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On star-dagger matrices and the core-EP decomposition*

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Abstract

The concept of star-dagger matrices was introduced in 1984 by Hartwig and Spindelböck. While they completely characterized the star-dagger matrices by using a block decomposition of the form $\begin{bmatrix} P & Q \\ 0 & 0 \end{bmatrix}$, they also proposed the following open problem:
"Can the triangular form $\begin{bmatrix} P & Q \\ 0 & R \end{bmatrix}$ be used to obtain further results on the star-dagger matrices?"
In this paper, we have attempted this open problem by using an upper-triangularization of Schur's type for a square matrix, namely, the core-EP decomposition. Furthermore, similar problems regarding bi-dagger and bi-EP matrices are investigated.

Keywords: Star-dagger matrix, partial isometry, bi-normal, bi-dagger, bi-EP, EP matrix, normal matrix, group matrix, Moore–Penrose inverse
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1. Introduction

A square complex matrix A is said to be star-dagger if its conjugate transpose A^* commutes with its Moore-Penrose inverse $A^\dagger$. Star-dagger matrices were formally defined in 1984 by Hartwig and Spindelböck [9]. This class includes certain well-known classes of matrices as special cases such as idempotent matrices, partial isometries, and normal matrices. The class of normal matrices includes in turn hermitian, skew-hermitian, and unitary matrices. Together with the star-dagger matrices, three other types of matrices were studied in [9], namely, bi-normal, bi-dagger, and bi-EP matrices. The first of these was introduced by Campbell [3] in order to extend the normal matrices. The other two classes are generalizations of the

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concept of EP matrix [11]. Various authors [2, 4, 5, 7] have studied the inter-relationship between the classes of normal, bi-normal, bi-dagger, star dagger, partial isometry, idempotent and orthogonal projectors.

The objective of this work is to obtain new properties and characterizations for star-dagger matrices by using the core-EP decomposition of a matrix. Furthermore, similar problems regarding bi-dagger and bi-EP matrices are also studied.

The paper is organized as follows. Section 2 provides the notation used in this paper and some preliminary results. Section 3 deals with partial isometries and star-dagger matrices by using the core-EP decomposition. Sections 4 and 5 offer new characterizations of the bi-dagger and bi-EP matrices, respectively. The main tool is the core-EP decomposition. Finally, Section 6 is devoted to the study of matrices of index 2. More precisely, it is proved that in this case, the concepts of 2-EP , 2-index EP, and bi-dagger matrices are equivalent.

2. Notation and preliminaries results

Throughout this paper, we denote the set of $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. The symbols A^*, $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\mathrm{rk}(A)$ will stand for the conjugate transpose, null space, range (column space), and rank of $A \in \mathbb{C}^{m\times n}$, respectively. Moreover, $A^\dagger \in \mathbb{C}^{n\times m}$ represents the Moore-Penrose inverse $A \in \mathbb{C}^{m\times n}$, i.e., the unique solution to the four equations [1]:

$$(1)\ AA^\dagger A = A, \quad (2)\ A^\dagger AA^\dagger = A^\dagger, \quad (3)\ (AA^\dagger)^* = AA^\dagger, \quad (4)\ (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse induces the orthogonal projectors $P_A := AA^\dagger$ and $Q_A := A^\dagger A$ onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

The index of $A \in \mathbb{C}^{n\times n}$, denoted by $\mathrm{Ind}(A)$, is the smallest nonnegative integer k such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$. When $\mathrm{Ind}(A) \le 1$, the matrix A is called a *group matrix* (GM, for short). A special subset of the GM matrices are the well-known EP matrices. Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is an *EP matrix* if $\mathcal{R}(A) = \mathcal{R}(A^*)$ or, equivalently, $P_A = Q_A$ [11].

For any two square complex matrices A and B of the same size, the commutator of A and B will be denoted by $[A, B] = AB - BA$.

Recall that $A \in \mathbb{C}^{m\times n}$ is a partial isometry if it verifies $A^\dagger = A^*$. As mentioned in the Introduction, in order to extend the square partial isometries, Hartwig and Spindelböck [9] defined the star-dagger (SD, for short) matrices as the square matrices for which A^* commutes with $A^\dagger$, that is, $A \in \mathbb{C}^{n\times n}$ is SD if $[A^*, A^\dagger] = 0$.

The following inclusions for $\mathbb{C}^{n\times n}$ were proved in [9]:

$$\{\text{Orthogonal projectors}\} \subseteq \{\text{Partial isometries}\} \subseteq \{\text{SD}\},$$

$$\{\text{Orthogonal projectors}\} \subseteq \{\text{Idempotents}\} \subseteq \{\text{SD}\}.$$

Together with the SD matrices, three other types of matrices were studied in [9]. The first of them are called bi-normal matrices. A matrix $A \in \mathbb{C}^{n\times n}$ is bi-normal if $[AA^*, A^*A] = 0$. This type of matrices are an extension of normal matrices to matrices of arbitrary index. The other two classes are called bi-dagger and bi-EP matrices. A matrix $A \in \mathbb{C}^{n\times n}$ is called bi-dagger and bi-EP, if $(A^\dagger)^2 = (A^2)^\dagger$ and $[P_A, Q_A] = 0$, respectively. Note that both bi-dagger and bi-EP are extensions of EP matrices to matrices of arbitrary index. The relationship between these matrix classes as given on [9] is given below:

$$\{\text{bi-normal}\} \subseteq \{\text{bi-dagger}\} \subseteq \{\text{bi-EP}\}. \tag{2.1}$$

