# Lagrangian and orthogonal splittings, quasitriangular Lie bialgebras and almost complex product structures

H. Montani[a)]

(*Electronic mail: hmontani@uaco.unpa.edu.ar)

(Dated: 7 March 2023)

We study complex product structures on quadratic vector spaces and on quadratic Lie algebras analyzing the Lagrangian and orthogonal splittingsassociated with them. We show that a Manin triple equipped with generalized metric $\mathscr{G}+\mathscr{B}$ such that $\mathscr{B}$ is an $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass -1 can be turned into another Manin triple that admits also an orthogonal splitting in Lie ideals. Conversely, a quadratic Lie algebra orthogonal direct sum of a pair anti-isomorphic Lie algebras, following similar steps as in the previous case, can be turned into a Manin triple admitting an orthogonal splitting into Lie ideals.

## I. INTRODUCTION

This work aims to study some algebraic aspects of quadratic vector spaces that admit a Lagrangian (maximally isotropic) and an orthogonal direct sum decomposition associated with complex product structures, mainly when they are endowed with a Lie algebra structure. In particular, we study a Manin triple with an orthogonal vector space splitting and use the (almost) complex product structure to build a new Manin triple where the orthogonal subspaces become Lie ideals. Conversely, starting from a pair of anti-isomorphic Lie algebras that are orthogonal relative to a non-degenerate symmetric bilinear form on the Lie algebra direct sum, a Manin triple is obtained that has these Lie algebras as Lie ideals. All of this is done by resorting to some tools in the realm of the modified classical Yang-Baxter equation.

Lagrangian and orthogonal splitting of vector spaces intervene as important ingredients in the formulation of Poisson-Lie T-duality: the Lagrangian decomposition is inherent to the Manin triple on which the T-dual sigma models are built, while projections on orthogonal subspaces give rise to the relevant dynamics[1],[2],[3],[4]. This orthogonal splitting, in subspaces of the same dimension, can be regarded as the eigenspaces decomposition of an involutive operator $\mathscr{E}$ that encodes the information of a generalized metric. Also, a linear operator like $\mathscr{E}$ appears in string theory T-duality, through Double Field Theory approach, disguised by a right multiplication with a null signature metric analogous to the bilinear form provided by the Manin triple (see for instance the review in ref.[5]).

Hence we study the coexistence of Lagrangian and orthogonal splitting in quadratic vector spaces and in Manin triples, when they arise from (almost) complex product structures[6] composed of a product structure $\mathscr{E}$ and an almost complex structure $\mathscr{J}$ naturally associated with $\mathscr{E}$. These operators are tied to a generalized metric on a Lagrangian splitting, or an anti-isomorphism between the subspaces of an orthogonal splitting. From a given pair $\{\mathscr{E},\mathscr{J}\}$, a family of Lagrangian or orthogonal splittings can be obtained by gauge transformations[7]. Then, in the framework of Lie bialgebras, appealing to the formulation by $\mathscr{O}$-operators[8] of the modified classical Yang-Baxter equation, we use quasitriangular factorizable solutions to get a Manin triple where the orthogonal subspaces become Lie subalgebras, Lie ideals in fact, and to the twilled extension procedure[9] to assemble the Lagrangian components in a bigger Lie algebra. The construction can be reversed, allowing the construction of a Manin triple with the same properties out of a pair of anti-isomorphic Lie algebras. Again, in this procedure, quasitriangular factorizable solutions of the modified classical Yang-Baxter equation are used.

We carry out this work in the following steps. In Section 2, we consider a $2n$-dimensional real quadratic vector space with a split bilinear form $(,)_{\mathrm{V}}:\mathrm{V}\otimes\mathrm{V}\longrightarrow\mathbb{R}$ endowed with a symmetric complex product structure $\{\mathscr{E},\mathscr{J}\}$ and study the direct sum decomposition (*splittings*) $\mathrm{V}=\mathrm{E}^{+}\oplus\mathrm{E}^{-}=\mathrm{F}_{+}\oplus\mathrm{F}_{-}$ with $\mathrm{E}^{\pm}$ an $n$-dimensional orthogonal subspace, and $\mathrm{F}_{\pm}$ a Lagrangian (maximally isotropic) subspace, arising from the involutive operators $\mathscr{E}$ and $\mathscr{J}\mathscr{E}$. In Section 3, we briefly describe the general framework to deal with Lie algebras and complex product structures. In Section 4, we start with a Manin triple $(\mathfrak{g},\mathfrak{g}_{+},\mathfrak{g}_{-})$ endowed with a generalized metric $\mathscr{G}+\mathscr{B}:\mathfrak{g}_{+}\longrightarrow\mathfrak{g}_{-}$, which gives rise to an orthogonal splitting $\mathfrak{g}=\mathscr{E}^{+}\oplus\mathscr{E}^{-}$, and we promote it to an $\mathscr{O}$-operator to get a new Lie algebra structure on $\mathfrak{g}_{+}$ by using a quasitriangular factorizable solution of the modified classical Yang-Baxter equation. Then we built a new Manin triple as a twilled extension of these Lie subalgebras, which has the notorious property of admitting the orthogonal subspaces as Lie ideals. In Section 5, we start with a pair of anti-isomorphic $n$-dimensional Lie algebras $\mathrm{E}^{+}$ and $\mathrm{E}^{-}$ and joint them together in a quadratic Lie algebra direct sum $\mathfrak{g}$, such that $\mathrm{E}^{+}$ and $\mathrm{E}^{-}$ are mutually orthogonal. From the associated operator $\mathscr{J}$, we construct a Lagrangian splitting $\mathfrak{g}=\mathrm{F}_{+}\oplus\mathrm{F}_{-}$. Again, from a metric $\mathscr{G}$ on $\mathrm{F}_{+}$ and a gauge transformation by a skew-symmetric map $\mathscr{B}$ we get a new orthogonal splitting of $\mathfrak{g}$. Following the analogous steps as in the previous section, we finally get a Manin triple admitting the orthogonal splitting as Lie ideals. Finally, in Section 6, some conclusion are summarized.

a)Departamento de Ciencias Exactas y Naturales,
Unidad Académica Caleta Olivia,
Universidad Nacional de la Patagonia Austral
and
CIT Golfo de San Jorge - CONICET,
(9011) Caleta Olivia, Argentina.

## II. COMPLEX PRODUCT STRUCTURE ON VECTOR SPACES: ORTHOGONAL AND LAGRANGIAN SPLITTINGS

Let us review some basic facts about real quadratic vector spaces (see for instance ref.[10]). A *quadratic vector space* $\mathrm{V}$ is a vector space endowed with non-degenerate symmetric bilinear form $(,)_{\mathrm{V}} : \mathrm{V}\otimes\mathrm{V}\longrightarrow\mathbb{R}$. Given a subspace $\mathrm{U}\subset\mathrm{V}$, $\mathrm{U}^{\perp}$ denotes the orthogonal complement of U. A subspace $\mathrm{U}\subset\mathrm{V}$ is isotropic if $\mathrm{U}\subseteq\mathrm{U}^{\perp}$. If V is even dimensional, the bilinear form is called *split* if the maximally isotropic subspaces are of dimension $\frac{1}{2}\dim\mathrm{V}$, in this case these subspaces are named *Lagrangian* subspaces.

We study the simultaneous decomposition of a quadratic vector space in a direct sum of two Lagrangian and a direct sum of two orthogonal subspaces of the same dimension. Let V be a $2n$-dimensional real quadratic vector space with a split bilinear form $(,)_{\mathrm{V}} : \mathrm{V}\otimes\mathrm{V}\longrightarrow\mathbb{R}$, and consider the direct sum decompositions $\mathrm{V}=\mathrm{E}^{+}\oplus\mathrm{E}^{-}=\mathrm{F}_{+}\oplus\mathrm{F}_{-}$ such that $\{\mathrm{E}^{+},\mathrm{E}^{-}\}$ are mutually orthogonal $n$-dimensional subspaces while $\{\mathrm{F}_{+},\mathrm{F}_{-}\}$ are Lagrangian subspaces. For the sake of brevity, we refer to this pair of simultaneous direct sum decompositions of V as a *double splitting*.

A *complex product structure* on the vector space V (see ref.[6]) is a pair of linear operators $\{\mathscr{E},\mathscr{J}\}:\mathrm{V}\longrightarrow\mathrm{V}$ such that

$$\mathscr{E}^2=\mathscr{I} \quad , \quad \mathscr{J}^2=-\mathscr{I} \quad , \quad \mathscr{E}\mathscr{J}+\mathscr{J}\mathscr{E}=0 \tag{1}$$

where $\mathscr{I}$ is the identity on V. The involutive operator $\mathscr{E}$ is called *product structure* on the vector space V and the linear operator $\mathscr{J}$ is a complex structure on V. Since $\mathscr{E}$ and $\mathscr{J}$ anticommute, $\mathscr{J}$ is a linear bijection between the eigenspaces $\mathscr{E}^{\pm}$, associated with the eigenvalues $\pm1$ of $\mathscr{E}$, so the eigenspaces $\mathscr{E}^{\pm}$ are $n$-dimensional and $\mathscr{E}$ is called a *para-complex structures*[11]. There is a third linear operator, namely $\mathscr{F}=\mathscr{J}\mathscr{E}$, which is involutive and anticommutes with $\mathscr{E}$ and $\mathscr{J}$. Thus $\{\mathscr{E},\mathscr{J},\mathscr{F}\}$ spans the algebra of linear homogeneous polynomials on these operators.

Let V be a real quadratic vector space, then we call $\{\mathscr{E},\mathscr{J}\}$ a *symmetric complex product structure* when $\mathscr{E}$ and $\mathscr{J}$ are symmetric relative to the bilinear form $(,)_{\mathrm{V}}$, implying that $\mathscr{F}$ is skew-symmetric. In this case, $\mathscr{J}$ is anti-compatible with $(,)_{\mathrm{V}}$ so $((,)_{\mathrm{V}},\mathscr{J})$ is an anti-Hermitian structure on V. However, from the complex product structure we may define a second quadratic bilinear form on V, namely $(,)_{\mathscr{E}}:\mathrm{V}\otimes\mathrm{V}\longrightarrow\mathbb{R}$ as $(X,Y)_{\mathscr{E}}:=(X,\mathscr{E}Y)_{\mathrm{V}}$ so that $((,)_{\mathrm{V}},\mathscr{J})$ is an Hermitian structure.

