Self-Induced Decoherence and the Classical Limit of Quantum Mechanics

Mario Castagnino and Olimpia Lombardi†

In this paper we argue that the emergence of the classical world from the underlying quantum reality involves two elements: self-induced decoherence and macroscopicity. Self-induced decoherence does not require the openness of the system and its interaction with the environment: a single closed system can decohere when its Hamiltonian has continuous spectrum. We show that, if the system is macroscopic enough, after self-induced decoherence it can be described as an ensemble of classical distributions weighted by their corresponding probabilities. We also argue that classicality is an emergent property that arises when the behavior of the system is described from an observational perspective.

1. Introduction. Both physicists and philosophers agree in believing that classical mechanics should be a limiting case of quantum mechanics: If quantum mechanics is correct, then its results must coincide with the results of classical mechanics in the appropriate limit. In this paper we shall argue that the classical limit of quantum mechanics involves two elements: the first one is the physical phenomenon of decoherence; the second one is macroscopicity. Although the role of decoherence in the emergence of classicality has been pointed out by many authors in the last years, we shall move away from the mainstream position with respect to the explanation of decoherence. Instead of appealing to the einselection approach, here the classical limit will be explained by means of the self-induced approach according to which decoherence does not require the openness of the system and its interaction with the environment. We shall show that, in this new scenario, the classical limit can be described by Figure 1.

Self-induced decoherence transforms quantum mechanics into a Bool-

†To contact the authors, please write to: Mario Castagnino, Casilla de Correos 67, Sucursal 28, 1428, Ciudad de Buenos Aires, Argentina; e-mail: mariocastagnino@citynet.net.ar, or to Olimpia Lombardi, C. Larralde 3440, 6°D, 1430, Ciudad de Buenos Aires, Argentina; e-mail: olimpiafilo@arnet.com.ar.

Philosophy of Science, 72 (December 2005) pp. 764–776. 0031-8248/2005/7205-0010$10.00

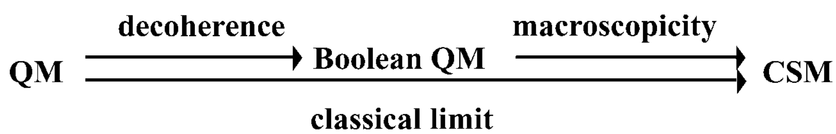

Figure 1.

ean quantum mechanics where the interference terms that preclude classicality have vanished. In turn, macroscopicity turns Boolean quantum mechanics into classical statistical mechanics formulated in phase space. This means that the classical limit of quantum mechanics is not classical mechanics but classical statistical mechanics: If the system is macroscopic enough, after decoherence it can be described in terms of an ensemble of classical densities, each one of them with its own probability of occurrence. We shall also argue that classicality is an emergent property that arises from the underlying quantum level when the system is described from an observational point of view. Finally, we shall consider the possibility of applying this account of the classical limit to the problem of quantum chaos.

2. Self-Induced Decoherence. In the original formulation of the algebraic formalism of quantum mechanics, the algebra of observables was a C^*-algebra which does not admit unbounded operators. For this reason, the self-induced approach to decoherence appeals to a *nuclear algebra* (see Treves 1967; Castagnino and Ordoñez 2004) whose elements are *nuclei* or *kernels*.[1] This nuclear algebra is used to generate two additional topologies by means of the Nelson operator: one of them corresponds to a *nuclear space* V_0 of generalized observables; the other topology corresponds to the space V_S of states, dual of V_0.

Following previous works (Antoniou et al. 1997; Laura and Castagnino 1998a, 1998b), we shall symbolize an observable belonging to V_O by a *round ket* $|O)$ and a state belonging to V_S by a *round bra* $(\rho|$. The result of the action of the round bra $(\rho|$ on the round ket $|O)$ is the expectation value of the observable $|O)$ in the state $(\rho|$:

$$\langle O \rangle_\rho = (\rho|O). \tag{1}$$

If the basis is discrete, $\langle O \rangle_\rho$ can be computed as usual, that is, as the trace of ρO. But if the basis is continuous, $\mathrm{Tr}(\rho O)$ is not well defined; never-

1. By means of a generalized version of the GNS theorem (Gel'fand-Naimark-Segal), it can be proved that this nuclear formalism has a representation in a rigged Hilbert space (Iguri and Castagnino 1999).

theless, $(\rho|O)$ can always be rigorously computed since $(\rho|$ is a linear functional belonging to V_S and acting on an operator $|O)$ belonging to V_O.

