Double field theory with matter and its cosmological application

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The phase space formulation of double field theory (DFT) indicates that statistical matter can be included in terms of (T-)duality multiplets. We propose the inclusion of a perfect fluid in the geometry of DFT through a generalized energy-momentum tensor written in terms of a DFT pressure, energy density, and velocity. The latter is an $O(D,D)$ vector and satisfies two invariant constraints in agreement with the on-shell constraints for the generalized momentum. We compute the conservation laws associated to the energy-momentum tensor considering general DFT backgrounds. Then we study cosmological backgrounds and we find an expression for the DFT cosmological dynamics with the perfect fluid coupled. This proposal reproduces the equations of string cosmology with nontrivial fixed dilaton charge upon parametrization of the DFT Einstein equations.

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I. INTRODUCTION

String theory (ST) is a very good candidate to describe nature from first principles. One of the cornerstones of ST is duality, which is necessary to show the equivalence between their different formulations. Moreover, T duality can be used as a guiding principle to construct a double geometry that describes strings in an $O(D,D)$ invariant way, $O(D,D)$ being an exact symmetry of ST. One possible framework to study the interplay between T duality and the low energy limit of ST is double field theory [1–3] (DFT).[1] The main idea of DFT is to accomplish $O(D,D)$ as a symmetry of the effective (or supergravity) action. The fundamental dimension of $O(D,D)$ is $2D$ and therefore the dimensions of the target space must be doubled as $\hat{X}^M = (\tilde{x}_\mu, x^\mu)$ with $\mu = 0, \ldots, D-1$.

The minimal formulation of DFT includes an invariant group (nondynamical) metric, $\hat{\eta}_{\hat{M}\hat{N}}$, and a field content given by a generalized metric $\hat{\mathcal{H}}_{\hat{M}\hat{N}}(\hat{X})$ and a generalized dilaton $\hat{d}(\hat{X})$, which are multiplets of the duality group. Standard diffeomorphisms and Abelian gauge transformations of the b field are replaced by generalized diffeomorphisms in order to preserve the invariance of the group metric, i.e., $\delta_{\hat{\xi}}\hat{\eta}_{\hat{M}\hat{N}} = 0$. This formulation describes the universal NS-NS (Neveu-Schwarz) sector of the low energy limit of ST upon suitable parametrization of the fields and parameters and imposing the strong constraint. The latter ensures the closure of the generalized diffeomorphisms and removes the dependence on half of the coordinates.

The construction of DFT is inspired in toroidal compactifications where T-duality transformations appear as a symmetry and the fundamental fields can be cast in multiplets of the duality group. These compactifications, in turn, are compatible with string cosmology [6], string gas cosmology [7], and other scenarios with matter terms [8]. At the level of the worldsheet, this issue was addressed in [9] for cosmological vacuum solutions and in [10,11] considering matter contributions and their relation with T duality. Moreover in [8] a duality covariant formulation was obtained for string cosmology. With these results, it is expected that matter fields can be coupled to the standard formulation of DFT, as it was recently stressed in [12].

When coupling matter fields to the vacuum DFT at the level of the action, the equations of motion derived from the variational principle can be recast into the form of a generalized Einstein equation [13]

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$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = \hat{\mathcal{T}}_{\hat{M}\hat{N}}, \tag{1.1}$$

where $\hat{\mathcal{G}}_{\hat{M}\hat{N}}$ is the symmetric DFT Einstein tensor,

$$\hat{\mathcal{T}}_{\hat{M}\hat{N}} = \hat{\mathcal{H}}_{\hat{M}\hat{N}}\left(\mathcal{L}_m - \frac{1}{2}\frac{\delta\mathcal{L}_m}{\delta\hat{d}}\right) - 2[\bar{P}_{\hat{M}\hat{K}}P_{\hat{N}\hat{L}} + \bar{P}_{\hat{N}\hat{K}}P_{\hat{M}\hat{L}}]\left(\frac{\delta\mathcal{L}_m}{\delta P_{\hat{K}\hat{L}}} - \frac{\delta\mathcal{L}_m}{\delta\bar{P}_{\hat{K}\hat{L}}}\right) \tag{1.2}$$

is the corresponding energy-momentum tensor with $\mathcal{L}_m$ the matter Lagrangian coupled to the double geometry and $P_{\hat{M}\hat{N}}$, $\bar{P}_{\hat{M}\hat{N}}$ are the usual DFT projectors [Eq. (2.8)]. Additionally the generalized Bianchi identities imply a vanishing divergence of the Einstein tensor, $\nabla^{\hat{M}}\hat{\mathcal{G}}_{\hat{M}\hat{N}} = 0$, and in turn the on-shell condition dictates a conservation law for matter, namely

$$\nabla^{\hat{M}}\hat{\mathcal{T}}_{\hat{M}\hat{N}} = 0. \tag{1.3}$$

The vacuum solutions for the DFT cosmological ansatz were studied in [14] and further solutions with matter were explored in [13,15]. Moreover, higher-derivative terms were included in [12,16–18] as well. Here again we observe that these works suggest that matter can be coupled to the standard construction of DFT. In turn having a well-understood framework of matter coupled to DFT would allow us to generalize the current computations of α' corrections using, for example, the results of [19] where a systematic procedure to access higher-derivative terms for vacuum DFT was introduced.

If one wants to describe matter from a statistical approach, as a gas or fluid in the double geometry, then it is possible to proceed from a double kinetic perspective [20].[2] Considering the phase space construction of DFT through the coordinates $(\hat{X}^{\hat{M}}, \hat{\mathcal{P}}^{\hat{M}})$, with $\hat{\mathcal{P}}^{\hat{M}}$ being a generalized momentum vector, the analogous equation to (1.2) is given by

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}}(\hat{X}) = \int d^{2D}\hat{\mathcal{P}}\, e^{-2\hat{d}}\hat{\mathcal{P}}^{\hat{M}}\hat{\mathcal{P}}^{\hat{N}}\hat{F}, \tag{1.4}$$

where $\hat{F} = \hat{F}(\hat{X}, \hat{\mathcal{P}})$ is a generalized one-particle distribution function. In this double kinetic framework, the conservation law (1.3) for the energy-momentum tensor (1.4) can be obtained from the transfer equations related to the generalized Boltzmann equation for the evolution of $\hat{F}$.

While in general relativity (GR) the explicit form of the energy-momentum tensor is well known for several statistical scenarios, e.g., the perfect fluid obtained by the integration of the relativistic version of the Maxwell-Boltzmann distribution function

$$T_{\mu\nu} = (p+e)u_\mu u_\nu + p g_{\mu\nu}, \tag{1.5}$$

this is not the case for DFT where both the generalized distribution function and the generalized tensor are not known. Moreover, the thermodynamics properties of this system and their equilibrium states are not fully understood. For this reason in this work we construct the energy-momentum tensor of the perfect fluid in the double geometry starting from a general ansatz in terms of the DFT metrics and a suitable generalization of the velocity of the fluid. The proposal for this tensor brings new features to rewrite the matter dynamics from a double geometry perspective and suggests new directions to construct a duality invariant thermodynamics, as we stress in Sec. V.

A. Main results

We consider a top-down perspective to build up a generalized energy-momentum tensor for the perfect fluid, proposing an effective construction based on symmetries, a generalized velocity of the fluid, and the fundamental fields of DFT.

First, we define a generalized version of the velocity for a point particle in the double geometry as [21]

$$\hat{U}^{\hat{M}} = \frac{D\hat{X}^{\hat{M}}}{D\tau}, \tag{1.6}$$

where τ is an affine parameter in the double geometry, which reduces to the standard proper time upon parametrization. In turn, this generalized velocity $\hat{U}^{\hat{M}}$ can be parametrized as

$$\hat{U}^{\hat{M}} = (\tilde{u}_\mu, u^\mu), \tag{1.7}$$

with $\tilde{u}_\mu$ a dual velocity.