A symmetric complex product structure $\{\mathscr{E},\mathscr{J}\}$ on a quadratic vector space V has associated a double splitting $\mathrm{V}=\mathrm{E}^{+}\oplus\mathrm{E}^{-}=\mathrm{F}_{+}\oplus\mathrm{F}_{-}$ defined by the eigenspaces $\mathrm{E}^{+}$ and $\mathrm{E}^{-}$ of the involutive symmetric operator $\mathscr{E}$ and the eigenspaces $\mathrm{F}_{+}$ and $\mathrm{F}_{-}$ of the involutive skew-symmetric operator $\mathscr{F}=\mathscr{J}\mathscr{E}$. This implies the existence of a linear bijections $\varphi:\mathrm{E}^{+}\longrightarrow\mathrm{E}^{-}$, with $\varphi^{\top}=-\varphi^{-1}$, such that the matrix block form of the operators $\{\mathscr{E},\mathscr{J},\mathscr{F}\}$ in a basis of eigenvectors of $\mathscr{E}$ are

$$\begin{aligned}\mathscr{E} &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \ , \ \mathscr{F}=\begin{pmatrix} 0 & \varphi^{-1} \\ \varphi & 0 \end{pmatrix} \\ \mathscr{J} &= \mathscr{F}\mathscr{E}=\begin{pmatrix} 0 & -\varphi^{-1} \\ \varphi & 0 \end{pmatrix}\end{aligned} \quad , \tag{2}$$

and, on the other hand, of a linear bijection $\mathscr{G}:\mathrm{F}_{+}\longrightarrow\mathrm{F}_{-}$, with $\mathscr{G}^{\top}=\mathscr{G}$, such that the matrix block form of the operators $\{\mathscr{E},\mathscr{J},\mathscr{F}\}$ in a basis of eigenvectors of $\mathscr{F}$ are

$$\begin{aligned}\mathscr{E} &= \begin{pmatrix} 0 & \mathscr{G}^{-1} \\ \mathscr{G} & 0 \end{pmatrix} \ , \ \mathscr{F}=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ \mathscr{J} &= \mathscr{F}\mathscr{E}=\begin{pmatrix} 0 & \mathscr{G}^{-1} \\ -\mathscr{G} & 0 \end{pmatrix}\end{aligned} \quad .$$

In the first case the eigenspace $\mathrm{F}_{\pm}$ can be identified as the graph of $\varphi$

$$\mathrm{F}_{\pm}=\left\{X^{+}\pm\varphi\left(X^{+}\right)/X^{+}\in\mathrm{E}^{+}\right\}=\mathrm{graph}\left(\pm\varphi\right)$$

and the linear map $\mathscr{G}:\mathrm{F}_{+}\longrightarrow\mathrm{F}_{-}$ is realized as

$$\mathscr{G}\left(X^{+}+\varphi\left(X^{+}\right)\right)=X^{+}-\varphi\left(X^{+}\right).$$

In the second case, the eigenspace $\mathrm{E}^{\pm}$ coincides with the graph of $\mathscr{G}$

$$\mathrm{E}^{\pm}=\left\{X_{+}\pm\mathscr{G}\left(X_{+}\right)/X_{+}\in\mathrm{F}_{+}\right\}=\mathrm{graph}\left(\pm\mathscr{G}\right).$$

and the linear map $\varphi:\mathrm{E}^{+}\longrightarrow\mathrm{E}^{-}$ turn to be

$$\varphi\left(X_{+}+\mathscr{G}\left(X_{+}\right)\right)=X_{+}-\mathscr{G}\left(X_{+}\right) \tag{3}$$

Thus, double splittings induced by complex product structures can be equivalently obtained from an orthogonal splitting supplied with a linear bijection $\varphi:\mathrm{E}^{+}\longrightarrow\mathrm{E}^{-}$ with $\varphi^{\top}=-\varphi^{-1}$ or, alternatively, from a Lagrangian splitting supplied with a linear bijection $\mathscr{G}:\mathrm{F}_{+}\longrightarrow\mathrm{F}_{-}$ with $\mathscr{G}^{\top}=\mathscr{G}$. Note that $\mathscr{G}$ can be regarded as a *metric* on the Lagrangian component $\mathrm{F}_{+}$.

In the next sections we shall exploit these alternative ways of building double splittings from a complex product structure to obtain a direct sum Lie algebra from an special Manin triple and, reciprocally, to obtain a Manin triple from a particular Lie algebra direct sum.

Starting from a Lagrangian splitting $\mathrm{V}=\mathrm{F}_{+}\oplus\mathrm{F}_{-}$ and a linear map $\mathscr{G}:\mathrm{F}_{+}\longrightarrow\mathrm{F}_{-}$ we can obtain a wide family of orthogonal splittings through a class of isometries named *gauge transformations*[7]. A *twisting* or *gauge transformation* on the vector space V is implemented on the Lagrangian decomposition $\mathrm{V}=\mathrm{F}_{+}\oplus\mathrm{F}_{-}$ by a skew-symmetric linear map $\mathscr{B}:\mathrm{F}_{+}\longrightarrow\mathrm{F}_{-}$ such that

$$\mathscr{B}\cdot\left(X_{+}+X_{-}\right)=X_{+}+X_{-}+\mathscr{B}\left(X_{+}\right). \tag{4}$$

So, we get the family of subspaces parametrized by $\mathscr{B}\in\mathrm{Skew}\left(\mathrm{F}_{+},\mathrm{F}_{-}\right)$

$$\mathrm{E}^{\pm}_{\mathscr{B}}=\left\{X_{+}\pm\left(\mathscr{B}\pm\mathscr{G}\right)X_{+}/X_{+}\in\mathrm{F}_{+}\right\}=\mathrm{graph}\left(\mathscr{B}\pm\mathscr{G}\right), \tag{5}$$

such that $\mathrm{V}=\mathrm{E}^{+}_{\mathscr{B}}\oplus\mathrm{E}^{-}_{\mathscr{B}}$ is an orthogonal splitting of V. Note that if $\mathscr{G}$ is positive definite, $\mathscr{G}\pm\mathscr{B}$ is invertible and can be regarded as a generalized metric on $\mathrm{F}_+$. However, we will abuse the language and keep this name even in the case where $\mathscr{G}\pm\mathscr{B}$ is non-invertible.

Gauge transformations led us to a broader framework in which the vector space V admit simultaneously the Lagrangian splitting $\mathrm{V}=\mathrm{F}_+\oplus\mathrm{F}_-$ and the orthogonal one $\mathrm{V}=\mathrm{E}^{+}_{\mathscr{B}}\oplus\mathrm{E}^{-}_{\mathscr{B}}$, for arbitrary skew-symmetric linear map $\mathscr{B}:\mathrm{F}_+\longrightarrow\mathrm{F}_-$. So we must review the above block matrix realization of the associated complex product structure $\{\mathscr{E}_{\mathscr{B}},\mathscr{J}_{\mathscr{B}}\}$.

**Proposition:** *The complex product structure* $\{\mathscr{E}_{\mathscr{B}},\mathscr{J}_{\mathscr{B}}\}$ *associated with the decompositions* $\mathrm{V}=\mathrm{E}^{+}_{\mathscr{B}}\oplus\mathrm{E}^{-}_{\mathscr{B}}=\mathrm{F}_+\oplus\mathrm{F}_-$ *is represented in the Lagrangian splitting as*

$$\mathscr{E}_{\mathscr{B}}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ \mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix},$$
$$\mathscr{J}_{\mathscr{B}}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ -\mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix} \tag{6}$$

*while in the orthogonal splitting is represented as*

$$\mathscr{E}_{\mathscr{B}}=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\ ,\ \mathscr{J}_{\mathscr{B}}=\begin{pmatrix}0 & -\varphi_{\mathscr{B}}^{-1}\\ \varphi_{\mathscr{B}} & 0\end{pmatrix}$$

*where* $\varphi_{\mathscr{B}}:\mathscr{E}^{+}_{\mathscr{B}}\longrightarrow\mathscr{E}^{-}_{\mathscr{B}}$ *defined as*

$$\varphi_{\mathscr{B}}(X_+ +(\mathscr{B}+\mathscr{G})X_+)=X_+ +(\mathscr{B}-\mathscr{G})X_+. \tag{7}$$

**Proof:** The proof is straightforward, it is easy to check that $(\mathscr{E}_{\mathscr{B}})^2=-(\mathscr{J}_{\mathscr{B}})^2=\mathscr{I}$, and $\mathscr{E}_{\mathscr{B}}\mathscr{J}_{\mathscr{B}}+\mathscr{J}_{\mathscr{B}}\mathscr{E}_{\mathscr{B}}=I$, and that the eigenspaces of $\mathscr{E}_{\mathscr{B}}$ are $\mathrm{E}^{+}_{\mathscr{B}}$ and $\mathrm{E}^{-}_{\mathscr{B}}$ in both direct sum decompositions. ■

The operators $\mathscr{E}_{\mathscr{B}}$ and $\mathscr{J}_{\mathscr{B}}$ introduced in eq. (6) are uniquely defined up to a conformal factor steaming from the metric change $\mathscr{G}\longrightarrow e^{\phi}\mathscr{G}$.

The operator $\mathscr{F}_{\mathscr{B}}$ in $\mathrm{V}=\mathrm{F}_+\oplus\mathrm{F}_-$ is

$$\mathscr{F}_{\mathscr{B}}=\begin{pmatrix}I & 0\\ 2\mathscr{B} & -I\end{pmatrix}$$

with the eigenspaces $\mathrm{F}_{\mathscr{B}-}=\mathrm{F}_-$ and $\mathrm{F}_{\mathscr{B}+}=\mathscr{B}\cdot\mathrm{F}_+$, providing a family of Lagrangian splittings parametrized by skew-symmetric maps from $\mathrm{F}_+$ to $\mathrm{F}_-$.

The operators (6) can be written as

$$\mathscr{E}=\mathrm{B}^{\top}\mathrm{GBA}\ \ ,\ \ \mathscr{J}=\mathrm{B}^{\top}\mathrm{JBA}$$

where

$$\mathrm{B}=\begin{pmatrix}\mathscr{I} & -\mathscr{B}\\ 0 & -\mathscr{I}\end{pmatrix}\ ,\ \mathrm{G}=\begin{pmatrix}\mathscr{G}^{-1} & 0\\ 0 & \mathscr{G}\end{pmatrix},$$

$$\mathrm{J}=\begin{pmatrix}\mathscr{G}^{-1} & 0\\ 0 & -\mathscr{G}\end{pmatrix}\ ,\ \mathrm{A}=\begin{pmatrix}0 & \mathscr{I}\\ \mathscr{I} & 0\end{pmatrix}.$$

It is interesting to note that linear operators like to $\mathscr{E}$, disguised as

$$\begin{pmatrix}\mathscr{G}^{-1} & -\mathscr{G}^{-1}\mathscr{B}\\ \mathscr{B}\mathscr{G}^{-1} & \mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B}\end{pmatrix}$$
$$=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ \mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},$$

is pervasive in string theory T-duality. There its occurrence can be traced back to references[12],[13] where it appears as a *metric* on a $2D$-dimensional manifold, then in early studies on T-duality as a *generalized metric* and, more recently, it becomes in the central object in the so called Double Field Theory approach to T-duality[14],[15]. It also plays a relevant role in generalized Kähler geometry[16]. Hence, all these problems can be endowed with an (almost) complex structure $\{\mathscr{E},\mathscr{J}\}$ like that given in eq. (6).