It was also proved

$$\{\text{SD}\} \cap \{\text{bi-normal}\} = \{\text{SD}\} \cap \{\text{bi-dagger}\} = \{\text{SD}\} \cap \{\text{bi-EP}\}.$$

In [12] H. Wang introduced a new triangular decomposition of Schur's type for a square matrix. It was proved that for any matrix $A \in \mathbb{C}^{n\times n}$ of index $k = \operatorname{Ind}(A)$, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \tag{2.2}$$

where T is a nonsingular matrix of size $t \times t$ whose diagonal entries are nonzero eigenvalues of A, and N is nilpotent with $\operatorname{Ind}(N) = k$. This representation of A is called the *core-EP decomposition* of A. Notice that if A is nonsingular (that is, $k = 0$) if and only if $t = n$, and A is nilpotent if and only if $t = 0$.

Henceforth, we can assume $\operatorname{Ind}(A) = k \geq 1$ whenever the core EP decomposition is used.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2), $m \in \mathbb{N}$, and $s = \lceil \frac{k}{m} \rceil$. Then $\operatorname{Ind}(A^m) = s$ and*

$$A^m = U \begin{bmatrix} T^m & \tilde{T}_m \\ 0 & N^m \end{bmatrix} U^*, \quad \tilde{T}_m := \sum_{i=0}^{m-1} T^i S N^{m-1-i}, \tag{2.3}$$

is the core-EP decomposition of A^m with $t = \operatorname{rk}(T^m) = \operatorname{rk}(A^{ms})$, and N^m is nilpotent of index s. In particular, the Moore-Penrose inverse of A^m is given by

$$(A^m)^\dagger = U \begin{bmatrix} (T^m)^*\Delta_m & -(T^m)^*\Delta_m \tilde{T}_m (N^m)^\dagger \\ \Omega_m^*\Delta_m & (N^m)^\dagger - \Omega_m^*\Delta_m \tilde{T}_m (N^m)^\dagger \end{bmatrix} U^*, \tag{2.4}$$

where

$$\Delta_m := (T^m (T^m)^* + \Omega_m \Omega_m^*)^{-1} \quad \textit{and} \quad \Omega_m := \tilde{T}_m (I_{n-t} - Q_{N^m}).$$

Proof. Let $r = ms - k$. Note that $\operatorname{Ind}(A^m) = s$. In fact, when $m < k$, we have $m(s-1) < k$ by definition of ceiling function. Therefore, $\operatorname{rk}((A^m)^s) = \operatorname{rk}(A^{k+r}) = \operatorname{rk}(A^{k+r+m}) = \operatorname{rk}((A^m)^{s+1})$ and $\operatorname{rk}((A^m)^s) =$

$\mathrm{rk}(A^{k+r}) = \mathrm{rk}(A^k) < \mathrm{rk}(A^{k-1}) \leq \mathrm{rk}((A^m)^{s-1})$.
On the other hand, if $k \leq m$, then $\mathrm{rk}(A^m) = \mathrm{rk}(A^{k+r}) = \mathrm{rk}(A^{k+r+m}) = \mathrm{rk}((A^m)^2)$.
Now, as $\mathrm{rk}(T^m) = \mathrm{rk}(T) = \mathrm{rk}(A^k) = \mathrm{rk}((A^m)^s)$ and clearly N^m is nilpotent of index s, we get that (2.3) is the core-EP decomposition of A^m. Finally, (2.4) follows from [6, Theorem 3.9]. □

Henceforth I_n will refer to the identity matrix of order n. From above lemma we derive the following expressions for the orthogonal projectors

$$P_{A^m} = U \begin{bmatrix} I_t & 0 \\ 0 & P_{N^m} \end{bmatrix} U^*, \quad Q_{A^m} = U \begin{bmatrix} (T^m)^*\Delta_m T^m & (T^m)^*\Delta_m\Omega_m \\ \Omega_m^*\Delta_m T^m & Q_{N^m} + \Omega_m^*\Delta_m\Omega_m \end{bmatrix} U^*, \quad m \in \mathbb{N}. \tag{2.5}$$

Remark 2.2. (i) $\tilde{T}_1 = S$ and $\Delta_1 = (TT^* + \Omega_1\Omega_1^*)^{-1}$, where $\Omega_1 = S(I_{n-t} - Q_N)$.

(ii) $\tilde{T}_2 = TS + SN$ and $\Delta_2 = (T^2(T^2)^* + \Omega_2\Omega_2^*)^{-1}$, where $\Omega_2 = (TS + SN)(I_{n-t} - Q_{N^2})$.

(iii) If $SN^* = 0$ then $\Omega_1 = S$ and $\Delta_1 = (TT^* + SS^*)^{-1}$.

(iv) If $S(N^*)^2 = 0$ then $\Omega_2 = TS + SN(I_{n-t} - Q_{N^2})$ and $\Delta_2 = (T^2(T^2)^* + \Omega_2\Omega_2^*)^{-1}$.

3. Star-dagger matrices and the core-EP decomposition

In this section, we provide some results concerning the characterizations of partial isometries and star-dagger matrices by using the core-EP decomposition.

Recall that the concept of unitary (isometry) matrices has been extended as partial isometry to rectangular matrices, using the Moore-Penrose inverse. Later, the concept of partial isometry was extended to SD matrices. Also, from [9, Remark 3] it follows that the class {normal} is a subset of {SD}. Further, the two classes coincide in case of nonsingular matrices.