Besides, both the generalized momentum $\hat{\mathcal{P}}^{\hat{M}}$ corresponding to the trajectory of a point particle and its generalized velocity $\hat{U}^{\hat{M}}$ are vectors that indicate the same geometric direction in the tangent space and then they must be proportional. Choosing the proportionality constant is, in fact, defining a specific affine parameter τ in (1.6). Indeed, we take the relation

$$\hat{\mathcal{P}}^{\hat{M}} = m\hat{U}^{\hat{M}} = mD\hat{X}^{\hat{M}}/D\tau, \tag{1.8}$$

with m the invariant mass. Since the generalized momenta satisfy a mass-shell condition and a strong constraintlike equation,

$$\hat{\mathcal{P}}^{\hat{M}}\hat{\mathcal{H}}_{\hat{M}\hat{N}}\hat{\mathcal{P}}^{\hat{N}} = -m^2, \tag{1.9}$$

$$\hat{\mathcal{P}}^{\hat{M}}\hat{\eta}_{\hat{M}\hat{N}}\hat{\mathcal{P}}^{\hat{N}} = 0, \tag{1.10}$$

the generalized velocity inherits the following relations

[2] See also [8,15] where a different procedure to include matter was implemented.

$$\hat{U}_{\hat{M}}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\hat{U}_{\hat{N}} = -1, \tag{1.11}$$

$$\hat{U}_{\hat{M}}\hat{\eta}^{\hat{M}\hat{N}}\hat{U}_{\hat{N}} = 0. \tag{1.12}$$

The constraints (1.10) and (1.12) get rid of half of the components of the generalized velocity/momentum in agreement with their actual number of degrees of freedom. Moreover, (1.11) and (1.12) allow one to consider a rest frame that basically sets the temporal direction upon a space-time splitting of the DFT coordinates.

Second, we introduce the index structure of the symmetric energy-momentum tensor of the perfect fluid considering the contributions of all the DFT projections of the generalized velocity vector $\hat{U}^{\hat{M}}$ and the fundamental metrics of DFT $\mathcal{H}_{\hat{M}\hat{N}}$ and $\hat{\eta}_{\hat{M}\hat{N}}$, namely

$$\hat{\mathcal{T}}_{\hat{M}\hat{N}} = b_1 U_{\underline{\hat{M}}}U_{\underline{\hat{N}}} + b_2 U_{\overline{\hat{M}}}U_{\overline{\hat{N}}} + b_3(U_{\underline{\hat{M}}}U_{\overline{\hat{N}}} + U_{\overline{\hat{M}}}U_{\underline{\hat{N}}}) + C\hat{\mathcal{H}}_{\hat{M}\hat{N}} + D\hat{\eta}_{\hat{M}\hat{N}}. \tag{1.13}$$

In the present analysis we do not allow nonvanishing off-diagonal components in the temporal sector after a space-time splitting; hence, the term proportional to the group metric is discarded ($D = 0$). Furthermore, in this first step we can simplify the expression (1.13) comparing with the generalized energy-momentum tensor for a generalized scalar field[3] in which only mixed components are present, consequently we discard the contributions coming from b_1 or b_2.

Finally and for later convenience we fix the remaining coefficients such that (1.13) yields

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}} = 2(\tilde{e}+\tilde{p})\hat{U}^{\underline{\hat{M}}}\hat{U}^{\overline{\hat{N}}} + 2(\tilde{e}+\tilde{p})\hat{U}^{\overline{\hat{M}}}\hat{U}^{\underline{\hat{N}}} + \tilde{p}\hat{\mathcal{H}}^{\hat{M}\hat{N}}, \tag{1.14}$$

where $\tilde{e}(\hat{X}) = \tilde{e}$ and $\tilde{p}(\hat{X}) = \tilde{p}$ are defined as the generalized energy density and pressure.

The expression (1.14) resembles the structure of the ordinary energy-momentum tensor in GR, which represents the metric sources [Eq. (1.5)]. However, its DFT generalization $\hat{\mathcal{T}}_{\hat{M}\hat{N}}$, includes not only the generalized metric sources but also the contributions from the generalized dilaton source [see Eq. (1.2)]; furthermore, the velocity term cannot be straightforwardly generalized due to the already mentioned DFT projections.

In addition to the construction of the generalization of the energy-momentum tensor for the perfect fluid (1.14) in the double geometry, we also explore the conservation equations derived from (1.3) and we elaborate on the DFT Einstein-type equations (1.1) sourced by (1.14) for a cosmological ansatz and their parametrization. As we have explained, this proposal must agree with the previous cosmological DFT constructions, such as [13,14], upon parametrization. Considering the space-time split and a rest frame indicating the temporal direction, the nontrivial equations coming from the generalized Einstein equation are given by

$$\begin{aligned} -\tilde{p} &= 2(\partial_0 d)^2 - 2\partial_{00}d - \frac{1}{16}\partial_0 A^{MN}\partial_0 A_{MN} + (\partial_0 \leftrightarrow \tilde{\partial}^0), \\ \frac{1}{2}(\tilde{e}+\tilde{p}) &= \partial_{00}d + \frac{1}{16}\partial_0 A^{KL}\partial_0 A_{KL} - (\partial_0 \leftrightarrow \tilde{\partial}^0), \\ 0 &= -P_M{}^P\left[\frac{1}{4}\partial_{00}A_{PQ} - \frac{1}{2}\partial_0 d\,\partial_0 A_{PQ} + (\partial_0 \leftrightarrow \tilde{\partial}^0)\right]\overline{P}^Q{}_N + (P \leftrightarrow \overline{P}). \end{aligned} \tag{1.15}$$

These equations determine the set of DFT cosmologies whose matter source corresponds to (1.14) and their parametrization reads

$$\begin{aligned} -2\tilde{p} &= 2(D-1)\dot{H} + D(D-1)H^2 - 4\ddot{\phi} + 4\dot{\phi}^2 - 4(D-1)H\dot{\phi} + \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl}, \\ \tilde{e}+\tilde{p} &= 2\ddot{\phi} - (D-1)(\dot{H}+H^2) - \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl}, \\ 0 &= (\dot{H} + (D-1)H^2 - 2H\dot{\phi})\delta^{ij} + \frac{1}{2a^4}\delta^{ik}\delta^{jl}\delta^{mn}\dot{b}_{km}\dot{b}_{ln}, \\ 0 &= \delta^{jl}\ddot{b}_{il} + [(D-5)H - 2\dot{\phi}]\delta^{jk}\dot{b}_{ik}. \end{aligned} \tag{1.16}$$

[3] A formal correspondence for these tensors exists in GR [22].

Here we keep the usual coordinates and eliminate the dual ones. However, other T-dual solutions can be easily obtained from (1.15). These cosmologies reduce to the results of [14] when the matter and the b field are neglected. Furthermore since DFT encodes the low energy limit of ST, this proposal is related to the fundamental equations of string cosmology. Particularly, it is possible to describe string cosmologies scenarios [6] by implementing the following field redefinitions:

$$\tilde{p} = e^{2\phi} p, \qquad \tilde{e} = e^{2\phi} e. \tag{1.17}$$

The succeed of DFT with matter yields in the fact that it is possible to obtain string cosmologies from the double geometry, whose only stringy ingredient is given by the duality group invariance. As we discuss in Sec. IV C, it is worth noting that the proposal (1.14) is a first step in the description of the generalized perfect fluid in the double geometry, which reproduces string cosmologies with a fixed dilaton source σ [cf. (4.18)] upon a suitable parametrization and imposing the strong constraint.

B. Outline

This paper is organized as follows: In Sec. II we introduce the basics of DFT and we describe the space-time split decomposition. In Sec. III we incorporate matter into DFT: First, we describe the construction of the DFT phase space from a kinetic theory point of view. Second, we propose an explicit form for the generalized energy-momentum tensor related to a perfect fluid in the double geometry. Finally we study the conservation laws from the divergence equation. We analyze the cosmological ansatz of DFT with matter given by the generalized Einstein equation, in Sec. IV. We also show the agreement with string cosmology considering field redefinitions of the DFT energy density and pressure. Conclusions are given in Sec. V.

II. VACUUM DOUBLE FIELD THEORY

In this section we introduce the basic aspects of the vacuum sector of DFT. We start by considering a formulation where the fundamental fields are the generalized metric and the generalized dilaton, which are in $O(D,D)$ representations. Then, we describe the space-time split framework. In both cases, the fundamental fields depend on both the ordinary coordinates and the dual ones.

A. Basics

DFT [1–3] is a proposal to rewrite the low energy limit of ST as a manifestly $O(D,D)$ invariant theory. All the DFT fields and parameters are duality covariant objects. The double geometry consists in a $2D$-dimensional space with coordinates $\hat{X}^{\hat{M}} = (\tilde{x}_\mu, x^\mu)$, $\hat{M} = 1, \ldots, 2D$. Here x^μ are coordinates on an embedded supergravity, and $\tilde{x}_\mu$ are dual coordinates. Considering a solution to the weak and strong constraints, namely

$$\begin{aligned} &\partial_{\hat{M}}(\partial^{\hat{M}}\star) = 0, \\ &(\partial_{\hat{M}}\star)(\partial^{\hat{M}}\star) = 0, \end{aligned} \tag{2.1}$$

where $\star$ is a generic field/parameter, half of the coordinates are taken away. For instance, simple solutions are $\tilde{\partial}^\mu = 0$ or $\partial_\mu = 0$ in which the fundamental fields depend only on x^μ or $\tilde{x}_\mu$, respectively. In (2.1), contractions are given by the $O(D,D)$ invariant metric, $\hat{\eta}_{\hat{M}\hat{N}}$.

