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# A "Prism" solid element for large strain shell analysis 

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#### Abstract

In this paper a triangular prism solid element for the analysis of thin/thick shells undergoing large elas-tic-plastic strains is developed. The element is based on a total Lagrangian formulation and uses as strain measure the logarithm of the right stretch tensor $(\mathbf{U})$ obtained from a modified right Cauchy-Green deformation tensor ( $\overline{\mathbf{C}}$ ). Three are the introduced modifications: (a) a classical assumed strain approach for transverse shear strains (b) an assumed strain approach for the in-plane components using information from neighbor elements and (c) an averaging of the volumetric strain over the element. The objective is to use this type of elements for the simulation of shells avoiding transverse shear locking, improving the membrane behavior of the in-plane triangle and to handle quasi-incompressible materials or materials with isochoric plastic flow. Several examples are presented that show the transverse-shear locking free behavior, the importance of the improvement in the membrane approach and the wide possibilities of the introduced element for the analysis of shell structures for both geometric and material non-linear behavior.


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## 1. Introduction

In the solid mechanics field, for the simulation of shell structures (i.e. when one of the dimensions of the solid is small compared with the other two), the use of finite elements that consider reasonable hypothesis about the shell normal behavior (Kirchhoff-Love or Reissner-Mindlin hypothesis) are preferred. This leads to elements where the geometric configuration is described by the movement of the middle surface only with an important economy in computer resources of both memory and CPU time.

However the use of solid elements for the simulation of shells has grown a lot in the last fifteen years promoted by the continuous improvement in computer facilities and also by the necessity to improve different aspects of the models to obtain more faithful simulations. Some of the advantages when using solid elements are: (a) no assumption of the stress state is needed so general tridimensional constitutive relations can be used; (b) contact forces effects are correctly included, in particular friction; (c) large transverse shear deformations and/or strain discontinuities across the thickness can be considered; (d) avoid special transition elements between shell elements and solid elements; (e) boundaries nonparallel to the shell normal or director can be correctly modeled; (f) Rotation vectors or local triads, that are in general costly and difficult to parameterize and update, are not needed.

[^0]The most used solid elements for shell analysis are the hexahedral elements, particularly the 8 -node brick. Solid elements based on the standard displacement formulation when used to simulate shells show different type of locking: transverse shear, membrane, curvature thickness and volumetric. Shear locking increases for slender shells or if the behavior is mainly bending. Volumetric locking appears when dealing with incompressible or nearly incompressible materials or elastic-plastic materials with isochoric plastic flow (metals typically). If just one element is used across the thickness it can not fit the Poisson effect at all points. If the shell is initially curved artificial transverse strains and stresses appear under pure bending due to curvature thickness locking. Finally membrane locking specially appears on initially curved shells when the behavior is mainly bending without middle surface stretching. On low interpolation order elements membrane locking does not appear, but the mesh density necessary to achieve convergence increases. These numerical locking problems indicate that the interpolation functions used (and their gradients) can not fit the solid behavior and many times make the solutions obtained useless.

There have been numerous advances in the development of solid elements aimed to cure the different locking problems. In many cases the objective is to generate models with just one element across the thickness. In that case the elements are denoted solidshell elements and mainly use assumed natural strains (ANS) and enhanced assumed strains (EAS) formulations, with the inclusion of internal degrees of freedom that are locally condensed. To cure the transverse shear locking, the classical approximation by Dvorkin and Bathe [5] is the most used when full integration is consid-
ered (see for example $[10,20]$ ) and a variation of it when reduced integration is preferred (see for example [3]). To cure membrane and curvature thickness locking elements with both (ANS and EAS) formulations have been proposed. The usual solutions for volumetric locking are to use selective reduced integration (SRI) or averaging the volumetric strain over the element. If one element is used across the thickness both techniques lead to a too flexible behavior, that is why also the EAS technique has been applied for the volumetric approach in solid-shell elements. One and four integration points are used on the middle surface (in the first case stabilization is needed to avoid hourglass modes), while across the shell thickness at least two points are needed to capture the bending effect (until 7 points are used in elastic-plastic problems). The implementation is quite more complex that the standard solid case and the element may show some instabilities when reduced integration is used or for large strains when the EAS technique is considered [17]. In Schwarze and Reese [17] a detailed state of the art for this type of elements can be found. In spite of the improvements solid-shell elements can not model discontinuities across the thickness, as typical appear in composite laminates, and the discretization across the thickness must be increased with a loss of some of their comparative advantages.

One of the motivations of this work is the simulation of composite laminates with non-linear behavior, including delamination, that requires more than one element across the thickness. In that line the present development intends to cure transverse shear locking in bending dominated problems, improve the membrane behavior to allow coarser meshes and to relieve the volumetric locking (for isotropic plastic flow), but it is not intended to improve the transverse strain variation (Poisson effect in bending dominated problems).

Curiously there are not developments for triangular prism solidshell elements. Probably the reason is the few possibilities given by the standard interpolation functions. One obvious and important advantage of a triangular prism element is that the triangular mesh generators are quite more efficient to give elements with good aspect ratio.

The behavior of the standard (displacement base) 6-node "prism" and 8-node "brick" is quite different, that's why the strategies to cure the different locking problems may be different. The transverse shear locking of the former is quite lower while the latter has a better in-plane behavior. Note also that for the same mesh density (measured in the number of nodes) a SRI strategy for the volumetric strain implies the double number of integration points for the prism than for the brick.

In this paper some improvements on the standard 6-node "prism" [22] are proposed. To avoid volumetric locking an averaging of the volumetric strain throughout the element is performed (restricted to use at least two elements in the thickness). Regarding the transverse shear we start from the proposal for quadratic triangular shell elements [14], thus in the direction normal to the shell an ANS approximation for some components of the metric tensor will be obtained. Finally for the in-plane metric tensor components we resort to the adjacent elements to define an ANS also [7]. The ultimate goal is to have a simple element that does not require any stabilization nor presents instabilities in large deformation and suitable for contact problems.

Next section summarizes the equations of solid mechanics most relevant to this work. Section 3 presents the formulation of the solid element. The improvements in the standard element standard are presented next, starting with the improvement in the in-plane behavior, followed by the transverse shear formulation and the strategy to avoid volumetric locking. Section 5 presents several examples showing the very good behavior of the element and finally some conclusions are summarized.

## 2. Lagrangian description of the solid

Consider a continuum body with reference configuration $\boldsymbol{\Omega}_{0}$ at the initial time $t=0$. We denote by $\mathbf{X}$ the original position of the material points of the solid with respect to an arbitrary global Cartesian system. By $\mathbf{R}(\mathbf{X})=\left[\mathbf{t}_{1}, \mathbf{t}_{2}, \mathbf{t}_{3}\right]$ we denote the local orthogonal triad where the material is mechanically characterized. The position of the material points at any instant time is described by $\mathbf{x}(\mathbf{X}, t)$. The deformation gradient $\mathbf{F}$ is defined as:

## $\mathbf{F}=\mathbf{\nabla} \mathbf{x}(\mathbf{X}, t)$

that can also be referred to the local material triad (we will use a hat (^) to denote variables referred to this system when necessary)
$\hat{\mathbf{F}}=\mathbf{F R}$
Then the corresponding right Cauchy-Green deformation $\mathbf{C}$ and Green-Lagrange strain $\mathbf{E}$ tensors are written as:
$\hat{\mathbf{C}}=\hat{\mathbf{F}}^{T} \hat{\mathbf{F}}=\mathbf{R}^{T} \mathbf{C} \mathbf{R}$
$\hat{\mathbf{E}}=\frac{1}{2}(\hat{\mathbf{C}}-\mathbf{1})$
The associated stress measure (through the virtual work principle) is the second Piola-Kirchhoff stress tensor $\hat{\boldsymbol{S}}$ that can be related to the Cauchy stress tensor $\boldsymbol{\sigma}$ by
$\hat{\mathbf{S}}=\operatorname{det}(\hat{\mathbf{F}}) \hat{\mathbf{F}}^{-1} \boldsymbol{\sigma} \hat{\mathbf{F}}^{-T}$
that allows to write the internal virtual work as:
$\int_{V_{0}} \hat{\mathbf{S}}: \delta \hat{\mathbf{E}} d V_{0}$
where $V_{0}$ is the undeformed original volume of the solid.
This conjugated stress-strain pair has some drawbacks when dealing with large strain elastic-plastic constitutive relations, then we introduce the Hencky deformation tensor $\mathbf{e}$, that requires the spectral decomposition of $\mathbf{C}$
$\mathbf{C}=\mathbf{L}^{T} \boldsymbol{\Lambda}^{2} \mathbf{L}$
where $\Lambda^{2}$ is a diagonal matrix collecting the eigenvalues $\lambda_{i}^{2}$ of $\mathbf{C}$ and $\mathbf{L}$ includes the associated (unit) eigenvectors. Then it is possible to write the Hencky deformations as:
$\hat{\mathbf{e}}=\hat{\mathbf{L}}^{T} \ln (\boldsymbol{\Lambda}) \hat{\mathbf{L}}$
This strain measure is a proper extension of the one-dimensional logarithmic (natural) strain and measure strains with respect to the original triad. With $\hat{\mathbf{T}}$ will be denoted the associated stress measure, that can be related to $\hat{\mathbf{S}}$ using the following expressions:
(a) defining the rotated tensors