Let us consider the simple case of a quantum system whose Hamiltonian has a continuous spectrum:

$$|H)|\omega\rangle = \omega|\omega\rangle, \quad \omega \in [0, \infty), \tag{2}$$

where ω and $|\omega\rangle$ are the generalized eigenvalues and eigenvectors of $|H)$, respectively. The task is now to express a generic observable $|O)$ in the eigenbasis of the Hamiltonian. Following the formalism of van Hove (1955), a generic observable $|O)$ belonging to the space V_O reads

$$|O) = \int O(\omega)|\omega)d\omega + \int\int O(\omega, \omega')|\omega; \omega')d\omega d\omega', \tag{3}$$

where $O(\omega)$ and $O(\omega, \omega')$ are generic distributions, and $|\omega) = |\omega\rangle\langle\omega|$ and $|\omega; \omega') = |\omega\rangle\langle\omega'|$ are the generalized eigenvectors of the observable $|O)$; $\{|\omega), |\omega; \omega')\}$ is a basis of V_O. On the other hand, states are represented by linear functionals belonging to the space V_S, which is the dual of V_O; therefore, a generic state $(\rho|$ belonging to V_S can be expressed as

$$(\rho| = \int \rho(\omega)(\omega|d\omega + \int\int \rho(\omega, \omega')(\omega; \omega'|d\omega d\omega', \tag{4}$$

where $\{(\omega|, (\omega; \omega'|\}$ is a basis of V_S, that is, the cobasis of $\{|\omega), |\omega; \omega')\}$, and it is defined by the relations[2] $(\omega|\omega') = \delta(\omega - \omega')$, $(\omega; \omega''|\omega'; \omega''') = \delta(\omega - \omega')\delta(\omega'' - \omega''')$, and $(\omega|\omega'; \omega'') = 0$. The only condition that the distribution $\rho(\omega)$ and $\rho(\omega, \omega')$ must satisfy is that of leading to a well-defined expectation value of the observable $|O)$ in the state $(\rho|$.

With respect to the formalism of van Hove, the new approach introduces two restrictions. The first one consists in considering only the observables $|O)$ whose $O(\omega, \omega')$ are regular functions; these observables define what we shall call "*van Hove space*," $V_O^{VH} \subset V_O$. Since the states are represented by linear functionals over the space of observables, in this case they belong to the dual of the space V_O^{VH}. The second restriction consists in considering only the states $(\rho|$ whose $\rho(\omega, \omega')$ are regular functions; these states belong to a convex space S included in the dual of V_O^{VH}, since they satisfy the usual conditions of quantum states: hermiticity,

2. These are just the generalization of the relationship between the basis $\{|i\rangle\}$ and the cobasis $\{\langle j|\}$ in the discrete case: $\langle j|i\rangle = \delta_{ij}$.

non-negativity and normalization of the diagonal terms.[3] Under these restrictions, decoherence follows in a straightforward way. According to the unitary von Neumann equation, $(\rho(t)| = e^{-iHt}(\rho_0|e^{iHt}$. Therefore, the expectation value of the observable $|O) \in V_O^{VH}$ in the state $(\rho(t)| \in S$ reads

$$\begin{aligned} \langle O\rangle_{\rho(t)} &= (\rho(t)|O) \\ &= \int \rho(\omega)O(\omega)d\omega + \int\int \rho(\omega, \omega')e^{-i(\omega-\omega')t}O(\omega, \omega')d\omega d\omega'. \end{aligned} \quad (5)$$

Since $O(\omega, \omega')$ and $\rho(\omega, \omega')$ satisfy the condition of being regular functions, when we take the limit for $t \to \infty$, we can apply the *Riemann-Lebesgue theorem*[4] according to which the second term of the right hand side of equation (5) vanishes. Therefore,

$$\lim_{t\to\infty} \langle O\rangle_{\rho(t)} = \lim_{t\to\infty} (\rho(t)|O) = \int \rho(\omega)O(\omega)d\omega. \quad (6)$$