Next, we characterize partial isometries and their powers by using the core-EP decomposition.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) and $m \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) *A^m is a partial isometry;*

(b) *N^m is a partial isometry, $\Delta_m = I_t$, and $\tilde{T}_m(N^m)^* = 0$.*

Proof. By definition, A^m is a partial isometry if $(A^m)^* = (A^m)^\dagger$. So, from (2.3) and (2.4) we have $(A^m)^* = (A^m)^\dagger$ if and only if the following conditions simultaneously hold:
(i) $(T^m)^*\Delta_m = (T^m)^*$;
(ii) $-(T^m)^*\Delta_m\tilde{T}_m(N^m)^\dagger = 0$;

(iii) $\Omega_m^* \Delta_m = \tilde{T}_m^*$;

(iv) $(N^m)^\dagger - \Omega_m^* \Delta_m \tilde{T}_m (N^m)^\dagger = (N^m)^*$.

As T and Δ_m are nonsingular, clearly conditions (i)-(iv) are equivalent to $\Delta_m = I_t$, $\tilde{T}_m (N^m)^* = 0$, and $(N^m)^* = (N^m)^\dagger$. Thus the conclusion. □

Corollary 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2). Then the following conditions are equivalent:*

(a) *A is a partial isometry;*

(b) *N is a partial isometry, $\Delta_1 = I_t$, and $SN^* = 0$.*

Proof. Follows from Theorem 3.1 for $m = 1$ and Remark 2.2 (i). □

From [9], we know that if A is a partial isometry, A^2 need not be a partial isometry.

Example 3.3. *Let* $A = \begin{bmatrix} 2/3 & -1/3 & 0 \\ 2/3 & 2/3 & 0 \\ -1/3 & 2/3 & 0 \end{bmatrix}$. *A straightforward computation yields*

$$A^\dagger = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}, \quad (A^2)^\dagger = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 8/9 \end{bmatrix}.$$

Thus, it is clear that A is a partial isometry but A^2 is not.

Next, we give necessary and sufficient conditions for the square of a partial isometry to be a partial isometry.

Corollary 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that A is a partial isometry. Then the following conditions are equivalent:*

(a) *A^2 is a partial isometry;*

(b) *N^2 is a partial isometry, $\Delta_2 = I_t$, and $SN(N^2)^* = 0$.*

Proof. It is a consequence from Theorem 3.1 for $m = 2$, Corollary 3.2, and Remark 2.2 (ii). □

Note that if A is written as in (2.2), a necessary condition for A to be a partial isometry is that N is also a partial isometry. Since the class of square partial isometries is a proper subset of the class of SD matrices, it is natural to ask whether a similar result is valid for the case of SD matrices. Next, we answer that question. Before, we need the following auxiliary lemma.

Lemma 3.5. *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) *A is* SD*;*

(b) $[AA^*A, A] = 0$*;*

(c) $A^m A^* A = AA^* A^m$*, for all integer $m \geq 2$;*

(d) $A^\dagger A^m A^* = A^* A^m A^\dagger$*, for all integer $m \geq 2$.*

Furthermore, if A is written as in (2.2), then $SN^* = 0$ and $\tilde{T}_m S^* = T^{m-1}SS^*$.

Proof. (a) $\Leftrightarrow$ (b). By [9, Proposition 2].
(b) $\Rightarrow$ (c). Note that $[AA^*A, A] = 0$ is equivalent to $A^2A^*A = AA^*A^2$. Now, the implication follows by induction on m.
(c) $\Rightarrow$ (b). Trivial.
(c) $\Rightarrow$ (d). It follows that by pre and post- multiplying by $A^\dagger$ in $A^mA^*A = AA^*A^m$ and by using the fact that $A^* = A^\dagger AA^* = A^*AA^\dagger$.
(d) $\Rightarrow$ (c). This follows by pre and post- multiplying by A in $A^\dagger A^m A^* = A^*A^mA^\dagger$ and by using the fact that $A^\dagger$ is an inner inverse of A.
Finally, we assume that A is written as in (2.2) and satisfies (c) for $m = k$. Then $A^k(A^*A) = (AA^*)A^k$, which is equivalent to

$$\begin{bmatrix} T^k & \tilde{T}_k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^*T & T^*S \\ S^*T & S^*S + N^*N \end{bmatrix} = \begin{bmatrix} TT^* + SS^* & SN^* \\ NS^* & NN^* \end{bmatrix} \begin{bmatrix} T^k & \tilde{T}_k \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} T^kT^*T + \tilde{T}_kS^*T & T^kT^*S + \tilde{T}_k(S^*S + N^*N) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (TT^* + SS^*)T^k & (TT^* + SS^*)\tilde{T}_k \\ NS^*T^k & NS^*\tilde{T}_k \end{bmatrix},$$

whence $SN^* = 0$, because T is nonsingular. Moreover, from (2.3) it is clear that $\tilde{T}_mS^* = T^{m-1}SS^*$. $\square$

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2). Then the following conditions are equivalent:*

(a) *A is* SD*;*

(b) *N is* SD*, $T\Delta_1 = \Delta_1 T$, $SN^* = 0$, and $SN = \Delta_1 SNN^*N$.*

