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Time-Reversal, Irreversibility and Arrow of Time in Quantum Mechanics

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The aim of this paper is to analyze time-asymmetric quantum mechanics with respect of its validity as a non time-reversal invariant, time-asymmetric theory as well as of its ability to determine an arrow of time.

KEY WORDS: irreversibility; arrow of time; rigged Hilbert spaces.

1. INTRODUCTION

The problem of the irreversibility in classical physics consisted in trying to find and adequate account of the compatibility between the irreversible macroscopic evolutions described by thermodynamics and the reversible microscopic evolutions resulting from classical mechanics (for historical details, see Ref. 1) given by the Newton laws. The solution to this problem has been traditionally presented in terms of the second law of thermodynamics. In the beginning of the 20th century, classical mechanics was replaced by quantum mechanics as the fundamental theory describing the underlying mechanical level. However, this fact did not affect the core of the original problem: quantum evolutions, considered independently of the measurement processes, turned out to be also reversible as they are governed by time-reversal invariant fundamental equations like Schrödinger and Dirac equations. In quantum mechanics, dynamical Hamiltonian processes usually lead to reversible evolutions. There are,

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nevertheless, processes considered irreversible like resonance phenomena. The irreversible character of these kind of phenomena is largely accepted (see Refs. 2, 3). However, there is not an equivalent of the second law of thermodynamics in quantum mechanics. This showed the need for a formulation of irreversibility in quantum mechanics. A "time-asymmetric" formulation of quantum mechanics was proposed by two groups, one lead by Arno Bohm,[4] and the other by Ilya Prigogine,[5–7] whose contributions were fundamental for the understanding of intrinsic irreversibility, that is, the irreversibility due not to the interaction between a system and its environment, but to the dynamics of the closed system. Traditionally, the problem of irreversibility has been associated to other two: non-time-reversal invariance and the existence of an arrow of time.

The aim of the present paper is to analyze the different aspects of this time-asymmetric formulation of quantum mechanics and to discuss its validity as a theory that, (i) breaks time-reversal invariance, (ii) accounts for a theory of *intrinsic irreversibility* in quantum mechanics, (iii) determines an arrow of time as a consequence of the theory itself.

This paper is organized as follows: In the next Section, we clarify what we understand by time-reversal invariance and irreversibility and give examples of different situations that can arise when we combine both concepts in all possible manners. We also want to establish which is the proper relation between irreversibility and time's arrow and, in particular, whether these two concepts are equivalent or not. In Sec. 3, we analyze the ideas of the time-asymmetric formulation of quantum mechanics and its consequences from the point of view of the irreversibility and the non time-reversal invariance of the theory. In Sec. 4, we try to get more insight in the nature of irreversibility in this formulation. Finally, in the last section, we intend to clarify the role of the formalism in the formulation of a quantum mechanical arrow of time.

2. DISENTANGLING CONCEPTS

When the problems of irreversibility and of the arrow of time are addressed, the main obstacle to be faced is conceptual confusion: the lack of consensus is primarily due to the fact that different concepts are identified and different questions are subsumed under the same label. In particular, the two problems are usually identified, as if irreversibility were the clue for understanding the origin and the nature of the arrow of time. In this section we shall clarify the concepts involved in the debate; as the result, both problems become evidently different.

2.1. Time-Reversal Invariance and Reversibility

The two central concepts involved in the discussions about what has been loosely called 'the problem of the direction of time' are time-reversal invariance and irreversibility.

Definition 1. A dynamical equation (law) is *time-reversal invariant* if it is invariant under the application of the time-reversal operator **T.**

The time-reversal operator **T** performs the transformation $t \to -t$ and reverses certain magnitudes which depend on the particular theory considered. Nevertheless, the central idea is that **T** must reverse all the dynamical variables whose definitions in function of t are non-invariant under the transformation $t \to -t$. As a result, given a time-reversal invariant equation L, if $e(t)$ is a solution of L, then $\mathbf{T}e(t)$ is also a solution.

On the other hand, it is well known that an attractor is defined as a subset of the phase space toward which a set of evolutions tend for $t \to \pm\infty$. We can extend this definition by considering a generalized concept of attractor as a subset of the set of the possible states of a system toward which a set of evolutions tend for $t \to \pm\infty$; this concept can be applied not only to phase spaces, but also to any kind of sets of states. Examples of generalized attractors are the attractors of classical dynamical systems (fixed point, limit cycle, fractal, etc.) and any classical or quantum equilibrium state. With this characterization, the concept of reversibility can be defined as follows:

Definition 2. A solution (evolution) $e(t)$ of a dynamical equation is *reversible* if it has no generalized attractors, for any representation of $e(t)$.

When the time dependent state $e(t)$ can be represented as an n-uple of dynamical variables, $e(t) = (v_1(t), \ldots, v_n(t))$, reversibility requires that, for any dynamical variable $v_i(t)$, the limit $\lim_{t\to\pm\infty} v_i(t)$ does not exist. In this case it can be said that the evolution $e(t)$ is reversible if it has no attractors in phase space.

Independently of the details of these two definitions (for further details, *cf.* Albert,[8]) Arntzenius,[9] it is quite clear that the concepts of time-reversal invariance and irreversibility are different to the extent that they apply to different mathematical (physical) entities: whereas time-reversal invariance is a property of dynamical equations and, *a fortiori*, of the sets of its solutions, reversibility is a property of a single solution of a dynamical equation. Furthermore, both properties are not even correlated; in fact, they can be combined with each other in the four possible cases.

These four cases have been analyzed with detail elsewhere[10] and, therefore, we just mention and recall them in here.