But this integral is equivalent to the expectation value of the observable $|O)$ in a new state $(\rho_*|$ where the off-diagonal terms have vanished:

$$(\rho_*| = \int \rho(\omega)(\omega|d\omega \Rightarrow \langle O\rangle_{\rho_*} = (\rho_*|O) = \int \rho(\omega)O(\omega)d\omega. \quad (7)$$

Therefore, for all $|O) \in V_O^{VH}$ and for all $(\rho| \in S$, we obtain the limit

$$\lim_{t\to\infty} \langle O\rangle_{\rho(t)} = \langle O\rangle_{\rho_*}. \quad (8)$$

This equation shows that the definition of self-induced decoherence involves the convergence of the expectation value of any observable belonging to V_O^{VH} to a value that can be computed as if the system were in a state represented by a diagonal density operator $(\rho_*|$. It can be proved that decoherence also obtains when the spectrum of the Hamiltonian has a single discrete value non-overlapping with the continuous part; but if the Hamiltonian's spectrum has more than one non-overlapping value in its discrete part, the system does not decohere (for details, see Castagnino and Lombardi 2004).

3. The restrictions on operators and states do not diminish the generality of the self-induced approach, since the observables not belonging to V_O^{VH} and the states not belonging to S are not experimentally accessible and, for this reason, in practice they are approximated, with the desired precision, by regular observables and states for which the self-induced approach works satisfactorily (for a full argument, see Castagnino and Lombardi 2004).

4. The Riemann-Lebesgue theorem states that $\lim_{x\to\infty}\int e^{ixy}f(y)dy = 0$ iff $f(y) \in L^1$ (that is, iff $\int |f(y)|dy < \infty$).

Up to this point we have considered a simplified case where the Hamiltonian was the only dynamical variable. But in a general case we must consider a CSCO $\{|H), |O_1), \ldots, |O_N)\}$, whose eigenvectors are $|\omega, o_1, \ldots, o_N\rangle$. In this case, $(\rho_*|$ will be diagonal in the variables ω, ω' but not in general in the remaining variables. Therefore, a further diagonalization of $(\rho_*|$ is necessary: as the result, a new set of eigenvectors $\{|\omega, r_1, \ldots, r_N\rangle\}$, corresponding to a new CSCO $\{|H), |R_1), \ldots, |R_N)\}$, emerges. This set defines the eigenbasis $\{|\omega, r_1, \ldots, r_N), |\omega, r_1, \ldots, r_N; \omega', r'_1, \ldots, r'_N)\}$ of the van Hove space of observables V_O^{VH}, where $|\omega, r_1, \ldots, r_N) = |\omega, r_1, \ldots, r_N\rangle\langle\omega, r_1, \ldots, r_N|$ and $|\omega, r_1, \ldots, r_N; \omega', r'_1, \ldots, r'_N) = |\omega, r_1, \ldots, r_N\rangle\langle\omega', r'_1, \ldots, r'_N|$. $(\rho_*|$ will be completely diagonal in the cobasis of states, $\{(\omega, r_1, \ldots, r_N|, (\omega, r_1, \ldots, r_N; \omega', r'_1, \ldots, r'_N|\}$ corresponding to the new eigenbasis of V_O^{VH} (for details, see Castagnino and Laura 2000a, Section II-B). This new CSCO can be called '*preferred CSCO*' since its eigenvectors define the basis that diagonalizes $(\rho_*|$.

Summing up, self-induced decoherence does not require the openness of the system of interest and its interaction with the environment: *A single closed system can decohere* since the diagonalization of the density operator does not depend on the openness of the system but on the *continuous spectrum* of the system's Hamiltonian.

3. Macroscopicity. In order to obtain the classical limit, the second step is to represent the diagonal state $(\rho_*|$ resulting from decoherence in the corresponding phase space and to apply the macroscopic limit $\hbar \to 0$.[5] As it is well known, the Wigner transformation maps states and operators into functions on phase space:

$$W\rho = \rho(q, p), \quad WO = O(q, p). \tag{9}$$

Moreover, we know that the Wigner transformation yields the correct expectation value of any observable in a given state when we are dealing with regular functions; however, it has not been defined when singular functions are involved. The peculiarity of our case is that $(\rho_*|$ is precisely the singular part of the initial state $(\rho|$ (compare (4) and (7)). Therefore, the Wigner transformation of singular states must be defined in this case.