Proof. (a)$\Rightarrow$(b). By Lemma 3.5 we know that A is SD if and only if $A(AA^*)A = (AA^*)A^2$. In particular, $SN^* = 0$. Moreover, from Remark 2.2 we have $\Delta_1^{-1} = TT^* + SS^*$. In consequence,

$$A(AA^*)A = (AA^*)A^2 \;\Leftrightarrow\; \begin{bmatrix} T\Delta_1^{-1}T & T\Delta_1^{-1}S + SNN^*N \\ 0 & N^2N^*N \end{bmatrix} = \begin{bmatrix} \Delta_1^{-1}T^2 & \Delta_1^{-1}\tilde{T}_2 \\ 0 & NN^*N^2 \end{bmatrix}, \quad (3.1)$$

where $\tilde{T}_2 = TS + SN$.
Since T is non-singular, it is clear that (3.1) holds if and only if

$$T\Delta_1 = \Delta_1 T, \quad T\Delta_1^{-1}S + SNN^*N = \Delta_1^{-1}(TS + SN), \quad \text{and} \quad N^2N^*N = NN^*N^2. \tag{3.2}$$

Since $T\Delta_1 = \Delta_1 T$, the second equality in (3.2) is equivalent to $SN = \Delta_1 SNN^*N$. Also, by Lemma 3.5 it is clear that the third equality in (3.2) is equivalent to N being a SD matrix. This completes the implication.
(b)⇒(a). Follows from Lemma 3.5. □

Now, we obtain conditions under which the square of an SD matrix is again SD.

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that A is* SD*. Then the following conditions are equivalent:*

(a) A^2 *is* SD*;*

(b) N^2 *is* SD, $[T^2, \Delta_1^{-1}TT^*+SNN^*S^*] = 0$, $SN(N^2)^* = 0$, *and* $(\Delta_1^{-1}TT^*+SNN^*S^*)\tilde{T}_2N^2 = \tilde{T}_2N^2(N^2)^*N^2$.

Proof. According to Lemma 2.1 we have

$$A^2 = U\begin{bmatrix} T^2 & \tilde{T}_2 \\ 0 & N^2 \end{bmatrix} U^*, \tag{3.3}$$

is the core EP decomposition of A^2.
(a)⇒(b). From (3.3) and Theorem 3.6 we obtain

$$A^2 \text{ is SD} \Leftrightarrow T^2\Delta_2 = \Delta_2 T^2, \quad \tilde{T}_2(N^2)^* = 0, \quad \tilde{T}_2N^2 = \Delta_2\tilde{T}_2N^2(N^2)^*N^2, \quad N^2 \text{ is SD}. \tag{3.4}$$

Moreover, as A is SD, by applying again Theorem 3.6 we have that $SN^* = 0$ and $T\Delta_1 = \Delta_1 T$. In consequence, from Remark 2.2 we have $\Delta_1^{-1} = TT^* + SS^*$, and so

$$\begin{aligned} \Delta_2^{-1} &= T^2(T^2)^* + \tilde{T}_2(\tilde{T}_2)^* \\ &= T^2(T^2)^* + (TS+SN)(TS+SN)^* \\ &= T^2(T^2)^* + TSS^*T^* + SNS^*T^* + TSN^*S^* + SNN^*S^* \\ &= T(TT^*)T^* + T(SS^*)T^* + SNN^*S^* \\ &= T\Delta_1^{-1}T^* + SNN^*S^* \\ &= \Delta_1^{-1}TT^* + SNN^*S^*. \end{aligned}$$

Thus, the first and third equations in (3.4) lead respectively to

$$[T^2, \Delta_1^{-1}TT^* + SNN^*S^*] = 0, \quad (\Delta_1^{-1}TT^* + SNN^*S^*)\tilde{T}_2N^2 = \tilde{T}_2N^2(N^2)^*N^2. \tag{3.5}$$

Also, the second equation in (3.4) leads to

$$SN(N^2)^* = 0. \tag{3.6}$$

Now, the implication follows from (3.5), (3.6) and the last condition in (3.4).

(b)⇒(a). Easy. □

Corollary 3.8. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that $SN = 0$ and A is* SD*. Then A^2 is* SD *if and only if N^2 is* SD *and $[T^2, T^*] = 0$.*

Proof. Since $SN = 0$, then $\tilde{T}_2 N = (TS + SN)N = 0$. Thus, as A is SD, by Theorem 3.7 it is clear that A^2 is SD if and only if N^2 is SD and $[T^2, \Delta_1^{-1}TT^*] = 0$. However, $[T^2, \Delta_1^{-1}TT^*] = 0$ is equivalent to $[T^2, T^*] = 0$, as Theorem 3.6 gives that $SN^* = 0$ and $T\Delta_1 = \Delta_1 T$. Now, the affirmation follows from the nonsingularity of T and Δ_1. □

Theorem 3.9. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that $SN^* = 0$. Then the Moore-Penrose inverse of A is given by*

$$A^\dagger = U \begin{bmatrix} T^*(TT^* + SS^*)^{-1} & 0 \\ S^*(TT^* + SS^*)^{-1} & N^\dagger \end{bmatrix} U^*. \tag{3.7}$$

In particular, if A is SD *then the Moore-Penrose inverse of A is as in* (3.7)*.*

Proof. The equality (3.7) is an immediate consequence from (2.4) for $m = 1$, and Remark 2.2.
In particular, if A is SD, the last affirmation of the theorem follows from Lemma 3.5. □

Hartwig and Spindelböck [9, Proposition 8] proved that if A is a partial isometry then:

$$A^2 \text{ is a partial isometry} \;\Leftrightarrow\; A \text{ is bi-normal} \;\Leftrightarrow\; A \text{ is bi-dagger} \;\Leftrightarrow\; A \text{ is bi-EP}.$$

Since the set of isometries partial is a proper subset of the class SD, it naturally leads one to think about the validity of such equivalences for the larger class of SD matrices. The following theorem clarifies this situation.