- *Time-reversal invariance and reversibility*. Let us consider the harmonic oscillator with Hamiltonian:

$$H = \frac{1}{2m}\,p^2 + \frac{1}{2}\,k^2\,q^2 \tag{1}$$

  The dynamical equations are time-reversal invariant as can be easily shown. As a result, the set of trajectories in phase space is symmetric with respect to the q-axis. The solutions $q(t)$ and $p(t)$ have no limit for $t \to \pm\infty$. In other words, each trajectory is reversible since it is a closed curve in phase space.

- *Time-reversal invariance and irreversibility*. Let us consider the pendulum with Hamiltonian:

$$H = \frac{1}{2m}\,p_\theta^2 - \frac{k^2}{2}\,\cos\theta \tag{2}$$

  Again the dynamical equations are time-reversal invariant since $\mathbf{T}\theta = \theta$. Therefore, the set of trajectories in phase space is symmetric with respect to the θ-axis. However, not all the solutions are reversible. In fact, when $H = \frac{k^2}{2}$, the evolution is irreversible.[11] For $H < \frac{k^2}{2}$ (oscillating pendulum) and $H > \frac{k^2}{2}$ (rotating pendulum), the evolutions are reversible.

- *Non time-reversal invariance and reversibility*. Let us now consider the modified oscillator with Hamiltonian:

$$H = \frac{1}{2m}\,p^2 + \frac{1}{2}\,K(p)^2\,q^2\,, \tag{3}$$

  where $K(p) = K_+$ when $p \geqslant 0$, $K(p) = K_-$ when $p < 0$, and K_+ and K_- are constants. This means that $\mathbf{T}K_+ = K_-$. As a consequence, if $K_+ \neq K_-$, the dynamical equations are non time-reversal invariant. Nevertheless, the continuous solutions $q(t)$ and $p(t)$ have no limit for $t \to \pm\infty$: each trajectory is reversible since it is a closed curve in phase space.

- *Non time-reversal invariance and irreversibility*. Let us consider a damped oscillator represented by the following dynamical equation:

$$\ddot{q} + k^2\,\dot{q} + A^2 q = 0\,, \tag{4}$$

which obviously is non time-reversal invariant. On the other hand, its solutions have the form $q(t) = q_0 \cos \omega t \; e^{-\gamma t}$. Here $\cos \omega t$ is the oscillating factor and $e^{-\gamma t}$ is the damping factor. As a consequence, the evolutions are irreversible since they tend to zero for $t \to \infty$.

Up to this point, we have presented the general definitions of time-reversal invariance and irreversibility. However, since here we are interested in quantum mechanics, we shall consider the following kind of evolutions:

$$e_t = U_t \; e_0 \tag{5}$$

where e_0 and e_t are vector states. The evolution operator is a unitary operator $U_t = e^{-iHt}$, which is an one parameter group on the time parameter t with $-\infty < t < \infty$. Observe that the evolutions of the form $e_t = U_t \; e_0$ are *always reversible*, since $e_t = e^{-iHt} e_0$ has no limit for $t \to \pm\infty$. In other words, since U_t is a unitary operator, it does not change the angle of separation (the inner product) or the distance (the square modulus of the difference) between vectors representing two different states. However, irreversible and, therefore, non-unitary evolutions can be obtained from the original reversible unitary dynamics by the introduction of some sort of *coarse-graining* (see a thourogly discussion in Ref. 12).

Once the concepts of time-reversal invariance and reversibility have been elucidated with precision, *the problem of irreversibility* can be stated in a simple way: *how to explain irreversible evolutions in terms of time-reversal invariant laws*. On the basis of such an elucidation, it also turns out to be clear that there is no conceptual puzzle in the problem of irreversibility: in principle, nothing prevents a time-reversal invariant equation from having irreversible solutions. However, difficulties arise when we are dealing with dynamical equations having unitary solutions: as we have seen, since unitary evolutions are always reversible, it is necessary to go to a different level of description in order to obtain irreversibility. This point will be relevant in the discussions about irreversibility as obtained by means the "time-asymmetric" formulation of quantum mechanics.

2.2. The Problem of the Arrow of Time

The problem of the arrow of time owes its origin to the intuitive asymmetry between past and future. We experience the time order of the world as 'directed': if two events are not simultaneous, one of them is earlier than the other. Moreover, we view our access to past and future quite differently: we remember the past and predict the future. From this point of view, several arrows of time have been considered in Physics: the thermodynamical arrow of time (the entropy in isolated systems increases to the future), the cosmological arrow of time (the universe expands toward the future), the electromagnetic arrow of time (retarded solutions of Maxwell equations are selected over advanced ones), the biological arrow of time (live entities birth and die) and the psicological arrow of time (the intuitive asymmetry between past and future). The question on the existence of a "quantum mechanical" arrow of time has been posed.[4–7]

Of course, the problem of the arrow of time arises when we seek a *physical correlate* of the intuitive asymmetry between past and future: do physical theories pick out a preferred direction of time?

The main difficulty to be encountered in answering this question relies on our anthropocentric perspective: the difference between past and future is so deeply rooted in our language and our thoughts that it is very difficult to shake off these temporally asymmetric assumptions. In fact, traditional discussions around the problem of the arrow of time are usually subsumed under the label 'the problem of the direction of time', as if we could find an exclusively physical criterion for singling out *the* direction of time, identified with what we call 'the future'. However, there is nothing in physical evolution laws that distinguishes, in a non-arbitrary way, between past and future as we conceive them in our ordinary language and our everyday life. It might be objected that physics implicitly assumes this distinction with the use of temporally asymmetric expressions, like 'future light cone', 'initial conditions', 'increasing time', and so on. However this is not the case, and the reason relies on the distinction between *conventional* and *substantial*.