The DFT action principle is invariant under a global $O(D,D)$ symmetry, which infinitesimally reads

$$\delta V_{\hat{M}} = V_{\hat{N}} \Omega^{\hat{N}}{}_{\hat{M}}, \tag{2.2}$$

where $V_{\hat{M}}$ is a generic $O(D,D)$ multiplet and $\Omega \in O(D,D)$ is the generic parameter of the transformation. A generalized notion of diffeomorphisms can be defined in the double space. These are given by the generalized Lie derivative,

$$\begin{aligned} \delta_{\hat{\xi}} V^{\hat{M}} &= \mathcal{L}_{\hat{\xi}} V^{\hat{M}} \\ &= \hat{\xi}^{\hat{N}} \partial_{\hat{N}} V^{\hat{M}} + (\partial^{\hat{M}} \hat{\xi}_{\hat{P}} - \partial_{\hat{P}} \hat{\xi}^{\hat{M}}) V^{\hat{P}} + \omega \partial_{\hat{N}} \hat{\xi}^{\hat{N}} V^{\hat{M}}, \end{aligned} \tag{2.3}$$

where $\hat{\xi}^{\hat{M}}$ a generic parameter and ω a density weight factor. The closure of these transformations is given by the C bracket

$$[\delta_{\hat{\xi}_1}, \delta_{\hat{\xi}_2}] V^{\hat{M}} = \delta_{\hat{\xi}_{21}} V^{\hat{M}}, \tag{2.4}$$

where

$$\hat{\xi}_{12}^{\hat{M}} = \hat{\xi}_1^{\hat{P}} \frac{\partial \hat{\xi}_2^{\hat{M}}}{\partial \hat{X}^{\hat{P}}} - \frac{1}{2} \hat{\xi}_1^{\hat{P}} \frac{\partial \hat{\xi}_{2\hat{P}}}{\partial \hat{X}_{\hat{M}}} - (1 \leftrightarrow 2). \tag{2.5}$$

The fundamental fields are the generalized dilaton $\hat{d}$ and the generalized metric $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$. In this work we consider that these fields integrate the vacuum sector of DFT. While the generalized metric transforms as a tensor under $O(D,D)$ transformations and generalized diffeomorphisms ($\omega = 0$), and it is an element of the duality group, i.e.,

$$\hat{\mathcal{H}}_{\hat{M}\hat{P}} \hat{\eta}^{\hat{P}\hat{Q}} \hat{\mathcal{H}}_{\hat{Q}\hat{N}} = \hat{\eta}_{\hat{M}\hat{N}}, \tag{2.6}$$

the generalized dilaton is an $O(D,D)$ scalar and transforms noncovariantly under generalized diffeomorphisms

$$\delta_{\hat{\xi}} \hat{d} = \hat{\xi}^{\hat{N}} \partial_{\hat{N}} \hat{d} - \frac{1}{2} \partial_{\hat{M}} \hat{\xi}^{\hat{M}}. \tag{2.7}$$

Both $\hat{\mathcal{H}}_{MN}$ and $\hat{\eta}_{MN}$ can be used to construct DFT projectors

$$P_{\hat{M}\hat{N}} = \frac{1}{2}(\hat{\eta}_{\hat{M}\hat{N}} - \hat{\mathcal{H}}_{\hat{M}\hat{N}}) \text{ and } \bar{P}_{\hat{M}\hat{N}} = \frac{1}{2}(\hat{\eta}_{\hat{M}\hat{N}} + \hat{\mathcal{H}}_{\hat{M}\hat{N}}). \tag{2.8}$$

The action principle of DFT is given by

$$S_{\rm DFT} = \frac{1}{2} \int d^{2D} \hat{X} e^{-2\hat{d}} \hat{\mathcal{R}}, \tag{2.9}$$

with $\hat{\mathcal{R}}$ as the generalized version of the Ricci scalar

$$\hat{\mathcal{R}} = \frac{1}{8}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{M}}\hat{\mathcal{H}}^{\hat{K}\hat{L}}\partial_{\hat{N}}\hat{\mathcal{H}}_{\hat{K}\hat{L}} - \frac{1}{2}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{N}}\hat{\mathcal{H}}^{\hat{K}\hat{L}}\partial_{\hat{L}}\hat{\mathcal{H}}_{\hat{M}\hat{K}} + 4\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{M}}\partial_{\hat{N}}\hat{d} + 4\partial_{\hat{M}}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{N}}\hat{d} - 4\hat{\mathcal{H}}^{\hat{M}\hat{N}}\partial_{\hat{M}}\hat{d}\partial_{\hat{N}}\hat{d} - \partial_{\hat{M}}\partial_{\hat{N}}\hat{\mathcal{H}}^{\hat{M}\hat{N}}, \quad (2.10)$$

which is a duality invariant scalar. Since this Lagrangian also transforms as a scalar under generalized diffeomorphisms, the action principle of DFT is fully invariant up to total derivatives.

In DFT it is also possible to construct a generalized Ricci tensor

$$\hat{\mathcal{R}}_{\hat{M}\hat{N}} = P_{\hat{M}}{}^{\hat{P}}\hat{\mathcal{K}}_{\hat{P}\hat{Q}}\bar{P}^{\hat{Q}}{}_{\hat{N}} + \bar{P}_{\hat{M}}{}^{\hat{P}}\hat{\mathcal{K}}_{\hat{P}\hat{Q}}P^{\hat{Q}}{}_{\hat{N}}, \quad (2.11)$$

where

$$\hat{\mathcal{K}}_{\hat{M}\hat{N}} = \frac{1}{8}\partial_{\hat{M}}\hat{\mathcal{H}}^{\hat{K}\hat{L}}\partial_{\hat{N}}\hat{\mathcal{H}}_{\hat{K}\hat{L}} - \frac{1}{4}(\partial_{\hat{L}} - 2\partial_{\hat{L}}\hat{d})(\hat{\mathcal{H}}^{\hat{L}\hat{K}}\partial_{\hat{K}}\hat{\mathcal{H}}_{\hat{M}\hat{N}}) + 2\partial_{\hat{M}}\partial_{\hat{N}}\hat{d} - \frac{1}{2}\partial_{(\hat{M}}\hat{\mathcal{H}}^{\hat{K}\hat{L}}\partial_{\hat{L}}\hat{\mathcal{H}}_{\hat{N})\hat{K}} + \frac{1}{2}(\partial_{\hat{L}} - 2\partial_{\hat{L}}\hat{d})(\hat{\mathcal{H}}^{\hat{K}\hat{L}}\partial_{(\hat{M}}\hat{\mathcal{H}}_{\hat{N})\hat{K}} + \hat{\mathcal{H}}^{\hat{K}}{}_{(\hat{M}}\partial_{\hat{K}}\hat{\mathcal{H}}^{\hat{L}}{}_{\hat{N})}).$$

It is straightforward to verify that these objects are fully covariant. Moreover, they can be constructed from a generalized Riemann tensor, but the latter cannot be entirely expressed in terms of the fundamental fields of DFT [23].

