ch^4,$$

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_j^2 \|\varepsilon^{-3} e^{-b_0 \frac{x+y}{\varepsilon}}\|_{0,R_{ij}} \right)^2 \leq Ch^4 \varepsilon^2,$$

which joint with (37) give

$$\sum_{R_{ij}\subset \Lambda_1} \beta_{min} \left(h_j^2 \|\partial_x \partial_y^2 u\|_{0,R_{ij}} \right)^2 \leq Ch^4 \varepsilon^{-2} \log \frac{1}{\varepsilon}. \tag{38}$$

Inequalities (34), (36) and (38) leave

$$A(\Lambda_1) \leq C \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} h^2.$$

2. On Λ_2 we use that $\beta \leq C/\varepsilon$. In order to estimate $A(\Lambda_2)$ we first note that since $h_i \leq Chx^\alpha$ for $(x, y) \in R_{ij} \subseteq \Lambda_2$ we have from Lemma 3 with $k = 3$ that

$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min} h_i^4 \|\partial_x^3 u\|_{0,R_{ij}}^2 \leq Ch^4\varepsilon^{-2}. \tag{39}$$

We use again (35) stated in Lemma 5. Using that for $R_{ij} \subseteq \Lambda_2$ the inequalities $h_i, h_j \leq Ch$, $h_i \leq hx^\alpha$ $h_j \leq hy^\alpha$ for $(x, y) \in R_{ij}$, $h_i \leq Ch\varepsilon \log \frac{1}{\varepsilon}$ and $|\Lambda_2| \leq C\varepsilon \log \frac{1}{\varepsilon}$ hold true, it can be checked that

$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min} h_i^2 h_j^2 \|(1+\varepsilon^{-1})\|_{0,R_{ij}}^2 \leq C\left(\log\frac{1}{\varepsilon}\right)^3 h^4,$$
$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min} h_i^2 h_j^2 \|\varepsilon^{-2} e^{-b_0\frac{x}{\varepsilon}}\|_{0,R_{ij}}^2 \leq C\varepsilon^{-2}h^4,$$
$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min} h_i^2 h_j^2 \|\varepsilon^{-1} e^{-b_0\frac{y}{\varepsilon}}\|_{0,R_{ij}}^2 \leq Ch^4,$$
$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min} h_i^2 h_j^2 \|\varepsilon^{-3} e^{-b_0\frac{(x+y)}{\varepsilon}}\|_{0,R_{ij}}^2 \leq C\varepsilon^{-1}h^4.$$

Therefore we obtain

$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min}\left(h_i h_j \|\partial_x^2\partial_y u\|_{0,R_{ij}}\right)^2 \leq Ch^4\varepsilon^{-2}. \tag{40}$$

We use now the etimate (37). Then, using that for $R_{ij} \subseteq \Lambda_2$ we have $h_j \leq h$ and $h_j \leq hy^\alpha$ for $(x, y) \in R_{ij}$ and since $|\Lambda_2| \leq C\varepsilon \log \frac{1}{\varepsilon}$ it follows

$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min}\left(h_j^2\|(1+\varepsilon^{-1}+\varepsilon^{-1}e^{-b_0\frac{x}{\varepsilon}})\|_{0,R_{ij}}\right)^2 \leq Ch^4\varepsilon^{-2}\log\frac{1}{\varepsilon},$$
$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min}\left(h_j^2\|\varepsilon^{-2}e^{-b_0\frac{y}{\varepsilon}}\|_{0,R_{ij}}\right)^2 \leq Ch^4,$$
$$\sum_{R_{ij}\subset\Lambda_2} \beta_{min}\left(h_j^2\|\varepsilon^{-3}e^{-b_0\frac{x+y}{\varepsilon}}\|_{0,R_{ij}}\right)^2 \leq C\varepsilon^{-2}h^4.$$

It follows that

$$\sum_{R_{ij}\subset S_1} \beta_{min}\left(h_j^2\|\partial_x\partial_y^2 u\|_{0,R_{ij}}\right)^2 \leq Ch^4\varepsilon^{-2}\log\frac{1}{\varepsilon}. \tag{41}$$