Definition 3. Two objects are *formally identical* when there is a permutation that interchanges the objects but does not change the properties of the system to which they belong.

In physics it is usual to work with formally identical objects: the two semicones of a light cone, the two spin senses, etc.

Definition 4. We shall say that we establish a *conventional* difference between two objects when we call two formally identical objects with two different names.

This is the case when we assign different signs to the two spin senses, or different names to the two light semicones, etc.

Definition 5. We shall say that the difference between two objects is *substantial* when we assign different names to two objects that are not formally identical (see Refs. 13, 14). In this case, although the particular names we choose are conventional, the difference is substantial.

Once this point is accepted, it turns to be clear that fundamental physics uses the labels 'past' and 'future' in a conventional way. Therefore, the problem cannot yet be posed in terms of singling out the future direction of time: the problem of the arrow of time becomes the problem of finding a *substantial difference between the two temporal directions*. It seems necessary to address the problem of the arrow of time from a perspective purged of our temporal intuitions. Thus, we must avoid the conclusions derived from subtly presupposing time-asymmetric notions. As claimed by Price,[(15)] it is necessary to stand at a point outside of time, and thence to regard reality in atemporal terms: this is the so called '*view from nowhen*'.

But then, what does 'the arrow of time' mean when we accept this constraint? We recognize the difference between the head and the tail of an arrow on the basis of its geometrical properties; therefore, we can substantially distinguish between both directions, head-to-tail and tail-to-head, *independently of our particular perspective*. Analogously, *the problem of the arrow of time* should be conceived in terms of *the possibility of establishing a substantial distinction between the two directions of time exclusively by means of arguments based on theoretical physics* (for a detailed discussion, see Refs. 16, 17).

If the problem is expressed in this way, the question is: why is time-reversal invariance an obstacle to solve the problem of the arrow of time? In order to answer this question we have to recall that, if e_t is a solution of a time-reversal invariant law L, then $\mathbf{T}e_t$ is also a solution of L. We shall call these two solutions '*time-symmetric twins*': they are twins because, in the context of the theory to which they belong and without presupposing a privileged direction of time, they are only conventionally different; they are time-symmetric because one is the temporal mirror image of the other. The traditional example of time-symmetric twins is given by electromagnetism, where dynamical equations always have advanced and retarded solutions. With this terminology we can say that a time-reversal invariant theory always produce time-symmetric twins: the obstacle to solve the problem of the arrow of time in this case relies on the fact that, in the context of the theory, the twins are only conventionally different.

The traditional arguments for discarding one of the twins and retaining the other invoke time-asymmetric notions which are not justified in the context of the theory. For instance, the retarded nature of radiation is usually explained by means of *de facto* arguments referred to initial conditions: advanced solutions of wave equations correspond to converging waves that require a miraculous cooperative emitting behavior of distant regions of space at the temporal origin of the process. A different but related argument is put forward by those who appeal to the impossibility (or high difficulty) of preparing time-reversed states in laboratory experiments like, for instance, experiments of scattering.[3] It seems quite clear that this kind of arguments, not based on theoretical considerations, are not legitimate in the context of the problem of the arrow of time to the extent that they put the arrow 'by hand' by presupposing the difference between the two directions of time from the beginning. In other words, they violate the 'nowhen' requirement of adopting an atemporal perspective purged of temporal intuitions like those related with the asymmetry between past and future or between initial and final conditions. Therefore, from an atemporal standpoint, the challenge consists in supplying a non-conventional criterion, based on theoretical arguments, for choosing one of the time-symmetric twins as the physically meaningful one: such a criterion will establish a substantial difference between the two members of the pair and, *a fortiori*, between the two directions of time (for a discussion of the concept of time-symmetric twins, see Ref. 18). This is the conceptual background we need in the discussion on whether the "time-asymmetric" quantum mechanics gives a solution to the problem of the arrow of time.

3. TIME-REVERSAL INVARIANCE AND IRREVERSIBILITY IN THE TIME-ASYMMETRIC VERSION OF QUANTUM MECHANICS

The purpose of the so called time-asymmetric quantum mechanics is to provide a formulation of quantum mechanics capable of describing irreversible quantum phenomena. This formulation makes use of the rigged Hilbert space formulation of quantum mechanics. This formulation is well known[19–22] as well is its application to resonance scattering,[21] therefore, here we briefly summarize it in order to recall its main ideas and introduce the notation we shall use along the present discussion.

Let us recall that we can construct two rigged Hilbert spaces

$$\mathbf{\Phi}_{\pm} \subset \mathcal{H} \subset \mathbf{\Phi}_{\pm}^{\times}, \tag{6}$$

such that there exists corresponding unitary representations $V_{\pm}$ such that[21]

$$V_{\pm}\,\mathcal{H} = L^2(\mathbb{R}^+) \tag{7}$$

$$V_{\pm}\,\mathbf{\Phi}_{\pm} = S \cap \mathcal{H}^2_{\pm}\Big|_{\mathbb{R}^+}\,, \tag{8}$$