Theorem 3.10. *Let $A \in \mathbb{C}^{n\times n}$. If A is* SD*, then any one of the following three statements implies A^2 is* SD*:*

(a) *A is bi-normal;*

(b) *A is bi-dagger;*

(c) *A is bi-EP.*

Proof. Since A is SD, it is well known that (a), (b), and (c) are equivalent [9]. So, it is sufficient to assume that one of the three statements holds. Suppose A is bi-dagger, that is, $(A^2)^\dagger = A^\dagger A^\dagger$. Since $A^* A^\dagger = A^\dagger A^*$, we obtain $(A^2)^*(A^2)^\dagger = A^* A^* A^\dagger A^\dagger = A^\dagger A^\dagger A^* A^* = (A^2)^\dagger (A^2)^*$. It follows that A^2 is SD. □

However, the reciprocal implications are false. In fact, for example take the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since A is idempotent, we know that it is SD, and so is A^2. However, A is not bi-EP. Similarly, since the class bi-EP contains both the bi-normal and bi-dagger matrix classes, the remaining reciprocal implications are also false.

4. Bi-EP matrices and the core-EP decomposition

In this section, we derive a characterization of bi-EP matrices by using the core-EP decomposition.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2). Then the following conditions are equivalent:*

(a) *A is bi-EP;*

(b) *$[P_N, Q_N + \Omega_1^* \Delta_1 \Omega_1] = 0$ and $\Omega_1(I_{n-t} - P_N) = 0$;*

(c) *N is bi-EP and $\Omega_1(I_{n-t} - P_N) = 0$;*

(d) *N is bi-EP and $S(I_{n-t} - P_N)(I_{n-t} - Q_N) = 0$.*

Proof. (a) ⇔ (b). From (2.5), direct calculations yield

$$Q_A P_A = \begin{bmatrix} T^*\Delta_1 T & T^*\Delta_1\Omega_1 P_N \\ \Omega_1^*\Delta_1 T & (Q_N + \Omega_1^*\Delta_1\Omega_1)P_N \end{bmatrix}, \quad P_A Q_A = \begin{bmatrix} T^*\Delta_1 T & T^*\Delta_1\Omega_1 \\ P_N\Omega_1^*\Delta_1 T & P_N(Q_N + \Omega_1^*\Delta_1\Omega_1) \end{bmatrix}.$$

By definition, A is bi-EP if and only if $Q_A P_A = P_A Q_A$ if and only if all the following three conditions hold:

(i) $T^*\Delta_1\Omega_1 P_N = T^*\Delta_1\Omega_1$;

(ii) $(Q_N + \Omega_1^*\Delta_1\Omega_1)P_N = P_N(Q_N + \Omega_1^*\Delta_1\Omega_1)$.

Since T and Δ_1 are nonsingular, (i) holds if and only if $\Omega_1(I_{n-t} - P_N) = 0$.
By definition of the commutator, it is clear that (ii) is equivalent to $[P_N, Q_N + \Omega_1^*\Delta_1\Omega_1] = 0$.
(b) ⇔ (c). Note that under condition $\Omega_1(I_{n-t} - P_N) = 0$, it follows that $[I_{n-t} - P_N, \Omega_1^*\Delta_1\Omega_1] = 0$.
Thus the equivalence follows from the identity

$$[P_N, Q_N + \Omega_1^*\Delta_1\Omega_1] = [P_N, Q_N] + [P_N, \Omega_1^*\Delta_1\Omega_1] = -[I_{n-t} - P_N, \Omega_1^*\Delta_1\Omega_1] + [P_N, Q_N].$$

(c) ⇔ (d). It directly follows from the equality $[P_N, Q_N] = 0$. □

Above theorem and Lemma 2.1 for $m = 2$ yield to the following corollaries.

Corollary 4.2. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that $\Omega_1 = 0$. Then A is bi-EP if and only if N is bi-EP.*

Corollary 4.3. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2). Then the following conditions are equivalent:*

(a) *A^2 is bi-EP;*

(b) *N^2 is bi-EP and $(TS + SN)(I_{n-t} - P_{N^2})(I_{n-t} - Q_{N^2}) = 0$.*

5. Bi-dagger matrices and the core-EP decomposition

In this section we characterize the bi-dagger matrices by using the core-EP decomposition. Before, we present several auxiliary results.

Lemma 5.1. *Let $A, B \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) *$Q_A BB^*$ is Hermitian;*

(b) *$[Q_A, BB^*] = 0$;*

(c) *$Q_A B = BQ_{AB}$.*

Proof. It is consequence of the proofs of Theorem 2 and Theorem 4 in [8]. □

Lemma 5.2. *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) *$Q_A AA^*$ is Hermitian;*