$$
\begin{align*}
& \mathbf{T}_{L}=\hat{\mathbf{L}}^{T} \hat{\mathbf{T}} \hat{\mathbf{L}} \\
& \mathbf{S}_{L}=\hat{\mathbf{L}}^{T} \hat{\mathbf{S}} \hat{L} \tag{9}
\end{align*}
$$

(b) the relationship between the 2nd Piola-Kirchhoff tensor and Hencky stress is (See for example Ref. [4]):

$$
\begin{align*}
& {\left[S_{L}\right]_{\alpha \alpha}=\frac{1}{\lambda_{\alpha}^{2}}\left[T_{L}\right]_{\alpha \alpha}} \\
& {\left[S_{L}\right]_{\alpha \beta}=\frac{\ln \left(\lambda_{\alpha} / \lambda_{\beta}\right)}{\frac{1}{2}\left(\lambda_{\alpha}^{2}-\lambda_{\beta}^{2}\right)}\left[T_{L}\right]_{\alpha \beta}} \tag{10}
\end{align*}
$$

(c) finally

$$
\begin{equation*}
\hat{\mathbf{S}}=\hat{\mathbf{R}}_{L} \mathbf{S}_{L} \hat{\mathbf{R}}_{L}^{T} \tag{11}
\end{equation*}
$$

The purpose of calculating $\hat{\mathbf{S}}$ is that the momentum equations are written as shown in Eq. (6) due to the computational complexity to obtain the variation of the Hencky strain (Eq. (8)).

With the aim of considering inelastic behavior, it will be assumed that the elastic strains are small and that it is feasible to decompose additively the Hencky strain tensor into an elastic and a plastic component
$\hat{\mathbf{e}}=\hat{\mathbf{e}}^{e}+\hat{\mathbf{e}}^{p}$
For isotropic materials it is also assumed that there is a linear relationship between the Hencky stress measure $\hat{\mathbf{T}}$ and the elastic strain component $\hat{\mathbf{e}}^{e}$ (defined by a constant constitutive tensor $\mathbf{D}$ ):

## $\hat{\mathbf{T}}=\mathbf{D}: \hat{\mathbf{e}}^{e}$

For a composite laminate, one possibility is to consider it as a unique equivalent material. In that case Eq. (13) can be applied directly. Such approach is generally valid only for the elastic range. A second possibility, computationally more costly but with a wider generality, is to analyze separately each component when necessary and their interaction (a simplified version of the serial/parallel mixing theory) [ 16,13 ]. In the later case it is convenient to express the components of the strain tensor $\hat{\mathbf{e}}$ in terms of the orthotropy principal directions of each component
$\hat{\mathbf{e}}_{i}=\mathbf{R}_{i}^{T} \hat{\mathbf{e}} \mathbf{R}_{i}$
where $\mathbf{R}_{i}=\left[\mathbf{t}_{1}^{i}, \mathbf{t}_{2}^{i}, \mathbf{t}_{3}^{i}\right]$ are the orthotropy principal directions of component $i$ with respect to the local system $\mathbf{R}$. The mixture theory allows to treat separately the evolution of each component, computing its stress state and then computing the equivalent stress state of the composite in terms of the volume fractions of each component.

## 3. Solid finite element

The kinematic described above has been implemented on a 6node triangular prism element. The reference and deformed element configurations are described by the standard isoparametric interpolations [22].
$\mathbf{X}(\xi)=\sum_{I=1}^{6} N^{I}(\xi) \mathbf{X}^{I}$
$\mathbf{x}(\xi)=\sum_{I=1}^{6} N^{I}(\xi) \mathbf{x}^{I}=\sum_{I=1}^{6} N^{I}(\xi)\left(\mathbf{X}^{I}+\mathbf{u}^{I}\right)$
where $\mathbf{X}^{\prime}, \mathbf{x}^{\mathbf{I}}$, are $\mathbf{u}^{\mathbf{1}}$ are respectively the original coordinates, the present coordinates and the displacements of node $I$. The shape functions $N^{I}(\xi)$ are the usual Lagrangian polynomials in terms of the local coordinates $\xi$ defined over the corresponding master element, that combine area coordinates $(\xi, \eta)$ on the triangular base with a linear interpolation ( $\zeta$ ) along the prism axis:
$N^{1}=z L^{1} \quad N^{4}=z L^{2}$
$N^{2}=\xi L^{1} \quad N^{5}=\xi L^{2}$
$N^{3}=\eta L^{1} \quad N^{6}=\eta L^{2}$
where we have used:
$z=1-\xi-\eta$
$L^{1}=\frac{1}{2}(1-\zeta)$
$L^{2}=\frac{1}{2}(1+\zeta)$
The computation of the Cartesian derivatives of the shape functions is performed in a standard way, defining the Jacobian matrix at each integration point
$\mathbf{J}=\frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}}$
then
$N_{\mathbf{X}}^{I}=\mathbf{J}^{-1} N_{\xi}^{I}$
At each element a local Cartesian triad, coincident with the principal orthotropy directions of the constitutive material, is defined
$\mathbf{R}=\left[\mathbf{t}_{1}, \mathbf{t}_{2}, \mathbf{t}_{3}\right]$
thus the Cartesian derivatives respect to this system $(\mathbf{Y})$ can be written as
$\hat{N}_{\mathbf{Y}}^{I}=\mathbf{R}^{T} N_{\mathbf{X}}^{I}$
this allows to compute the deformation gradient $\hat{\mathbf{F}}$ as a function of the present nodal coordinates
$\hat{F}_{i j}=\sum_{I=1}^{N N} \hat{N}_{j}^{I} x_{i}^{I}$
and the components of tensor $\hat{\mathbf{C}}$
$\hat{C}_{i j}=\hat{F}_{k i} \hat{F}_{k j}$
To evaluate the constitutive equations the strain tensor is decomposed in its volumetric and deviatoric parts. This decomposition is performed in a multiplicative form at each integration point
$\hat{\mathbf{C}}=[\operatorname{det}(\hat{\mathbf{C}})]^{\frac{1}{3}} \hat{\mathbf{C}}_{D}=J^{\frac{2}{3}} \hat{\mathbf{C}}_{D}$
where the volumetric and deviatoric strains are defined as
$\Delta=\ln J$
$\hat{\mathbf{e}}_{D}=\ln \left(\hat{\mathbf{C}}_{D}^{\frac{1}{2}}\right)$
as a result
$\hat{\mathbf{e}}=\frac{\Delta}{3} \mathbf{1}+\hat{\mathbf{e}}_{D}$
The volumetric strain component is averaged over the element ( $\bar{\Delta}$ ) when volumetric locking may appear due to isochoric plastic flow, leading to a modified strain tensor at each integration point
$\overline{\mathbf{e}}=\frac{\bar{\Delta}}{3} \mathbf{1}+\hat{\mathbf{e}}_{D}$
Adopting the hypothesis of additivity of elastic and plastic strain components, the strain tensor is written as:
$\overline{\mathbf{e}}=\overline{\mathbf{e}}^{p}+\overline{\mathbf{e}}^{e}$
For materials with yield surface independent of the mean press, the trace of the plastic component is null so $\bar{\Delta}$ is purely elastic. The associated stress tensor is derived from an hyperelastic constitutive law, assuming a linear relation between stress tensor and the elastic component of the strain tensor

## $\hat{\mathbf{T}}=\mathbf{D} \overline{\mathbf{e}}^{e}$

For an isotropic material with isochoric plastic strain, the elastic relation may be written as
$\hat{\mathbf{T}}=2 G \overline{\mathbf{e}}_{D}^{e}+K \bar{\Delta}$
with $G$ and $K$ the usual shear and bulk moduli respectively.
If the von Mises or the Hill yield criterion is used it is then possible to work separately with deviatoric and volumetric components, thus facilitating the integration of the constitutive equations.