B. Space-time split of double field theory

We start by splitting the DFT coordinates as

$$\hat{X}^{\hat{M}} = (\tilde{x}_0, x^0, X^M), \quad (2.12)$$

where $M = 3, \ldots, 2D$, $\tilde{x}_0 = \tilde{t}$, $x^0 = t$ and the partial derivatives are

$$\partial_{\hat{M}} = (\tilde{\partial}^0, \partial_0, \partial_M). \quad (2.13)$$

The weak constraint now implies

$$\partial_M\partial^M\star = -2\tilde{\partial}^0\partial_0\star, \quad (2.14)$$

while the strong constraint is

$$\partial_M\Box\partial^M\star = -\partial_0\Box\tilde{\partial}^0\star - \partial_0\star\tilde{\partial}^0\Box, \quad (2.15)$$

where $\star$ and $\Box$ are arbitrary fields/parameters. The $O(D,D)$ invariant metric decomposes as

$$\hat{\eta}_{\hat{M}\hat{N}} = \begin{pmatrix} \hat{\eta}^{00} & \hat{\eta}^{0}{}_{0} & \hat{\eta}^{0}{}_{N} \\ \hat{\eta}_{0}{}^{0} & \hat{\eta}_{00} & \hat{\eta}_{0N} \\ \hat{\eta}_{M}{}^{0} & \hat{\eta}_{M0} & \hat{\eta}_{MN} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \eta_{MN} \end{pmatrix}, \quad (2.16)$$

while the generalized metric is given by

$$\hat{\mathcal{H}}_{\hat{M}\hat{N}} = \begin{pmatrix} -N^{-2} & \alpha & N^{-2}\mathcal{N}_N \\ \alpha & -\frac{1}{2}\alpha\mathcal{N}^K\mathcal{N}_K - N^2 + \mathcal{H}_{PK}\mathcal{N}^P\mathcal{N}^K & -\alpha\mathcal{N}_N + \mathcal{H}_{NK}\mathcal{N}^K \\ N^{-2}\mathcal{N}_M & -\alpha\mathcal{N}_M + \mathcal{H}_{MK}\mathcal{N}^K & \mathcal{H}_{MN} - N^{-2}\mathcal{N}_M\mathcal{N}_N \end{pmatrix}, \quad (2.17)$$

where $\alpha = \frac{1}{2}N^{-2}\mathcal{N}^M\mathcal{N}_M$ [24]. In (2.17) $\mathcal{N}_M$ is a generalized shift vector and N is a generalized lapse function. The generalized dilaton, $\hat{d}$, can be redefined in terms of the generalized lapse function,

$$e^{-2\hat{d}} = Ne^{-2d}. \quad (2.18)$$

Since we are interested in cosmological ansatz, we consider $\mathcal{N}_M = 0$ and $N = 1$. So far, the line element is given by

$$dS^2 = -d\tilde{t}^2 - dt^2 + \mathcal{H}_{MN}dX^M dX^N, \quad (2.19)$$

with $\mathcal{H}_{MN} = \mathcal{H}_{MN}(\tilde{t}, t, X)$. The dependence in all the dual coordinates is crucial to preserve the $O(D,D)$ invariance. Since we are considering $N = 1$, the generalized dilaton is not redefined and we get $\hat{d}(\hat{X}) = d(\tilde{t}, t, X)$.

We can easily decompose both the DFT Lagrangian

$$\mathcal{L}_{\rm DFT} = \frac{1}{8}\mathcal{H}^{PQ}\partial_P\mathcal{H}^{MN}\partial_Q\mathcal{H}_{MN} - \frac{1}{8}\partial_0\mathcal{H}^{MN}\partial_0\mathcal{H}_{MN} - \frac{1}{8}\tilde{\partial}^0\mathcal{H}^{MN}\tilde{\partial}^0\mathcal{H}_{MN} - \frac{1}{2}\mathcal{H}^{PQ}\partial_Q\mathcal{H}^{MN}\partial_N\mathcal{H}_{PM} + 4\mathcal{H}^{MN}\partial_M d\partial_N d - 4\partial_0 d\partial_0 d - 4\tilde{\partial}^0 d\tilde{\partial}^0 d - 2\partial_M\mathcal{H}^{MN}\partial_N d, \quad (2.20)$$

and the generalized curvatures

$$\hat{\mathcal{R}} = \frac{1}{8}\mathcal{H}^{PQ}\partial_P\mathcal{H}^{MN}\partial_Q\mathcal{H}_{MN} - \frac{1}{8}\partial_0\mathcal{H}^{MN}\partial_0\mathcal{H}_{MN} - \frac{1}{8}\tilde{\partial}^0\mathcal{H}^{MN}\tilde{\partial}^0\mathcal{H}_{MN} - \frac{1}{2}\mathcal{H}^{PQ}\partial_Q\mathcal{H}^{MN}\partial_N\mathcal{H}_{PM} + 4\mathcal{H}^{MN}\partial_{MN}d - 4\partial_{00}d - 4\tilde{\partial}^{00}d + 4\partial_M\mathcal{H}^{MN}\partial_N d - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_0 d\partial_0 d + 4\tilde{\partial}^0 d\tilde{\partial}^0 d - \partial_{MN}\mathcal{H}^{MN}, \quad (2.21)$$

$$\hat{\mathcal{K}}_{00} = \frac{1}{8}\partial_0\mathcal{H}^{KL}\partial_0\mathcal{H}_{KL} + 2\partial_{00}d, \quad (2.22)$$

$$\hat{\mathcal{K}}^{00} = \frac{1}{8}\tilde{\partial}^0\mathcal{H}^{KL}\tilde{\partial}^0\mathcal{H}_{KL} + 2\tilde{\partial}^{00}d, \quad (2.23)$$

$$\hat{\mathcal{K}}^0{}_0 = \hat{\mathcal{K}}_0{}^0 = \frac{1}{8}\tilde{\partial}^0\mathcal{H}^{KL}\partial_0\mathcal{H}_{KL} + 2\tilde{\partial}^0\partial_0 d, \quad (2.24)$$

and

$$\hat{\mathcal{K}}_{MN} = \frac{1}{8}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{4}\partial_L\mathcal{H}^{KL}\partial_K\mathcal{H}_{MN} + \frac{1}{4}\partial_{00}\mathcal{H}_{MN} + \frac{1}{4}\tilde{\partial}^{00}\mathcal{H}_{MN} - \frac{1}{4}\mathcal{H}^{LK}\partial_{LK}\mathcal{H}_{MN} - \frac{1}{2}\partial_0 d\partial_0\mathcal{H}_{MN} - \frac{1}{2}\tilde{\partial}^0 d\partial^0\mathcal{H}_{MN} + \frac{1}{2}\partial_L d\mathcal{H}^{LK}\partial_K\mathcal{H}_{MN} + 2\partial_{MN}d - \frac{1}{2}\partial_{(M}\mathcal{H}^{KL}\partial_L\mathcal{H}_{N)K} + \frac{1}{2}\partial_L\mathcal{H}^{KL}\partial_{(M}\mathcal{H}_{N)K} + \frac{1}{2}\mathcal{H}^{KL}\partial_{L(M}\mathcal{H}_{N)K} - \partial_L d\mathcal{H}^{KL}\partial_{(M}\mathcal{H}_{N)K} + \frac{1}{2}\partial_L\mathcal{H}^K{}_{(M}\partial_K\mathcal{H}^L{}_{N)} + \frac{1}{2}\mathcal{H}_{0(M}\partial_L\tilde{\partial}^0\mathcal{H}^L{}_{N)} + \frac{1}{2}\mathcal{H}^0{}_{(M}\partial_L\partial_0\mathcal{H}^L{}_{N)} + \frac{1}{2}\mathcal{H}^K{}_{(M}\partial_{LK}\mathcal{H}^L{}_{N)} - \frac{1}{2}\partial_L d\mathcal{H}^0{}_{(M}\partial_0\mathcal{H}^L{}_{N)} - \frac{1}{2}\partial_L d\mathcal{H}_{0(M}\tilde{\partial}^0\mathcal{H}^L{}_{N)} - \frac{1}{2}\partial_L d\mathcal{H}^K{}_{(M}\partial_K\mathcal{H}^L{}_{N)}, \quad (2.25)$$

where the rounded brackets in the indexes mean symmetrization and $\partial_{MN}(\star) = \partial_M(\partial_N(\star))$. The previous expressions (2.21)–(2.25) are useful to obtain closed expressions for rewriting cosmological solutions from the DFT framework with matter. Moreover, considering a cosmological principle [14], we can impose $\mathcal{H}_{MN}(\tilde{t}, t, X) = A_{MN}(\tilde{t}, t)$ and $d(\tilde{t}, t, X) = d(\tilde{t}, t)$ and the expressions for the curvatures drastically simplify. In this case, the remaining symmetry is $O(1,1) \times O(D-1, D-1)$. We discuss about this point in Sec. IV.

III. INCORPORATING MATTER INTO DOUBLE FIELD THEORY

The original construction of DFT [2] is based on a rewriting of the low energy limit of ST where the invariance under $O(D,D)$ is accomplished before compactification. In fact, the dynamics of the supergravity NS-NS vacuum sector regarding the metric $g_{\mu\nu}$, the b field and the dilaton ϕ can be completely rewritten within the framework of DFT and, particularly, in terms of $O(D,D)$ multiplets. In this case a generalized vacuum Einstein equation can be constructed as

$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = 0, \quad (3.1)$$

where the lhs is given by the symmetric generalized Einstein tensor

$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = -\frac{1}{2}\hat{\mathcal{H}}_{\hat{M}\hat{N}}\hat{\mathcal{R}} - 2\hat{\mathcal{R}}_{\hat{M}\hat{N}}, \quad (3.2)$$

with $\hat{\mathcal{R}}$ and $\hat{\mathcal{R}}_{\hat{M}\hat{N}}$ given in Eqs. (2.10) and (2.11), respectively. In order to get the full dynamics, Eq. (3.1) must be complemented with the equation of motion coming from the variation of the vacuum action (2.9) with respect to the dilaton, namely $\hat{\mathcal{R}} = 0$ [4,23].