### A. Dual construction

An analogue of the above construction can be achieved using the inverse map $\varphi^{-1}:\mathrm{E}^-\longrightarrow\mathrm{E}^+$ in place of $\varphi:\mathrm{E}^+\longrightarrow\mathrm{E}^-$, in such a way that the eigenspaces of F are now defined as

$$\mathrm{F}_{\pm}=\left\{X^-\pm\varphi^{-1}\left(X^-\right)/X^-\in\mathrm{E}^-\right\}$$

This construction comes to be dual of the former, relative to the bilinear form $(,)_{\mathrm{V}}$ in the following sense

$$\left(X^-\pm\varphi^{-1}\left(X^-\right),Y^+\right)_{\mathrm{V}}=\left(X^-,Y^+\mp\varphi\left(Y^+\right)\right)_{\mathrm{V}},$$

for $X^-\in\mathrm{E}^-,Y^+\in\mathrm{E}^+$. The analogous of the map $\mathscr{G}$ is now the linear bijection $\tilde{\mathscr{G}}:\mathrm{F}_-\longrightarrow\mathrm{F}_+$ defined as

$$\tilde{\mathscr{G}}\left(X^- -\varphi^{-1}\left(X^-\right)\right)=X^- +\varphi^{-1}\left(X^-\right)$$

and from it we get

$$\mathrm{E}^{\mp}=\left\{X_-\pm\tilde{\mathscr{G}}(X_-)/X_-\in\mathrm{F}_-\right\}=\mathrm{graph}\left(\pm\tilde{\mathscr{G}}\right).$$

By writting $X^-=-\varphi(X^+)$ one may see that $\tilde{\mathscr{G}}=\mathscr{G}^{-1}$. In the Lagrangian decomposition $\mathrm{V}=\mathrm{F}_+\oplus\mathrm{F}_-$ the operators $\tilde{\mathscr{E}}$ and $\tilde{\mathscr{J}}$ are represented by the block matrices

$$\tilde{\mathscr{E}}=\begin{pmatrix}0 & \tilde{\mathscr{G}}\\ \tilde{\mathscr{G}}^{-1} & 0\end{pmatrix}\ ,\ \tilde{\mathscr{J}}=\begin{pmatrix}0 & \tilde{\mathscr{G}}\\ -\tilde{\mathscr{G}}^{-1} & 0\end{pmatrix}.$$

Things become more interesting after applying the gauge transformation isometry

$$\tilde{\mathscr{B}}\cdot(X_+ +X_-)=\left(X_+ +\tilde{\mathscr{B}}(X_-)+X_-\right)$$

with $\tilde{\mathscr{B}}:\mathrm{F}_-\longrightarrow\mathrm{F}_+$ a skew-symmetric linear map, which allows to get the family of orthogonal subspaces

$$\mathrm{E}^{\pm}_{\tilde{\mathscr{B}}}=\left\{X_-\pm\left(\tilde{\mathscr{B}}\pm\tilde{\mathscr{G}}\right)X_-/X_-\in\mathrm{F}_-\right\}=\mathrm{graph}\left(\tilde{\mathscr{B}}\pm\tilde{\mathscr{G}}\right),$$

parametrized by $\tilde{\mathscr{B}} \in \mathrm{Skew}(\mathrm{F}_-, \mathrm{F}_+)$. Hence, the Lagrangian and orthogonal splitting $\mathrm{V} = \mathrm{F}_+ \oplus \mathrm{F}_- = \mathrm{E}^+_{\tilde{\mathscr{B}}} \oplus \mathrm{E}^-_{\tilde{\mathscr{B}}}$ have associated the complex product structure $\{\tilde{\mathscr{E}}_{\mathscr{B}}, \tilde{\mathscr{J}}_{\mathscr{B}}\}$

$$\mathscr{E}_{\tilde{\mathscr{B}}} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \ , \ \mathscr{J}_{\tilde{\mathscr{B}}} = \begin{pmatrix} 0 & -\tilde{\varphi}^{-1} \\ \tilde{\varphi} & 0 \end{pmatrix},$$

refered to $\mathrm{V} = \mathrm{E}^+_{\tilde{\mathscr{B}}} \oplus \mathrm{E}^-_{\tilde{\mathscr{B}}}$, where $\tilde{\varphi} : \mathscr{E}^+_{\mathscr{B}} \longrightarrow \mathscr{E}^-_{\mathscr{B}}$ now defined as

$$\tilde{\varphi}\left(X_- + \left(\tilde{\mathscr{B}} + \tilde{\mathscr{G}}\right) X_-\right) = X_- + \left(\tilde{\mathscr{B}} - \mathscr{G}\right) X_-.$$

In the Lagrangian splitting $\mathrm{V} = \mathrm{F}_+ \oplus \mathrm{F}_-$ they are represented by block matrices[3]

$$\mathscr{E}_{\tilde{\mathscr{B}}} = \begin{pmatrix} \tilde{\mathscr{B}}\tilde{\mathscr{G}}^{-1} & \tilde{\mathscr{G}} - \tilde{\mathscr{B}}\tilde{\mathscr{G}}^{-1}\tilde{\mathscr{B}} \\ \tilde{\mathscr{G}}^{-1} & -\tilde{\mathscr{G}}^{-1}\tilde{\mathscr{B}} \end{pmatrix},$$

$$\tilde{\mathscr{J}}_{\tilde{\mathscr{B}}} = \begin{pmatrix} \tilde{\mathscr{B}}\tilde{\mathscr{G}}^{-1} & -\tilde{\mathscr{G}} - \tilde{\mathscr{B}}\tilde{\mathscr{G}}^{-1}\tilde{\mathscr{B}} \\ \tilde{\mathscr{G}}^{-1} & -\tilde{\mathscr{G}}^{-1}\tilde{\mathscr{B}} \end{pmatrix}.$$

The eigenspace $\mathrm{E}^{\pm}_{\tilde{\mathscr{B}}}$ of $\tilde{\mathscr{E}}_{\tilde{\mathscr{B}}}$ coincides with $\mathrm{E}^{\pm}_{\mathscr{B}}$ (5) if and only if

$$(\mathscr{B} \pm \mathscr{G})\left(\tilde{\mathscr{B}} \pm \tilde{\mathscr{G}}\right) = \mathscr{I} \ \text{ and } \ \left(\tilde{\mathscr{B}} \pm \tilde{\mathscr{G}}\right)(\mathscr{B} \pm \mathscr{G}) = \mathscr{I} \ ,$$

which in turn implies the relations

$$\tilde{\mathscr{G}} = \left(\mathscr{G} - \mathscr{B}\mathscr{G}^{-1}\mathscr{B}\right)^{-1}$$

$$\begin{aligned} \tilde{\mathscr{B}} &= -\mathscr{G}^{-1}\mathscr{B}\left(\mathscr{G} - \mathscr{B}\mathscr{G}^{-1}\mathscr{B}\right)^{-1} \\ &= -\left(\mathscr{G} - \mathscr{B}\mathscr{G}^{-1}\mathscr{B}\right)^{-1}\mathscr{B}\mathscr{G}^{-1} \end{aligned}$$

making $\mathscr{E}_{\tilde{\mathscr{B}}} = \mathscr{E}_{\mathscr{B}}$. These relations also hold after interchanging $(\mathscr{G}, \mathscr{B}) \longleftrightarrow \left(\tilde{\mathscr{G}}, \tilde{\mathscr{B}}\right)$. Note that both descriptions lead to the same double splitting provided the generalized metric $\mathscr{G} \pm \mathscr{B}$ provided it is invertible which, for instance, is warranted if $\mathscr{G}$ gives rise to a positive definite metric on $\mathrm{F}_+$, or $\tilde{\mathscr{G}}$ on $\mathrm{F}_-$.

## III. COMPLEX PRODUCT STRUCTURE ON LIE ALGEBRAS

A *quadratic Lie algebra* is a Lie algebra $\mathfrak{g}$ equipped with an invariant, non-degenerate symmetric bilinear form $(,)_{\mathfrak{g}}$. Let $\mathfrak{g}$ be a quadratic Lie algebra with a split bilinear form $(,)_{\mathfrak{g}}$, and assume that the underlying vector space is supplied with a double decomposition so it has associated a pair of operators $\mathscr{E}, \mathscr{J} : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\mathscr{E}^2 = \mathscr{I}$, $\mathscr{J}^2 = -\mathscr{I}$ and $\mathscr{E}\mathscr{J} + \mathscr{J}\mathscr{E} = 0$. In presence of a Lie algebra structure, it is important to pay attention to integrability issues: on a Lie algebra $\mathfrak{g}$ a linear operator $\mathscr{E} : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $\mathscr{E}^2 = \mathscr{I}$ is called an *almost product structure*, and it is said integrable if the Nijehuis condition is satisfied, namely

$$[\mathscr{E}X, \mathscr{E}Y] - \mathscr{E}([\mathscr{E}X, Y] + [X, \mathscr{E}Y]) + [X, Y] = 0$$

for all $X, Y \in \mathfrak{g}$. Equivalently, the linear operator $\mathscr{E}$ is integrable iff its eigenspaces $\mathrm{E}_+$ and $\mathrm{E}_-$ are Lie subalgebras of $\mathfrak{g}$. An integrable almost product structure is called a *product structure*. If the eigenspaces $\mathrm{E}_+$ and $\mathrm{E}_-$, associated with the eigenvalues $+1$ and $-1$, respectively, have the same dimension, the product structure is called a *paracomplex structure*.[11]

A linear operator $\mathscr{J} : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $\mathscr{J}^2 = -\mathscr{I}$ is called an *almost complex structure* and it is integrable if the Nijenhuis condition

$$[\mathscr{J}X, \mathscr{J}Y] - \mathscr{J}([\mathscr{J}X, Y] + [X, \mathscr{J}Y]) - [X, Y] = 0$$

is satisfied. In this case, it is called a *complex structure*.

A *complex product structure* on a Lie algebra $\mathfrak{g}$ is given by a product structure $\mathscr{E}$ and a complex structure $\mathscr{J}$ such that $\mathscr{E}\mathscr{J} + \mathscr{J}\mathscr{E} = 0$. Complex product structures on Lie algebras are exhaustively studied in ref.[6]. In this work we are involved with almost complex structures, and the product structures become integrable only in some particular case.