As an alternative to the logarithmic strain, the spectral decomposition (7) allows to easily deal with large strain hyperelastic materials (elastomers), using models such as Ogden, Mooney-Rivlin, neo-Hookean, etc., that are usually defined in terms of a strain energy written in terms of the principal stretches.

## 4. Improvements in the standard element

If the triangular prism element described above is to be used for large strain elastic-plastic analysis of shells including contact the element must be improved substantially. With that aim different modifications are made over the metric tensor $\mathbf{C}$. In what follows the deformation gradient, the Cauchy Green deformation tensor and the Hencky strain tensor will be written in the local system but to alleviate notation the above hat will be dropped.

The discretization of a shell using triangular prism elements implies two steps (a) a discretization of the shell middle surface with three-node triangles and (b) a discretization in the thickness direction using one or more solid prism elements based on the triangles defined before. In a standard way it will be assumed that the 6node connectivity associates nodes 1-3 and nodes 4-6 with planes nearly parallel to shell middle surface and that the later ones are above the former ones along the shell normal at a distance equal to the layer thickness. Thus middle surface normal direction (local $y_{3}$ ) is almost coincident with local natural coordinate $\zeta$.

As the kinematic formulation of the element is based on the right Cauchy-Green deformation tensor, an interesting possibility is to directly modify the components of $\mathbf{C}$ associated to the behavior intended to improve
$\mathbf{C}=\left[\begin{array}{lll}C_{11}^{\#} & C_{12}^{\#} & C_{13}^{*} \\ C_{21}^{\#} & C_{22}^{\#} & C_{23}^{*} \\ C_{31}^{*} & C_{32}^{*} & C_{33}\end{array}\right]$
where the components with the upper index \# are those that have the main influence on the in-plane behavior of the shell and those denoted with an $*$ are those associated with the transverse shear. Then the deformation tensor may be divided into three parts
$\mathbf{C}=\mathbf{C}_{1}+\mathbf{C}_{2}+\mathbf{C}_{3}$
where
$\mathbf{C}_{1}=C_{11} \mathbf{t}^{1} \otimes \mathbf{t}^{1}+C_{22} \mathbf{t}^{2} \otimes \mathbf{t}^{2}+C_{12}\left(\mathbf{t}^{1} \otimes \mathbf{t}^{2}+\mathbf{t}^{2} \otimes \mathbf{t}^{1}\right)$
corresponds with the components on the tangent plane
$\mathbf{C}_{2}=C_{13}\left(\mathbf{t}^{1} \otimes \mathbf{t}^{3}+\mathbf{t}^{3} \otimes \mathbf{t}^{1}\right)+C_{23}\left(\mathbf{t}^{2} \otimes \mathbf{t}^{3}+\mathbf{t}^{3} \otimes \mathbf{t}^{2}\right)$
are those components mainly associated with the transverse shear strains and
$\mathbf{C}_{3}=C_{33} \mathbf{t}^{3} \otimes \mathbf{t}^{3}$
is used to compute the through the thickness strain.
4.1. Improvements on the in-plane behavior using the adjacent elements

To enhance the in-plane behavior we resort to the adjacent elements (in-plane neighbors) to define a four-elements patch involving 12 nodes (see Fig. 1(a)). Thus an in-plane quadratic interpolation (linear along $\zeta$ coordinate) can be defined. Here we follow the convective option used in rotation-free shell elements [7] that averages at the element center the metric tensor components computed at each mid-side points. In this case exactly the same computations can be performed at both upper and lower faces (see Fig. 1(b) with the notation of the lower face), and a subsequent
interpolation to the integration points. The local quadratic shape functions associated to the lower face are:

| $I$ | $N$ | $N_{\ell \xi}$ | $N_{\eta}$ |
| :--- | :--- | :--- | :--- |
| 1 | $(z+\xi \eta)$ | $(-1+\eta)$ | $(-1+\xi)$ |
| 2 | $(\xi+\eta z)$ | $(1-\eta)$ | $(z-\eta)$ |
| 3 | $(\eta+z \xi)$ | $(z-\xi)$ | $(1-\xi)$ |
| 7 | $\frac{z}{2}(z-1)$ | $\left(\frac{1}{2}-z\right)$ | $\left(\frac{1}{2}-z\right)$ |
| 8 | $\frac{\xi}{2}(\xi-1)$ | $\left(\xi-\frac{1}{2}\right)$ | 0 |
| 9 | $\frac{\eta}{2}(\eta-1)$ | 0 | $\left(\eta-\frac{1}{2}\right)$ |

Then, over both upper and lower element face defined by the three nodes of the element and another three from the adjacent elements:

1. We compute a local system $\left(\mathbf{t}_{1}, \mathbf{t}_{2}\right)$ on the shell tangent plane, with $\mathbf{t}_{3}$ normal to the face.
2. At each mid-side point ( $G_{K}$ ) we evaluate the in-plane Jacobian ( $\mathbf{X}_{\xi}, \mathbf{X}_{\eta}$ ) and we project it over the directions $\left(\mathbf{t}_{1}, \mathbf{t}_{2}\right)$

$$
\mathbf{J}=\left[\begin{array}{ll}
\mathbf{X}_{\xi} \cdot \mathbf{t}_{1} & \mathbf{X}_{\eta} \cdot \mathbf{t}_{1}  \tag{39}\\
\mathbf{X}_{\xi} \cdot \mathbf{t}_{2} & \mathbf{X}_{\eta} \cdot \mathbf{t}_{2}
\end{array}\right]
$$

3. We compute the shape function derivatives, that involve four nodes only, at each mid-side point $G_{K}$

$$
\left[\begin{array}{l}
N_{1}^{I}  \tag{40}\\
N_{2}^{I}
\end{array}\right]^{K}=\mathbf{J}_{K}^{-1}\left[\begin{array}{l}
N_{\xi}^{I} \\
N_{\eta}^{I}
\end{array}\right]^{K}
$$

4. This allows to compute the in-plane deformation gradient $\left(\mathbf{f}_{1}^{K}, \mathbf{f}_{2}^{K}\right)$ and with it $C_{i j}^{K}(i, j=1,2)$ that is afterward averaged over the face $\bar{C}_{i j}^{f}$ ( $f=1,2$ for lower and upper face respectively).
5. When and adjacent element is missing (boundary), as originally proposed for rotation-free shells, the values of the components of $C_{i j}$ computed from the 3 -node central triangle are included for the averaging.

For the prism element, two integration points are used along the normal direction $\left(\zeta= \pm 3^{-\frac{1}{2}}\right)$. At these points the in-plane components of the Cauchy-Green tensor are interpolated using

$$
\begin{equation*}
\bar{C}_{i j}(\zeta)=L^{1} \bar{C}_{i j}^{1}+L^{2} \bar{C}_{i j}^{2} \tag{41}
\end{equation*}
$$