where S is the Schwartz space of infinitely differentiable functions vanishing at infinite faster than the inverse of any polynomial, $\mathcal{H}_{\pm}$ are the spaces of Hardy functions on the upper (+) and lower (−) halves of the complex plane, and the space $L^2(\mathbb{R}^+)$ is the Hilbert space of the square integrable Lebesgue functions on the positive real axis. The symbol $\big|_{\mathbb{R}^+}$ means that we consider the restriction of the functions in $S \cap \mathcal{H}^2_{\pm}$ on the real axis. The spaces $\mathbf{\Phi}^{\times}_{\pm}$ are the respective antiduals (antilinear continuous functionals) of the locally convex spaces $\mathbf{\Phi}_{\pm}$.[21] The unitary operators $V_{\pm}$ diagonalize the absolutely continuous parts of the total Hamiltonian H in the sense that $V_{\pm}\,H\,V_{\pm}^{-1}$ is the multiplication operator on $L^2(\mathbb{R}^+)$ (for simplicity, we assume that the spectrum of H is nondegenerate in this sector).[21] The vector $\phi_{\pm} \in \mathbf{\Phi}_{\pm}$ represents the same state than the wave function in the energy representation, given by $\phi_{\pm}(\omega) = \langle\omega|\phi_{\pm}\rangle := V_{\pm}\phi_{\pm}, \phi_{\pm}(\omega) \in S \cap \mathcal{H}^2_{\pm}\big|_{\mathbb{R}^+}$.

Let $U_{-t} := e^{itH}$ be the adjoint of the evolution operator U_t. Then, one can readily show that[21]

$$U_{-t}\,\mathbf{\Phi}_{-} \subset \mathbf{\Phi}_{-}\,, \quad \text{if } t \leqslant 0 \tag{9}$$

$$U_{-t}\,\mathbf{\Phi}_{+} \subset \mathbf{\Phi}_{+}\,, \quad \text{if } t \geqslant 0\,. \tag{10}$$

Now, let $\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^{\times}$ be a rigged Hilbert space, and U an operator on $\mathcal{H}$ such that $U^{\dagger}\mathbf{\Phi} \subset \mathbf{\Phi}$, where $U^{\dagger}$ is the adjoint of U. Then, we can extend U into $\mathbf{\Phi}^{\times}$ by means of the following duality formula[21]

$$\langle U^{\dagger}\phi|F\rangle = \langle\phi|U^{\times}F\rangle\,, \ \forall\phi \in \mathbf{\Phi}\,, \ \forall F \in \mathbf{\Phi}^{\times}\,, \tag{11}$$

where $U^{\times}$ is the extension of U to $\mathbf{\Phi}^{\times}$. Using (9, 10, 11), we conclude that

$$U_t^{-\times}\mathbf{\Phi}^{\times}_{-} \subset \mathbf{\Phi}^{\times}_{-}\,, \quad t \leqslant 0\,, \tag{12}$$

$$U_t^{+\times}\mathbf{\Phi}^{\times}_{+} \subset \mathbf{\Phi}^{\times}_{+}\,, \quad t \geqslant 0\,. \tag{13}$$

Here, we have introduced the signs $\pm$ in the extensions $U_t^{\pm\times}$ in order to distinguish these two extensions, which are defined on different spaces ($\mathbf{\Phi}^{\times}_{\pm}$).

Due to the construction of the triplets (6), the antiduals $\mathbf{\Phi}_\pm^\times$ contain two kinds of functionals interesting from the point of view of physics: the Dirac kets (eigenfunctionals of H with eigenvalues in the continuous spectrum of H) and the Gamow vectors. The latter are eigenfunctionals of H, whose respective eigenvalues coincide with the resonance poles for the Hamiltonian pair (H_0, H).[21] Resonance poles appear as complex conjugate pairs $z_R = \omega_R - i\frac{\Gamma}{2}$ and $z_R^* = \omega + i\frac{\Gamma}{2}$, with $\Gamma > 0$. The model has been constructed such that $H\mathbf{\Phi}_\pm \subset \mathbf{\Phi}_\pm$ and, therefore, H can be extended to the antiduals $\mathbf{\Phi}_\pm^\times$ using the duality formula (11). If we call the extensions of H into $\mathbf{\Phi}_-^\times$ and $\mathbf{\Phi}_+^\times$, $H_-^\times$ and $H_+^\times$, respectively, the existence of two functionals Ψ^G, *the growing Gamow vector*, and Ψ^D, *the decaying Gamow vector*, can be proved such that

$$H_+^\times\,\Psi^D = z_R\,\Psi^D = \left(\omega_R - i\frac{\Gamma}{2}\right)\Psi^D \tag{14}$$

$$H_-^\times\,\Psi^G = z_R^*\,\Psi^G = \left(\omega_R + i\frac{\Gamma}{2}\right)\Psi^G\,. \tag{15}$$

In addition, a semigroup time evolution can be defined on these eigenfunctionals via the duality formula (11).[21] This gives:

$$\langle U_{-t}\phi_-|\Psi^G\rangle = \langle\phi_-|U_t^{-\times}\,\Psi^G\rangle = e^{-i\omega_R t}\,e^{\Gamma t/2}\,\langle\phi_-|\Psi^G\rangle\,, \tag{16}$$

$\forall\,\phi_- \in \Phi_-\,, t \leqslant 0.$

$$\langle U_{-t}\phi_+|\Psi^D\rangle = \langle\phi_+|U_t^{+\times}\,\Psi^D\rangle = e^{-i\omega_R t}\,e^{-\Gamma t/2}\,\langle\phi_+|\Psi^D\rangle\,, \tag{17}$$

$\forall\,\phi_+ \in \Phi_+\,, t \geqslant 0.$

Observe that (17) represents an exponentially decaying process with lifetime $\tau = \frac{2}{\Gamma}$, whose limit when t goes to infinity results:

$$\lim_{t\to\infty}\langle\phi_+|U_t^{+\times}\Psi^D\rangle\ = \lim_{t\to\infty}\langle\phi_+|\Psi^D\rangle\,e^{-i\omega_R t}\,e^{-\frac{\Gamma}{2}t} = 0 \tag{18}$$

This means that, for $t \to \infty$, the decaying Gamow vector Ψ^D exponentially decays in a weak sense. Analogously, the growing Gamow vector Ψ^G exponentially decays in a weak sense for $t \to -\infty$.

