ALL SEQUENTIAL ALLOTMENT RULES ARE OBVIOUSLY STRATEGY-PROOF

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Abstract: For division problems with single-peaked preferences (Sprumont, 1991) we show that all sequential allotment rules, identified by Barberà, Jackson and Neme (1997) as the class of strategy-proof, efficient and replacement monotonic rules, are also obviously strategy-proof. Although obvious strategy-proofness is in general more restrictive than strategy-proofness, this is not the case in this setting.

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1 INTRODUCTION

Barberà, Jackson and Neme (1997) consider the class of division problems where agents might begin with natural claims to minimal or maximal assignments, or might be treated with different priorities, due for example to their seniorities, and these initial entitlements should be attended as far as possible. They characterize the class of strategy-proof, efficient and replacement monotonic rules on the domain of single-peaked preferences as the family of sequential allotment rules. In this paper we ask: How might efficient allotments be implemented while, at the same time, promoting solidarity among agents who may have problems with contingent reasoning? Specifically, what would happen if we demanded that the rule be obviously strategy-proof rather than just strategy-proof? Li (2017) proposes the stronger incentive notion of obvious strategy-proofness under which agents, in order to identify that truth-telling is an optimal decision, do not need to reason contingently about other agents' decisions.

2 PRELIMINARIES

Agents are the elements of a finite set $N = \{1, \ldots, n\}$, where $n \geq 2$. They have to share k indivisible units of a good, where $k \geq 2$ is a positive integer. An *allotment* is a vector $x = (x_1, \ldots, x_n) \in \{0, \ldots, k\}^N$ such that $\sum_{i=1}^{n} x_i = k$. We refer to $x_i \in \{0, \ldots, k\}$ as agent i's *assignment*. Let X be the set of allotments. Each agent $i \in N$ has a (weak) *preference* R_i over $\{0, \ldots, k\}$, the set of i's possible assignments. Let P_i be the strict preference associated with R_i. The preference R_i is *single-peaked* if (i) it has a unique most-preferred assignment $\tau(R_i)$, the *top* of R_i, such that for all $x_i \in \{0, \ldots, k\} \setminus \{\tau(R_i)\}$, $\tau(R_i)\ P_i\ x_i$, and (ii) for any pair $x_i, y_i \in \{0, \ldots, k\}$, $y_i < x_i < \tau(R_i)$ or $\tau(R_i) < x_i < y_i$ implies $x_i\ P_i\ y_i$. We assume that agents have single-peaked preferences. Often, only $\tau(R_i)$ about R_i will be relevant and if R_i is obvious, we will refer to its top as τ_i. We denote by $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{k}$ the vectors $(0, \ldots, 0), (1, \ldots, 1), (k, \ldots, k) \in \{0, \ldots, k\}^N$ and, given $S \subset N$, by $\mathbf{0}_S$, $\mathbf{1}_S$ and $\mathbf{k}_S$ the corresponding subvectors where all agents in S receive the assignment 0, 1 or k, respectively. Given $x = (x_1, \ldots, x_n)$, we denote $(x_i)_{i \in S}$ as x_S and $(x_i - 1)_{i \in S}$ as $(x - \mathbf{1})_S$.

Let $\mathcal{R}$ be the set of all single-peaked preferences. *Profiles*, denoted by $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$, are n-tuples of single-peaked preferences. To stress the role of agent i's or agents in S, we will represent a profile R by (R_i, R_{-i}) or by (R_S, R_{-S}), respectively.

A solution of the division problem (k, N) is a *rule* $\Phi : \mathcal{R}^N \to X$ that selects, for each profile $R \in \mathcal{R}^N$, an allotment $\Phi(R) \in X$.

A desirable requirement on rules is *efficiency*. A rule $\Phi : \mathcal{R}^N \to X$ is *efficient* if, for each $R \in \mathcal{R}^N$, there is no $y \in X$ such that $y_i\ P_i\ \Phi_i(R)$ for all $i \in N$ and $y_j\ P_j\ \Phi_j(R)$ for at least one $j \in N$.