The modified tangent matrix $\overline{\mathbf{B}}$ relating the incremental displacements $\delta \mathbf{u}$ with the incremental tensor components can be written as:

$$
\delta\left[\begin{array}{l}
\frac{1}{2} \bar{C}_{11}  \tag{42}\\
\frac{1}{2} \bar{C}_{22} \\
\bar{C}_{12}
\end{array}\right]=L^{1} \delta\left[\begin{array}{l}
\frac{1}{2} \bar{C}_{11}^{1} \\
\frac{1}{2} \bar{C}_{22}^{1} \\
\bar{C}_{12}^{1}
\end{array}\right]+L^{2}\left[\begin{array}{l}
\frac{1}{2} \bar{C}_{11}^{2} \\
\frac{1}{2} \bar{C}_{22}^{2} \\
\bar{C}_{12}^{2}
\end{array}\right]=\delta\left[\begin{array}{l}
E_{11} \\
E_{22} \\
2 E_{12}
\end{array}\right]
$$

and

$$
\delta\left[\begin{array}{c}
\frac{1}{2} \widetilde{C}_{11}^{f}  \tag{43}\\
\frac{1}{2} \bar{C}_{22}^{f} \\
\bar{C}_{12}^{f}
\end{array}\right]=\frac{1}{3} \sum_{K=1}^{3} \delta\left[\begin{array}{c}
\frac{1}{2} C_{11}^{K} \\
\frac{1}{2} C_{22}^{K} \\
C_{12}^{K}
\end{array}\right]=\frac{1}{3} \sum_{K=1}^{3} \sum_{J=1}^{4}\left[\begin{array}{c}
\mathbf{f}_{1}^{K} N_{1}^{(K)} \\
\mathbf{f}_{2}^{K} N_{2}^{(K)} \\
\left(\mathbf{f}_{1}^{K} N_{2}^{J(K)}+\mathbf{f}_{2}^{K} N_{1}^{(K)}\right)
\end{array}\right] \delta \mathbf{u}^{(K)}=\overline{\mathbf{B}}_{3 \times 18}^{f} \delta \mathbf{u}^{f}
$$

where array $\delta \mathbf{u}^{\mathbf{f}}$ include only the nodes on each face $f$ (lower or upper). Note that index $J=1 \ldots 4$ because at each mid-side point only four nodes contribute to the gradient, that is why the special notation $\delta \mathbf{u}^{\left({ }^{(K)}\right)}$.

### 4.1.1. Geometric stiffness matrix

The geometric stiffness matrix results from:


Fig. 1. Patch of elements.

$$
\begin{align*}
\delta \mathbf{u}^{T} \mathbf{K}_{g} \Delta \mathbf{u} & =\int_{V} \frac{\partial}{\partial \mathbf{u}}\left(\delta\left[\begin{array}{l}
\frac{1}{2} \bar{C}_{11} \\
\delta \\
\frac{1}{2} \bar{C}_{22} \\
\bar{C}_{12}
\end{array}\right]\right)^{T}\left[\begin{array}{l}
S_{11} \\
S_{22} \\
S_{12}
\end{array}\right] \Delta \mathbf{u} d V \\
& =\sum_{G=1}^{2} \frac{1}{3} V_{0} l_{G} \sum_{f=1}^{2} L^{f} \sum_{K=1}^{3} \sum_{l=1}^{4} \sum_{j=1}^{4}\left\{\delta \mathbf{u}^{\prime}\left[N_{1}^{J} N_{2}^{J}\right]\left[\begin{array}{ll}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{array}\right]\left[\begin{array}{l}
N_{1}^{J} \\
N_{2}^{J}
\end{array}\right] \Delta \mathbf{u}^{\prime}\right\}^{(K)} \tag{44}
\end{align*}
$$

where the sum on $G$ is the numerical integration with 2 Gauss points along direction $\zeta$.

### 4.2. Transverse shear formulation

The transverse shear approach is based as usual on an interpolation of mixed tensorial components. Here we will adopt the general methodology presented in Oñate et al. [14] for the cure of transverse shear locking on Reissner-Mindlin shell elements. There a 6 -node quadratic triangular element is proposed with a linear variation of the transverse shear strain tangent to the side. This approximation can be easily particularized for linear elements [21] assuming a constant value of the transverse shear strain tangent to the side. The relevant components of the right Cau-chy-Green tensor can be written as:

$$
\left[\begin{array}{l}
C_{\xi 3}  \tag{45}\\
C_{\eta 3}
\end{array}\right]=\left[\begin{array}{lll}
-\eta & -\eta & 1-\eta \\
\xi & \xi-1 & \xi
\end{array}\right]\left[\begin{array}{l}
\sqrt{2} C_{t 3}^{1} \\
-C_{\eta 3}^{2} \\
C_{\xi 3}^{3}
\end{array}\right]=\mathbf{P}(\xi, \eta) \tilde{\mathbf{c}}
$$

where the transverse shear components $\left(C_{\xi 3}, C_{\eta 3}\right)$ has been defined with respect to a mixed coordinate system, i.e. that includes the inplane natural coordinates $(\xi, \eta)$ and the spatial local coordinate in the transverse direction $\left(y_{3}\right)$. The components of interest are computed at each side $\left(1\left(\xi=\eta=\frac{1}{2}\right), 2\left(\xi=0, \eta=\frac{1}{2}\right)\right.$ and $3\left(\xi=\frac{1}{2}\right.$, $\eta=0$ ), see Fig. 2) as a function of the deformation gradient along the side and the gradient in the normal direction $\mathbf{y}_{3}$ :

On the other hand, the numerical integration is performed with two points along the prism axis ( $\xi=\eta=\frac{1}{3}$ ), whereby:


Fig. 2. Points for the computation of the transverse shear strains.

$$
\left[\begin{array}{l}
C_{\xi 3}  \tag{46}\\
C_{\eta 3}
\end{array}\right]=\left[\begin{array}{lll}
-\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\
\frac{1}{3} & -\frac{2}{3} & \frac{1}{3}
\end{array}\right]\left[\begin{array}{l}
\sqrt{2} C_{t 3}^{1} \\
-C_{\eta 3}^{2} \\
C_{\xi 3}^{3}
\end{array}\right]=\frac{\sqrt{2} C_{t 3}^{1}-C_{\eta 3}^{2}+C_{\xi 3}^{3}}{3}\left[\begin{array}{l}
-1 \\
+1
\end{array}\right]+\left[\begin{array}{l}
C_{\xi 3}^{3} \\
C_{\eta 3}^{2}
\end{array}\right]
$$

For the purpose of having an element without spurious zero-energy deformation modes, it is necessary to have at least four strain values, therefore it is not enough to compute just two values over the middle surface (similar to a selected reduced integration). Here we will compute the three natural strains for the two values of $\zeta$ corresponding to the integration points (another option is to compute them at $\zeta= \pm 1$ and an interpolation to the Gauss points). With this scheme, using (46), the number of components of the Cauchy-Green tensor will be the minimum necessary, thus it will be unnecessary to use any type of stabilization.

The mixed components computed at the sampling points $1-3$ are:
$\tilde{\mathbf{c}}=\left[\begin{array}{l}\sqrt{2} C_{t 3}^{1} \\ -C_{\eta 3}^{2} \\ C_{\xi 3}^{3}\end{array}\right]=\left[\begin{array}{l}\sqrt{2} \mathbf{f}_{t}^{1} \cdot \mathbf{f}_{3}^{1} \\ -\mathbf{f}_{\eta}^{2} \cdot \mathbf{f}_{3}^{2} \\ \mathbf{f}_{\xi}^{3} \cdot \mathbf{f}_{3}^{3}\end{array}\right]$
where the $\mathbf{f}_{i}^{K}$ are the present configuration derivatives respect to the coordinates indicated by the index at each sampling point $K$. This allows to compute the mixed tensor:
$\overline{\mathbf{C}}_{2}=C_{\xi 3}\left(\mathbf{t}^{\xi} \otimes \mathbf{t}^{3}+\mathbf{t}^{3} \otimes \mathbf{t}^{\xi}\right)+C_{\eta 3}\left(\mathbf{t}^{\eta} \otimes \mathbf{t}^{3}+\mathbf{t}^{3} \otimes \mathbf{t}^{\eta}\right)$
where $\left[\begin{array}{ccc}\mathbf{t}^{\xi} & \mathbf{t}^{\eta} & \mathbf{t}^{3}\end{array}\right]$ are the dual base vectors of the local base $\left[\begin{array}{ll}\mathbf{t}_{\xi} & \mathbf{t}_{\eta} \mathbf{t}_{3}\end{array}\right]=\left[\frac{\partial \mathbf{X}}{\partial \xi} \frac{\partial \mathbf{X}}{\partial \eta} \frac{\partial \mathbf{X}}{\partial y_{3}}\right]$, and then to compute the modified Cartesian components (again denoted by an over bar):
$\bar{C}_{13}=\mathbf{t}_{1} \cdot \overline{\mathbf{C}}_{2} \cdot \mathbf{t}_{3}=\mathbf{t}_{1} \cdot\left[C_{\xi 3}\left(\mathbf{t}^{\xi} \otimes \mathbf{t}^{3}+\mathbf{t}^{3} \otimes \mathbf{t}^{\xi}\right)+C_{\eta 3}\left(\mathbf{t}^{\eta} \otimes \mathbf{t}^{3}+\mathbf{t}^{3} \otimes \mathbf{t}^{\eta}\right)\right] \cdot \mathbf{t}_{3}$
denoting by $a_{i}^{j}=\mathbf{t}_{i} \cdot \mathbf{t}^{j}$ (with $i=1,2,3$ and $\left.j=\xi, \eta, 3\right)$
$\bar{C}_{13}=C_{\xi 3}\left(a_{1}^{\xi} a_{3}^{3}+a_{1}^{3} a_{3}^{\xi}\right)+C_{\eta 3}\left(a_{1}^{\eta} a_{3}^{3}+a_{1}^{3} a_{\eta}^{3}\right)=C_{\xi 3} a_{1}^{\xi}+C_{\eta 3} a_{1}^{\eta}$
and similarly for the other component of interest. Thus using the condition $a_{i}^{j}=\delta_{i}^{j}$ results
$\left[\begin{array}{l}\bar{C}_{13} \\ \bar{C}_{23}\end{array}\right]=\left[\begin{array}{ll}a_{1}^{\xi} & a_{1}^{\eta} \\ a_{2}^{\xi} & a_{2}^{\eta}\end{array}\right]\left[\begin{array}{l}C_{\xi 3} \\ C_{\eta 3}\end{array}\right]=\mathbf{J}_{p}^{-1}\left[\begin{array}{l}C_{\xi 3} \\ C_{\eta 3}\end{array}\right]$
where $\mathbf{J}_{p}^{-1}$ is the inverse of the isoparametric mapping restricted to the surface tangent plane. Note that due to the way in which the local system has been defined the components in (51) are null in the reference configuration.