One of the purposes of these ideas is to obtain a formulation of quantum mechanics capable of explaining irreversible quantum phenomena; according to this view, the use of rigged Hilbert spaces is what turns standard quantum mechanics into a time-asymmetric theory where irreversible quantum descriptions are possible. However, we have seen that non-time-reversal invariance and irreversibility are different concepts; therefore, it is worth while to ask how and by means of which formal resources the new formalism accounts for these two different features.

In the many works on "time-asymmetric" quantum mechanics,[4–7] it is suggested that, in the rigged Hilbert space formalism, the fact that evolutions are described by means of semigroups rather than groups is what permits irreversible behavior to be modeled in a natural way. In fact, irreversibility is introduced by the fact that processes that exponentially decay (grow) as $e^{-\frac{\Gamma}{2}t}$ ($e^{\frac{\Gamma}{2}t}$) can be obtained: they have a well defined limit for $t \to \infty$ ($t \to -\infty$). This is a direct consequence of the choice of the spaces $\mathbf{\Phi}_{\pm}$ constructed in (8) and the properties of the Hardy functions.

An alternative description of resonances by Gamow vectors could have been given as follows: Let V any unitary operator such that it diagonalizes the total Hamiltonian H in the sense that VHV^{-1} be the multiplication operator in $L^2(\mathbb{R}^+)$. Let us define a rigged Hilbert space as follows: consider the space $\mathcal{D}(\mathbb{R})$ of all infinitely differentiable functions with compact support on the real line $\mathbb{R}$, and the Fourier transform of this space $\mathcal{F}(\mathcal{D}(\mathbb{R}))$. Each function on $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ is entire analytic and, considered as complex functions on the real line, Schwartz functions.

Proposition. The space of restrictions $\mathcal{F}(\mathcal{D}(\mathbb{R}))\Big|_{\mathbb{R}^+}$ to the positive semiaxis $\mathbb{R}^+$ of the functions in $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ is dense in $L^2(\mathbb{R}^+)$.

Proof. Let us consider the space $\mathcal{D}(\mathbb{R}^+)$ of infinitely differentiable functions with compact support in the positive semiaxis $\mathbb{R}^+$. This is a subset of $L^2(\mathbb{R}^+)$ and, therefore, the Paley Wienner theorem[23] asserts that the space $\mathcal{F}(\mathcal{D}(\mathbb{R}^+))$ of their Fourier transforms is dense in the space $\mathcal{H}^2_-$ of Hardy functions in the lower half plane. This means that for each $\varphi_-(\omega) \in \mathcal{H}^2_-$ and each $\varepsilon > 0$, there exists an $f(\omega) \in \mathcal{F}(\mathcal{D}(\mathbb{R}^+))$ such that

$$\int_{-\infty}^{\infty} |\varphi_-(\omega) - f(\omega)|^2 \, d\omega < \varepsilon \,. \tag{19}$$

This obviously implies that

$$\int_{0}^{\infty} |\varphi_-(\omega) - f(\omega)|^2 \, d\omega < \varepsilon \,, \tag{20}$$

which means that the space $\mathcal{F}(\mathcal{D}(\mathbb{R}^+))\Big|_{\mathbb{R}^+}$ of the restrictions of functions to the positive semiaxis $\mathbb{R}^+$ is dense in $\mathcal{H}^2_-\Big|_{\mathbb{R}^+}$. As the space $\mathcal{H}^2_-\Big|_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$, the space $\mathcal{F}(\mathcal{D}(\mathbb{R}^+))\Big|_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$ (with the topology on the latter). Since,

$$\mathcal{F}(\mathcal{D}(\mathbb{R}^+))\Big|_{\mathbb{R}^+} \subset \mathcal{F}(\mathcal{D}(\mathbb{R}))\Big|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+)\,, \tag{21}$$

we conclude that $\mathcal{F}(\mathcal{D}(\mathbb{R}))\Big|_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$. Note that any function in $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ is entire analytic and, therefore, is uniquely determined by its values on the positive semiaxis. □

Let $\boldsymbol{\Phi} := V^{-1}\left(\mathcal{F}(\mathcal{D}(\mathbb{R}))\Big|_{\mathbb{R}^+}\right)$ and endow $\boldsymbol{\Phi}$ with the topology derived from the topology on $\mathcal{D}(\mathbb{R})$ exactly as we derived the topology on $\boldsymbol{\Phi}_\pm$ from the topology on the Schwartz space.[21] Then, $\boldsymbol{\Phi} \subset \mathcal{H} \subset \boldsymbol{\Phi}^\times$ is a rigged Hilbert space such that there exist two functionals $\Psi^G, \Psi^D \in \boldsymbol{\Phi}^\times$ with the following properties:

(i) Both are eigenvectors of the extension $H^\times$ of the total Hamiltonian H into $\boldsymbol{\Phi}^\times$,

$$H^\times \Psi^G = z_R^*\, \Psi^G; \quad H^\times \Psi^D = z_R\, \Psi^D\,, \tag{22}$$

where z_R and z_R^* are the pair of complex conjugate poles (of the S-matrix or the reduced resolvent, see Refs. 21, 24) that determine a resonance for the Hamiltonian pair (H_0, H).