5. In the rest of the paper we shall use the counterfactual limit $\hbar \to 0$ in order to simplify expressions, but in any case such a limit represents the factual limit $\hbar/S \to 0$ corresponding to the situations in which the characteristic action S of the system is much greater than $\hbar$. For an explanation of the difference between factual and counterfactual limits, see Rohrlich 1989 and, with a different terminology, Bruer 1982.

In fact, the task is to find the classical distribution $\rho_c(q, p)$ resulting from applying the limit $\hbar \to 0$ to the Wigner transformation of $(\rho_*|$:

$$\rho_c(q, p) = \lim_{\hbar \to 0} W(\rho_*| = \int \rho(\omega)\left[\lim_{\hbar \to 0} W(\omega|\right] d\omega. \tag{10}$$

Under the only reasonable requirement that the Wigner transformation leads to the correct expectation value of any observable in a given state also in the singular case, it can be proved that, when H is the only dynamical variable (see Castagnino 2004):

$$\lim_{\hbar \to 0} W(\omega| = \delta(H(q, p) - \omega), \tag{11}$$

where $H(q, p)$ is the Wigner transformation of the quantum Hamiltonian H, $H(q, p) = WH$. As a consequence, the classical distribution $\rho_c(q, p)$ becomes

$$\rho_c(q, p) = \int \rho(\omega)\delta(H(q, p) - \omega)d\omega. \tag{12}$$

This result has a clear physical interpretation. We know that the classical distribution $\rho_c(q, p)$ is defined in a 2-dimensional phase space, and $H(q, p) = \omega$ is the only global constant of motion. As a consequence, $\rho_c(q, p)$ represents an infinite sum of classical densities $\delta(H(q, p) - \omega)$ represented by the hypersurfaces $H(q, p) = \omega$ in phase space, and averaged by the corresponding value of the function $\rho(\omega)$. On the other hand, $\rho(\omega)$ is normalized and non-negatively defined due to its origin, since it represents the diagonal components of the initial quantum state $(\rho|$ (see (4)); this fact is what permits it to be interpreted as a probability function. Therefore, $\rho_c(q, p)$ can be conceived as the infinite sum of the classical densities defined by the global constants of motion $H(q, p) = \omega$ and weighted by their corresponding probabilities given by the quantum initial condition $(\rho(t = 0)|$.

In the general case H is not the only dynamical variable, but the system has a CSCO consisting of $N + 1$ observables $\{|H), |O_1), \ldots, |O_N)\}$. As we have seen, in this case a preferred CSCO $\{|H), |R_1), \ldots, |R_N)\}$ emerges, in such a way that $(\rho_*|$ becomes diagonal in the new basis of states $\{(\omega, r_1, \ldots, r_N|, (\omega, r_1, \ldots, r_N; \omega', r_1', \ldots, r_N'|\}$. In this case, $(\rho_*|$ results:

$$(\rho_*| = \int_{r_1} \cdots \int_{r_N} \int_{\omega} \rho(\omega, r_1, \ldots, r_N)(\omega, r_1, \ldots, r_N| d\omega dr_1 \cdots dr_N, \tag{13}$$

and the classical distribution $\rho_c(\phi)$ becomes (for a detailed proof, see Castagnino and Gadella 2007):

$$\rho_c(\phi) = \int_{r_1} \cdots \int_{r_N} \int_{\omega} \rho(\omega, r_1, \ldots, r_N)\delta(H(\phi) - \omega)\delta(R_1(\phi) - r_1) \cdots \delta(R_N(\phi) - r_N) d\omega dr_1 \cdots dr_N, \qquad (14)$$

where now $\phi = (\boldsymbol{q}, \boldsymbol{p}) = (q_1, \ldots, q_{N+1}, p_1, \ldots, p_{N+1})$ and $R_i(\phi) = WR_i$, with $i = 1$ to N. This means that, in this case, $\rho_c(\phi)$ is defined on a $2(N + 1)$–dimensional phase space, and there are $N + 1$ global constants of motion $H(\phi) = \omega$, $R_i(\phi) = r_i$. Therefore, in this general case the classical distribution $\rho_c(\phi)$ defined on $\square^{2(N+1)}$ can be conceived as an infinite sum of classical densities defined by the corresponding global constants of motion $H(\phi) = \omega$, $R_1(\phi) = r_1, \ldots, R_N(\phi) = r_N$ and weighted by the corresponding probability $\rho(\omega, r_1, \ldots, r_N)$.[6]