Next we will apply the study of section II to a Manin triple endowed with a generalized metric. A Manin triple consists of a triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ where $\mathfrak{g}$ is equipped with an invariant non-degenerate symmetric bilinear form $(,)_{\mathfrak{g}}$ and $\mathfrak{g}_+, \mathfrak{g}_-$ are Lagrangian (maximally isotropic) Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. There is one-to-one correspondence between Manin triples on $\mathfrak{g}$ and Lie bialgebra structures on the Lie algebra $\mathfrak{g}_\pm$[17], which in turn are in one-to-one correspondence with Poisson-Lie structures on the connected and simply-connected Lie group $G_\pm$ associated with $\mathfrak{g}_\pm$. The main idea behind the correspondence between Manin triples on $\mathfrak{g}$ and Lie bialgebra structures on the Lie algebra $\mathfrak{g}_\pm$ is the identification $\mathfrak{g}_\pm \simeq \mathfrak{g}^*_\mp$ through the linear bijection induced by the bilinear form $(,)_{\mathfrak{g}}$, such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \simeq \mathfrak{g}_\pm \oplus \mathfrak{g}^*_\pm = \mathscr{D}(\mathfrak{g}_\pm)$, where $\mathscr{D}(\mathfrak{g}_\pm)$ is called the double Lie algebra of $\mathfrak{g}_\pm$, or the classical double of $\mathfrak{g}_\pm$. In fact, a Lie bialgebra on the Lie algebra $\mathfrak{g}_\pm$ is defined by a Lie cobracket $\delta : \mathfrak{g}_\pm \longrightarrow \mathfrak{g}_\pm \otimes \mathfrak{g}_\pm$ such that the dual map $\delta^* : \mathfrak{g}^*_\pm \otimes \mathfrak{g}^*_\pm \longrightarrow \mathfrak{g}^*_\pm$ defines a Lie bracket on $\mathfrak{g}^*_\pm$. There is also a compatibility condition between the Lie cobracket and the Lie bracket on $\mathfrak{g}_\pm$ which implies that $\delta$ is 1-cocycle with values in $\mathfrak{g}_\pm \otimes \mathfrak{g}_\pm$. Then, building a cobracket from a coboundary $r \in \mathfrak{g}_\pm \otimes \mathfrak{g}_\pm$ (the $r$-matrix) fulfills automatically this requirement. It remains the co-Jacobi condition (or the Jacobi identity for the induced Lie bracket in $\mathfrak{g}^*_\pm$) which is a hard restriction on the $r$-matrix that boils down to asking for the ad-invariance of the objects $r_{12} + r_{21} \in \mathfrak{g}_\pm \otimes \mathfrak{g}_\pm$ and $[r_{13}, r_{23}] + [r_{12}, r_{13}] + [r_{12}, r_{23}] \in \mathfrak{g}_\pm \otimes \mathfrak{g}_\pm \otimes \mathfrak{g}_\pm$, giving rise to the classical Yang-Baxter equation and the modified classical Yang-Baxter equation.

## IV. FROM MANIN TRIPLES TO LIE ALGEBRAS ORTHOGONAL DIRECT SUM

Starting from a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, that is a quadratic Lie algebra, and a generalized metric $\mathscr{G} + \mathscr{B} : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_-$ we build a new Manin triple structure admitting an orthogonal direct sum decomposition in terms of Lie algebra ideals associated with the graphs of $\mathscr{G} + \mathscr{B}$ and its orthogonal complement, namely the graph of $\mathscr{B}-\mathscr{G}$. This will be achieved by turning $\mathscr{B}$ into an $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $\kappa=-1$.

### A. Generalized metrics and almost complex product structures

Let $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ be a Manin triple with $\dim\mathfrak{g}=2n$. We assume that $\mathfrak{g}_+$ is equipped with a generalized metric realized through a linear map $\mathscr{H}:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-$ such that, relative to the bilinear form $(,)_{\mathfrak{g}}$, it can be decomposed as $\mathscr{H}=\mathscr{G}+\mathscr{B}$ where $\mathscr{G}$ is the symmetric component, assumed to be invertible, and $\mathscr{B}$ is the skew-symmetric one.

Let $\mathscr{H}^\top:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-$ be the transpose of $\mathscr{H}:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-$, $(\mathscr{H}X_+,Y_+)_{\mathfrak{g}}=\left(X_+,\mathscr{H}^\top Y_+\right)_{\mathfrak{g}}$, and consider the subspaces $\mathscr{E}^+$ and $\mathscr{E}^-$ of $\mathfrak{g}$ defined by the graphs of the linear maps $\mathscr{H}$ and $-\mathscr{H}^\top$,

$$\mathscr{E}^{\pm}=\{X_++(\mathscr{B}\pm\mathscr{G})X_+/X_+\in\mathfrak{g}_+\}. \qquad (8)$$

$\mathscr{E}^+$ and $\mathscr{E}^-$ are transversal orthogonal subspaces of dimension $n$ such that $\mathfrak{g}=\mathscr{E}^+\oplus\mathscr{E}^-$, and the vector space $\mathfrak{g}$ admits a double splitting, namely $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-=\mathscr{E}^+\oplus\mathscr{E}^-$. We describe these orthogonal subspaces by introducing a the linear operators $\mathscr{E},\mathscr{J}:\mathfrak{g}\longrightarrow\mathfrak{g}$ such that

$$\mathscr{E}\left(X_++\mathscr{B}X_+\pm\mathscr{G}X_+\right)=\pm\left(X_++\mathscr{B}X_+\pm\mathscr{G}X_+\right)$$

$$\mathscr{J}\left(X_++\mathscr{B}X_+\pm\mathscr{G}X_+\right)=\pm\left(X_++\mathscr{B}X_+\mp\mathscr{G}X_+\right)$$

implying that $\mathscr{E}$ and $\mathscr{J}$ are symmetric operators satisfying the properties

$$\mathscr{E}^2=\mathscr{I}\ \ ,\ \ \mathscr{J}^2=-\mathscr{I}\ \ ,\ \ \mathscr{E}\mathscr{J}+\mathscr{J}\mathscr{E}=0\ ,$$

that is, they conform an almost complex product structure on $\mathfrak{g}$. $\mathscr{J}$ is anti-compatible with the bilinear form $(,)_{\mathfrak{g}}$, turning $\mathfrak{g}$ in a complex vector space with an anti-Hermitian metric.

In the direct sum decomposition $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$, the block matrix form of these operators are

$$\mathscr{E}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ \mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix},$$

$$\mathscr{J}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ -\mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix}$$

while in the orthogonal decomposition they take the form given in eq. (2). They span a tridimensional algebra with basis $\{\mathscr{E},\mathscr{J},\mathscr{F}\}$, where $\mathscr{F}=\mathscr{J}\mathscr{E}$ is an involutive skew-symmetric operator. In the present framework, the linear operators $\{\mathscr{E},\mathscr{J}\}$ are not integrable in general.

### B. Generalized metrics and $\mathscr{O}$-operators

The next step is to bring the generalized metric $\mathscr{H}=\mathscr{G}+\mathscr{B}$ on $\mathfrak{g}_+$ into the context of the $r$-matrix method[18],[19]. We consider the generalization of the $r$-matrix method introduced in[9],[20] and the further extensions of ref.[8]-[21], by regarding the $r$-matrices as linear maps from a representation space of a Lie algebra to the Lie algebra itself. In the last references, the method is extended to the case in which the representation space is also a Lie algebra and the action is a derivation. Although the setting of refs.[9],[20] would be enough for the current developments, we will use the more general and versatile method of $\mathscr{O}$-operators of reference[8], combined with the *twilled* extension of Lie algebras of ref.[9].

Let's make a brief review of some basic definitions of the $\mathscr{O}$-operators limited to what is needed in this work. Let $\mathfrak{k}$ be a finite dimensional Lie algebra and $\mathrm{V}$ a representation space of $\mathfrak{k}$, and denote by $\sigma:\mathfrak{k}\longrightarrow\mathrm{End}(\mathrm{V})$ the linear map such that $X\longmapsto\sigma_X$ for $X\in\mathfrak{k}$. A linear map $\beta:\mathrm{V}\longrightarrow\mathfrak{k}$ is said *antisymmetric* if

$$\sigma_{\beta(V)}W+\sigma_{\beta(W)}V=0$$

for $V,W\in\mathrm{V}$, and it is said $\mathfrak{k}$-*invariant* if

$$\beta\left(\sigma_XV\right)=\left[X,\beta\left(V\right)\right]$$

for $X\in\mathfrak{k},Y\in\mathrm{V}$. Then, if $\beta:\mathrm{V}\longrightarrow\mathfrak{k}$ be a linear map, antisymmetric, $\mathfrak{k}$-invariant, a linear map $r:\mathrm{V}\longrightarrow\mathfrak{k}$ is called an *extended* $\mathscr{O}$*-operator with extension* $\beta$ *of mass* $(-1)$ if it fulfills the equation

$$\left[r\left(V\right),r\left(W\right)\right]-r\left(\sigma_{r(V)}W-\sigma_{r(W)}V\right)=-\left[\beta\left(V\right),\beta\left(W\right)\right] \quad (9)$$

for $V,W\in\mathrm{V}$.

The main result for our purpose is established in the following theorem (Theorem 2.18 in ref.[20], and a restricted version of Theorem 2.13 in ref.[8]).

**Theorem:** *Let* $\mathfrak{k}$ *be a finite dimensional Lie algebra,* $\mathrm{V}$ *be a a representation space of* $\mathfrak{k}$*,* $r,\beta:\mathrm{V}\rightarrow\mathfrak{k}$ *be linear maps.*

i. *If* $r:\mathrm{V}\rightarrow\mathfrak{k}$ *is an extended* $\mathscr{O}$*-operator with extension* $\beta$ *of mass* $\kappa$*, then the bracket*

$$[V,W]_r=\sigma_{r(V)}W-\sigma_{r(W)}V$$

*defines a Lie algebra on* $\mathrm{V}$*. We denote this Lie algebra* $(\mathrm{V},[,]_r)$ *as* $\mathrm{V}^r$*.*

ii. *If* $\beta:\mathrm{V}\rightarrow\mathfrak{k}$ *is* $\mathfrak{k}$*-invariant of mass* $\kappa$ *=-1, then* $r$ *satisfies*

$$\left[r\left(V\right),r\left(W\right)\right]-r\left(\sigma_{r(V)}W-\sigma_{r(W)}V\right)=-\left[\beta\left(V\right),\beta\left(W\right)\right]$$

*if and only if* $(r\pm\beta):\mathrm{V}^r\longrightarrow\mathfrak{k}$ *is a Lie algebra homomorphism, namely*

$$\left(r\pm\beta\right)[V,W]_r=\left[(r\pm\beta)V,(r\pm\beta)W\right]$$

$$\forall V,W\in\mathrm{V}^r.$$

Equation (9) comes to play the role of the modified classical Yang-Baxter equation. Under these conditions, the new Lie bracket on V defines the coboundary cobracket on $\delta_r$ on $\mathfrak{k}$.