(b) *$[Q_A, AA^*] = 0$;*

(c) *$Q_A A = AQ_{A^2}$;*

(d) *A is bi-EP and $[P_A Q_A, AA^*] = 0$.*

Furthermore, if A is written as in (2.2), then any of the above conditions is equivalent to $T^*\Delta_1 = T(T^2)^*\Delta_2 + S\Omega_2^*\Delta_2$, $\Delta_1^{-1}(T^*)^{-1}SQ_{N^2} = TS + \Omega_1 N - \Omega_2$, $\Omega_1^*\Delta_1 = N\Omega_2^*\Delta_2$, and $\Omega_1^*(T^*)^{-1}SQ_{N^2} = NQ_{N^2} - Q_N N$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). It is consequence of Lemma 5.1 with $A = B$.
(b) $\Rightarrow$ (d). Firstly, we will prove that A is bi-EP which is equivalent to $A^2(A^\dagger)^2A^2 = A^2$ [9, Corollary 3]. Further, we note that this condition holds if and only if $A^2(A^\dagger)^2A^2(A^2)^* = A^2(A^2)^*$ since $\mathcal{R}(A^2(A^2)^*) = \mathcal{R}(A^2)$. Thus, it is sufficient to prove this last equality. In fact, as $Q_AAA^* = AA^*Q_A$, we obtain

$$A^2(A^\dagger)^2A^2(A^2)^* = AP_A(Q_AAA^*)A^* = AP_A(AA^*Q_A)A^* = A(P_AA)A^*(Q_AA^*) = A^2(A^2)^*,$$

whence A is bi-EP.
Now, as $P_AQ_A = Q_AP_A$ and $Q_AAA^* = AA^*Q_A$ we have

$$P_AQ_AAA^* = Q_AP_AAA^* = Q_AAA^* = AA^*Q_A = AA^*P_AQ_A,$$

that is, $[P_AQ_A, AA^*] = 0$.
(d) $\Rightarrow$ (b). As A is bi-EP, i.e., $P_AQ_A = Q_AP_A$, and $[P_AQ_A, AA^*] = 0$ we have

$$Q_AAA^* = Q_AP_AAA^* = P_AQ_AAA^* = AA^*P_AQ_A = AA^*Q_A.$$

Finally, assume that A is written as in (2.2). From (2.5) for $m = 1, 2$, we have that $Q_AA = AQ_{A^2}$ if and only if the following conditions simultaneously hold:

(i) $T^*\Delta_1 = T(T^2)^*\Delta_2 + S\Omega_2^*\Delta_2$;

(ii) $T^*\Delta_1TS + T^*\Delta\Omega_1N = T(T^2)^*\Delta_2\Omega_2 + SQ_{N^2} + S\Omega_2^*\Delta_2\Omega_2$;

(iii) $\Omega_1^*\Delta_1 = N\Omega_2^*\Delta_2$;

(iv) $\Omega_1^*\Delta_1TS + Q_NN + \Omega_1^*\Delta_1\Omega_1N = NQ_{N^2} + N\Omega_2^*\Delta_2\Omega_2$.

According to (i) we have that (ii) is equivalent to $T^*\Delta_1TS + T^*\Delta\Omega_1N = SQ_{N^2} + T^*\Delta_1\Omega_2$, that is,

$$\Delta_1^{-1}(T^*)^{-1}SQ_{N^2} = TS + \Omega_1N - \Omega_2 \tag{5.1}$$

since T and Δ_1 are nonsingular.
Using (iii) we see that (iv) is equivalent to $\Omega_1^*\Delta_1(TS+\Omega_1N-\Omega_2) = NQ_{N^2} - Q_NN$, that is, $\Omega_1^*(T^*)^{-1}SQ_{N^2} = NQ_{N^2} - Q_NN$ by (5.1). The proof is complete. □

Lemma 5.3. *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(a) A^*AP_A *is Hermitian;*

(b) $[A^*A, P_A] = 0$*;*

(c) $P_A A^* = A^* P_{A^2}$;

(d) A *is bi-EP and* $[Q_A P_A, A^* A] = 0$.

Furthermore, if A *is written as in (2.2), then any of the above conditions is equivalent to* $P_N N^* = N^* P_{N^2}$ *and* $S(I_{n-t} - P_N) = 0$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). It directly follows from Lemma 5.2 by taking A^* instead of A, and using the fact that A is bi-EP if and only if A^* is bi-EP.
Finally, if A is written as in (2.2), from (2.5) for $m = 1, 2$, we have that $P_A A^* = A^* P_{A^2}$ if and only if $P_N N^* = N^* P_{N^2}$ and $S(I_{n-t} - P_N) = 0$. The proof is complete. □

Now, we are able to obtain some new characterizations of bi-dagger matrices.

Theorem 5.4. *Let* $A \in \mathbb{C}^{n\times n}$. *Then the following conditions are equivalent:*

(a) A *is bi-dagger;*

(b) $Q_A A A^*$ *and* $A^* A P_A$ *are Hermitian;*

(c) $[Q_A, AA^*] = 0$ *and* $[A^* A, P_A] = 0$;

(d) $Q_A A = A Q_{A^2}$ *and* $P_A A^* = A^* P_{A^2}$;

(e) A *is bi-EP,* $[P_A Q_A, AA^*] = 0$, *and* $[Q_A P_A, A^* A] = 0$.

Furthermore, if A *is written as in (2.2), then any of the above conditions is equivalent to* $T^*\Delta_1 = T(T^2)^*\Delta_2 + S\Omega_2^*\Delta_2$, $\Delta_1^{-1}(T^*)^{-1} S Q_{N^2} = TS + \Omega_1 N - \Omega_2$, $\Omega_1^*\Delta_1 = N\Omega_2^*\Delta_2$, $\Omega_1^*(T^*)^{-1} S Q_{N^2} = N Q_{N^2} - Q_N N$, $P_N N^* = N^* P_{N^2}$, *and* $S(I_{n-t} - P_N) = 0$.