The necessary components of the deformation gradient with respect to the natural coordinates at the sampling points (1-3) are obtained valuing the shape functions derivatives:
$\left[\begin{array}{l}\mathbf{f}_{t}^{1} \\ -\mathbf{f}_{\eta}^{2} \\ \mathbf{f}_{\xi}^{3}\end{array}\right]=\left[\begin{array}{c}\mathbf{f}_{t}\left(\xi=\eta=\frac{1}{2}\right) \\ -\mathbf{f}_{\eta}\left(\xi=0, \eta=\frac{1}{2}\right) \\ \mathbf{f}_{\xi}\left(\xi=\frac{1}{2}, \eta=0\right)\end{array}\right]$
Recalling the shape functions (17) of the six-node triangular prism element, the tangent gradients evaluated at the sampling points results:
$\left[\begin{array}{l}\sqrt{2} \mathbf{f}_{t}^{1} \\ -\mathbf{f}_{\eta}^{2} \\ \mathbf{f}_{\xi}^{3}\end{array}\right]=\left[\begin{array}{l}\left(-\mathbf{x}^{2}+\mathbf{x}^{3}\right) L^{1}+\left(-\mathbf{x}^{5}+\mathbf{x}^{6}\right) L^{2} \\ \left(\mathbf{x}^{1}-\mathbf{x}^{3}\right) L^{1}+\left(\mathbf{x}^{4}-\mathbf{x}^{6}\right) L^{2} \\ \left(-\mathbf{x}^{1}+\mathbf{x}^{2}\right) L^{1}+\left(-\mathbf{x}^{4}+\mathbf{x}^{5}\right) L^{2}\end{array}\right]$
Until now we have written the gradients of $\mathbf{x}$ (with respect to natural coordinates) at the sampling points in terms of the nodal coordinates. The gradients with respect to the transverse direction can be expressed as:
$\mathbf{f}_{3}=\left[\begin{array}{lll}\mathbf{f}_{\xi} & \mathbf{f}_{\eta} & \mathbf{f}_{\zeta}\end{array}\right]\left[\begin{array}{l}\xi_{3} \\ \eta_{3} \\ \zeta_{3}\end{array}\right]=\left[\begin{array}{lll}\mathbf{f}_{\xi} & \mathbf{f}_{\eta} & \mathbf{f}_{\zeta}\end{array}\right]\left[\begin{array}{c}\frac{\partial \xi}{\partial y_{3}} \\ \frac{\partial \eta}{\partial y_{3}} \\ \frac{\partial \zeta}{\partial y_{3}}\end{array}\right]=\nabla_{\xi}(\mathbf{x}) \mathbf{j}_{3}^{-T}$
where $\left(\frac{\partial}{\partial y_{3}}\right)$ are the components in direction $y_{3}$ of the inverse of the Jacobian of the isoparametric mapping (third column $\mathbf{j}_{3}^{-1}$ ), also
$\mathbf{f}_{3}=\sum_{I=1}^{6} N_{3}^{I} \mathbf{x}^{I}$
with
$N_{3}^{I}=\left[\begin{array}{lll}N_{\xi}^{I} & N_{\eta}^{I} & N_{\zeta}^{I}\end{array}\right] \mathbf{j}_{3}^{-T}$
Once the transverse shear components have been obtained, they can be expressed in the Cartesian system as shown in (51)

$$
\left[\begin{array}{l}
\bar{C}_{13}  \tag{57}\\
\bar{C}_{23}
\end{array}\right]=\left[\begin{array}{ll}
\frac{\partial \xi}{\partial y_{1}} & \frac{\partial \eta}{\partial y_{1}} \\
\frac{\partial \xi}{\partial y_{2}} & \frac{\partial \eta}{\partial y_{2}}
\end{array}\right]\left[\begin{array}{l}
C_{\xi 3} \\
C_{\eta 3}
\end{array}\right]=\mathbf{J}_{p}^{-1}\left[\begin{array}{l}
C_{\xi 3} \\
C_{\eta 3}
\end{array}\right]
$$

The transverse shear components of tensor $\mathbf{C}_{2}$ respect to the Cartesian system stem from replacing Eqs. (47) into (45) and these into (51)

$$
\left[\begin{array}{l}
\bar{C}_{13}  \tag{58}\\
\bar{C}_{23}
\end{array}\right](\xi, \eta)=\mathbf{J}_{p}^{-1} \mathbf{P}(\xi, \eta)\left[\begin{array}{l}
\mathbf{f}_{t}^{1} \cdot \mathbf{f}_{3}^{1} \\
-\mathbf{f}_{\eta}^{2} \cdot \mathbf{f}_{3}^{2} \\
\mathbf{f}_{\xi}^{3} \cdot \mathbf{f}_{3}^{3}
\end{array}\right]=\mathbf{J}_{p}^{-1} \mathbf{P}(\xi, \eta) \tilde{\mathbf{c}}
$$

The tangent matrix $\overline{\mathbf{B}}_{s}$, relating displacement increments with strain increments, results from first computing at the sampling points

$$
\tilde{\mathbf{B}}_{s} \delta \mathbf{u}^{e}=\left[\begin{array}{l}
\delta \mathbf{f}_{t}^{1} \cdot \mathbf{f}_{3}^{1}+\mathbf{f}_{t}^{1} \cdot \delta \mathbf{f}_{3}^{1}  \tag{59}\\
-\delta \mathbf{f}_{\eta}^{2} \cdot \mathbf{f}_{3}^{2}-\mathbf{f}_{\eta}^{2} \cdot \delta \mathbf{f}_{3}^{2} \\
\delta \mathbf{f}_{\xi}^{3} \cdot \mathbf{f}_{3}^{3}+\mathbf{f}_{\xi}^{3} \cdot \delta \mathbf{f}_{3}^{3}
\end{array}\right]
$$

where (with $\delta \mathbf{x}^{e}=\delta \mathbf{u}^{e}$ )

$$
\begin{align*}
\delta\left[\begin{array}{l}
\sqrt{2} \mathbf{f}_{t}^{1} \\
-\mathbf{f}_{\eta}^{2} \\
\mathbf{f}_{\xi}^{3}
\end{array}\right]= & {\left[\begin{array}{l}
-L^{1} \delta \mathbf{x}^{2}+L^{1} \delta \mathbf{x}^{3}-L^{2} \delta \mathbf{x}^{5}+L^{2} \delta \mathbf{x}^{6} \\
L^{1} \delta \mathbf{x}^{1}-L^{1} \delta \mathbf{x}^{3}+L^{2} \delta \mathbf{x}^{4}-L^{2} \delta \mathbf{x}^{6} \\
-L^{1} \delta \mathbf{x}^{1}+L^{1} \delta \mathbf{x}^{2}-L^{2} \delta \mathbf{x}^{4}+L^{2} \delta \mathbf{x}^{5}
\end{array}\right] } \\
& =\left[\begin{array}{lllll}
+0 & -L^{1} & +L^{1}+0 & -L^{2} & +L^{2} \\
+L^{1} & +0 & -L^{1} & +L^{2} & +0 \\
-L^{1} \\
-L^{1} & +L^{1} & 0 & -L^{2} & +L^{2}
\end{array}\right] \delta\left[\begin{array}{l}
\mathbf{u}^{1} \\
\mathbf{u}^{2} \\
\mathbf{u}^{3} \\
\mathbf{u}^{4} \\
\mathbf{u}^{5} \\
\mathbf{u}^{6}
\end{array}\right] \tag{60}
\end{align*}
$$