### C. $\mathscr{B}$ as an extended $\mathscr{O}$-operator with extension $\mathscr{G}$

Let's start with a Manin triple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ with $\dim\mathfrak{g}=2n$ equipped with an almost product structure $\mathscr{E}$ and an almost complex structure $\mathscr{J}$ such that $\mathscr{E}\mathscr{J}+\mathscr{J}\mathscr{E}=0$ so, besides the Lagrangian splitting $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$, it admits a vector space orthogonal splitting $\mathfrak{g}=\mathscr{E}^+\oplus\mathscr{E}^-$. Now, we will use the linear operators $\mathscr{G},\mathscr{B}:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-$ to build a new Lie bialgebra structure on $\mathfrak{g}_+$ by asking for $\mathscr{B}$ to be an extended $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $\kappa=-1$. In doing so, we consider the vector space $\mathfrak{g}_+$ as a representation space of $\mathfrak{g}_-$ by the map $\sigma:\mathfrak{g}_-\longrightarrow\mathrm{End}(\mathfrak{g}_+)/X_-\longmapsto\sigma_{X_-}$ defined as

$$\sigma_{X_-}Y_+=\Pi_{\mathfrak{g}_+}[X_-,Y_+] \tag{10}$$

where $\Pi_{\mathfrak{g}_\pm}:\mathfrak{g}\longrightarrow\mathfrak{g}_\pm$ is the projector. It is just the dressing action of $\mathfrak{g}_-$ on $\mathfrak{g}_+$. In addition we assume that $\mathscr{G}$ is $\mathfrak{g}_-$-invariant which means that

$$\mathscr{G}\left(\sigma_{X_-}Y_+\right)=[X_-,\mathscr{G}Y_+] \tag{11}$$

for $X_-\in\mathfrak{g}_-$ and $Y_+\in\mathfrak{g}_+$. It also implies that $\mathscr{G}$ is antisymmetric

$$\sigma_{\mathscr{G}X_+}Y_++\sigma_{\mathscr{G}Y_+}X_+=0.$$

Then we take $\mathscr{B}$ as an extended $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $\kappa=-1$, therefore $\mathscr{B}$ and $\mathscr{G}$ fulfill the condition

$$[\mathscr{B}X_+,\mathscr{B}Y_+]-\mathscr{B}\left(\sigma_{\mathscr{B}X_+}Y_+-\sigma_{\mathscr{B}Y_+}X_+\right)=-[\mathscr{G}X_+,\mathscr{G}Y_+] \tag{12}$$

for $X_+,Y_+\in\mathfrak{g}_+$. In turn, this implies that the bracket

$$[X_+,Y_+]_{\mathscr{B}}=\sigma_{\mathscr{B}X_+}Y_+-\sigma_{\mathscr{B}Y_+}X_+ \tag{13}$$

defines a new Lie algebra structure $(\mathfrak{g}_+,[,]_{\mathscr{B}})$ and eq. (12) is equivalent to stating that $(\mathscr{B}\pm\mathscr{G}):\mathfrak{g}_+^{\mathscr{B}}\longrightarrow\mathfrak{g}_-$ is a Lie algebra homomorphism

$$(\mathscr{B}\pm\mathscr{G})[X_+,Y_+]_{\mathscr{B}}=[(\mathscr{B}\pm\mathscr{G})X_+,(\mathscr{B}\pm\mathscr{G})Y_+] \tag{14}$$

for $X_+,Y_+\in(\mathfrak{g}_+,[,]_{\mathscr{B}})$. We denote the Lie algebra $(\mathfrak{g}_+,[,]_{\mathscr{B}})$ as $\mathfrak{g}_+^{\mathscr{B}}$.

### D. Twilled extensions

We shall use a *twilled extension* of Lie algebras, introduced in reference[9], to obtain a new Lie algebra structure from $\mathfrak{g}_-$ and $\mathfrak{g}_+^{\mathscr{B}}$, so let's have a brief review of how it works. Consider a couple of Lie algebras $\mathfrak{g}_-$ and $\mathfrak{g}_+$, equipped with linear maps $\sigma:\mathfrak{g}_-\longrightarrow\mathrm{End}(\mathfrak{g}_+)/X_-\longmapsto\sigma_{X_-}$ and $\rho:\mathfrak{g}_+\longrightarrow\mathrm{End}(\mathfrak{g}_-)/X_+\longmapsto\rho_{X_-}$ turning $\mathfrak{g}_+$ into a representation space of $\mathfrak{g}_-$ and $\mathfrak{g}_-$ into a representation space of $\mathfrak{g}_+$, respectively. With this input, the twilled extension $\mathfrak{g}$ of $\mathfrak{g}_-$ and $\mathfrak{g}_+$ is the vector space $\mathfrak{g}_+\oplus\mathfrak{g}_-$ endowed with the skew-symmetric bracket

$$\begin{aligned}&[X_++X_-,Y_++Y_-]_{\mathfrak{g}_{\mathscr{B}}}\\&=[X_+,Y_+]+\sigma_{X_-}Y_+-\sigma_{Y_-}X_++[X_-,Y_-]+\rho_{X_+}Y_--\rho_{Y_+}X_-\end{aligned}$$

which is a Lie algebra structure on $\mathfrak{g}_+\oplus\mathfrak{g}_-$ iff $\rho$ and $\sigma$ satisfy a pair of constraints in order to ensure the validity of the Jacobi identity. These constraints are most conveniently expressed in terms of so called *orbit maps* defined as the assignments $Y_-\in\mathfrak{g}_-\longmapsto\rho_{Y_-}\in\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)$ and $Y_+\in\mathfrak{g}_+\longmapsto\sigma_{Y_+}\in\mathrm{Hom}_{lin}(\mathfrak{g}_-,\mathfrak{g}_+)$ defined as

$$\rho_{Y_-}X_+:=\rho_{X_+}Y_-\quad,\quad\sigma_{Y_+}X_-:=\sigma_{X_-}Y_+\ .$$

Thus the constraints take the form

$$\begin{aligned}\rho_{[X_-,Y_-]}&=ad_{X_-}\circ\rho_{Y_-}-\rho_{Y_-}\circ\sigma_{X_-}+\rho_{X_-}\circ\sigma_{Y_-}-ad_{Y_-}\circ\rho_{X_-}\\ \sigma_{[X_+,Y_+]}&=ad_{X_+}\circ\sigma_{Y_+}-\sigma_{Y_+}\circ\rho_{X_+}+\sigma_{X_+}\circ\rho_{Y_+}-ad_{Y_+}\circ\sigma_{X_+}\end{aligned} \tag{15}$$

As shown in ref.[9], these relations mean that $\rho$ is 1-cocycles on $\mathfrak{g}_-$ with values in $\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)$, and $\sigma$ is a 1-cocycle on $\mathfrak{g}_+$ with values in $\mathrm{Hom}_{lin}(\mathfrak{g}_-,\mathfrak{g}_+)$.

#### 1. *The twilled extension of* $\mathfrak{g}_+^{\mathscr{B}},\mathfrak{g}_-$

Now we specialize this method to the setting of subsection IV C where $\mathfrak{g}_+$ is a representation space of $\mathfrak{g}_-$ by the map $\sigma:\mathfrak{g}_-\longrightarrow\mathrm{End}(\mathfrak{g}_+)/X_-\longmapsto\sigma_{X_-}$ defined in (10), and $\mathfrak{g}_+$ is endowed with the Lie bracket $[,]_{\mathscr{B}}$ of eq. (13). So, we need to define a map $\rho:\mathfrak{g}_+\longrightarrow\mathrm{End}(\mathfrak{g}_-)/X_+\longmapsto\rho_{X_-}$ turning $\mathfrak{g}_-$ into a representation space of $\mathfrak{g}_+$. Since $\sigma$ is given, we focus on finding a linear map $\rho$ fulfilling the constraints (15) following path of *exact* twilled extension which ensures the constraints are satisfied[9]. Regarding $\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)$ as a $\mathfrak{g}_-$-module under the left action $\mathfrak{g}_-\times\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)\longrightarrow\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)$ defined as

$$\left(X_-,\rho_{Y_-}\right)\longmapsto X_-\cdot\rho_{Y_-}=\rho_{Y_-}\circ\sigma_{X_-}-ad_{X_-}\circ\rho_{Y_-}$$

for $\left(X_-,\rho_{Y_-}\right)\in\mathfrak{g}_-\times\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)$, a 1-form $\rho$ on $\mathfrak{g}_-$ with values in the $\mathfrak{g}_-$-module $\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)$ is a 1-cocycle if

$$d\rho(X_-,Y_-)=X_-\cdot\rho_{Y_-}-Y_-\cdot\rho_{X_-}-\rho_{[X_-,Y_-]}=0$$

which is equivalent to the first constraint in eq. (15). An obvious solution to the first constraint is to choose a map $\rho$ as a coboundary, i.e., we take $\rho=d\mathscr{B}$ with $\mathscr{B}\in\mathrm{Hom}_{lin}(\mathfrak{g}_+,\mathfrak{g}_-)$ regarded as a 0-form, then

$$\rho_{X_-}=d\mathscr{B}(X_-)=\mathscr{B}\circ\sigma_{X_-}-ad_{X_-}\circ\mathscr{B}. \tag{16}$$

and it produces the linear action

$$\rho_{Y_+}X_-=\mathscr{B}\sigma_{X_-}Y_+-[X_-,\mathscr{B}Y_+]$$

Moreover, the second constraint in eq. (15) reduce to a trivial identity. Thus, we get a *right exact twilled extension* $\mathfrak{g}_{\mathscr{B}}$ of $\mathfrak{g}_+^{\mathscr{B}}$ and $\mathfrak{g}_-$, namely the vector space $\mathfrak{g}_+\oplus\mathfrak{g}_-$ endowed with the Lie bracket

$$\begin{aligned}&[X_++X_-,Y_++Y_-]_{\mathfrak{g}_{\mathscr{B}}}\\&=[X_+,Y_+]_{\mathscr{B}}+\sigma_{X_-}Y_+-\sigma_{Y_-}X_+\\&\quad+[X_-,Y_-]+\mathscr{B}\sigma_{X_-}Y_+-[X_-,\mathscr{B}Y_+]\\&\quad-\mathscr{B}\sigma_{Y_-}X_+-[Y_-,\mathscr{B}X_+]\,.\end{aligned}$$

### 2. The twilled extension of $\left(\mathfrak{g}_+^{\mathscr{B}}\right)^{op}, \mathfrak{g}_-$ and the orthogonal subspace $\mathscr{E}^{\pm}$ as Lie ideals

