A CONSTRAINED SHAPE OPTIMIZATION PROBLEM IN ORLICZ-SOBOLEV SPACES

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Abstract: In this manuscript we study the following optimization problem: given a bounded and regular domain $\Omega \subset \mathbb{R}^N$ we look for an optimal shape for the "W−vanishing window" on the boundary with prescribed measure over all admissible profiles in the framework of the Orlicz-Sobolev spaces associated to constant for the "Sobolev trace embedding". In this direction, we establish existence of minimizer profiles and optimal sets, as well as we obtain further properties for such extremals. Finally, we also place special emphasis on analyzing the corresponding optimization problem involving an "A−vanishing hole" (inside the domain) with volume constraint.

Keywords: *Nonlinear partial differential equations, Orlicz-Sobolev spaces, Shape optimization problems.*
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1 INTRODUCTION

Shape optimization problems constitute an important landmark concerning the modern development of the mathematical theory of optimization. Such issues are a longstanding subject of investigation. Some enlightening examples of such issues appear in eigenvalue problems with geometric constraints, optimization problems with constrained perimeter or volume, optimal design problems, problems in structural optimization, free boundary optimization problems, just to mention a few.

A shape optimization problem can be mathematically written as follows:

$$\min \{\mathcal{J}(\mathcal{O}) : \mathcal{O} \subset \Omega \ \text{ with } \ \mathcal{O} \ \text{ fulfilling a certain property } \ \mathbb{P}\},$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\mathcal{O}$ is an *a priori* unknown configuration (in general satisfying a specific property related to some constraint) and $\mathcal{J}$ is a "cost functional", which in several situations has an explicit integral representation, whose link with the competing configuration $\mathcal{O}$ arises via a solution of a PDE (cf. [10] for nice surveys with a number of illustrative examples, we also recommend the reading of [2], [6], [8], [9] and [12] for other references with regard to free boundary and shape optimization problems).

In the scope of the modern Analysis and PDE's theory, the *Sobolev Trace Embedding Theorem*, namely

$$W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$$

with the associated estimate for a constant $\mathcal{S} > 0$ (*Sobolev trace constant*)

$$\mathcal{S}\|u\|^p_{L^q(\partial\Omega)} \le \|u\|^p_{W^{1,p}(\Omega)} \quad \text{(Sobolev trace inequality)}$$

constitutes a fundamental tool in order to study certain issues in mathematics such as eigenvalue and Steklov type problems, functional type inequalities, existence and solvability of boundary-value problems among others.

Historically, optimization problems associated to the *best constant for Sobolev trace embedding*, namely

$$\mathcal{S}_{p,q} = \inf \left\{ \frac{\int_\Omega |\nabla u|^p + |u|^p dx}{\left(\int_{\partial\Omega} |u|^q d\mathcal{H}^{N-1}\right)^{\frac{p}{q}}} : u \in W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega) \right\},$$

have received a warm attention. Particularly, we must highlight that in [3], [4] and [7] the authors studied for the $p-$Laplacian operator the problem of finding an optimal hole/window into the domain (resp. on the boundary) with prescribed measure associate to best constant for the Sobolev trace embedding. More precisely, they analyze the following quantity:

$$\mathcal{S}(\Gamma) = \inf_{u \in \mathrm{X}_\Gamma} \frac{\int_\Omega |\nabla u|^p + |u|^p dx}{\left(\int_{\partial\Omega} |u|^q d\mathcal{H}^{N-1}\right)^{\frac{p}{q}}}, \qquad \textbf{(S.E.C.)}$$

where $1 \le q < p_\star = \frac{p(N-1)}{N-p}$ (the critical exponent in the Sobolev trace embedding) and

$$\mathrm{X}_\Gamma = \left\{ u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega);\ u = 0 \text{ −a.e. in } \Gamma \ \text{(resp. } \mathcal{H}^{N-1} \text{ a.e. on } \partial\Gamma) \right\},$$

where $\mathcal{L}^N$ (resp. $\mathcal{H}^{N-1}$) stands for the $N-$dimensional Lebesgue measure (resp. $(N-1)-$dimensional Hausdorff measure). Furthermore, another important issue in these works regards to the following shape optimization problem: for any fixed $0 < \alpha < 1$ the optimization problem

$$\mathcal{S}(\alpha) = \inf \left\{ \mathcal{S}(\Gamma) : \Gamma \subset \Omega \ \text{(resp. } \Gamma \subset \partial\Omega) \text{ s.t. } \frac{(\Gamma)}{(\Omega)} = \alpha \left(\text{resp. } \frac{\mathcal{H}^{N-1}(\Gamma)}{\mathcal{H}^{N-1}(\partial\Omega)} = \alpha \right) \right\}$$

is achieved by a pair (u_0, Γ_0) (an existence result). Moreover, under suitable regularity assumptions on the boundary, they obtain that $\Gamma_0 = \{u = 0\}$ (an explicit characterization result).