4. The Emergent Character of Classicality. In its traditional form, a coarse-grained description arises from a partition of a phase space into discrete and disjoint cells. This mathematical procedure defines a projector (see Mackey 1989). In other words, a traditional coarse-graining amounts to a projection whose action is to eliminate some components of the state vector corresponding to the fine-grained description—only certain components are retained as "relevant". If this idea is generalized, coarse-graining can be conceived as an operation that reduces the number of components of a generalized vector representing a state. From this viewpoint, taking a partial trace is a particular case of coarse-graining, since a partial trace reduces the number of the components of the density operator on which it is applied. Therefore, a reduced density operator ρ_r, which describes the open system in the einselection approach to decoherence (Paz and Zurek 2002, Zurek 2003), is a coarse-grained state since it is an improper mixture (d'Espagnat 1995) resulting from tracing over the environmental degrees of freedom (for a detailed discussion, see Castagnino and Lombardi 2004).

In the context of the self-induced approach, decoherence is not produced by the interaction between the system of interest and its environment, but results from the own dynamics of the whole quantum system governed by a Hamiltonian with continuous spectrum. Of course, this characterization does not contradict the fact that the off-diagonal terms of a density

6. This account of the classical limit has been applied to the so called 'Mott problem' (Castagnino and Laura 2000b) and to the description of a closed quantum universe (Castagnino and Lombardi 2003).

operator representing a quantum state will never vanish through the unitary evolution described by the von Neumann equation: What decoherence shows is that the expectation value $\langle O\rangle_{\rho(t)}$ of any observable $|O) \in V_O^{VH}$ will evolve in such a way that, for $t \to \infty$, it can be computed as the expectation value of $|O)$ in a diagonal state $(\rho_*|$. Formally this is expressed by the fact that, even though we can strictly obtain the limit

$$\lim_{t\to\infty} \langle O\rangle_{\rho(t)} = \langle O\rangle_{\rho_*}, \tag{15}$$

the state $(\rho(t)|$ has only a *weak limit* (Castagnino and Laura 2000a):

$$w - \lim_{t\to\infty} (\rho(t)| = (\rho_*|. \tag{16}$$

This weak limit means that, although the off-diagonal terms of the density operator $(\rho(t)|$ never vanish through the unitary evolution, the system decoheres *from an observational point of view*, that is, from the viewpoint given by the observable $|O)$, for all $|O) \in V_O^{VH}$.

In other words, $\langle O\rangle_\rho = (\rho|O)$ can be thought as representing the state $(\rho|$ of the system 'viewed' from the perspective given by the observable $|O)$ and, in this sense, $\langle O\rangle_\rho$ also involves a sort of coarse-graining. Of course, since in this case we are dealing with continuous variables, we cannot strictly speak of reducing the number of components of a vector state. However, the action of the functional $(\rho| \in S$ onto the observable $|O) \in V_O^{VH}$ can be characterized in terms of a generalized notion of projection, which permits $\langle O\rangle_\rho$ to be conceived as the result of a projection of the state $(\rho|$. In fact, we can define a projector belonging to the space $V_O^{VH} \otimes S$:

$$\Pi = |O)(\rho_O|, \tag{17}$$

where $(\rho_O| \in S$ satisfies

$$(\rho_O|O) = 1 \tag{18}$$

in order to guarantee that Π be a projector:

$$\Pi^2 = |O)(\rho_O|O)(\rho_O| = |O)(\rho_O| = \Pi. \tag{19}$$

In this case

$$(\rho_{\text{rel}}| = (\rho|\Pi = (\rho|O)(\rho_O|, \tag{20}$$

where $(\rho_{\text{rel}}|$ is the projected part of $(\rho|$, relevant for decoherence. This means that $\langle O\rangle_\rho = (\rho|O)$ is the result of the projection of $(\rho|$ onto a subspace of S defined by the state $(\rho_O|$ corresponding to the observable $|O)$. On this basis we can understand why $\langle O\rangle_\rho$ can be conceived as a coarse-grained magnitude, that gives us the partial description of $(\rho|$ from the perspective given by $|O)$. Therefore, decoherence is a coarse-grained process, resulting in this case from the coarse-graining introduced by the observable of interest in the underlying unitary dynamics.