There is a second twilled construction in the framework of the previous subsection steaming from the following observation: if $(\sigma,\rho)$ fulfill the conditions (15) for the Lie algebras $\mathfrak{g}_-$ and $\mathfrak{g}_+^{\mathscr{B}}$, then $(\sigma,-\rho)$ fulfills the Jacobi conditions for the Lie algebras $\mathfrak{g}_-$ and $(\mathfrak{g}_+)^{op}$. Thus we get a right exact twilled extension $\tilde{\mathfrak{g}}_{\mathscr{B}}$ of $\left(\mathfrak{g}_+^{\mathscr{B}}\right)^{op}$ and $\mathfrak{g}_-$, namely $\tilde{\mathfrak{g}}_{\mathscr{B}} = \left(\mathfrak{g}_+^{\mathscr{B}}\right)^{op} \oplus \mathfrak{g}_-$ with Lie bracket

$$
\begin{aligned}
&[X_+ + X_-, Y_+ + Y_-]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \\
&= -[X_+,Y_+]_{\mathscr{B}} + \sigma_{X_-}Y_+ - \sigma_{Y_-}X_+ \\
&\quad + [X_-,Y_-] + \mathscr{B}\sigma_{X_-}Y_+ - [X_-,\mathscr{B}Y_+] \\
&\quad - \mathscr{B}\sigma_{Y_-}X_+ + [Y_-,\mathscr{B}X_+] .
\end{aligned}
$$

Recalling that $\sigma : \mathfrak{g}_- \longrightarrow \mathrm{End}(\mathfrak{g}_+)$ was given in eq. (10), the map $\rho : \left(\mathfrak{g}_+^{\mathscr{B}}\right)^{op} \otimes \mathfrak{g}_- \longrightarrow \mathfrak{g}_-$ turns in

$$
\rho(X_+,Y_-) = \mathscr{B}\Pi_{\mathfrak{g}_+}[X_+,Y_-] - [\mathscr{B}X_+,Y_-]
$$

and the Lie bracket in $\tilde{\mathfrak{g}}_{\mathscr{B}} = \left(\mathfrak{g}_+^{\mathscr{B}}\right)^{op} \oplus \mathfrak{g}_-$ gets the explicit form

$$
\begin{aligned}
&[X_+ + X_-, Y_+ + Y_-]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \qquad (17)\\
&= -[X_+,Y_+]_{\mathscr{B}} + \Pi_{\mathfrak{g}_+}[X_-,Y_+] - \Pi_{\mathfrak{g}_+}[Y_-,X_+] \\
&\quad + [X_-,Y_-] + \mathscr{B}\Pi_{\mathfrak{g}_+}[X_+,Y_-] - [\mathscr{B}X_+,Y_-] \\
&\quad - \mathscr{B}\Pi_{\mathfrak{g}_+}[Y_+,X_-] + [\mathscr{B}Y_+,X_-] .
\end{aligned}
$$

This double Lie algebra has a remarkable property, which is expressed in the following theorem.

**Theorem::** *The orthogonal subspaces $\mathscr{E}^+$ and $\mathscr{E}^-$ defined in eq. (8) are Lie ideals in $\tilde{\mathfrak{g}}_{\mathscr{B}}$, and they are (anti) isomorphic to $\mathfrak{g}_-$.*

**Proof:** To prove that $\mathscr{E}^{\pm}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}_{\mathscr{B}}$, we note that the $\mathfrak{g}_-$-component of the Lie bracket on $\mathscr{E}^{\pm}$ can be written as

$$
\begin{aligned}
&\Pi_{\mathfrak{g}_-}[X_+ + (\mathscr{B}\pm\mathscr{G})X_+, Y_+ + (\mathscr{B}\pm\mathscr{G})Y_+]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \\
&= -[(\mathscr{B}\pm\mathscr{G})X_+,(\mathscr{B}\pm\mathscr{G})Y_+] \\
&\quad + \mathscr{B}\sigma_{(\mathscr{B}\pm\mathscr{G})X_+}Y_+ - \mathscr{B}\sigma_{(\mathscr{B}\pm\mathscr{G})Y_+}X_+ \\
&\quad \pm [(\mathscr{B}\pm\mathscr{G})X_+,\mathscr{G}Y_+] \mp [(\mathscr{B}\pm\mathscr{G})Y_+,\mathscr{G}X_+]
\end{aligned}
$$

and, by virtue of the $\mathfrak{g}_-$-symmetry (11) applied in the last line and because of the relation (14), it reduces to

$$
\begin{aligned}
&\Pi_{\mathfrak{g}_-}[X_+ + (\mathscr{B}\pm\mathscr{G})X_+, Y_+ + (\mathscr{B}\pm\mathscr{G})Y_+]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \\
&= (\mathscr{B}\pm\mathscr{G})\left(\Pi_{\mathfrak{g}_+}[X_+ + (\mathscr{B}\pm\mathscr{G})X_+, Y_+ + (\mathscr{B}\pm\mathscr{G})Y_+]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\right)
\end{aligned}
$$

Therefore, the full Lie bracket between elements of $\mathscr{E}^{\pm}$ can be written as

$$
\begin{aligned}
&[(I+\mathscr{B}\pm\mathscr{G})X_+,(I+\mathscr{B}\pm\mathscr{G})Y_+]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \\
&= (I+\mathscr{B}\pm\mathscr{G})\left(\Pi_{\mathfrak{g}_+}[(I+\mathscr{B}\pm\mathscr{G})X_+,(I+\mathscr{B}\pm\mathscr{G})Y_+]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\right)
\end{aligned}
$$

proving that $\mathscr{E}^{\pm}$ is a Lie subalgebra. After the evaluation of the $\mathfrak{g}_+$-component of the Lie bracket we get

$$
\begin{aligned}
&[(I+\mathscr{B}\pm\mathscr{G})X_+,(I+\mathscr{B}\pm\mathscr{G})Y_+]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \\
&= \pm 2(I+\mathscr{B}\pm\mathscr{G})\mathscr{G}^{-1}[\mathscr{G}X_+,\mathscr{G}Y_+] . \qquad (18)
\end{aligned}
$$

To prove that $\mathscr{E}^{\pm}$ is an ideal we evaluate the crossed Lie bracket between elements of $\mathscr{E}^+$and $\mathscr{E}^-$, where we use again the $\mathfrak{g}_-$-invariance of $\mathscr{G}$ so that after some handling we arrive to

$$
\begin{aligned}
&[X_+ + (\mathscr{B}+\mathscr{G})X_+, Y_+ + (\mathscr{B}-\mathscr{G})Y_+]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \\
&= -[\mathscr{B}X_+,\mathscr{B}Y_+] - [\mathscr{G}X_+,\mathscr{G}Y_+] + \mathscr{B}[X_+,Y_+]_{\mathscr{B}}
\end{aligned}
$$

which vanishes because of the $\mathscr{O}$-operator relation (12), showing that $\mathscr{E}^{\pm}$ is a Lie algebra ideal.

Note that each $X_+ \in \mathfrak{g}_+$ can be written as $X_+ = \mathscr{G}^{-1}X_-$ for some $X_- \in \mathfrak{g}_-$, then eq. (18) can be written as

$$
\begin{aligned}
&\left[\frac{1}{2}(I+\mathscr{B}\pm\mathscr{G})\mathscr{G}^{-1}X_-, \frac{1}{2}(\mathscr{I}+\mathscr{B}\pm\mathscr{G})\mathscr{G}^{-1}Y_-\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}} \\
&= \pm\frac{1}{2}(I+\mathscr{B}\pm\mathscr{G})\mathscr{G}^{-1}[X_-,Y_-]
\end{aligned}
$$

therefore $(\mathscr{I}+\mathscr{B}\pm\mathscr{G})\mathscr{G}^{-1}/2 : \mathfrak{g}_- \longrightarrow \mathscr{E}^{\pm}$ is a Lie algebra (anti) homomorphism with a trivial kernel and, because $\mathfrak{g}_-$ and $\mathscr{E}^{\pm}$ have the same dimension, it is a bijection.■

**Corollary:** *The Lie algebra $\tilde{\mathfrak{g}}_{\mathscr{B}}$ admits the double splitting*

$$
\tilde{\mathfrak{g}}_{\mathscr{B}} = (\mathfrak{g}_+)^{op} \oplus \mathfrak{g}_- = \mathscr{E}^+ \oplus \mathscr{E}^- .
$$

This Lie bracket leaves invariant the bilinear form $(,)_{\mathfrak{g}}$

$$
\begin{aligned}
&\left([X_+ + X_-, Y_+ + Y_-]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}, Z_+ + Z_-\right)_{\tilde{\mathfrak{g}}} \\
&= -\left(Y_+ + Y_-, [X_+ + X_-, Y_+ + Y_-]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\right)_{\mathfrak{g}}
\end{aligned}
$$

so we get the following result.

**Proposition:** $\left(\tilde{\mathfrak{g}}_{\mathscr{B}}, \left(\mathfrak{g}_+^{\mathscr{B}}\right)^{op}, \mathfrak{g}_-\right)$ *equipped with the bilinear form $(,)_{\mathfrak{g}}$ is a Manin triple.*

It is worth highlighting that the above results imply that the product structure $\mathscr{E}$ is integrable, so it is a para-complex structure on $\tilde{\mathfrak{g}}_{\mathscr{B}}$. On the other hand, the non-abelian character of the ideals $\mathscr{E}^+$ and $\mathscr{E}^-$ prevents the integrability of the linear operator $\mathscr{J}$, other wise $\mathscr{E}^+$ and $\mathscr{E}^-$ would be abelian ideals (see ref.[6]).

In reference to the operator $\mathscr{J}$, we can recover the antiisomorphism $\varphi_{\mathscr{B}} : \mathscr{E}^+ \longrightarrow \mathscr{E}^-$, see eq. (7), as the composition

$$
\varphi_{\mathscr{B}} = \frac{1}{2}(I+\mathscr{B}\mp\mathscr{G})\mathscr{G}^{-1} \circ \left(\frac{1}{2}(I+\mathscr{B}\pm\mathscr{G})\mathscr{G}^{-1}\right)^{-1},
$$

with

$$
\varphi_{\mathscr{B}}(X_+ + (\mathscr{B}+\mathscr{G})X_+) = X_+ + (\mathscr{B}-\mathscr{G})X_+ .
$$

It is interesting to note the relation between the Lie bracket (18) and the integrability of the complex structure $\mathscr{J}_{\mathscr{B}}$. In fact, since in the $\mathscr{E}^+\oplus\mathscr{E}^-$ splitting, $\mathscr{J}_{\mathscr{B}}$ takes the form

$$\mathscr{J}_{\mathscr{B}}=\begin{pmatrix}0 & -\varphi_{\mathscr{B}}^{-1}\\ \varphi_{\mathscr{B}} & 0\end{pmatrix}$$

then, the Nijenhuis tensor reduce to[6],

$$\begin{aligned}&\mathscr{N}_{\varphi}\left(X^+,Y^+\right)\\ &=\varphi_{\mathscr{B}}\left[X^+,Y^+\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}+\varphi_{\mathscr{B}}^{-1}\left[\varphi_{\mathscr{B}}\left(X^+\right),\varphi_{\mathscr{B}}\left(Y^+\right)\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\\ &-\left[\varphi_{\mathscr{B}}\left(X^+\right),Y^+\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}-\left[X^+,\varphi_{\mathscr{B}}\left(Y^+\right)\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}},\end{aligned}$$

for $X^+=(I+\mathscr{B}\pm\mathscr{G})X_+, Y^+=(I+\mathscr{B}\pm\mathscr{G})Y_+\in\mathscr{E}^+$. Then, after some computations, we get

$$\mathscr{N}_{\varphi}\left((I+\mathscr{B}\pm\mathscr{G})X_+,(I+\mathscr{B}\pm\mathscr{G})Y_+\right)=-4\left[\mathscr{G}X_+,\mathscr{G}Y_+\right]$$

and the Lie bracket (18) can be written as

$$\left[X^+,Y^+\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}=\mp\frac{1}{2}\left(\mathscr{I}+\mathscr{B}\pm\mathscr{G}\right)\mathscr{G}^{-1}\mathscr{N}_{\varphi}\left(X^+,Y^+\right)$$

making clear that $\mathscr{J}$ is integrable iff $\mathscr{E}^{\pm}$ is abelian.