The inclusion of matter in the dynamics at the level of supergravity is also well understood both through a variational principle from an specific matter action or through the usual Einstein equation with a proper energy-momentum tensor. While the relation between supergravity, matter, and T duality was studied in several works, the generalized Einstein equation in terms of $O(D,D)$ multiplets was constructed in [13] and takes the following form:

$$\hat{\mathcal{G}}_{\hat{M}\hat{N}} = \hat{\mathcal{T}}_{\hat{M}\hat{N}}, \quad (3.3)$$

where $\hat{\mathcal{T}}_{\hat{M}\hat{N}}$ is the generalized symmetric energy-momentum tensor that incorporates the effects of matter into the dynamics at the level of DFT.

The aim of this section is to find an explicit expression for $\hat{\mathcal{T}}_{\hat{M}\hat{N}}$ written only in terms of manifestly $O(D,D)$-covariant quantities that recovers the well-known string cosmology equations upon suitable parametrization. So that we propose the inclusion of matter into the DFT formulation based on kinetic theory extending the results of [20], where the generalized energy-momentum tensor was defined as the second moment of a generalized one-particle distribution function.

In the first part of this section we give a brief review of the phase space for the point particle in the double geometry. Then we construct the generalized energy-momentum tensor for a perfect fluid in the double geometry. This proposal is given in terms of a generalized velocity, which is consistently constrained. We present the conservation laws for the generalized energy-momentum tensor, which are given by the generalized Euler equations and the generalized relativistic energy conservation equation.

A. Double phase space and the point particle

The phase space of DFT can be constructed considering an extension of the double geometry with coordinates

$$\{\hat{X}^{\hat{M}}, \hat{\mathcal{P}}^{\hat{M}}\}, \tag{3.4}$$

where $\hat{\mathcal{P}}^{\hat{M}}$ is the generalized momentum. These coordinates transform as vectors with respect to $O(D,D)$ and satisfy

$$\frac{\partial \hat{\mathcal{P}}^{\hat{M}}}{\partial \hat{X}^{\hat{N}}} = 0. \tag{3.5}$$

The strong constraint must also be extended as

$$\left(\frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}}\star\right)\left(\frac{\partial}{\partial \hat{\mathcal{P}}_{\hat{M}}}\star\right) = \frac{\partial}{\partial \hat{\mathcal{P}}^{\hat{M}}}\left(\frac{\partial}{\partial \hat{\mathcal{P}}_{\hat{M}}}\star\right) = 0, \tag{3.6}$$

$$\left(\frac{\partial}{\partial \hat{X}^{\hat{M}}}\star\right)\left(\frac{\partial}{\partial \hat{\mathcal{P}}_{\hat{M}}}\star\right) = \frac{\partial}{\partial \hat{X}^{\hat{M}}}\left(\frac{\partial}{\partial \hat{\mathcal{P}}_{\hat{M}}}\star\right) = 0. \tag{3.7}$$

Equations (3.6) and (3.7) guarantee that the generalized diffeomorphisms on the phase space,

$$\delta_{\hat{\xi}} V^{\hat{Q}}(\hat{X},\hat{\mathcal{P}}) = \mathcal{L}_{\hat{\xi}} V^{\hat{Q}}(\hat{X},\hat{\mathcal{P}}) + \hat{\mathcal{P}}^{\hat{N}} \frac{\partial \hat{\xi}^{\hat{M}}}{\partial \hat{X}^{\hat{N}}} \frac{\partial \hat{V}^{\hat{Q}}(\hat{X},\hat{\mathcal{P}})}{\partial \hat{\mathcal{P}}^{\hat{M}}} - \hat{\mathcal{P}}^{\hat{N}} \frac{\partial \hat{\xi}_{\hat{N}}}{\partial \hat{X}_{\hat{M}}} \frac{\partial V^{\hat{Q}}(\hat{X},\hat{\mathcal{P}})}{\partial \hat{\mathcal{P}}^{\hat{M}}}, \tag{3.8}$$

satisfy a closure condition,

$$[\delta_{\hat{\xi}_1}, \delta_{\hat{\xi}_2}] V^{\hat{M}}(\hat{X},\hat{\mathcal{P}}) = -\delta_{\hat{\xi}_{12}} V^{\hat{M}}(\hat{X},\hat{\mathcal{P}}), \tag{3.9}$$

where $\hat{\xi}_{12}^{M}(\hat{X})$ is given by the C bracket [Eq. (2.5)]. In (3.8) each diffeomorphism parameter depends only on the space-time coordinates, $\hat{\xi}^{\hat{M}} = \hat{\xi}^{\hat{M}}(\hat{X})$, and $V^{\hat{Q}}(\hat{X},\hat{\mathcal{P}})$ is a generic vector on the double phase space.

The generalized energy-momentum tensor is a tensor in the double space,

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}}(\hat{X}) = \int \hat{\mathcal{P}}^{\hat{M}} \hat{\mathcal{P}}^{\hat{N}} \hat{F} e^{-2\hat{d}} d^{2D}\hat{\mathcal{P}}, \tag{3.10}$$

where $\hat{F} = \hat{F}(\hat{X},\hat{\mathcal{P}})$ is the one-particle generalized distribution function. This function transforms as a phase space scalar. Indeed the generalized energy-momentum tensor fulfills a conservationlike equation

$$\nabla_{\hat{M}} \hat{\mathcal{T}}^{\hat{M}\hat{N}} = 0, \tag{3.11}$$

as it is shown in [20] by computing the divergence of the second moment of the generalized Boltzmann equation for an equilibrium state.

On the other hand, the generalized momentum of a particle encodes the physics of an ordinary D-dimensional momentum $p_\mu = g_{\mu\nu} p^\nu$ so we need to impose a strong-constraint-like equation,

$$\hat{\mathcal{P}}^{\hat{M}} \hat{\eta}_{\hat{M}\hat{N}} \hat{\mathcal{P}}^{\hat{N}} = 0, \tag{3.12}$$

in order to get the correct number of degrees of freedom when parametrizing. Furthermore, we also impose a mass-shell-like condition for the particle in the double geometry given by

$$-\hat{\mathcal{P}}^{\hat{M}} \hat{\mathcal{H}}_{\hat{M}\hat{N}} \hat{\mathcal{P}}^{\hat{N}} = m^2. \tag{3.13}$$

In fact Eqs. (3.12) and (3.13) are analogous to the level matching condition and the mass-squared operator in toroidal compactifications [4]. Despite this analogy, the constraints for the DFT momentum presented here are conceptually different. In other words, a generalized momentum can be used to arrange the D-dimensional momentum and winding modes for a closed string in a toroidal background, the generalized momentum being an $O(D,D,Z)$ multiplet [25] and Eq. (3.13) reproducing the spectrum of the massless states of the string. Here the generalized momentum describes the momentum of a particle in the double geometry and, therefore, it is an $O(D,D,R)$ multiplet. Furthermore, the parametrization of the phase-space diffeomorphism (3.8) requires $\hat{\mathcal{P}}^{\hat{M}} = (\tilde{p}_\mu, p^\mu) = (0, p^\mu)$, which is a solution of (3.12) and it is also in agreement with the strong constraint in the phase space (3.6) and (3.7). Additionally the generalized velocity for the particle in the double geometry is given by [21]

$$\hat{U}^{\hat{M}} = \frac{D\hat{X}^{\hat{M}}}{D\tau}, \tag{3.14}$$

and τ is an affine parameter in the double geometry, which reduces to the standard proper time upon parametrization.

Both the generalized momentum $\hat{\mathcal{P}}^{\hat{M}}$, evaluated on the trajectory of a point particle, and the generalized velocity $\hat{U}^{\hat{M}}$ are vectors that indicate the same geometric direction in the tangent space and then they must be proportional. Choosing the proportionality constant is, in fact, defining an specific affine parameter τ in (3.14). Indeed, we take the relation

$$\hat{\mathcal{P}}^{\hat{M}} = m\hat{U}^{\hat{M}} = mD\hat{X}^{\hat{M}}/D\tau, \tag{3.15}$$

where m is the invariant mass in (3.13), with the generalized velocity parametrized as

$$\hat{U}^{\hat{M}} = (\tilde{u}_\mu, u^\mu), \tag{3.16}$$

with $\tilde{u}_\mu$ a dual velocity. It is straightforward to rewrite the properties of the generalized momentum in terms of the generalized velocity as

$$\hat{U}_{\hat{M}}\hat{\mathcal{H}}^{\hat{M}\hat{N}}\hat{U}_{\hat{N}} = -1 \tag{3.17}$$

$$\hat{U}_{\hat{M}}\hat{\eta}^{\hat{M}\hat{N}}\hat{U}_{\hat{N}} = 0. \tag{3.18}$$

In the next part of this section we take advantage of the phase-space construction of DFT and we give a proposal for the generalized energy-momentum tensor of a perfect fluid in the double space. This is a simple proposal to effectively describe matter contributions in DFT such as fermionic [26] or Ramond-Ramond contributions [27], among others.