At this point, it is worth stressing the role played by $(\rho_*|$ in this explanation of decoherence. The diagonal operator $(\rho_*|$ does not denote the real quantum state of the system in the infinite-time limit. The quantum state of the system is always represented by $(\rho(t)|$, which does not strongly converges to $(\rho_*|$. It only approaches to $(\rho_*|$ in a weak sense, that is, in a weak topology. In other words, $(\rho(t)|$ always describes an unitary evolution and, therefore, it does not tend to a definite strong limit for $t \to \infty$. The only fact that we can strictly assert is that, in the infinite-time limit, the expectation value of the observable $|O)$ can be computed as if the whole system were in the quantum state $(\rho_*|$. It is interesting to note that $\langle O \rangle_\rho$ could also be computed in the Heisenberg picture, where the observable $|O(t))$ evolves with time whereas the state $(\rho|$ is constant; in this case we would obtain a diagonal operator $|O_*)$. This fact clearly shows that the fundamental magnitude in the explanation of decoherence is the expectation value $\langle O \rangle_\rho$ and not the state $(\rho|$. In fact, $\langle O \rangle_{\rho(t)}$ is the magnitude that approaches a definite limit for $t \to \infty$, and no quantum law prevents it from having this kind of behavior. This situation is analogous to the familiar case of unstable dynamical systems, where it is completely natural to obtain a non-unitary coarse-grained evolution from an underlying unitary dynamics.

These considerations are particularly relevant to the interpretation of the classical statistical description resulting from the classical limit. The classical density distribution $\rho_c(q, p)$ arising from decoherence and macroscopicity tells us that the system behaves as a classical statistical system from the perspective given by any observable $|O) \in V_O^{VH}$. This means that our predictions about the expectation value of any relevant observable on the quantum system will lead us to the same results as those we would obtain on a classical system described as an ensemble of classical densities weighted by their corresponding probabilities. Since those classical densities are defined by the global constants of motion $H(\phi) = \omega$ and $R_i(\phi) = r_i$, with $i = 1$ to N in a $2(N + 1)$–dimensional phase space, each one of them can be conceived as a set of classical point-like states defined by those constants of motion and the value of the remaining variables of the phase space. But this fact does not mean that there actually exist classical point-like states in the quantum level. The classical density distribution $\rho_c(q, p)$ does not correspond to the quantum state of the system; it is a coarse-grained magnitude that describes the behavior of the system from the observational point of view given by the observable of interest. As a consequence, classicality is an emergent property that arises in a coarse-grained level of description from an underlying quantum level.

5. Perspectives: The Problem of Quantum Chaos. At present it is well known that chaotic behavior is an ubiquitous feature of classical systems. This contrasts with the fact that chaos in quantum systems seems to be the exception rather than the rule. Some authors even consider that there is no quantum chaos, since the models resulting from quantization of chaotic classical systems do not exhibit chaotic behavior. However, on the other hand, the opinion that there is some kind of conflict between quantum mechanics and chaos sounds surprising in the light of the increasing attention that quantum chaos has received from the community of physicists during the last years. Perhaps the main reason for this confusing situation is the disagreement about what 'quantum chaos' means. If we consider that the problem of quantum chaos consists in explaining how classical chaotic properties can emerge from the quantum realm, then such a problem becomes a particular case of the classical limit of quantum mechanics. Therefore, a detailed account of the classical limit turns out to be a relevant element for the solution of the quantum chaos problem.

It is interesting to note the points of contact between this way of conceiving the problem and the program of Belot and Earman (1997) about this question. In the first place, we agree with Belot and Earman about the kind of quantum systems considered in the problem. Whereas many authors look for quantum chaos in open systems in order to obtain non-unitary time evolutions (see Kronz 1998; Paz and Zurek 2002; Zurek 2003), Belot and Earman restrict their attention to the standard quantum-mechanical treatments of closed quantum systems: they focus exclusively on Schrödinger evolutions and ignore the measurement problem. As we have seen, our account of the classical limit also applies to closed quantum systems to the extent that self-induced decoherence does not require the openness of the system and its interaction with the environment; this explanation permits quantum systems to behave classically even in situations in which there is no measurement involved.