**Lemma:** *The linear map $\varphi_{\mathscr{B}}:\mathscr{E}^+\longrightarrow\mathscr{E}^-$ defining the almost complex structure $\mathscr{J}$ is a Lie algebra antihomomorphism.*

**Proof:** Let $X^+=X_++(\mathscr{B}+\mathscr{G})X_+$ and $Y^+=Y_++(\mathscr{B}+\mathscr{G})Y_+$ in $\mathscr{E}^+$, then

$$\begin{aligned}&\varphi_{\mathscr{B}}\left(\left[X^+,Y^+\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\right)\\ &=2\left(\mathscr{G}^{-1}\left[\mathscr{G}X_+,\mathscr{G}Y_+\right]+(\mathscr{B}-\mathscr{G})\mathscr{G}^{-1}\left[\mathscr{G}X_+,\mathscr{G}Y_+\right]\right)\\ &=-\left[X_++(\mathscr{B}-\mathscr{G})X_+,Y_++(\mathscr{B}-\mathscr{G})Y_+\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\end{aligned}$$

then

$$\varphi_{\mathscr{B}}\left(\left[X^+,Y^+\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\right)=-\left[\varphi_{\mathscr{B}}\left(X^+\right),\varphi_{\mathscr{B}}\left(Y^+\right)\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}.$$

as stated above.■

This property can be also expressed as

$$\mathscr{E}\left[\mathscr{J}X,\mathscr{J}Y\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}=\mathscr{J}\left[X,Y\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}.$$

### E. $\mathscr{E}^{\pm}$ as $\mathfrak{g}_-$-Lie algebras

It is interesting to note that the Lie algebra $\mathfrak{g}_-$ is a common component in the Manin triples $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ and $\left(\tilde{\mathfrak{g}}_{\mathscr{B}},\mathfrak{g}_+^{\mathscr{B}},\mathfrak{g}_-\right)$, and acts as a derivation on the subalgebras $\mathscr{E}^+$ and $\mathscr{E}^-$ of $\tilde{\mathfrak{g}}_{\mathscr{B}}$.

**Proposition::** *The Lie subalgebra $\mathscr{E}^{\pm}\subset\tilde{\mathfrak{g}}_{\mathscr{B}}$ is a $\mathfrak{g}_-$-Lie algebra under the adjoint action in $\mathfrak{g}_{\mathscr{B}}$.*

**Proof:** Consider the Lie bracket (17) of $\tilde{\mathfrak{g}}_{\mathscr{B}}$ restricted to $\mathfrak{g}_-\times\mathscr{E}^{\pm}$, then the adjoint action of $\mathfrak{g}_-$ on $\mathscr{E}^{\pm}$ is

$$\begin{aligned}&\mathrm{ad}_{X_-}^{\tilde{\mathfrak{g}}_{\mathscr{B}}}\left(Y_++(\mathscr{B}\pm\mathscr{G})Y_+\right)\\ &=\left[X_-,Y_++(\mathscr{B}\pm\mathscr{G})Y_+\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}\\ &=\Pi_{\mathfrak{g}_+}\left[X_-,Y_+\right]+\left[X_-,(\mathscr{B}\pm\mathscr{G})Y_+\right]\\ &\quad+\mathscr{B}\Pi_{\mathfrak{g}_+}\left[X_-,Y_+\right]+\left[\mathscr{B}Y_+,X_-\right]\end{aligned}$$

and recalling that $\mathscr{G}:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-$ is $\mathfrak{g}_-$-invariant, it reduces to

$$\mathrm{ad}_{X_-}^{\tilde{\mathfrak{g}}_{\mathscr{B}}}\left(I+\mathscr{B}\pm\mathscr{G}\right)Y_+=\left(I+\mathscr{B}\pm\mathscr{G}\right)\Pi_{\mathfrak{g}_+}\left[X_-,Y_+\right]$$

therefore $\mathscr{E}^{\pm}$ are invariant subspaces under the adjoint action of $\mathfrak{g}_-$. Writing it in terms of the dressing action of $\mathfrak{g}_-$ on $\mathfrak{g}_+$ in the Manin triple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$, it turns in

$$\mathrm{ad}_{X_-}^{\tilde{\mathfrak{g}}_{\mathscr{B}}}\left(I+\mathscr{B}\pm\mathscr{G}\right)Y_+=\left(I+\mathscr{B}\pm\mathscr{G}\right)\left(Y_+\right)^{X_-}$$

which means that $(I+\mathscr{B}\pm\mathscr{G})$ intertwines between the dressing action of $\mathfrak{g}_-$ on $\mathfrak{g}_+$ and the adjoint action of $\mathfrak{g}_-$ on $\mathscr{E}^{\pm}$.

Moreover, because of the Jacobi identity we get

$$\mathrm{ad}_{X_-}^{\tilde{\mathfrak{g}}_{\mathscr{B}}}\left[X^{\pm},Y^{\pm}\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}=\left[\mathrm{ad}_{X_-}^{\tilde{\mathfrak{g}}_{\mathscr{B}}}X^{\pm},Y^{\pm}\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}+\left[X^{\pm},\mathrm{ad}_{X_-}^{\tilde{\mathfrak{g}}_{\mathscr{B}}}Y^{\pm}\right]_{\tilde{\mathfrak{g}}_{\mathscr{B}}}$$

therefore $\mathscr{E}^{\pm}$ are $\mathfrak{g}_-$-Lie algebras.■

This action $\mathrm{ad}^{\tilde{\mathfrak{g}}_{\mathscr{B}}}:\mathfrak{g}_-\times\mathscr{E}^{\pm}\longmapsto\mathscr{E}^{\pm}$ can be promoted to an action of the connected simply connected Lie group $G_-\subset\tilde{G}_{\mathscr{B}}$ associated with the Lie algebra $\mathfrak{g}_-$ $\mathrm{Ad}^{\tilde{G}_{\mathscr{B}}}:G_-\times\mathscr{E}^{\pm}\longrightarrow\mathscr{E}^{\pm}$ $/(g_-,Y^{\pm})\longmapsto\mathrm{Ad}_{g_-}^{\tilde{G}_{\mathscr{B}}}Y^{\pm}$ such that

$$\mathrm{Ad}_{g_-}^{\tilde{G}_{\mathscr{B}}}\left(I+\mathscr{B}\pm\mathscr{G}\right)Y_+=\left(I+\mathscr{B}\pm\mathscr{G}\right)\Pi_{\mathfrak{g}_+}\mathrm{Ad}_{g_-}^{G}Y_+$$

meaning that the orbit of $G_-\subset\tilde{G}_{\mathscr{B}}$ through $X^{\pm}\in\mathscr{E}^{\pm}$ is the image of the dressing orbit of $G_-\subset G$ through $X_+\in\mathfrak{g}_+$ by the isomorphism $(I+\mathscr{B}\pm\mathscr{G}):\mathfrak{g}_+\longrightarrow\mathscr{E}^{\pm}$.

### F. $\mathscr{B}$ and $\mathscr{G}$ from factorizable quasitriangular $r$-matrices

Let $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ be a Manin triple whose bilinear form is $(,)_{\mathfrak{g}}$, and let $r\in\mathfrak{g}_-\otimes\mathfrak{g}_-$ be a *factorizable quasitriangular* solution of the modified classical Yang-Baxter equation, with $r^+$and $r^-$ the symmetric and skew-symmetric components of $r$. Define the linear operators $\hat{r}^-,\hat{r}^+:\mathfrak{g}_-^*\longrightarrow\mathfrak{g}_-$ as $\hat{r}^{\pm}(\xi)=\langle\xi\otimes I,r^{\pm}\rangle$, where $\hat{r}^+$ is invertible, then the $\mathfrak{g}_-$-invariance of $r^+$ becomes in

$$\hat{r}^+\circ ad_{X_-}^*+ad_{X_-}\circ\hat{r}^+=0.$$

Here $ad_{X_-}$ denotes the adjoint action of $\mathfrak{g}_-$ on itself, and in the whole work, $ad^*$ just means the transpose operator, so the coadjoint action is the $-ad^*$.

By using the linear bijection $\psi:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-^*$ induced by the bilinear form $(,)_{\mathfrak{g}}$, we introduce the linear operators

$$\mathscr{B}=\hat{r}^-\circ\psi:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-\quad,\quad\mathscr{G}=\hat{r}^+\circ\psi:\mathfrak{g}_+\longrightarrow\mathfrak{g}_-$$

then the $\mathfrak{g}_-$-invariance turns in

$$\mathscr{G}\left(\Pi_{\mathfrak{g}_+}\left[X_-,Y_+\right]\right)=\left[X_-,\mathscr{G}Y_+\right].\tag{19}$$

The ad-invariance of the bilinear form means that $\psi^{-1}\circ\mathrm{ad}_{X_+}^{\mathfrak{g}^*}+\mathrm{ad}_{X_+}^{\mathfrak{g}}\circ\psi^{-1}=0$, therefore the action $\sigma:\mathfrak{g}_-\longrightarrow\mathrm{End}(\mathfrak{g}_+)$ introduced in eq. (10) can be written as

$$\sigma_{X_-}Y_+=\Pi_{\mathfrak{g}_+}\left[X_-,Y_+\right]=-\psi^{-1}\left(ad_{X_-}^*\psi\left(Y_+\right)\right)$$

then

$$\sigma_{\mathscr{B}X_+}Y_+ = -\psi^{-1}\left(ad^*_{\mathscr{B}X_+}\psi(Y_+)\right)$$

or, equivalently,

$$\hat{r}^- ad^*_{\mathscr{B}X_+}\psi(Y_+) = -\mathscr{B}\Pi_{\mathfrak{g}_+}[\mathscr{B}X_+, Y_+].$$

This allows to write the classical Yang-Baxter equation

$$[r_{13}, r_{23}] + [r_{12}, r_{13}] + [r_{12}, r_{23}] = 0$$

as[20]

$$\begin{aligned}&[\mathscr{B}X_+, \mathscr{B}Y_+] - \mathscr{B}(\sigma(\mathscr{B}X_+, Y_+) - \sigma(\mathscr{B}Y_+, X_+))\\ &= -[\mathscr{G}X_+, \mathscr{G}Y_+]\end{aligned}$$

so $\mathscr{B}$ is an $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $-1$.