B. The perfect fluid in the double geometry

A relevant ingredient for including a perfect fluid in DFT is the generalized symmetric energy-momentum tensor $\mathcal{T}_{\hat{M}\hat{N}}$. In our case its index structure is given only by the fundamental 2-index tensors of DFT, $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$ and $\hat{\eta}_{\hat{M}\hat{N}}$, and the generalized velocity vector of the fluid $\hat{U}_{\hat{M}}$, which defines the temporal direction upon the split (2.12). Hence considering every projection of the velocity separately we have

$$\hat{\mathcal{T}}_{\hat{M}\hat{N}} = b_1 U_{\underline{\hat{M}}} U_{\underline{\hat{N}}} + b_2 U_{\overline{\hat{M}}} U_{\overline{\hat{N}}} + b_3 (U_{\underline{\hat{M}}} U_{\overline{\hat{N}}} + U_{\overline{\hat{M}}} U_{\underline{\hat{N}}}) + C\hat{\mathcal{H}}_{\hat{M}\hat{N}} + D\hat{\eta}_{\hat{M}\hat{N}}. \tag{3.19}$$

In this work we analyze a first proposal for the perfect fluid and therefore we do not include nonvanishing off-diagonal components in the temporal sector, and consequently we consider $D = 0$.

The expression (3.19) resembles the structure of the ordinary energy-momentum tensor in GR, namely

$$T_{\mu\nu} = (p + e)u_\mu u_\nu + p g_{\mu\nu}, \tag{3.20}$$

which takes into account metric sources. However, its DFT generalization $\hat{\mathcal{T}}_{\hat{M}\hat{N}}$, include not only generalized metric sources but also contributions form the generalized dilaton source [see Eq. (1.2)]. Furthermore, the velocity term cannot be straightforwardly generalized due to the DFT projections.

The dynamics of the perfect fluid in Riemannian geometries is equivalent to the dynamics of a scalar field [22]. Since the dynamics for the generalized scalar field is given by a straightforward generalization of the Klein-Gordon equation it is expected that the minimal proposal for the generalized energy-momentum tensor of the perfect fluid contains only nontrivial b_3, thus including the mixed projections present in the generalized energy-momentum tensor of the generalized scalar field [see Eq. (4.24) in [20]]. Therefore our proposal for the generalized energy-momentum tensor reads

$$\hat{\mathcal{T}}_{\hat{M}\hat{N}} = B(\hat{U}_{\underline{\hat{M}}}\hat{U}_{\overline{\hat{N}}} + \hat{U}_{\overline{\hat{M}}}\hat{U}_{\underline{\hat{N}}}) + C\hat{\mathcal{H}}_{\hat{M}\hat{N}}, \tag{3.21}$$

where we have neglected the b_1 and b_2. Each term of (3.21) transforms covariantly under T duality through (2.2), so these transformations mix the components of the energy-momentum tensor as pointed out in [11]. The expression (3.21) hence captures all these (T-dual) tensors in a unified approach.

Using the space-time split formulation of DFT, the generalized velocity $\hat{U}_{\hat{M}}$ splits according to

$$\hat{U}^{\hat{M}} = (\tilde{u}_0, u^0, U^M). \tag{3.22}$$

Furthermore, from the relations (3.17)–(3.18) we get

$$\hat{U}_{\overline{\hat{M}}}\hat{U}^{\overline{\hat{M}}} = -\frac{1}{2}, \qquad \hat{U}_{\underline{\hat{M}}}\hat{U}^{\underline{\hat{M}}} = \frac{1}{2}. \tag{3.23}$$

Finally, we identify the quantities B and C in the generalized energy-momentum tensor (3.21) as

$$B = 2[\tilde{e}(X) + \tilde{p}(X)], \quad C = \tilde{p}(X). \tag{3.24}$$

Here we define $\tilde{e}(X)$ and $\tilde{p}(X)$ as the generalized notions of the energy density and pressure from a formal analogy between (3.21) and the structure of the usual energy-momentum tensor of the relativistic perfect fluid (3.20). The tilded variables in these expressions may be related to the ordinary energy density and pressure through a field redefinition as we show in Sec. IV C.

On the other hand, we extend the DFT projectors in the following way,

$$\bar{h}_{\hat{M}\hat{N}} = \overline{P}_{\hat{M}\hat{N}} + 2\overline{P}_{\hat{M}}{}^{\hat{P}}\overline{P}_{\hat{N}}{}^{\hat{Q}}\hat{U}_{\hat{P}}\hat{U}_{\hat{Q}}, \tag{3.25}$$

$$\underline{h}_{\hat{M}\hat{N}} = P_{\hat{M}\hat{N}} - 2P_{\hat{M}}{}^{\hat{P}}P_{\hat{N}}{}^{\hat{Q}}\hat{U}_{\hat{P}}\hat{U}_{\hat{Q}}. \tag{3.26}$$

The new projectors satisfy $\bar{h}_{\hat{M}\hat{P}}\bar{h}^{\hat{P}}{}_{\hat{N}} = \bar{h}_{\hat{M}\hat{N}}$ and $\underline{h}_{\hat{M}\hat{P}}\underline{h}^{\hat{P}}{}_{\hat{N}} = \underline{h}_{\hat{M}\hat{N}}$. These projectors also satisfy the orthogonality conditions

$$\bar{h}_{\hat{M}\hat{N}}\hat{U}^{\hat{M}} = 0, \tag{3.27}$$

$$\underline{h}_{\hat{M}\hat{N}}\hat{U}^{\hat{M}} = 0. \tag{3.28}$$

The conservation law for the generalized energy-momentum tensor was deduced in [20] and takes the form

$$\nabla_{\hat M}(2(\tilde e+\tilde p)\hat U^{\underline{\hat M}}\hat U^{\overline{\hat N}}+2(\tilde e+\tilde p)\hat U^{\overline{\hat M}}\hat U^{\underline{\hat N}}+\tilde p\hat{\mathcal H}^{\hat M\hat N})=0. \tag{3.29}$$

We can extract the generalization of the energy conservation equation for a perfect fluid and the generalization of the relativistic Euler equation in the double space from (3.29). Indeed the Eq. (3.29) may be projected using the generalized velocities, $\hat U_{\overline{\hat N}}, \hat U_{\underline{\hat N}}$ and the projectors $\bar h_{\hat N\hat P}, \underline{h}_{\hat N\hat P}$. Thus we get four equations, one per each type of projection.

First we project (3.29) using the different projections of the generalized velocity and we get

$$\hat U_{\overline{\hat N}}\nabla_{\hat M}(2(\tilde e+\tilde p)\hat U^{\underline{\hat M}}\hat U^{\overline{\hat N}}+2(\tilde e+\tilde p)\hat U^{\overline{\hat M}}\hat U^{\underline{\hat N}}+\tilde p\hat{\mathcal H}^{\hat M\hat N})=0, \tag{3.30}$$

$$\hat U_{\underline{\hat N}}\nabla_{\hat M}(2(\tilde e+\tilde p)\hat U^{\underline{\hat M}}\hat U^{\overline{\hat N}}+2(\tilde e+\tilde p)\hat U^{\overline{\hat M}}\hat U^{\underline{\hat N}}+\tilde p\hat{\mathcal H}^{\hat M\hat N})=0. \tag{3.31}$$