(ii) The space $\boldsymbol{\Phi}$ is left invariant under the action of the whole group $U_t = e^{-itH}$ and therefore this group can be extended to the dual $\boldsymbol{\Phi}^\times$. As a consequence, if we call this extension $U_t^\times$, the functionals evolve *for all values of t* as

$$U_t^\times \Psi^G = e^{-i\omega_R t}\, e^{\Gamma t/2}\, \Psi^G \quad \text{and} \quad U_t^\times \Psi^D = e^{-i\omega_R t}\, e^{-\Gamma t/2}\, \Psi^D. \tag{23}$$

We do not include the proofs of these assertions as they are a repetition of the proofs for the corresponding results on $\boldsymbol{\Phi}_\pm^\times$ in Ref. 21. The conclusion of these results is very simple: the very use of rigged Hilbert spaces in the description of resonance scattering does not lead by itself to a semigroup description. This semigroup description is the consequence of the choice of the spaces $\boldsymbol{\Phi}_\pm$, constructed via Hardy functions.

On the other hand, an evolution is irreversible if it has a limit for $t \to \pm\infty$. In the formalism under discussion, irreversibility is introduced by the fact that processes that exponentially decay (grow) as $e^{-\frac{\Gamma}{2}t}$ ($e^{\frac{\Gamma}{2}t}$) can be obtained: they have a well defined limit for $t \to \infty$ ($t \to -\infty$) (Observe that this is true even if we use the rigged Hilbert space defined as in the last paragraph, instead of $\mathbf{\Phi}_\pm \subset \mathcal{H} \subset \mathbf{\Phi}_\pm^\times$ with $\mathbf{\Phi}_\pm$ as in (8)). It is worthy to mention that the existence of the functionals Ψ^G and Ψ^D is a consequence of the use of analytically continuable functions, a certain fact in all the choices of rigged Hilbert spaces here discussed.

In the case that we use the description based in the rigged Hilbert spaces in where the basic (or test vector) spaces are defined by the $\mathbf{\Phi}_\pm$ as in (8), one can readily show[25] that if T is the time-reversal operator on the Hilbert space $\mathcal{H}$, we have that

$$T\,\mathbf{\Phi}_\pm = \mathbf{\Phi}_\mp\,, \tag{24}$$

and therefore, $\mathbf{\Phi}_\pm$ are time-symmetric twins in the sense of the previous section. Furthermore, with the aid of the duality formula (11), we can extend T to the antiduals. If, for simplicity, we also call T to the extensions of the time-reversal operator to the antiduals, we obtain:

$$T\,\mathbf{\Phi}_\pm^\times = \mathbf{\Phi}_\mp^\times\,, \tag{25}$$

which shows that the rigged Hilbert spaces $\mathbf{\Phi}_+ \subset \mathcal{H} \subset \mathbf{\Phi}_+^\times$ and $\mathbf{\Phi}_- \subset \mathcal{H} \subset \mathbf{\Phi}_-^\times$ are time-reversed of each other. They are time-symmetric twins. In addition, we want to remark that $T\Psi^G = \Psi^D$ and $T\Psi^D = \Psi^G$, so that both Gamow vectors are time-symmetric of each other.[25]

4. THE COARSE-GRAINED NATURE OF IRREVERSIBILITY

In the rigged Hilbert space description of resonances, the Gamow vectors have the following property:

$$\lim_{t\mapsto -\infty} \Psi^G(t) = \lim_{t\mapsto -\infty} e^{-i\omega_R t}\, e^{\Gamma t/2}\, \Psi^G = 0 \tag{26}$$

$$\lim_{t\mapsto \infty} \Psi^D(t) = \lim_{t\mapsto \infty} e^{-i\omega_R t}\, e^{-\Gamma t/2}\, \Psi^D = 0\,, \tag{27}$$

where $\Psi^G = \Psi^G(0)$ and $\Psi^D = \Psi^D(0)$. This result is true no matter which rigged Hilbert spaces, from the above described, we use. The real meaning of Eqs. (26) and (27) is obtained by taking the corresponding limits in (16) and (17) (and their obvious implementation for the later rigging). This means that the equations (26) and (27) are not correct from a mathematical point of view: what decays as t goes to infinity (minus infinity) is not $\Psi^D(t)$ ($\Psi^G(t)$) but instead $\langle\phi_+|\Psi^D(t)\rangle$ ($\langle\phi_-|\Psi^G(t)\rangle$). As a consequence, if we want to conceive $\Psi^D(t)$ as a generalized state,[4] we only can strictly say that the expectation value of the observable $A = |\phi_+\rangle\langle\phi_+|$ in the state $\Psi^D(t)$ decays exponentially:

$$\langle A\rangle_{\Psi^D(t)} = |\langle\phi_+|\Psi^D(t)\rangle|^2 = |\langle\phi_+|\Psi^D\rangle|^2\, e^{-\Gamma t}\,. \tag{28}$$

Observe that (28) is well defined for $t > 0$, as $\phi_+ \in \mathbf{\Phi}_+$.