Proof. (a) $\Leftrightarrow$ (b). Since A is bi-dagger if and only if $(A^2)^\dagger = (A^\dagger)^2$, the equivalence can be derived by taking $A = B$ in [8, Theorem 2].
The remainder of the proof directly follows from Lemma 5.2 and Lemma 5.3. □

Corollary 5.5. *Let* $A \in \mathbb{C}^{n\times n}$ *be written as in (2.2) such that* N *is bi-dagger. Then the following conditions are equivalent:*

(a) A *is bi-dagger;*

(b) $T^*\Delta_1 = T(T^2)^*\Delta_2 + S\Omega_2^*\Delta_2$, $\Delta_1^{-1}(T^*)^{-1} S Q_{N^2} = TS + \Omega_1 N - \Omega_2$, $\Omega_1^*\Delta_1 = N\Omega_2^*\Delta_2$, $\Omega_1^*(T^*)^{-1} S Q_{N^2} = 0$, *and* $S(I_{n-t} - P_N) = 0$.

Proof. As N is bi-dagger, by Theorem 5.4 (d) we have $Q_N N = NQ_{N^2}$ and $P_N N^* = N^* P_{N^2}$. So, by applying again Theorem 5.4 the result follows. □

Unlike what happens with partial isometries, SD matrices and bi-EP matrices, in the case of a bi-dagger matrix A written as in (2.2), it can be seen that N does not inherit the property of being also bi-dagger. The following results provide certain conditions under which N is also bi-dagger.

Corollary 5.6. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that A is bi-dagger. Then N is bi-dagger if and only if $\Omega_1^*(T^*)^{-1}SQ_{N^2} = 0$.*

Proof. As A is bi-dagger, according to Theorem 5.4 we have $\Omega_1^*(T^*)^{-1}SQ_{N^2} = NQ_{N^2} - Q_N N$ and $P_N N^* = N^* P_{N^2}$. Now, Theorem 5.4 completes the proof. □

Corollary 5.7. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that $S(N^*)^2 = 0$. Then the following conditions are equivalent:*

(a) *A is bi-dagger;*

(b) *N is bi-dagger, $T^*\Delta_1 = T(T^2)^*\Delta_2 + S\Omega_2^*\Delta_2$, $\Omega_1^*\Delta_1 = N\Omega_2^*\Delta_2$, $TS + \Omega_1 N = \Omega_2$, and $S(I_{n-t} - P_N) = 0$.*

Proof. By hypothesis $SQ_{N^2} = 0$. Hence $\Omega_1^*(T^*)^{-1}SQ_{N^2} = \Delta_1^{-1}(T^*)^{-1}SQ_{N^2} = 0$. Now, Corollary 5.5 and Corollary 5.6 complete the proof. □

Corollary 5.8. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that $SN^* = 0$. Then the following conditions are equivalent:*

(a) *A is bi-dagger;*

(b) *N is bi-dagger, $T^*\Delta_1 = T(T^2)^*\Delta_2 + S\Omega_2^*\Delta_2$, $\Omega_1^*\Delta_1 = N\Omega_2^*\Delta_2$, and $S(I_{n-t} - P_N) = 0$.*

Proof. By hypothesis $SQ_N = 0$. Moreover, since $S(N^*)^2 = 0$, we have $SQ_{N^2} = 0$. Therefore $\Omega_1^*(T^*)^{-1}SQ_{N^2} = \Delta_1^{-1}(T^*)^{-1}SQ_{N^2} = TS + \Omega_1 N - \Omega_2 = 0$. Now, Corollary 5.5 and Corollary 5.6 complete the proof. □

Corollary 5.9. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that $S = 0$. Then A is bi-dagger if and only if N is bi-dagger.*

Proof. Since $S = 0$, $\tilde{T}_2 = 0$. Thus, $\Omega_1 = \Omega_2 = 0$, $\Delta_1 = (TT^*)^{-1}$, and $\Delta_2 = (T^2(T^2)^*)^{-1}$. Now, the result follows from Corollary 5.8. □

6. Some considerations about matrices of index 2

In this section, we study the concepts of k-EP matrix and k-index EP matrix when $k = 2$. More precisely, in this case we prove that both 2-EP matrices and 2-index matrices are equivalent to the matrix being bi-dagger.

The concepts of k-EP matrices and k-index-EP matrices were introduced to extend the class of EP matrices to square complex matrices of arbitrary index. A matrix A with $\text{Ind}(A) = k$ is called k-index EP if A^k is EP [13]. The class of all k-index EP matrices is denoted by $\mathbb{C}_n^{k,iEP}$. A matrix $A \in \mathbb{C}^{n\times n}$ with $\text{Ind}(A) = k$ is called k-EP matrix if $A^k A^\dagger = A^\dagger A^k$ [10]. The class of all k-EP matrices is denoted by $\mathbb{C}_n^{k,\dagger}$. Both classes of matrices can be characterized by using core-EP decomposition given in (2.2):

$$A \in \mathbb{C}_n^{k,iEP} \quad \Leftrightarrow \quad S = 0, \quad [13, \text{Theorem } 2.3] \tag{6.1}$$

$$A \in \mathbb{C}_n^{k,\dagger} \quad \Leftrightarrow \quad S(I_{n-t} - Q_N) = 0 \text{ and } \tilde{T}_k(I_{n-t} - P_N) = 0, \quad [6, \text{Theorem } 3.10] \tag{6.2}$$

where $\tilde{T}_k = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$.

By using (6.1) and (6.2), one can easily verify that [13, Remark 2.1]

$$\mathbb{C}_n^{k,iEP} \subseteq \mathbb{C}_n^{k,\dagger}. \tag{6.3}$$

The following examples show that the class $\mathbb{C}_n^{k,iEP}$ is strictly included in $\mathbb{C}_n^{k,\dagger}$.