Therefore, the generalized relativistic energy conservation equation reads

$$\begin{aligned}
&-(\tilde e+\tilde p)\nabla_{\underline{\hat M}}\hat U^{\underline{\hat M}}-U^{\underline{\hat M}}\nabla_{\underline{\hat M}}(\tilde e+\tilde p)\\
&\quad+2(\tilde e+\tilde p)\hat U_{\overline{\hat N}}(\hat U^{\overline{\hat M}}\nabla_{\overline{\hat M}}U^{\underline{\hat N}}+\hat U^{\underline{\hat M}}\nabla_{\underline{\hat M}}U^{\overline{\hat N}})=-\hat U^{\overline{\hat M}}\nabla_{\overline{\hat M}}\tilde p,\\
&-(\tilde e+\tilde p)\nabla_{\overline{\hat M}}\hat U^{\overline{\hat M}}-U^{\overline{\hat M}}\nabla_{\overline{\hat M}}(\tilde e+\tilde p)\\
&\quad-2(\tilde e+\tilde p)\hat U_{\underline{\hat N}}(\hat U^{\underline{\hat M}}\nabla_{\underline{\hat M}}U^{\overline{\hat N}}+\hat U^{\overline{\hat M}}\nabla_{\overline{\hat M}}U^{\underline{\hat N}})=-\hat U^{\underline{\hat M}}\nabla_{\underline{\hat M}}\tilde p.
\end{aligned}$$

On the other hand the projection (3.29) with $\bar h_{\hat N\hat P}$ is

$$\bar h_{\hat N\hat P}\nabla_{\hat M}(2(\tilde e+\tilde p)\hat U^{\underline{\hat M}}\hat U^{\overline{\hat N}}+2(\tilde e+\tilde p)\hat U^{\overline{\hat M}}\hat U^{\underline{\hat N}}+\tilde p\hat{\mathcal H}^{\hat M\hat N})=0. \tag{3.32}$$

Consequently, the generalized Euler equation projected onto $\bar h_{\hat N\hat P}$, written in a duality covariant way, is

$$2(\tilde e+\tilde p)\bar h_{\hat N\hat P}(\hat U^{\overline{\hat M}}\nabla_{\overline{\hat M}}\hat U^{\underline{\hat N}}+\hat U^{\underline{\hat M}}\nabla_{\underline{\hat M}}\hat U^{\overline{\hat N}})=-\bar h_{\hat N\hat P}\nabla^{\overline{\hat N}}\tilde p. \tag{3.33}$$

Analogously we obtain the complementary generalized Euler equation projected with $\underline{h}_{\hat N\hat P}$,

$$2(\tilde e+\tilde p)\underline{h}_{\hat N\hat P}(\hat U^{\overline{\hat M}}\nabla_{\overline{\hat M}}\hat U^{\underline{\hat N}}+\hat U^{\underline{\hat M}}\nabla_{\underline{\hat M}}\hat U^{\overline{\hat N}})=\underline{h}_{\hat N\hat P}\nabla^{\hat N}\tilde p. \tag{3.34}$$

So far we have described how to incorporate matter into the standard formulation of DFT through the generalized energy-momentum tensor for a perfect fluid. In the next section we use this framework to explore the cosmological ansatz with matter.

IV. COSMOLOGICAL ANSATZ

DFT is a powerful framework to rewrite the low energy limit of ST in a T-duality covariant way, and, particularly, it is possible to make contact with the vacuum solutions of string cosmology using a suitable parametrization of the DFT fields and parameters. The pioneer work [14] introduced the cosmological equations for the dilaton and the metric tensor that came from the generalized Einstein tensor, while in [13] a generalized energy-momentum tensor for the matter coupled to the double geometry was included. Our main goal in this work is to achieve the manifestly covariant system of equations that couples the vacuum fields of the double geometry to matter using a cosmological ansatz at the DFT level. Indeed, in this section we elaborate on a common cosmological DFT framework considering the matter source as a perfect fluid through (3.21).

We first perform a space-time split of all the multiplets and then we derive the full dynamics regarding the matter and vacuum DFT fields (4.11). Second, we parametrize the vacuum fields as usual and write down the parametrized cosmological equations including $g_{\mu\nu}$, ϕ, and b-field contributions (4.16). In addition we find a parametrization of the DFT matter description, (4.22) and (4.23), which is compatible with the actual pressure and energy density in the string cosmology limit [6].

A. Space-time split DFT with matter

Considering the space-time split (2.17) with $\mathcal N_M=0$, $N=1$ and a cosmological principle for the fundamental fields, namely $\mathcal H_{MN}=A_{MN}(\tilde t,t)$ and $d=d(\tilde t,t)$, it is straightforward to decompose the generalized Ricci scalar

$$\begin{aligned}\hat{\mathcal R}=&-\frac{1}{8}\partial_0A^{MN}\partial_0A_{MN}-\frac{1}{8}\tilde\partial^0A^{MN}\tilde\partial^0A_{MN}-4\partial_{00}d\\&-4\tilde\partial^{00}d+4\partial_0d\partial_0d+4\tilde\partial^0d\tilde\partial^0d,\end{aligned} \tag{4.1}$$

the nonvanishing components of the generalized Ricci tensor

$$\begin{aligned}\hat{\mathcal R}_{00}&=-\hat{\mathcal R}^{00}=-\frac{1}{2}\hat{\mathcal K}^{00}+\frac{1}{2}\hat{\mathcal K}_{00},\\ \hat{\mathcal R}_{MN}&=P_M{}^P\hat{\mathcal K}_{PQ}\bar P^Q{}_N+\bar P_M{}^P\hat{\mathcal K}_{PQ}P^Q{}_N,\end{aligned} \tag{4.2}$$

and the nonvanishing components of $\mathcal K_{\hat M\hat N}$

$$\hat{\mathcal K}_{00}=\frac{1}{8}\partial_0A^{KL}\partial_0A_{KL}+2\partial_{00}d, \tag{4.3}$$

$$\hat{\mathcal K}^{00}=\frac{1}{8}\tilde\partial^0A^{KL}\tilde\partial^0A_{KL}+2\tilde\partial^{00}d, \tag{4.4}$$

$$\hat{\mathcal{K}}_0{}^0 = \hat{\mathcal{K}}^0{}_0 = \frac{1}{8}\partial_0 A^{KL}\tilde{\partial}^0 A_{KL}, \tag{4.5}$$

$$\hat{\mathcal{K}}_{MN} = \frac{1}{4}\partial_{00}A_{MN} + \frac{1}{4}\tilde{\partial}^{00}A_{MN} - \frac{1}{2}\partial_0 d\partial_0 A_{MN} - \frac{1}{2}\tilde{\partial}^0 d\tilde{\partial}^0 A_{MN}, \tag{4.6}$$

with (2.14) already imposed.

On the other hand, we consider that the matter is given by a perfect fluid in the double space, i.e.,

$$\hat{\mathcal{T}}^{\hat{M}\hat{N}} = 2(\tilde{e}+\tilde{p})\hat{U}^{\underline{\hat{M}}}\hat{U}^{\overline{\hat{N}}} + 2(\tilde{e}+\tilde{p})\hat{U}^{\overline{\hat{M}}}\hat{U}^{\underline{\hat{N}}} + \tilde{p}\hat{\mathcal{H}}^{\hat{M}\hat{N}}. \tag{4.7}$$

We choose a rest frame indicating the temporal direction such that $\hat{U}^{\hat{M}} = (0,1,0)$ and, therefore, the nonvanishing components of the generalized energy-momentum tensor are

$$\hat{\mathcal{T}}^{00} = (\tilde{e}+\tilde{p}) - \tilde{p}, \tag{4.8}$$

$$\hat{\mathcal{T}}_{00} = -(\tilde{e}+\tilde{p}) - \tilde{p}, \tag{4.9}$$

$$\hat{\mathcal{T}}_{MN} = \tilde{p}A_{MN}. \tag{4.10}$$

From (3.3), we get

$$\begin{gathered}
-\tilde{p} = 2(\partial_0 d)^2 - 2\partial_{00}d - \frac{1}{16}\partial_0 A^{MN}\partial_0 A_{MN} + (\partial_0 \leftrightarrow \tilde{\partial}^0),\\
\frac{1}{2}(\tilde{e}+\tilde{p}) = \partial_{00}d + \frac{1}{16}\partial_0 A^{KL}\partial_0 A_{KL} - (\partial_0 \leftrightarrow \tilde{\partial}^0),\\
0 = -P_M{}^P\left[\frac{1}{4}\partial_{00}A_{PQ} - \frac{1}{2}\partial_0 d\partial_0 A_{PQ} + (\partial_0 \leftrightarrow \tilde{\partial}^0)\right]\bar{P}^Q{}_N + (P \leftrightarrow \bar{P}).
\end{gathered} \tag{4.11}$$

These are the space-time split DFT equations that can be used to determine cosmological solutions with matter. Strictly speaking, the expressions (4.11) are combinations of the different components of the generalized Einstein equation.