A rule $\Phi : \mathcal{R}^N \to X$ is *strategy-proof* if for all $R \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$,

$$\Phi_i(R_i, R_{-i}) \; R_i \; \Phi_i(R'_i, R_{-i}).$$

A rule $\Phi : \mathcal{R}^N \to X$ is *replacement monotonic* if for all $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$,

$$\Phi_i(R_i, R_{-i}) \leq \Phi_i(R'_i, R_{-i}) \text{ implies } \Phi_j(R_i, R_{-i}) \geq \Phi_j(R'_i, R_{-i}) \text{ for all } j \neq i.$$

Individual rationality with respect to an allotment $q \in X$ guarantees that each agent i receives an assignment that is weakly preferred to q_i. A rule $\Phi : \mathcal{R}^N \to X$ is *individually rational* with respect to an allotment $q \in X$ if for all $R \in \mathcal{R}^N$ and $i \in N$,

$$\Phi_i(R) \; R_i \; q_i.$$

An Individually Rational Sequential allotment rules allot the k units sequentially, using guaranteed allotments for the agents that evolve throughout the process and that are compared to their tops. We describe the general procedure that any sequential allotment rule follows. The rule has to specify an initial guaranteed allotment for the agents, q.

The *scarcity* allotment $\overline{q} \in X$, to be used when the sum of the tops is strictly larger than k, and the *excess* allotment $\underline{q} \in X$, to be used when the sum of the tops is strictly smaller than k.

To define a sequential allotment rule Φ, let q the guaranteed allotment, and let $\tau = (\tau_1, \ldots, \tau_n) \in \{0, \ldots, k\}^N$ be an arbitrary vector of tops.

Suppose $\sum_{i=1}^{n} \tau_i = k$. Then, since τ is the unique efficient allotment at τ, $\Phi(\tau) = \tau$.

Suppose $\sum_{i=1}^{n} \tau_i > k$ (the case $\sum_{i=1}^{n} \tau_i < k$ is symmetric). If $\tau_j \geq q_j$ for all j, then $\Phi(\tau) = q$. Otherwise, each j with $\tau_j \leq q_j$ receives τ_j and leaves the process with τ_j units, while the other agents remain. The guaranteed assignments of the remaining agents are weakly increased by distributing among them the not yet allotted units. Agents with a top smaller than or equal to the new guaranteed assignment receive the top and leave the process, while the others remain. The process proceeds this way until all units have been already allotted, with the remaining agents receiving their last guaranteed assignment.

At the end of the process, each agent i receives either τ_i or i's final guaranteed assignment which has been moving towards τ_i throughout the process.

Theorem 1 $(BJN1997)$ *Let (k, N) be a division problem. A rule $\Phi : \mathcal{R}^N \to X$ is strategy-proof, efficient, replacement monotonic and individually rational if and only if Φ is a individually rational sequential allotment rule.*

3 Obviously strategy-proof implementation

We briefly describe the notion of obvious strategy-proofness. Li (2017) proposes this notion with the aim of reducing the contingent reasoning that agents have to carry out to identify that, given a rule, truth-telling is always a weakly dominant strategy. A rule Φ is obviously strategy-proof if there exists an extensive game form with two properties. First, for each profile $R \in \mathcal{R}^N$ one can identify a behavioral strategy profile, associated to truth-telling, such that if agents play according to such strategy the outcome is $\Phi(R)$, the allotment selected by the rule Φ at R; that is, the extensive game form induces Φ. Second, whenever agent i with preferences R_i has to play, i evaluates the consequence of choosing the action prescribed by i's truth-telling strategy according to the *worst* possible outcome among all outcomes that may occur as an effect of later actions made by agents throughout the rest of the game. In contrast, i evaluates the consequence of choosing an action different from the one prescribed by i's truth-telling strategy according to the *best* possible outcome among all outcomes that may occur again as an effect of later actions throughout the rest of the game. Then, i's truth-telling strategy is obviously dominant in the game in extensive form if, whenever i has to play, its pessimistic outcome is at least as preferred as the optimistic outcome associated to any other strategy. If the extensive game form induces Φ and for each agent truth-telling is obviously dominant, then Φ is obviously strategy-proof.