and (with a notation abuse)

$$
\delta\left[\begin{array}{l}
\mathbf{f}_{3}^{1}  \tag{61}\\
\mathbf{f}_{3}^{2} \\
\mathbf{f}_{3}^{3}
\end{array}\right]=\left[\begin{array}{llllll}
N_{3}^{1(1)} & N_{3}^{2(1)} & N_{3}^{3(1)} & N_{3}^{4(1)} & N_{3}^{5(1)} & N_{3}^{6(1)} \\
N_{3}^{1(2)} & N_{3}^{2(2)} & N_{3}^{3(2)} & N_{3}^{4(2)} & N_{3}^{5(2)} & N_{3}^{6(2)} \\
N_{3}^{1(3)} & N_{3}^{2(3)} & N_{3}^{3(3)} & N_{3}^{4(3)} & N_{3}^{5(3)} & N_{3}^{6(3)}
\end{array}\right] \delta\left[\begin{array}{l}
\mathbf{u}^{1} \\
\mathbf{u}^{2} \\
\mathbf{u}^{3} \\
\mathbf{u}^{4} \\
\mathbf{u}^{5} \\
\mathbf{u}^{6}
\end{array}\right]
$$

then interpolate to the integration points using (46) and finally convert to the Cartesian system
$\overline{\mathbf{B}}_{s}(\xi, \eta)=\mathbf{J}_{p}^{-1} \mathbf{P}(\xi, \eta) \tilde{\mathbf{B}}_{s}$
The Green-Lagrange strain components associated to the transverse shear are directly the interpolated values

$$
\begin{align*}
{\left[\begin{array}{l}
2 E_{13} \\
2 E_{23}
\end{array}\right] } & =\left[\begin{array}{l}
\bar{C}_{13} \\
\bar{C}_{23}
\end{array}\right](\xi, \eta)=\mathbf{J}_{p}^{-1}\left[\begin{array}{lll}
-\eta & -\eta & 1-\eta \\
\xi & \xi-1 & \xi
\end{array}\right] \tilde{\mathbf{c}} \\
& =\mathbf{J}_{p}^{-1}\left\{\frac{\sqrt{2} C_{t 3}^{1}-C_{\eta 3}^{2}+C_{\xi 3}^{3}}{3}\left[\begin{array}{c}
-1 \\
+1
\end{array}\right]+\left[\begin{array}{c}
C_{\xi 3}^{3} \\
C_{\eta 3}^{2}
\end{array}\right]\right\} \tag{63}
\end{align*}
$$

While the nodal equivalent forces can be expressed as:
$\mathbf{r}^{T}=\int_{V}\left[\begin{array}{l}S_{13} \\ S_{23}\end{array}\right]^{T}\left[\overline{\mathbf{B}}_{s}\right]_{2 \times 18} d V=\int_{y_{3}} A \mathbf{Q}^{T} \mathbf{J}_{p}^{-1}\left[\begin{array}{lll}-\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ +\frac{1}{3} & -\frac{2}{3} & \frac{1}{3}\end{array}\right]\left[\tilde{\mathbf{B}}_{s}\right]_{3 \times 18} d y_{3}$
where the generalized shear forces $\overline{\mathbf{Q}}$ at each Gauss point are defined as:
$\overline{\mathbf{Q}}_{3 \times 1}=\left[\begin{array}{cc}-\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3}\end{array}\right]\left[\begin{array}{ll}J_{11}^{-1} & J_{12}^{-1} \\ J_{21}^{-1} & J_{22}^{-1}\end{array}\right]\left[\begin{array}{l}S_{13} \\ S_{23}\end{array}\right] A$

### 4.2.1. Geometric stiffness matrix

The above expressions allow to advance in obtaining the geometric stiffness matrix:
$\delta \mathbf{u}^{T} \mathbf{K}_{S G} \Delta \mathbf{u}=\delta \mathbf{u}^{T} \int_{y_{3}} \Delta\left[\tilde{\mathbf{B}}_{s}\right]_{18 \times 3}^{T} \overline{\mathbf{Q}} d y_{3}$
where

$$
\Delta\left(\tilde{\mathbf{B}}_{s} \delta \mathbf{u}\right) \overline{\mathbf{Q}}=\overline{\mathbf{Q}}^{T} \Delta\left[\begin{array}{l}
\delta \mathbf{f}_{t}^{1} \cdot \mathbf{f}_{3}^{1}+\mathbf{f}_{t}^{1} \cdot \delta \mathbf{f}_{3}^{1}  \tag{67}\\
-\delta \mathbf{f}_{\eta}^{2} \cdot \mathbf{f}_{3}^{2}-\mathbf{f}_{\eta}^{2} \cdot \delta \mathbf{f}_{3}^{2} \\
\delta \mathbf{f}_{\xi}^{3} \cdot \mathbf{f}_{3}^{3}+\mathbf{f}_{\xi}^{3} \cdot \delta \mathbf{f}_{3}^{3}
\end{array}\right]
$$

that, for example, for the first side is:

$$
\begin{align*}
& \bar{Q}_{1} \times\left[\left(-L^{1} \delta \mathbf{u}^{2}+L^{1} \delta \mathbf{u}^{3}-L^{2} \delta \mathbf{u}^{5}+L^{2} \delta \mathbf{u}^{6}\right) \cdot \sum_{I=1}^{6} N_{3}^{I(1)} \Delta \mathbf{u}^{I}\right. \\
& \left.\quad+\sum_{I=1}^{6} N_{3}^{I(1)} \delta \mathbf{u}^{I}\left(-L^{1} \Delta \mathbf{u}^{2}+L^{1} \Delta \mathbf{u}^{3}-L^{2} \Delta \mathbf{u}^{5}+L^{2} \Delta \mathbf{u}^{6}\right)\right] \tag{68}
\end{align*}
$$

### 4.3. Volumetric behavior

To avoid volumetric locking (for quasi-incompressible materials or isochoric plastic flow) an averaging of the volumetric strain over the two integration points is performed. This leads to an excessively flexible behavior when just one element is used across the thickness. Thus for those materials at least two elements will be used through the thickness. Besides that to capture the details of the elastic-plastic behavior four points are at least necessary thus one element is insufficient.

## 5. Numerical examples

In the set of examples shown below we denote by Wedge the standard triangular prism described in Section 3. The same element but incorporating any of the improvements described above, to improve membrane behavior or to cure shear or volumetric locking, is denoted by Prism and different suffixes. The suffix B indicates that the volumetric averaging is performed; suffix $S$ indicates that includes ANS for transverse shear and suffix $Q$ indicates that the improvement in the membrane behavior has been activated.

For large scale simulations with strong non-linearities associated to geometrical instabilities, complex constitutive models or frictional contact, it is very common to use explicit algorithms for the integration of the momentum equations. These algorithms are conditionally stable and when shells are discretized with solid elements the critical time increment depends on the thickness of the layers. This may lead to very low critical time increments with prohibitively large CPU times. For these type of models Olovsson et al. [15] proposed a simple selective mass scaling strategy that makes the critical time increment independent of the through the thickness discretization. Such a technique has been used here.

Results obtained with other elements developed by the author have been used with comparative purposes. Solag: 8-node solid element based on the formulation described in Section 3 with volumetric strain average; Solag-S: same as above but includes an assumed strain approach for transverse shear [9]. LBST and BBST are rotation-free thin shell triangular elements, the former [6] uses the standard constant strain triangle for the membrane part, while the later includes an assumed strain approach for the membrane part [8] almost identical to the formulation presented above.