In the same way that in the classical Sobolev spaces, such trace embedding also plays a significant role in more general contexts governed by spaces with non-standard growth, for which naturally we can quote the well-known *Orlicz-Sobolev spaces* (cf. [5] for such subjects). Such spaces extend the classical notion of Sobolev spaces to a context with non-power nonlinearities (cf. [13]), and currently such spaces are fully understood and studied in Analysis, PDE's, Free boundary problems, etc (cf. [1] and [11] for some surveys. According to our knowledge, up to the date, there is no research concerning such optimization problems (**S.E.C.**) in general sceneries with non-standard growth. For this very reason, such lack of investigations was one of our main starting points in considering shape optimization problems associate to the Sobolev trace embedding in the framework of Orlicz-Sobolev spaces.

2 Main results

We define the *Orlicz-Sobolev embedding constant* as follows:

$$S_{G,H} := \inf_{u \in X} \frac{\int_\Omega G(|\nabla u|) + G(|u|) dx}{\int_{\partial\Omega} H(|u|) d\mathcal{H}^{N-1}}, \qquad \textbf{(O.S.E.C.)}$$

where $X = W^{1,G}(\Omega) \setminus W_0^{1,G}(\Omega)$ is an admissible functional class defined under Orlicz-Sobolev spaces, and G and H are suitable Young functions. It is worth highlighting that such a quantity (**O.S.E.C.**) is linked to some extent with the compact trace embedding $W^{1,G}(\Omega) \hookrightarrow L^H(\partial\Omega)$, where H and G fulfill a certain compatibility condition.

Different from the classical Rayleigh quotient, our definition employs an inhomogeneous quotient. This imposes an extra difficulty in our problem, which will be overcame by asking a "normalization" of boundary term (the associated modular), and then, we consider the class of admissible functions subject to such a constraint. Furthermore, we point out that different from its $p-$power counterpart (cf. [3]), this version involves further extensions and difficulties that are treated and resolved.

The first purpose consists in analyzing the shape optimization problem related to the "analogue" trace embedding constant associated to the Orlicz-Sobolev spaces. In this direction, we consider a regular and bounded domain $\Omega \subset \mathbb{R}^N$ and a subset of the boundary $\mathrm{W} \subset \partial\Omega$ (a "window") such that $\mathrm{W} \neq \partial\Omega$. We define the minimization problem

$$S_{G,H}(\mathrm{W}) := \inf \left\{ \int_\Omega G(|\nabla u|) + G(|u|) dx \colon \int_{\partial\Omega} H(|u|) d\mathcal{H}^{N-1} = 1 \right\}, \qquad \textbf{(Min)}$$

where the infimum is taken over the set $\mathrm{X_W} := \{u \in \mathrm{X} \colon u = 0 \;\; \mathcal{H}^{N-1} - \text{a.e. on W}\}$.

In our researches, the constant $S_{G,H}(\mathrm{W})$ represents the counterpart for the first *H-Steklov eigenvalue* of the "$G-$Laplacian operator", which is defined as

$$\Delta_G u := \left(\frac{G'(|\nabla u|)}{|\nabla u|} \cdot \nabla u\right).$$

Furthermore, notice that if $G(t) = H(t) = \frac{1}{p}t^p$ for $p > 1$, then we fall into the well-known case of the Steklov eigenvalue for the $p-$Laplacian operator.

Next, let $0 < \alpha < \mathcal{H}^{N-1}(\partial\Omega)$ be a fixed constant. Taking into account (**Min**), we define the following shape optimization problem:

$$\mathcal{S}(\alpha) := \inf\{S_{G,H}(W) \colon W \subset \partial\Omega \text{ and } \mathcal{H}^{N-1}(W) = \alpha\}. \qquad (\alpha-\textbf{Window})$$

In this framework, a set $W \subset \partial\Omega$ in which the above infimum is achieved so-called *optimal window* for the constant $\mathcal{S}(\alpha)$.

Our first result provides the existence of minimizers and optimal shapes for our optimization problem, with a lower bound estimate for the null set of minimizers.