Collecting (39)–(41) we find

$$A(\Lambda_2) \leq C\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}} h^2.$$

3. We consider the estimate on Λ_3. We note that R_{11} is exterior to Λ_3 and then we have $h_i \leq hx^\alpha$ for all $(x, y) \in R_{i1} \subseteq \Lambda_3$. Since $h \leq e^{-1}$ we have

$$h_1 = h^{2\log\frac{1}{\varepsilon}} = \varepsilon^{2\log\frac{1}{h}} < \varepsilon^2$$

and then we also have $|\Lambda_3| \le C\varepsilon^2$. We will also use that $\beta \le \frac{C}{\varepsilon}$ on Λ_3. Then, from the estimate for $\partial_x^3 u$ from Lemma 3 we have

$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_i^2\|_{0,R_{i1}}\partial_x^3 u\|\right)^2 \le Ch^4\int_0^{h_1}\int_0^1\left(1+\varepsilon^{-3}x^{2\alpha}e^{-b_0\frac{x}{\varepsilon}}\right)^2 dx\,dy \le Ch^4. \quad (42)$$

Now we again take into account the estimate (35). Following the previous argument and since $h_i \le h$, $h_1 \le h\varepsilon$ we have

$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_ih_1\|1+\varepsilon^{-1}+\varepsilon^{-1}e^{-b_0\frac{y}{\varepsilon}}\|_{0,R_{i1}}\right)^2 \le Ch^4\varepsilon,$$
$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_ih_1\|\varepsilon^{-2}e^{-b_0\frac{x}{\varepsilon}}\|_{0,R_{i1}}\right)^2 \le Ch^4\varepsilon^2,$$

and since $h_i \le hx^\alpha$ for all $(x,y)\in R_{i1}\subseteq\Lambda_3$ we also have

$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_ih_1\|\varepsilon^{-3}e^{-b_0\frac{x+y}{\varepsilon}}\|\right)^2 \le Ch^4.$$

Thus we arrive at

$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_ih_1\|\partial_x^2\partial_y u\|_{0,R_{i1}}\right)^2 \le h^4. \quad (43)$$

On the other hand, we now use the estimate (37). Since again $h_1 \le h\varepsilon$ we obtain

$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_1^2\|1+\varepsilon^{-1}+\varepsilon^{-1}e^{-b_0\frac{x}{\varepsilon}}\|_{0,R_{i1}}\right)^2 \le Ch^4\varepsilon^3,$$
$$\sum_{R_{i1}\subset S_8} \beta_{min}\left(h_1^2\|\varepsilon^{-2}e^{-b_0\frac{y}{\varepsilon}}\|_{0,R_{i1}}\right)^2 \le Ch^4\varepsilon.$$

With all the previous arguments we also can check that

$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_1^2\|\varepsilon^{-3}e^{-b_0\frac{x+y}{\varepsilon}}\|\right)^2 \le Ch^4\varepsilon^4.$$

The last three inequalities give us

$$\sum_{R_{i1}\subset\Lambda_3} \beta_{min}\left(h_1^2\|\partial_x\partial_y^2 u\|_{0,R_{i1}}\right)^2 \le h^4\varepsilon. \quad (44)$$

Finally, from (42)–(44) leave

$$A(\Lambda_3) \le Ch^2\varepsilon.$$

4. Now, we consider the estimate on Λ_4. We note that

$$h_1 = h^{2\log\frac{1}{\varepsilon}} = h^{\log\frac{1}{\varepsilon}} h^{\log\frac{1}{\varepsilon}} = h^{\log\frac{1}{\varepsilon}} \varepsilon^{\log\frac{1}{h}} \le h\varepsilon.$$

Furthermore, as we proved in the previous item, we also have $h_1 < \varepsilon^2$, and as a consequence $|\Lambda_4| \le \varepsilon^2$. Then, we can simply use that $\partial_x^3 u \le C\varepsilon^{-3}$, which follows from Lemma 3 to obtain

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_1^2\|\partial_x^3 u\|_{0,R_{1j}}\right)^2 \le Ch^4\varepsilon^{-1}. \tag{45}$$