### 5.1. Cantilever beam with a point load

This first example (extracted from [15]) considers the dynamic behavior of a cantilever beam with length, width and thickness $L=1, b=0.1$ and $t=0.01$ respectively. The mechanical properties are Young's modulus $E=1 \times 10^{2} \mathrm{GPa}$, Poisson's ratio $v=0$ and mass density $\delta=1000 \mathrm{~kg} / \mathrm{m}^{3}$. The point load applied at the free side has a value of 100 N with a Heaviside step time function. As the problem is elastic with $v=0$ there is no Poisson's effect across the thickness nor volumetric locking. The behavior is purely bending and it is useful to evaluate the shear locking and assess the proposed cure. The discretization includes ten uniform divisions along the length, one in the width and one element through the thickness (see Fig. 3). The volumetric approach is standard, i.e. we are not considering an averaging in the element or a selective reduced integration (as this leads in this case to a very flexible behavior).

If the standard 8-node solid element (Solag) is used a strong shear locking occurs making the solution completely invalid. Fig. 4 shows that the amplitude of the displacements is just $2 \%$ of the correct value. Curiously if the standard prism element (Wedge)


Fig. 3. Cantilever beam under point load.


Fig. 4. Cantilever beam with a point load. Free-edge displacements.
is used the shear locking is not so severe. This is due to the particular characteristics of the present example and the mesh used. If for the 8 -node element four integration points are used as suggested by Liu et al. [11] the same severe locking is observed. However if the selected four integration points belongs to a plane (such an element presents hourglass modes), the results do not show locking for this example. Note that the four integration points of two adjacent prism element belong to a plane in this simple example but in contrast the element is correctly integrated and does not present spurious modes. Both elements including ANS for transverse shear (8-node Solag-S and 6-node Prism-S) give results almost identical to those obtained with the triangular shell element BBST using the same discretization.

### 5.2. Scordelis-Lo cylindrical roof

The second example is a classical cylindrical roof under self weight. The shell is supported by rigid diaphragms at the curved sides and is free along the straight sides. Only one-quarter of the roof is modeled due to symmetry. Structured meshes with the same number of elements along each side (see Fig. 5(a)) were used. Of the two possible mesh orientations the one shown in the figure is considered with two elements across the thickness.

The Fig. 5(b) shows the convergence of the vertical displacement of point $A$ at the mid-point of the free side (reference value is $w_{A}=3.610$ ) as a function of the number of elements per side. This is a membrane dominated problem then a good membrane approach is crucial for a fast convergence. To emphasize this the results obtained with shell elements LBST and BBST, which main difference is the membrane approach, are included. Note that the later rapidly reaches convergence to the target value. The figure also includes four curves corresponding to different combinations of the modifications proposed over the standard solid element. Clearly the standard Wedge locks severely and it can also be seen that modifying only the membrane behavior (Prism-Q) there are no noticeable changes. When the modification in the transverse shear is included (Prism-S) the results obtained are similar to those of the triangular shell element with constant strain membrane (LBST). Combining both shear and membrane improvements (Prism-SQ) the results rapidly converge to the reference value.

### 5.3. Semi-spherical shell with a $18^{\circ}$ hole

This is a well-known double curved shell problem in the context of large elastic displacements. The Fig. 6(a) shows the geometry


Fig. 5. Scordelis cylindrical roof. (a) geometry (b) displacement of point A.


Fig. 6. Semi-spherical shell with a hole. Original and deformed geometry.
considered using symmetry conditions and the loads applied. The discretization used includes two elements across the thickness and 32 elements along each side. The mesh is relatively fine on the shell surface due to the double curvature and because this is an almost inextensional problem where the membrane behavior is not significant. The middle surface radius is $R=10 \mathrm{~mm}$, and the mechanical properties of the material are Young's modulus $E=6.825 \times 10^{4} \mathrm{GPa}$ and Poisson ratio $v=0.3$. Two different values of the thickness have been considered in order to assess the element in thin $(R / t=250)$ and very thin $(R / t=1000)$ shell problems. The maximum element aspect ratios are 25 and 100 for the thin and very thin case respectively.

The Fig. 6(b) shows the deformed configuration for an inward displacement of the loaded point equal to $60 \%$ of the shell radius. Whilst the Fig. 7 plots the displacement (absolute values) of the loaded points where the largest displacement corresponds to the inward load. For the case $R / t=250$ the results presented in Simo et al. [19] using a shear deformable shell element are included while for the case $R / t=1000$ those extracted from Ref. [17] obtained with a sophisticated solid-shell brick element (Q1STs) are plotted. Also for both thicknesses results obtained with thin shell finite element BBST are shown. For the solid prism element four different results are plotted. Those obtained with the standard displacement formulation (Wedge) that locks severely and three combinations of the proposed modifications, all including the ANS for transverse shear (index $s$ ) that is the most significant for this example. For the present problem and mesh it may be seen that the membrane formulation has a minor influence and when the
volumetric strain average is considered a more flexible behavior is again obtained due to the relaxation of the Poisson effect. Note that for the standard formulation the locking increase notoriously for the very thin case, while for the modified version the results do not show substantial differences for the different thickness values.

### 5.4. Simulation of the ply drop-off test

This example is intended to show how the element formulation described in Section 3 combined with a simplified serial/parallel mixing theory can be used to model non-linear behavior of composites including delamination and failure. The Fig. 8 shows schematically the test setup and the loads applied. The experimental specimen has 18 layers in the thinner part and 27 layers in the thicker part (layer thickness is 0.78 mm ). The tensile force generates bending in the section with thickness variation and a strong shear stress between the upper continuous and the lower discontinuous layers that produces delamination between then. The details of the constitutive model used, the properties and orientation of the layers and a detailed discussion of this test can be found in Martinez et al. [13].

The finite element discretization includes 6 elements across the thickness for the thinner part and 9 for the thicker part, i.e. each element spreads over 3 layers of composite. Due to axial symmetry half of the specimen is modeled with two divisions in width direction. Finally 33 non-uniform divisions are considered in the axial direction, leading to a mesh with 1251 nodes and 1452 prism elements.


Fig. 7. Semi-spherical shell with a hole. Displacements of the loaded points. (a) $R / t=250$ (b) $R / t=1000$.


Fig. 8. Ply drop-off test specimen and load applied.

Displacements are imposed at both ends until a total elongation of 1.8 mm . A first sight the problem can be classified as static, however the delamination is basically a dynamic problem that releases energy with a strong stress redistribution than may lead to nonconvergence if an implicit static non-linear strategy is used (continuation method). Here an explicit integrator is considered with a special strategy that allows the delamination process take place before continuing the straining.

The Fig. 9 plots the load applied versus the total length change of the specimen. The results included are the experimental ones and those obtained with the standard element (wedge) and including the transverse shear approach (Prism_S). Also for the numerical simulations the vertical displacement of point A (indicated in Fig. 8) is plotted. The first part of the curves (before $\Delta L=1 \mathrm{~mm}$ ) show an excellent agreement between the experimental and the numerical results. This implies that the specimen tangent stiffness $\left(\frac{\Delta F}{\Delta L}\right)$ is correctly determined by the constitutive model. Note that the vertical displacement $W$ is larger for the element with the modified transverse shear than for the standard element. This is due to the transverse shear locking of the later and although it does not seem to be very strong it implies larger shear stresses so an earlier failure may be expected for the standard element. The experimental values indicate a change in strength between $\Delta L=1 \mathrm{~mm}$ and 1.1 mm that is associated with the onset of delamination. Also for the second part of the experimental data the tan-


Fig. 9. Ply drop-off test. Load-displacement results.
gent stiffness diminish to an almost constant value as if the delamination had reached the right end instantaneously. For the numerical simulations the propagation of the delamination is more gradual. The onset of delamination for the model using the Wedge element occurs for $\Delta L=1 \mathrm{~mm}$ where a peak in the curve can be seen while for the Prism-S element the delamination also stars for $\Delta L=1 \mathrm{~mm}$ but the curve peak occurs for $\Delta L=1.15 \mathrm{~mm}$ but with a faster delamination than for the standard element.

Finally Fig. 10 shows three different stages of the delamination process where the contour field of an internal variable of the model (a sort of equivalent deformation) has been drawn. Note that the discontinuity produced by the delamination is here treated in a continuous form.