The second point of agreement is related to the formalism adopted for addressing the problem. Belot and Earman point out that physicists are able to derive testable predictions from quantum mechanics with no reference to the measurement problem, and this fact can be justified by a reliance on the notion of expectation values of observables and their evolutions. On this basis, Belot and Earman develop their argumentation in the language provided by the algebraic formalism of quantum mechanics. Our account of the classical limit also agrees with Belot-Earman's approach regarding this point since it is completely expressed in the context of the algebraic formalism and relies on the time behavior of the expectation values of the relevant observables of the quantum system.

But the most striking coincidence is the closeness between Belot-Earman's definition of quantum mixing and our definition of self-induced decoherence. In fact, according to the authors, an abstract dynamical system $(\boldsymbol{A}, \theta)$, where $\boldsymbol{A}$ is a C^*-algebra representing the algebra of observables, and θ is an automorphism of $\boldsymbol{A}$ representing the time evolution of the system, is *mixing* if there exists a θ-invariant state φ_E such that (Belot and Earman 1997, 161)

$$\lim_{n\to\infty} \varphi(\theta_n A) = \varphi_E(A) \tag{21}$$

for any φ acting on $\boldsymbol{A}$ and for any $A \in \boldsymbol{A}$. It is not difficult to see that we can replace $\varphi(\theta_n A)$ with $(\rho(t)|A)$ since these two expressions represent the expectation value of the observable A in the time-dependent state φ in the first case and $(\rho(t)|$ in the second case, and we can also replace $\varphi_E(A)$ with $(\rho_*|A)$ because both φ_E and $(\rho_*|$ represent the equilibrium state where the expectation value of A does not undergo further changes; we thus obtain

$$\lim_{t\to\infty} (\rho(t)|A) = (\rho_*|A). \tag{22}$$

Leaving aside the fact that Belot and Earman work with a C^*-algebra instead of a nuclear algebra, (22) is precisely the definition of self-induced decoherence. Furthermore, it is easy to prove that $(\rho_*|$ is unique for each set of values of the observables of the preferred CSCO.

At this stage, it seems right to conclude that if a quantum system satisfies the conditions for decohering, it is a quantum mixing system according to Belot-Earman's definition, and vice versa. In fact, as the authors point out, if a quantum system is mixing according their definition, then 1 is the only eigenvalue of the transformation θ_n and is a simple eigenvalue. It is quite clear that this property is also satisfied in the case of decoherence: $(\rho_*|$ is the only eigenvector of U_t and its corresponding eigenvalue is 1. Can we conclude that self-induced decoherence is a necessary and sufficient condition for quantum mixing?

The answer to this question is negative, because the classical system arising from the classical limit of a quantum system that fulfills the conditions imposed by Belot-Earman's definition may be not classically mixing. In fact, when the quantum system is endowed with a CSCO of $N + 1$ observables, which is sufficient for defining an eigenbasis for the system's states, the classical distribution $\rho_c(\phi)$ resulting from the classical limit is defined on a phase space $\square^{2(N+1)}$ and has $N + 1$ global constants of motion in involution. But, as it is well known, a classical system with n degrees of freedom and n global constants of motion in involution is *integrable* and, as a consequence, non-mixing.

This argument makes clear that the problem of quantum chaos requires to study how non-integrability can be expressed in quantum mechanics.

In particular, it is necessary to consider quantum systems endowed with a CSCO of $A + 1$ observables, with $0 \leq A < N$, that is a CSCO that does not define an eigenbasis in terms of which the state of the system can be expressed. This is the case of the systems, as the Helium atom, whose quantum numbers are not sufficient for defining their states. The program is, then, to extend our formalism in such a way that this kind of systems can be theoretically described and their classical limit can be obtained.