In other words, whereas $\langle A\rangle_{\Psi^D(t)}$ exponentially decays as t goes to infinity:

$$\lim_{t\to\infty} \langle A\rangle_{\Psi^D(t)} = 0\,, \tag{29}$$

the generalized state $\Psi^D(t)$ has only a *weak limit*:

$$w - \lim_{t\to\infty} \Psi^D(t) = 0\,. \tag{30}$$

As we have argued elsewhere,[(12)] this weak limit means that the generalized state $\Psi^D(t)$ decays *from an observational point of view*, that is from the perspective given by the observable $A = |\phi^+\rangle\langle\phi^+|$, for any $\phi^+ \in \mathbf{\Phi}_+$. In this sense, $\langle A\rangle_{\Psi^D}$ involves a generalized coarse-graining, that is, a generalized projection of the vector $\Psi^D(t)$ onto a subspace defined by the operator A. In fact, since $A^2 = A$, the observable A can be conceived as a projector Π:

$$A = |\phi_+\rangle\langle\phi_+| = \Pi \tag{31}$$

Then, we can define a coarse-grained state Ψ^D_{cg} as:

$$\Psi^D_{cg} = \Pi\,\Psi^D = |\phi_+\rangle\langle\phi_+|\Psi^D\rangle \tag{32}$$

With this definition:

$$\begin{aligned}|\Psi^D_{cg}\rangle\langle\Psi^D_{cg}| &= |\phi_+\rangle\langle\phi_+|\Psi^D\rangle\langle\Psi^D|\phi_+\rangle\langle\phi_+| \\ &= |\langle\phi_+|\Psi^D\rangle|^2\,|\phi_+\rangle\langle\phi_+| = \langle A\rangle_{\Psi^D}\,|\phi_+\rangle\langle\phi_+|\,.\end{aligned} \tag{33}$$

This means that the expectation value of the observable $A = |\phi_+\rangle\langle\phi_+|$ in the state Ψ^D can be viewed as the result of the action of the projector

[4] This may have some difficulties, see for instance Refs. 26, 27. For other arguments, see Ref. 28.

$\Pi = A$ on the vector Ψ^D. On this basis we can understand why $\langle A \rangle_{\Psi^D(t)}$ is a coarse-grained magnitude: strictly speaking, this coarse-grained magnitude is what decays for $t \to \infty$, and not the generalized state $\Psi^D(t)$ as Eq. (27) seems to suggest. A similar discussion is possible for Ψ^G and time going to minus infinite.

The conclusion of this argument is that, in the case of the evolution described by Gamow vectors, the coarse-grained magnitude that decays as t goes to infinity is the expectation value of the observable $A = |\phi_+\rangle\langle\phi_+|$ in the generalized state Ψ^D, for any $\phi_+ \in \mathbf{\Phi}_+$, and there is no quantum law that prevents it from having this kind of behavior. But to interpret the decaying Gamow vector as a truly decaying state can only be the result of a philosophically biased rejection of coarse-graining, which misdirects the interpretation of the irreversible processes described by the theory.

5. THE PROBLEM OF THE ARROW OF TIME IN TIME-ASYMMETRIC QUANTUM MECHANICS

As we have seen, two rigged Hilbert spaces arise as a consequence of the use of Hardy functions in the particular realization of the space $\mathbf{\Phi}$, which determines the properties of the antidual $\mathbf{\Phi}^\times$. Thus, two antiduals arise, $\mathbf{\Phi}^\times_\pm$, which contain not only all the physically realizable states, but also generalized states like the Gamow vectors and the Dirac kets.[(21)] Time evolution has a semigroup structure on the antiduals: the semigroups $U_t^{+\times}$ with $t \geqslant 0$ and $U_t^{-\times}$ with $t \leqslant 0$, govern the time evolutions on $\mathbf{\Phi}^\times_+$ and $\mathbf{\Phi}^\times_-$, respectively. We have also shown that the fact that $U_t^{+\times}$ and $U_t^{-\times}$ form semigroups is what would break down the original time-reversal invariance of quantum mechanics in its standard version. On the basis of these previous results, we have seen that those two rigged Hilbert spaces with their corresponding evolution operators lead to pairs of time-symmetric twins, since they are two non time-reversal invariant formalisms, one the temporal mirror image of the other. If we want to distinguish between the two directions of time, $t \geqslant 0$ and $t \leqslant 0$, the challenge consists in supplying a non-conventional criterion, based on theoretical considerations, for choosing one of the twins of each pair as the physically meaningful one.

In their analysis of the Friedrich's model for quantum scattering, Antoniou and Prigogine[(6)] adopt the following interpretation: $U_t^{+\times}$ carries states into the *forward direction of time* and then, describes evolutions reaching equilibrium *in the future*; $U_t^{-\times}$ carries states into the *backward direction of time* and then, describes evolutions reaching equilibrium *in the past*. Even if with a different terminology, the authors are admitting that these two evolutions are only conventionally different to the extent that the theory by itself gives no basis for selecting one of the elements of the pair as the physically relevant. Therefore, they adopt an observational criterion for retaining one of the semigroups and discarding the other: since no physical system has ever been observed evolving to equilibrium toward the past, the physically relevant semigroup of evolution operators is the semigroup corresponding to $U_t^{+\times}$, valid for $t \geqslant 0$.(6,7) Although this appeal to observational considerations is a legitimate move in the everyday work of physicists, it is not acceptable when the problem at issue is to explain the arrow of time, since the fact that our observations are time-directed was known from the very beginning: the real problem consists in accounting for the difference between the two directions of time by means of theoretical arguments.