Fix an extensive game form $\Gamma \in \mathcal{G}$ and a preference profile $R \in \mathcal{R}^N$. Let (Γ, R) denote the game in extensive form where each agent $i \in N$ evaluates strategy profiles in Γ according to R_i. A strategy σ_i is *weakly dominant* in (Γ, R) if, for all σ_{-i} and all σ_i',

$$g_i(\sigma)\ R_i\ g_i(\sigma_i', \sigma_{-i}).$$

We are now ready to define obvious strategy-proofness in the context of division problems.

Definition 1 Let (k, N) be given. A rule $\Phi : \mathcal{R}^N \to X$ is *obviously strategy-proof* if there is an extensive game form $\Gamma \in \mathcal{G}$ associated to (k, N) such that, for each $i \in N$ and $R_i \in \mathcal{R}$,
(i) there exists $\sigma_i^{R_i} \in \Sigma_i$ such that $\Phi(R) = g(\sigma^R)$, where $R = (R_1, \ldots, R_n)$ and $\sigma^R = (\sigma_1^{R_1}, \ldots, \sigma_n^{R_n})$, and
(ii) $\sigma_i^{R_i}$ is weakly dominant in (Γ, R).[1]

When (i) holds we say that Γ *induces* Φ. When (i) and (ii) hold we say that Γ OSP-*implements* Φ and refer to $\sigma_i^{R_i}$ as i's *truth-telling* strategy.

Our main result states that all sequential allotment rules are obviously strategy-proof. Namely, in the two statements of Proposition 1, strategy-proofness can be replaced by obvious strategy-proofness. The proof of our result is constructive, and based on the Monotonous and Individualized Algorithm (MIA).

Theorem 2 *All individually rational sequential allotment rule are obviously strategy-proof.*

3.1 The Monotonous and Individualized Algorithm (MIA)

Our aim here is to define, for the division problem (k, N), a family of extensive game forms in $\mathcal{G}$, which we will refer to as Monotonous and Individualized Games ($\mathcal{MIG}$), with the properties that (i) in each $\Gamma \in \mathcal{MIG}$, truth-telling is always weakly dominant and (ii) for each sequential allotment rule Φ, one can identify a $\Gamma \in \mathcal{MIG}$ that OSP-implements Φ. We define the family through the Monotonous and Individualized Algorithm (MIA).

Let $j \in N$ and $\beta_j \in \{0, \ldots, k\}$. Define $\beta_j^- = \max\{\beta_j - 1, 0\}$ and $\beta_j^+ = \min\{\beta_j + 1, k\}$.

3.2 Monotonous and Individualized Algorithm (the MIA)

Stage A. *Input:* A feasible allotment q
Each agent in $i \in N$ choices an actions a_i in $A_i = \{q_i^-, q_i, q_i^+\}$.
Set $N_u = \{i : a_i = q_i + 1\}$, $N_s = \{i : a_i = q_i\}$, $N_d = \{i : a_i = q_i - 1\}$
output of **Stage A** , N_u, N_d, N_s and q

Stage B. Step B.t (t ≥ 1**).**
Input: Partition N_u, N_d, N_s and q, output of **Stage A** if **t** $= 1$, or **Stage B.t-1** if **t** > 1.
Choose agents $j \in N_u$ and $r \in N_d$.
Set $\beta_j = q_j + 1$ and $\beta_r = q_r - 1$.$^{(*)}$
Step B.t.a. Agent $j \in N_u$ has to choose an action a_j from the set $A_j = \{\beta_j, \beta_j^+\}$.
Step B.t.b. Agent $r \in N_d$ has to choose an action a_r from the set $A_r = \{\beta_r^-, \beta_r\}$.
Set

$$N_u := \begin{cases} N_u \setminus \{j\} & \text{if } a_j = \beta_j \\ N_u & \text{if } a_j = \beta_j + 1, \end{cases} \qquad N_d := \begin{cases} N_d \setminus \{r\} & \text{if } a_r = \beta_r \\ N_d & \text{if } a_j = \beta_r - 1, \end{cases}$$

$$N_s := \begin{cases} N_s \cup \{j\} & \text{if } a_j = \beta_j \text{ and } a_r = \beta_r - 1 \\ N_s \cup \{r\} & \text{if } a_j = \beta_j + 1 \text{ and } a_r = \beta_r \\ N_s \cup \{j, r\} & \text{if } a_j = \beta_j \text{ and } a_r = \beta_r \\ N_s & \text{if } a_j = \beta_j + 1 \text{ and } a_r = \beta_r - 1, \end{cases}$$

[1] Recall that by Mackenzie (2020), requiring weak dominance is equivalent to requiring obvious dominance.