Now, take into account again (35)

$$\left|\partial_x^2\partial_y u(x,y)\right| \le C\left(1+\varepsilon^{-1}\right) + \varepsilon^{-2}e^{-b_0\frac{x}{\varepsilon}} + \varepsilon^{-1}e^{-b_0\frac{y}{\varepsilon}} + \varepsilon^{-3}e^{-b_0\frac{x+y}{\varepsilon}}.$$

We firstly note that, since $h_j \le h$, we have

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_1 h_j\|(1+\varepsilon^{-1}+\varepsilon^{-1}e^{-b_0\frac{y}{\varepsilon}})\|_{0,R_{1j}}\right)^2 \le Ch^4\varepsilon,$$

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_1 h_j\|\varepsilon^{-2}e^{-b_0\frac{x}{\varepsilon}}\|_{0,R_{1j}}\right)^2 \le Ch^4\varepsilon^{-1}.$$

and secondly, since $h_j \le hy^\alpha$ for all $(x,y)\in R_{1j}\subseteq\Lambda_4$, $j\neq 1$ we have

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_1 h_j\|\varepsilon^{-3}e^{-b_0\frac{x+y}{\varepsilon}}\|_{0,R_{1j}}\right)^2 \le Ch^4.$$

From the last three inequalities we obtain

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_1 h_j\|\partial_x^2\partial_y u\|_{0,R_{1j}}\right) \le Ch^4\varepsilon^{-1}. \tag{46}$$

Now we use the estimate (37)

$$\left|\partial_x\partial_y^2 u(x,y)\right| \le C\left(1+\varepsilon^{-1}\right) + \varepsilon^{-1}e^{-b_0\frac{x}{\varepsilon}} + \varepsilon^{-2}e^{-b_0\frac{y}{\varepsilon}} + \varepsilon^{-3}e^{-b_0\frac{x+y}{\varepsilon}}.$$

Since $|\Lambda_4| \le \varepsilon^2$ and $h_j \le h$ we have

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_j^2\|(1+\varepsilon^{-1}+\varepsilon^{-1}e^{-b_0\frac{x}{\varepsilon}})\|_{0,R_{1j}}\right)^2 \le Ch^4\varepsilon^{-1}.$$

We also have

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_j^2\|\varepsilon^{-2}e^{-b_0\frac{y}{\varepsilon}}\|_{0,R_{1j}}\right)^2 \le Ch^4\varepsilon^2,$$

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min}\left(h_j^2\|\varepsilon^{-3}e^{-b_0\frac{x+y}{\varepsilon}}\|_{0,R_{1j}}\right)^2 \le Ch^4,$$

where we used again $h_1 \leq \varepsilon^2$ and $h_j \leq hy^\alpha$ for $(x,y) \in R_{1j} \subseteq \Lambda_4$, $j \neq 1$. Then we obtain

$$\sum_{\substack{R_{1j}\subset\Lambda_4\\ j\neq 1}} \beta_{min} \left(h_j^2 \|\partial_x \partial_y^2 u\|_{0,R_{1j}} \right) \leq Ch^4 \varepsilon^{-1}. \tag{47}$$

Finally, since

$$|\partial_x^3 u|, |\partial_x^2 \partial_y u|, |\partial_x \partial_y^2 u| \leq C\varepsilon^{-3}$$

and using $h_1 \leq h\varepsilon$ and $h_1 \leq \varepsilon^2$, and so $|R_{11}| \leq \varepsilon^4$, we obtain

$$\beta_{min} h_1^4 \left(\|\partial_x^3 u\|_{0,R_{11}} + \|\partial_x^2 \partial_y u\|_{0,R_{11}} + \|\partial_x \partial_y^2\|_{0,R_{11}} \right)^2 \leq Ch^4 \varepsilon. \tag{48}$$

Therefore, inequalities (45)–(48) leave

$$A(\Lambda_4) \leq Ch^2 \varepsilon^{\frac{1}{2}}.$$

In this way we obtain (33) when the indices i, j are restricted to the ones for which $R_{ij} \subset \Omega_s$. The proof concludes by symmetry arguments. $\square$