Example 6.1. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $\text{Ind}(A) = 3$. *Denoting* $T = 1$, $S = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$ *and* $N = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, *we have* $I_3 - Q_N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $I_3 - P_N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Therefore,*

$$S(I_3 - Q_N) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad (T^2 S + TSN + SN^2)(I_3 - P_N) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Consequently $A \in \mathbb{C}_n^{k,\dagger}$ *by* (6.2). *However, since* $S \neq 0$, *according to* (6.1) *we have* $A \notin \mathbb{C}_n^{k,iEP}$.

As can be seen in the above example $\text{Ind}(A) = 3$. Next, we show that if $\text{Ind}(A) = 2$, the concepts of 2-EP, 2-index EP matrix, and bi-dagger are equivalent.

Remark 6.2. *Clearly, if* $\text{Ind}(A) = k \leq 1$, *the concepts of k-EP , k-index EP matrix, and bi-dagger are equivalent since they coincide with the concept of EP matrix.*

Theorem 6.3. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\text{Ind}(A) = 2$. *Then the following conditions are equivalent:*

(a) *A is 2-index EP (or equivalently* A^2 *is EP);*

(b) A^2 *is bi-dagger;*

(c) A^2 *is bi-EP;*

(d) *A is 2-EP (or equivalently* $A^2A^\dagger = A^\dagger A^2$*);*

(e) *A is bi-dagger.*

Proof. (a)⇒(b) As A^2 is EP and $\text{Ind}(A) = 2$, clearly $\text{Ind}(A^2) \leq 1$. Thus, A^2 is bi-dagger.
(b) ⇒(c) Follows from (2.1).
(c) ⇒(a) As $\text{Ind}(A^2) \leq 1$ and A^2 is bi-EP then A^2 is EP.
(a) ⇒(d) Follows from (6.3) for $k = 2$.
(d)⇒(e) By [10, Proposition 2.13].
(e)⇒(a) By [10, Lemma 2.4]. □

Corollary 6.4. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\text{Ind}(A) = 2$. *Then any one of conditions (a)-(e) in Theorem 6.3 implies A is bi-EP.*

Proof. Follows from (2.1). □

The following example shows that the reciprocal of above result is false.

Example 6.5. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $\text{Ind}(A) = 2$, *A is bi-EP but A is not bi-dagger.*

However, for the case of nilpotent matrices of index 2, the reciprocal of Corollary 6.4 is true.

Theorem 6.6. *Let $A \in \mathbb{C}^{n\times n}$ a nilpotent matrix with* $\mathrm{Ind}(A) = 2$. *Then the following conditions are equivalent:*

(a) *A is 2-index EP (or equivalently A^2 is EP);*

(b) *A^2 is bi-dagger;*

(c) *A^2 is bi-EP;*

(d) *A is 2-EP (or equivalently $A^2A^\dagger = A^\dagger A^2$);*

(e) *A is bi-dagger.*

(f) *A is bi-EP.*

Proof. Equivalences (a) to (e) were proved in Theorem 6.3.
(e)⇒(f) By (2.1).
(f)⇒(e) Clearly, $(A^2)^\dagger = 0$. Further, A is bi-EP (i.e. $P_AQ_A = Q_AP_A$) we have $A(A^\dagger)^2A = 0$ whence $(A^\dagger)^2 = 0$. Thus, $(A^2)^\dagger = (A^\dagger)^2$, that is, A is bi-dagger. □

Corollary 6.7. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that* $\mathrm{Ind}(A) = 2$. *Then A is bi-EP if and only if N satisfies any one of conditions (a)-(e) in Theorem 6.6 and $S(I_{n-t} - Q_N - P_N) = 0$.*

Proof. From Theorem 4.1 we know that A is bi-EP if and only if N is bi-EP and $S(I_{n-t}-Q_N)(I_{n-t}-P_N) = 0$. Thus, by using the fact that N is nilpotent with $\mathrm{Ind}(N) = 2$, the result follows by Theorem 6.6. □

We finish this section with two results concerning bi-dagger matrices and SD matrices of index 2.

Theorem 6.8. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that* $\mathrm{Ind}(A) = 2$. *Then A is bi-dagger if and only if N is bi-dagger, $T^*\Delta_1 = T(T^2)^*\Delta_2 + S\tilde{T}_2^*\Delta_2$, $\Omega_1^*\Delta_1 = N\tilde{T}_2^*\Delta_2$, and $S(I_{n-t} - P_N) = 0$.*

Proof. As $\mathrm{Ind}(A) = 2$, clearly $N^2 = 0$. Therefore, $\Omega_1 N = SN$ and $\Omega_2 = \tilde{T}_2 = TS + SN$. Thus, $TS + \Omega_1 N - \Omega_2 = 0$. Now, as $S(N^*)^2 = 0$, Corollary 5.7 completes the proof. □

Remark 6.9. From Theorem 6.8, one can note that if A is written in its core-EP decomposition (2.2) and $\mathrm{Ind}(A) = 2$, a necessary condition for A to be bi-dagger is that N is also bi-dagger.

Theorem 6.10. *Let $A \in \mathbb{C}^{n\times n}$ be written as in (2.2) such that* $\mathrm{Ind}(A) = 2$ *and A is* SD. *Then A^2 is* SD *if and only if* $[T^2, \Delta_1^{-1}TT^* + SNN^*S^*] = 0$.

Proof. It is a direct consequence from Theorem 3.7. □

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