B. Parametrization

In this section we parametrize all the fundamental quantities in order to analyze the dynamics of the usual supergravity fields. Indeed the parametrization of the $O(D-1,D-1)$ generalized metric reads

$$A_{MN} = \frac{1}{a^2}\begin{pmatrix} \delta^{ij} & -\delta^{ik}b_{kj} \\ b_{ik}\delta^{kj} & a^4\delta_{ij} - b_{ik}\delta^{kl}b_{lj} \end{pmatrix}, \tag{4.12}$$

where the spatial indices are $i,j = 1,\ldots,D-1$. Every component only depends on the ordinary time t due to the solution to the strong constraint $\tilde{\partial}^0 = 0$. Naturally, it is possible to inspect the dual solution imposing $\partial_0 = 0$, or any other T-dual combination. The parametrization of the generalized dilaton is given by

$$e^{-2d} = a^{D-1}e^{-2\phi}, \tag{4.13}$$

and, consequently, $d = \phi - \frac{(D-1)}{2}\ln(a)$. The spatial DFT projectors reads

$$P_{MN} = \frac{1}{2}(\eta_{MN} - A_{MN}), \quad \bar{P}_{MN} = \frac{1}{2}(\eta_{MN} + A_{MN}), \tag{4.14}$$

while

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix}. \tag{4.15}$$

Applying the expressions (4.12)–(4.15) to the system (4.11) is a straightforward computation, but breaking the duality group in terms of $Gl(D)$ representations generates large expressions that are suitable to manage with a software for symbolic algebraic tensor manipulation [28]. Using this code, the parametrization of the dynamical DFT cosmological equations is given by

$$\begin{gathered}
-2\tilde{p} = 2(D-1)\dot{H} + D(D-1)H^2 - 4\ddot{\phi} + 4\dot{\phi}^2 - 4(D-1)H\dot{\phi} + \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl},\\
\tilde{e} + \tilde{p} = 2\ddot{\phi} - (D-1)(\dot{H} + H^2) - \frac{1}{4a^4}\delta^{ij}\delta^{kl}\dot{b}_{ik}\dot{b}_{jl},\\
0 = (\dot{H} + (D-1)H^2 - 2H\dot{\phi})\delta^{ij} + \frac{1}{2a^4}\delta^{ik}\delta^{jl}\delta^{mn}\dot{b}_{km}\dot{b}_{ln},\\
0 = \delta^{jl}\ddot{b}_{il} + [(D-5)H - 2\dot{\phi}]\delta^{jk}\dot{b}_{ik},
\end{gathered} \tag{4.16}$$

where $H = \dot{a}/a$, and we keep the same notation for the pressure and the energy density at this supergravity level. The equations in (4.16) describe the family of cosmologies that can be rewritten in a DFT framework using the generalized Einstein equation (3.3) with the generalized energy-momentum tensor (4.7) representing matter. When $\tilde{e} = 0$, $\tilde{p} = 0$, and $b_{ij} = 0$, the vacuum solutions exactly reproduces the results of [14]. The relation between this model and string cosmologies with matter is analyzed in Sec. IV C.

C. From DFT to string cosmology

Here we analyze the relation between the DFT cosmology with matter and string cosmologies. One possible string cosmology scenario that incorporates matter is based on the following action

$$S_{\rm SC} = \frac{1}{2}\int d^D x\sqrt{-|g|}e^{-2\phi}[R + 4(\nabla\phi)^2] + S_M,$$
$$= \int d^D x\sqrt{-|g|}e^{-2\phi}\left[\frac{1}{2}(R + 4(\nabla\phi)^2) + L_{\rm mat}\right], \quad (4.17)$$

whose fundamental gravitational fields are the metric tensor $g_{\mu\nu}$ and the dilaton ϕ, while $b_{ij} = 0$. Moreover R is the usual Ricci scalar and S_M is the action term of matter. The equations of motion for the gravitational sector come from the variation of the action with respect to the metric tensor $\delta_g S_{\rm SC} = 0$ and to the dilaton $\delta_\phi S_{\rm SC} = 0$. In both variations the matter term acts as a source, indeed the source $\delta_g S_M$ is related to the energy-momentum tensor $T_{\mu\nu}$ and, analogously, a dilaton charge (or a dilaton source) σ is given by

$$\sigma = \frac{-1}{\sqrt{-g}}\frac{\delta S_M}{\delta\phi} = -e^{-2\phi}\left[\frac{\delta L_{\rm mat}}{\delta\phi} - 2L_{\rm mat}\right]. \quad (4.18)$$

Regarding the usual cosmological ansatz, the equations of motion in this string cosmology framework with matter sources become [6]

$$-e^{2\phi}\sigma = 2(D-1)\dot{H} + D(D-1)H^2 - 4\ddot{\phi} + 4\dot{\phi}^2$$
$$- 4(D-1)H\dot{\phi}, \quad (4.19)$$

$$-e^{2\phi}\left(e + \frac{\sigma}{2}\right) = (D-1)(\dot{H} + H^2) - 2\ddot{\phi}, \quad (4.20)$$

$$e^{2\phi}\left(p - \frac{\sigma}{2}\right) = \dot{H} + (D-1)H^2 - 2H\dot{\phi}, \quad (4.21)$$

where e and p represent the usual energy density and pressure of the perfect fluid.

It turns out that these equations can be obtained from our model of DFT with matter [Eqs. (3.3) and (4.7)] when

$$\tilde{p} = e^{2\phi}p, \quad (4.22)$$

$$\tilde{e} = e^{2\phi}e, \quad (4.23)$$

and $b_{ij} = 0$. In this case the consistency between both approaches forces

$$\sigma = 2p. \quad (4.24)$$

In consequence the proposal (4.7) allows one to write string cosmologies in which their dilaton charge is fixed according to (4.24). From the point of view of the action principle, the interplay between the dilaton charge and the variables of the fluid is expected since the generalized dilaton is a fundamental field which takes part of the DFT measure and indeed guarantees the invariance of the action under generalized diffeomorphisms. Another equivalent way to understand this phenomena is that the generalized energy-momentum tensor, $\hat{\mathcal{T}}^{\hat{M}\hat{N}}$, incorporates both the dilaton and metric sources contributions at once [see e.g., Eq. (1.22) in [20]].

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we have extended the standard vacuum DFT construction in order to include matter through a generalized energy-momentum tensor, which mimics the dynamics of a perfect fluid in the double geometry. We propose an ansatz for this tensor (3.21) in terms of a generalized notion of pressure, energy density, and velocity, the latter satisfying a strong constraintlike equation and a mass-shell-like condition (3.17)–(3.18). We explicitly derive the conservation laws for this tensor (3.32) and (3.33)–(3.34), which enable us the possibility to explore aspects of hydrodynamics at the DFT level.

We perform a space-time split where the dependence of the fundamental fields over the dual coordinates is preserved. We impose that the generalized shift vector and the generalized lapse function satisfy $\mathcal{N}_M = 0$ and $N = 1$, respectively, while the generalized metric and the generalized dilaton satisfy a cosmological ansatz according to $\mathcal{H}_{MN} = A_{MN}(\tilde{t}, t)$ and $d = d(\tilde{t}, t)$. This framework preserves the duality group $O(1,1) \times O(D-1, D-1)$ and allows us to rewrite string cosmology equations in a duality covariant language as shown in (4.11). Interestingly enough, the matter contributions do not affect all the components of the generalized Einstein equation, and the DFT cosmologies can be directly related to string cosmologies upon field redefinitions of $\tilde{p}$ and $\tilde{e}$ according to (4.22) and (4.23) when $b_{ij} = 0$. In a more general case, the present construction (4.11) allows one to obtain a family of duality covariant b-field contributions to string cosmology as in (4.16). The results of this work pave the way for the understanding of the effective inclusion of statistical matter in DFT with applications to ST.

We finish this section with a list of possible follow-up projects:

(1) Duality invariant distribution function: Making use of the construction given in [20] and considering a generalization of the Maxwell-Juttner distribution function, the generalized energy-momentum tensor for the perfect fluid might be computed from a kinetic approach. This is a broad line of investigation since the duality invariant distribution function would allow the inclusion of thermodynamic concepts in the double geometry such as the generalized entropy current or the study of equilibrium states.

(2) Beyond perfect fluids in DFT: The inclusion of a generalized entropy current from the phase space formulation should be the next step to interpret duality transformations on the generalized energy-momentum tensor as imperfect terms. At the level of the low energy limit of string theory, this issue was studied in [11].

(3) α' corrections: A systematic way to introduce these corrections in a DFT background was given in [19]. The corrections are related to higher-derivative terms that deform the generalized Einstein tensor when matter terms are neglected. It would be interesting to test this procedure when matter deformations are taken into account in the double geometry. This might bring new perspectives for string cosmology.

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