$q_j := \beta_j$ and $q_r := \beta_r$.

Output: The partition N_u, N_d, N_s and $q = (q_i)_{i \in N}$.

If $N_u \neq \emptyset$ and $N_d \neq \emptyset$, go to **Step B.t+1**.

If $N_u = \emptyset$ or $N_d = \emptyset$, stop, and the outcome of the MIA is the allotment q.

Denote by $\mathcal{MIG}$ the family of all extensive game forms defined by the MIA once, at each step B, a the agents j and r are selected. Let $\Gamma \in \mathcal{MIG}$ and let σ be a strategy in Γ.

3.3 Truth-telling is weakly dominant

The *truth-telling* strategy $\sigma_i^{R_i}$ (relative to R_i) is the strategy where, whenever agent i is called to play, i chooses the best action in A_i according to R_i.

Proposition 1 *For each agent i, the strategy $\sigma_i^{R_i}$ is weakly dominant in the game in extensive form* (Γ, R)

Proof Let Γ be defined by the MIA. Fix arbitrary $i \in N$, $R_i \in \mathcal{R}$ and σ_{-i}, and consider any $\sigma_i' \neq \sigma_i^{R_i}$. Let N_u, N_d, N_s and $(q_i)_{i \in N}$ be the output of the run of the MIA when agents play according to $(\sigma_i^{R_i}, \sigma_{-i})$ and let N_u', N_d', N_s' and $(q_i')_{i \in N}$ be the output of the run of the MIA when agents play according to (σ_i', σ_{-i}). We verify that $q_i \; R_i \; q_i'$. Assume first that $i \in N_s$. Then, by (R1.1) in Remark 1, $\tau(R_i) = q_i$ and, accordingly, $q_i \; R_i \; q_i'$. Assume now that $q_i \neq q_i'$. There exists a step at which for the first time $\sigma_i^{R_i}$ and σ_i' select different actions, say a_i and a_i', and q_i follows after a_i and q_i' after a_i'. We distinguish between two symmetric cases.

Assume that $i \in N_u$ (the case $i \in N_d$ is symmetric). As $i \in N_u$, by definition of $\sigma_i^{R_i}$, $\tau(R_i) > q_i$. By the definition of $\sigma_i^{R_i}$, $a_i = \max_{R_i} A_i \leq \tau(R_i)$. Since $i \in N_u$, the guaranteed assignment has weakly increased from $a_i - 1$ (the guaranteed assignment at the step where i could choose a_i' as well) to q_i until the end of the MIA. Hence, $a_i - 1 \leq q_i$ and

$$\max\nolimits_{R_i} A_i - 1 \leq q_i < \tau(R_i). \tag{1}$$

Similarly, and as $a_i \neq a_i'$,

$$a_i' \leq \max\nolimits_{R_i} A_i - 1. \tag{2}$$

By (2), $i \in N_d' \cup N_s'$, and the guaranteed assignment has weakly decreased from a_i' to q_i' until the end of the MIA. Hence, $q_i' \leq a_i'$ and, together with (1) and (2), $q_i' \leq q_i < \tau(R_i)$. By single-peakedness, $q_i \; R_i \; q_i'$.

Hence, for all σ_{-i} and σ_i', $g_i(\sigma_i^{R_i}, \sigma_{-i}) \; R_i \; g_i(\sigma_i', \sigma_{-i})$, which means that $\sigma_i^{R_i}$ is weakly dominant in (Γ, R).

Theorem 3 *For each (IR) sequential allotment rule, there exists a $\mathcal{MIG}$ that OSP-implements it.*

References

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