### 5.5. Spherical dome under uniform step load

This is also a classical example [2] of the elastic and elasticplastic dynamic behavior of shells that appears in most of the commercial non-linear finite element codes manuals (see for example [1]). The comparisons are made against a converged finite element solution obtained with an axisymmetric quadrilateral solid element Q4P1 (4 elements across the thickness and 100 elements along the meridian).

Two discretizations have been considered over one quarter of the dome. A rather coarse one with 96 elements over the shell surface and two element across the thickness. And a fine discretization with 2888 elements on the middle surface and four


Fig. 10. Ply drop-off test. Deformed configurations.


Fig. 11. Spherical dome with uniform pressure. (a) elastic, (b) elastic-plastic.
elements across the thickness. For the elastic case the volumetric approach is standard while for elastic-plastic case the element averaging is considered.

The Fig. 11(a), corresponding to the elastic model, plots the apex displacement of the reference solution and those obtained with the coarse mesh. The results show an important improvement of the modified element respect to the standard one. The Fig. 11(b) plots the results of the elastic-plastic case. Again the results of the present element are quite better than the standard one but the mesh must be refined to obtain results similar to the reference solution.

### 5.6. Thin-walled elbow under in-plane bending and internal pressure

A pipe of radius $r=19.83 \mathrm{~cm}$ and thickness $t=1.041 \mathrm{~cm}$ is formed by a straight cylinder of length 182.9 cm, a $90^{\circ}$ elbow with radius 60.95 cm and a second cylinder of length 60.96 cm . The pipe
is fully clamped at one end and subjected to an imposed rotation at the other end. This rotation involves all the nodes at the end section keeping it plane and circular. The pipe may be internally pressurized, so two cases are considered (a) $p=0$ and (b) $p=3.45 \mathrm{MPa}$. The bending ovalize the cross section specially in the zone of the elbow. The mechanical properties of the constituent material are: $E=194 \mathrm{GPa}, v=0.264, \delta=7800 \mathrm{~kg} / \mathrm{m}^{3}$ and associative plasticity (von Mises) with isotropic hardening defined by the relation $\sigma_{y}=5.71 \times 10^{8}\left(e_{p}+0.006\right)^{0.1}$. For comparison purposes two models were considered: one with 3 -node triangular shell elements (BBST) only and another that couples shell elements for the cylindrical parts and solid elements for the elbow. The discretization used is rather coarse with 672 triangular elements over the middle surface. For the coupled shell-solid model, three elements were used across the thickness. The element aspect ratio (length/thickness) of the solid elements ranges from 15 to 30 . For the shell elements 6 integration points were used through the thickness to


Fig. 12. In plane bending of an elbow. Final configuration $(X=1)$ with contour fill of the effective plastic strain over the external surface.


Fig. 13. In plane bending of an elbow. Driving Moment versus cross section rotation, (a) without internal pressure, (b) with internal pressure.


Fig. 14. Geometry of the tools (dimensions in mm ) for the Numisheet ' 93 benchmark.
have the same number of integration points for both shell and solid elements and to capture with detail the elastic-plastic bending.

The Fig. 12 shows the final deformed geometry of the tube for the coupled mesh and the zone of the elbow for the shell only


Fig. 15. Punch force versus punch travel.


Fig. 16. Equivalent plastic strain for the final punch travel.
model. The contours of equivalent plastic strain have been drawn in both models for comparative purposes. A larger spread of the plastic strain can be seen for the shell model than can be due to the discontinuities that exist at the coupling zone. In both models only half the pipe has been discretized due to symmetry.

The Fig. 13 plots the driving moment as a function of the rotation of the end cross section for both load cases (without and with internal pressure) and for both shell-only and coupled shell-solid models. For comparison results with the coupled model using the standard element and the two versions of the 8-node brick Solag element are also plotted. For the present element two possibilities have been considered, all of them with volumetric strain average: (a) using just the volumetric strain average (Prism-B) and (b) with both the membrane improvement and the ANS for transverse shear (Prism-BSQ).

The coupled model including the standard solid element with and without volumetric strain averaging is notoriously rigid due to the shear locking (again this locking is less severe for the 6-node element). There are no significant differences between the shell model and the coupled model when the ANS approach transverse shear is used. The differences between the results obtained with the 8-node solid element and present element Prism-BSQ are mainly due to the better membrane behavior of the later.

### 5.7. Deep drawing of a square sheet

The last example considered is the deep drawing of a thin sheet corresponding to one of the benchmarks proposed in NUMISHEET'93 [12]. The Fig. 14 shows the geometry of the tools. The undeformed sheet is square with side length 150 mm . The elastic mechanical properties of the mild steel considered are: elastic modulus $E=206 \mathrm{GPa}$ and Poisson ratio $v=0.3$. For the plastic behavior the classical Hill's yield function with constant coefficients $F=0.283, G=0.358, \quad H=0.642, \quad L=1.065, \quad M=1.179$, $N=1.289$ was assumed. These coefficients were computed from the Lankford ratios $R_{0}=1.79, R_{90}=2.27$ and $R_{45}=1.51$. Isotropic hardening is defined by the yield stress along rolling direction ( X direction) $\sigma_{0}=567.3\left(0.00713+e_{p}\right)^{0.264}$.

The symmetry conditions shown in the figure have been considered, then just one quarter of the geometry has been included in the model. The discretization of the sheet includes 30 elements on each side ( 1800 elements in the plane) and 2 or 4 elements in the thickness direction (a total of 3600 or 7200 elements). The blank holder force used is 19.6 kN and the adopted friction coefficient is $\mu=0.144$. The simulation considered a punch stroke of 40 mm .

The Fig. 15 plots the punch force versus the punch travel. This figure includes the results obtained with the 3-node triangular

Table 1
Draw-in at punch stroke 40 mm . Simulation vs experimental values [mm].

| Direction | Experimental |  |  | BBST | Q1STs | Prism |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | Min. | Max. | Aver. |  | Ref. [18] |  | 2 Lay. |
|  | 4 Lay. |  |  |  |  |  |
| $0^{\circ}$ | 26.75 | 29.60 | 27.96 | 27.94 | 29.00 | 27.64 | 27.78 |
| $90^{\circ}$ | 26.75 | 29.59 | 27.95 | 27.03 | 29.00 | 27.69 | 27.94 |
| $45^{\circ}$ | 14.60 | 16.31 | 15.36 | 17.43 | 16.41 | 16.80 | 16.82 |

shell element BBST with 5 through the thickness integration points and with the standard element (denoted by We-2 and We-4) for comparison. The results obtained with both thickness discretization are included (denoted by P2 and P4) and four combination of the proposed improvements, all including volumetric strain averaging. It may be seen that only the combination of both membrane and transverse shear improvement ( SQ ) lead to the correct results. Note also that using 2 or 4 elements through the thickness does not show noticeable differences. Besides that it must be said that for the last part of the simulation not only the drawing forces are incorrect, also an spurious localized increment of the plastic strains appear for all but the combination SQ.

The Fig. 16 shows the contour fills of the effective plastic strain for the solid models with two and four elements through the thickness and for the model using shell element BBST. No differences are noticeable between using 2 or 4 elements in the thickness discretization, note that the later implies roughly the double CPU time than the former. The differences between the solid models and the shell model are quite small.

Finally Table 1 compares the draw-in measurements for the three simulations mentioned above with the maximum, minimum and average experimental values presented in the conference [12] and with those obtained with the eight-node solid shell element described in Schwarze and Reese [17] using a similar in-plane mesh but just one element across the thickness with seven integration points. Present results are well in the experimental range and compare quite well with the experimental average for both rolling and transverse directions but are rather large along the diagonal.

## 6. Conclusions

In this paper we have developed a triangular prism solid-shell element suitable for nonlinear analysis with elastic-plastic large strains. In the formulation assumed strain techniques have been used to prevent transverse shear locking and to improve the membrane behavior. To avoid volumetric locking the averaging of the volumetric strain in the element has been used. The formulation
is simple and can effectively achieve the objectives. The main conclusions are:

- Transverse shear locking disappears completely in all cases analyzed.
- For elastic problems it is even possible to use just one element across the thickness without resorting to averaging the volumetric strain (for membrane dominated problems).
- In elastic-plastic problems, use of two elements through the thickness give very good results.
- In membrane dominated problems (e.g. deep drawing) the improvement of the membrane behavior is crucial to obtain the correct results.
- The element did not show any problem in large elastic-plastic strain cases.


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