Bohm's response to the problem of the arrow of time is subtler than the solution proposed by Prigogine and his coworkers. In his detailed description of scattering processes, Bohm breaks the symmetry between the two twins by appealing to the so-called '*preparation-registration arrow of time*', expressed by the slogan '*no registration before preparation*'.(4)[5] On this basis, Bohm replaces the representational postulate of standard quantum mechanics—according to which states are represented by the vectors of a separable Hilbert space and observables are represented by self-adjoint operators on that space—by a new one that distinguishes between the mathematical descriptions of states ψ and of observables φ:

$$\{\psi\} \equiv \mathbf{\Phi}_- \subset \mathcal{H} \subset \mathbf{\Phi}_-^\times \tag{34}$$

$$\{\varphi\} \equiv \mathbf{\Phi}_+ \subset \mathcal{H} \subset \mathbf{\Phi}_+^\times \tag{35}$$

In particular, the new representational postulate asserts that the vectors $|\psi\rangle \in \mathbf{\Phi}_-$ represent the states of the system and the vectors $|\varphi\rangle \in \mathbf{\Phi}_+$ represent the observables of the system in the sense that a state is $\rho = |\psi\rangle\langle\psi|$ and an observable is $A = |\varphi\rangle\langle\varphi|$. States and observables must undergo a time evolution as states and observables. In this sense, time evolution for an observable given by $\psi \in \mathbf{\Phi}_-$ must be of the form:

$$\rho(t) = e^{-itH}\rho(0)e^{itH} = e^{-itH}|\psi\rangle\langle\psi|e^{itH}\,. \tag{36}$$

This equation makes sense if and only if $t \geqslant 0$, since $e^{itH}\psi \in \mathbf{\Phi}_-$ for each $\psi \in \mathbf{\Phi}_-$ if and only if $t \leqslant 0$. The evolution for observables must be of the form

[5] Bohm acknowledges that the origin of the idea of a preparation-registration arrow can be traced back to the works of Günther Ludwig.(29)

$$A(t) = e^{itH} A(0) e^{-itH} = e^{itH} |\varphi\rangle\langle\varphi| e^{-itH} \,. \tag{37}$$

Observe, that, since $e^{itH}\varphi \in \mathbf{\Phi}_+$ for each $\varphi \in \mathbf{\Phi}_+$ if and only if $t \geqslant 0$, Eq. (37) makes sense if and only if $t \geqslant 0$. Both, states and observables evolve now forward in time and therefore, a privileged direction of time appears.

As we can see, Bohm's approach breaks the symmetry between the time-symmetric twins arising from $\mathbf{\Phi}_- \subset \mathcal{H} \subset \mathbf{\Phi}_-^\times$ and $\mathbf{\Phi}_+ \subset \mathcal{H} \subset \mathbf{\Phi}_+^\times$ by means of a representational postulate based on the preparation–registration arrow. However, this preparation registration arrow puts the arrow of time 'by hand' by adding a particular temporal relationship to the ontological priority of states with respect to observables. In fact, without the time-asymmetric intuition introduced by the preparation–registration arrow, nothing prevents us from reversing the representational postulate by stating that $\mathbf{\Phi}_- \subset \mathcal{H} \subset \mathbf{\Phi}_-^\times$ is the rigged Hilbert space for the representation of observables and $\mathbf{\Phi}_+ \subset \mathcal{H} \subset \mathbf{\Phi}_+^\times$ is the rigged Hilbert space for the representation of states. In this case, we would obtain the temporal mirror image of Bohm's theory. This new representational postulate restores the symmetry because now we have two postulates leading to two non time-reversal invariant theories, one the temporal mirror image of the other: they also lead to time-symmetric twins and the challenge consists, again, in supplying a theoretical and non-conventional criterion for retaining one of them and discarding the other. Bohm's decision of selecting the future directed alternative is based on presupposing the arrow of time from the very beginning on the basis of pretheoretical intuitions.

In conclusion, the proposal of time-asymmetric quantum mechanics, either in Prigogine's or in Bohm's version, does not supply an acceptable answer to the problem of the arrow of time. We have argued elsewhere[16,18] that this conclusion is not surprising to the extent that the problem cannot be solved in local terms: the substantial distinction between the two directions of time requires a global perspective that permits the irreversible processes of the universe to be correlated in such a way that all of them parallely decay in the same temporal direction. But the detailed discussion of this point goes beyond the limits of the present paper.

6. CONCLUSIONS

Although in principle nothing prevents a time-reversal invariant theory from describing irreversible evolutions, in the case of dynamical equations with unitary solutions, time-reversal invariance and reversibility seem to go hand-in-hand. Nevertheless, even in this case both properties are different to the extent that they are related with distinct features of the formalism: whereas time-reversal invariance implies the *group structure* of the evolution operators, reversibility is a consequence of the *unitary* character of such operators. Therefore, even if the time-reversal invariance of a theory is broken down by means of semigroup evolution laws, this fact does not affect the reversible character of the evolutions if they are still described by unitary operators. The only way to extract irreversibility from unitary processes is by means of some mathematical procedure that leads to a level of description different from the original dynamical level, where non-unitary evolutions can be obtained. This is precisely the effect of rigged Hilbert space formulation: as we extend evolutions to the antiduals of a rigged Hilbert space (in order to define the time evolution for generalized states like Gamow vectors), the concept of unitarity is lost.

Thus, the conclusions of this study can be summarized as follows:

- The non time-reversal invariance of the theory is due to the semigroup structure of the evolution laws which, in turn, is a consequence of the use of a particular realization of the rigged Hilbert space based on Hardy functions.
- The irreversibility of the evolutions is due to the existence of Gamow vectors, which do not directly depend on the choice of rigged Hilbert spaces using Hardy functions. The use of functions with adequate analytical properties will give equally well Gamow vectors with the correct time evolution.
- Although time-asymmetric quantum mechanics is a powerful theory for the description of intrinsic irreversibility, the formalism does not determine an arrow of time free of preconceptions.

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