6. Conclusions. On the basis of the assumption that the problem of the classical limit amounts to the question of how the classical world arises from an underlying quantum reality, the aim of this paper was to present a theoretical and general answer to the question. Our account of the classical limit involves two elements: self-induced decoherence, conceived as a process that depends on the own dynamics of a closed quantum system governed by a Hamiltonian with continuous spectrum, and macroscopicity, that allows the result of decoherence to turn into an ensemble of classical densities on phase space weighted by their corresponding probabilities. When these formal results are considered in the light of a generalized concept of coarse-graining, decoherence turns out to be a coarse-grained process that, in the infinite-time limit and the macroscopic limit, leads to classicality.

REFERENCES

Antoniou, Ioannis, Zdislaw Suchanecki, Roberto Laura, and Syuichi Tasaki (1997), "Intrinsic Irreversibility of Quantum Systems with Diagonal Singularity", *Physica A* 241: 737–772.

Belot, Gordon, and John Earman (1997), "Chaos out of Order: Quantum Mechanics, the Correspondence Principle and Chaos", *Studies in History and Philosophy of Modern Physics* 28: 147–182.

Bruer, John (1982), "The Classical Limit of Quantum Theory", *Synthese* 50: 167–212.

Castagnino, Mario (2004), "The Classical-Statistical Limit of Quantum Mechanics", *Physica A* 335: 511–517.

Castagnino, Mario, and Manuel Gadella (2007), "The Problem of the Classical Limit of Quantum Mechanics and the Role of Self-Induced Decoherence", *Foundations of Physics*, forthcoming (Los Alamos National Laboratory, quant-ph/0306014).

Castagnino, Mario, and Roberto Laura (2000a), "Functional Approach to Quantum Decoherence and the Classical Final Limit", *Physical Review A* 62: 022107.

——— (2000b), "Functional Approach to Quantum Decoherence and the Classical Limit: The Mott and Cosmological Problems", *International Journal of Theoretical Physics* 39: 1737–1765.

Castagnino, Mario, and Olimpia Lombardi (2003), "The Self-Induced Approach to Decoherence in Cosmology", *International Journal of Theoretical Physics* 42: 1281–1299.

——— (2004), "Self-Induced Decoherence: A New Approach", *Studies in History and Philosophy of Modern Physics* 35: 73–107.

Castagnino, Mario, and Adolfo Ordoñez (2004), "Algebraic Formulation of Quantum Decoherence", *International Journal of Theoretical Physics* 43: 695–719.

d'Espagnat, Bernard (1995), *Veiled Reality: An Analysis of Present-Day Quantum Mechanical Concepts*. Reading, MA: Addison-Wesley.

Iguri, Sergio, and Mario Castagnino (1999), "The Formulation of Quantum Mechanics in Terms of Nuclear Algebras", *International Journal of Theoretical Physics* 38: 143–164.

Kronz, Frederick (1998), "Nonseparability and Quantum Chaos", *Philosophy of Science* 65: 50–75.

Laura, Roberto, and Mario Castagnino (1998a), "Minimal Irreversible Quantum Mechanics: Mixed States and the Diagonal Singularity", *Physical Review A* 57: 4140–4152.

——— (1998b), "Functional Approach for Quantum Systems with Continuous Spectrum", *Physical Review E* 57: 3948–3961.

Mackey, Michael (1989), "The Dynamic Origin of Increasing Entropy", *Review of Modern Physics* 61: 981–1015.

Paz, Juan Pablo, and Wojciech Zurek (2002), "Environment-Induced Decoherence and the Transition from Quantum to Classical", in A. J. Leggett (ed.), *Lecture Notes in Physics*, vol. 587. Heidelberg-Berlin: Springer (Los Alamos National Laboratory, quant-ph/0010011).

Rohrlich, Fritz (1989), "The Logic of Reduction: The Case of Gravitation", *Foundations of Physics* 19: 1151–1170.

Treves, Francois (1967), *Topological Vector Spaces, Distributions and Kernels*. New York: Academic Press.

van Hove, Leon (1955), "Energy Corrections and Persistent Perturbation Effects in Continuous Spectra", *Physica* 21: 901–923.

Zurek, Wojciech (2003), "Decoherence, Einselection and the Quantum Origins of the Classical", *Reviews of Modern Physics* 75: 715–776 (Los Alamos National Laboratory, quant-ph/0105127).