In contrast with previous result, now we establish that minimizers have an $\alpha-$sharp measure provided we assume enough regularity on the boundary.

We put special attention to the shape optimization problem for finding an optimal interior hole $A \subseteq \Omega$ with prescribed volume associated to the "Orlicz-Sobolev embedding" constant, i.e.,

$$\mathbb{S}_{G,H}(\mathrm{A}) := \inf\left\{\int_\Omega G(|\nabla u|) + G(|u|)dx \colon \int_{\partial\Omega} H(|u|)d\mathcal{H}^{N-1} = 1\right\}, \tag{1}$$

where the infimum is taken in the class $\mathrm{X_A} := \left\{u \in W^{1,G}(\Omega) \setminus W_0^{1,G}(\Omega) \colon u = 0 \text{ a.e. in A}\right\}$.

In the same way we can consider an optimal design problem associated to the constant $\mathbb{S}_{G,H}(A)$, as follows: for $\alpha \in (0, \mathcal{L}^N(\Omega))$ we define

$$\mathbb{S}(\alpha) = \inf\left\{\mathbb{S}_{G,H}(\mathrm{A}) : \mathrm{A} \subset \Omega \quad \text{and} \quad (\mathrm{A}) = \alpha\right\}. \qquad (\alpha-\textbf{Hole})$$

A set $A \subset \Omega$ in which the above infimium is achieved is called *optimal interior hole*.

The following result is the analogous one for the optimization of a hole into the domain instead on the boundary.

Theorem 1 *Given* $0 < \alpha < (\Omega)$. *There exists a set* $\mathrm{A}_0 \subset \Omega$ *such that* $(\mathrm{A}_0) = \alpha$ *and* $\mathbb{S}_{G,H}(\mathrm{A}_0) = \mathbb{S}(\alpha)$. *Moreover, every corresponding extremal* u_0 *to* $(\alpha-$**Hole**$)$ *verifies that* $(\{u_0 = 0\}) = \alpha$.

In the next result we prove that there is no upper bound for $\mathbb{S}_{G,H}(\mathrm{A})$, where $\mathrm{A} \subset \Omega$ is an optimal interior hole.

Theorem 2 *Let* $0 < \alpha < (\Omega)$ *be a fixed quantity. Then, the following statement holds true:*

$$\sup\{\mathbb{S}_{G,H}(\mathrm{A}) : \mathrm{A} \subset \Omega \;\textit{and}\; (\mathrm{A}) = \alpha\} = +\infty.$$

Next, in order to give sense to "Orlicz-Sobolev trace constant" for functions vanishing in a negligible subset (zero $N-$dimentional Lebesgue measure) we will need to consider the space

$$W_{\mathrm{A}}^{1,G}(\Omega) = \overline{C_0^\infty(\overline{\Omega} \setminus \mathrm{A})}$$

where the closure is taken in $W^{1,G}-$norm, i.e., $W_{\mathrm{A}}^{1,G}(\Omega)$ are the functions that can be approximated by smooth functions that vanish in a neighborhood of A.

In this context the "Orlicz-Sobolev constant" is defined as

$$\mathbb{S}_{\mathrm{A}} = \inf_{W_A^{1,G}(\Omega)} \left\{ \int_\Omega G(|u|) + G(|\nabla u|)dx\colon \int_{\partial\Omega} H(|u|)d\mathcal{H}^{N-1} = 1 \right\}.$$

At this point, it is important to question when $\mathbb{S}_{\mathrm{A}}$ recovers the usual "Orlicz-Sobolev trace constant", i.e., when $\mathbb{S}_{\mathrm{A}} = \mathbb{S}_\emptyset$. A key ingredient for this result is the notion of $G-$capacitary sets. We prove that $\mathbb{S}_{\mathrm{A}} = \mathbb{S}_\emptyset$ if and only if $\mathrm{Cap}_G(\mathrm{A}) = 0$.

A natural issue is what can be inferred about the extremals u and "the optimal set" $\{u = 0\} \subset \partial\Omega$ when the domain has certain symmetry. In our last result, we prove that (when Ω is a unity ball) there exists an extremal (resp. an optimal window) spherically symmetric.

Theorem 3 *Let $\Omega = B_1$ and let $0 < \alpha < \mathcal{H}^{N-1}(\partial\Omega)$ fixed. Then, there exists an optimal window which is a spherical cap.*

3 Keywords

Nonlinear partial differential equations, Orlicz-Sobolev spaces, Shape optimization problems.

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