With the help of (19) and writing $X_+ = \mathscr{G}^{-1}X_-$ and $Y_+ = \mathscr{G}^{-1}Y_-$, it turns in

$$\begin{aligned}&[\mathscr{B}\mathscr{G}^{-1}X_-, \mathscr{B}\mathscr{G}^{-1}Y_-]\\ &-\mathscr{B}\mathscr{G}^{-1}\left[\mathscr{B}\mathscr{G}^{-1}X_-, Y_-\right] - \mathscr{B}\mathscr{G}^{-1}\left[X_-, \mathscr{B}\mathscr{G}^{-1}Y_-\right]\\ &= -[X_-, Y_-]\end{aligned}$$

showing that $\mathscr{B}\mathscr{G}^{-1}$ is a solution of the modified classical Yang-Baxter equation in the form of ref.[19].

### G. Orthogonal splitting of a factorizable quasitriangular Lie bialgebra

Here we set aside the original Manin triple and apply the latter results on a factorizable quasitriangular Lie bialgebra $\mathfrak{k}$, providing a rather simple nontrivial example of the construction of this section. Thus, the Lie bialgebra is defined by a quasitriangular factorizable $r$-matrix which also plays the role of generalized metric and provides the orthogonal splitting of the underlying vector space. In fact, if $\mathfrak{k}$ is of coboundary type with quasitriangular factorizable $r$-matrix defining the Lie bracket on $\mathfrak{k}^*$ through the cobracket in $\mathfrak{k}$, $\delta_r : \mathfrak{k} \longrightarrow \mathfrak{k}\otimes\mathfrak{k}$, as

$$\langle [\xi, \lambda]_r, X\rangle = \langle \xi\otimes\lambda, \delta_r X\rangle$$

then

$$[\xi, \lambda]_r = ad^*_{\hat{r}^-(\lambda)}\xi - ad^*_{\hat{r}^-(\xi)}\lambda,$$

and the classical double of $\mathfrak{k}$, namely $\mathfrak{k}\oplus\mathfrak{k}^*$, is defined by the Lie bracket

$$\begin{aligned}&[X+\xi, Y+\lambda]\\ &= [X,Y] - ad^{r*}_{\xi}Y + ad^{r*}_{\lambda}X + [\xi,\lambda]_r - ad^*_X\lambda + ad^*_Y\xi\end{aligned}$$

As twilled extension, it corresponds to $\sigma_X\lambda = -ad^*_X\lambda$ and $\rho_\xi Y = -ad^{r*}_{\xi}Y$, where $ad^{r*}$ means the transpose of the adjoint action by the Lie structure $[,]_r$ on $\mathfrak{k}^*$. Also we have that

$$ad^{r*}_{\lambda}X = \hat{r}^-(ad^*_X\lambda) + ad_X\hat{r}^-(\lambda)$$

that has the form of the 1-cocycle introduced en (16), so it is an exact twilled extension. Then the Lie bracket in $\mathfrak{k}\oplus\mathfrak{k}^*$ turns in

$$\begin{aligned}&[X+\xi, Y+\lambda]\\ &= [X,Y] + \left[\hat{r}^-\xi, Y\right] - \hat{r}^- ad^*_Y\xi - \left[\hat{r}^-\lambda, X\right] + \hat{r}^- ad^*_X\lambda\\ &\quad + [\xi,\lambda]_r - ad^*_X\lambda + ad^*_Y\xi.\end{aligned} \tag{20}$$

Note that together with the natural bilinear form

$$\langle (X,\xi), (Y,\lambda)\rangle = \langle \xi, Y\rangle + \langle X, \lambda\rangle,$$

$(\mathfrak{k}\oplus\mathfrak{k}^*, \mathfrak{k}, \mathfrak{k}^*)$ is a Manin triple.

On the other side, the twilled extension of $\mathfrak{k}$ and $(\mathfrak{k}^*)^{op}$ is defined by the Lie bracket

$$\begin{aligned}&[X+\xi, Y+\lambda]'\\ &= [X,Y] - \left[\hat{r}^-\xi, Y\right] + \hat{r}^- ad^*_Y\xi + \left[\hat{r}^-\lambda, X\right] - \hat{r}^- ad^*_X\lambda\\ &- [\xi,\lambda]_r - ad^*_X\lambda + ad^*_Y\xi\end{aligned}$$

The orthogonal subspaces $\mathscr{E}^+, \mathscr{E}^-$ of decomposition (8) are defined as

$$\mathscr{E}^\pm = \left\{X + \left(\hat{r}^- \pm \hat{r}^+\right)\xi / \xi \in \mathfrak{k}^*\right\}$$

and the crossed Lie bracket is

$$\begin{aligned}&\left[\left(\xi, \left(\hat{r}^- \pm \hat{r}^+\right)\xi\right), \left(\lambda, \left(\hat{r}^- \pm \hat{r}^+\right)\lambda\right)\right]'\\ &= \pm 2\left(ad^*_{\hat{r}^+\lambda}\xi, \left(\hat{r}^- \pm \hat{r}^+\right)ad^*_{\hat{r}^+\lambda}\xi\right).\end{aligned}$$

while

$$\left[\left(\xi, \left(\hat{r}^- + \hat{r}^+\right)\xi\right), \left(\lambda, \left(\hat{r}^- - \hat{r}^+\right)\lambda\right)\right] = 0.$$

Next we apply these results to a concrete example.

#### 1. Example: $\mathfrak{sl}_2$

Let us consider the Lie algebra $\mathfrak{sl}_2$ spanned by the basis $\{H, X_+, X_-\}$ with the Lie brackets (Example 8.1.10 in[22])

$$[H, X_+] = 2X_+ \quad , \quad [H, X_-] = -2X_- \quad , \quad [X_+, X_-] = H\ .$$

Here we have the quasitriangular factorizable solution of the classical Yang-Baxter equation $r \in \mathfrak{sl}_2\otimes\mathfrak{sl}_2$

$$r = X_+\otimes X_- + \frac{1}{4}H\otimes H \tag{21}$$

whose symmetric and skew-symmetric parts are

$$r_+ = \frac{1}{2}X_+\otimes X_- + \frac{1}{2}X_-\otimes X_+ + \frac{1}{4}H\otimes H$$

$$r_- = \frac{1}{2}X_+\otimes X_- - \frac{1}{2}X_-\otimes X_+$$

The Lie cobracket $\delta_r : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2\otimes\mathfrak{sl}_2$ is $\delta Z = ad_Z r = ad_Z r_-$, giving

$$\delta H = 0$$

$$\delta X_+ = \tfrac{1}{2}(X_+\otimes H - H\otimes X_+)\ .$$

$$\delta X_- = \tfrac{1}{2}(X_-\otimes H - H\otimes X_-)$$

Let $\mathfrak{sl}_2^*$ be the dual vector space of $\mathfrak{sl}_2$, then it turns into a Lie algebra with the Lie bracket defined as

$$\langle [\eta,\xi]_r, Z\rangle = \langle \eta\otimes\xi, \delta_r Z\rangle$$

therefore, if $\{h,x_+,x_-\}\subset\mathfrak{sl}_2^*$ is the dual basis, we have that

$$[h,x_+]_r = -\tfrac{1}{2}x_+ \quad , \quad [h,x_-]_r = -\tfrac{1}{2}x_- \quad , \quad [x_+,x_-]_r = 0 \; . \tag{22}$$

This Lie algebra $\mathfrak{sl}_2^*$ is also a bialgebra with the cobracket $\delta : \mathfrak{sl}_2^* \longrightarrow \mathfrak{sl}_2^*\otimes\mathfrak{sl}_2^*$ giving rise to the Lie algebra in $\mathfrak{sl}_2$, then

$$\delta h = x_+\otimes x_- - x_-\otimes x_+$$

$$\delta x_+ = 2\left(h\otimes x_+ - x_+\otimes h\right)$$

$$\delta x_- = 2\left(x_-\otimes h - h\otimes x_-\right)$$

A nondegenerate symmetric bilinear form on $\mathfrak{sl}_2\oplus\mathfrak{sl}_2^*$, making the summands maximally isotropic subalgebras, is defined by the pairing

$$\langle H,h\rangle = 1 \quad , \quad \langle X_+,x_+\rangle = 1 \quad , \quad \langle X_-,x_-\rangle = 1$$

with all the other bracket vanishing.

The twilled extension giving rise to the classical double Lie algebra on $\mathfrak{sl}_2\oplus\mathfrak{sl}_2^*$ is defined by the Lie bracket (20), which we write in terms of the cobracket $\delta_r$ for ease of calculation

$$\begin{aligned}[(X,\xi),(Y,\lambda)] = [X,Y] - (\xi\otimes I)\,\delta Y + (\lambda\otimes I)\,\delta X \\ + [\xi,\lambda]_r - (X\otimes I)\,\delta\lambda + (Y\otimes I)\,\delta\xi\end{aligned}$$

From here we get the non-vanishing crossed Lie brackets

$$\begin{aligned}
&[X_+,h] = -\tfrac{1}{2}X_+ - x_-, && [X_-,h] = -\tfrac{1}{2}X_- + x_+\\
&[H,x_+] = -2x_+, && [X_+,x_+] = \tfrac{1}{2}H + 2h\\
&[H,x_-] = 2x_-, && [X_-,x_-] = \tfrac{1}{2}H - 2h
\end{aligned}$$

Let us now to build the generalized metric $\mathscr{G}\pm\mathscr{B} : \mathfrak{sl}_2^* \longrightarrow \mathfrak{sl}_2$ on $\mathfrak{sl}_2^*$ with

$$\mathscr{G}\xi = \langle \xi\otimes Id, r_+\rangle \quad \text{and} \quad \mathscr{B}\xi = \langle \xi\otimes Id, r_-\rangle$$

Thus we get

$$\mathscr{G}h = \tfrac{1}{4}H \quad , \quad \mathscr{G}x_+ = \tfrac{1}{2}X_- \quad , \quad \mathscr{G}x_- = \tfrac{1}{2}X_+ \; .$$

The skew-symmetric part $\mathscr{B}$ is

$$\mathscr{B}h = 0 \quad , \quad \mathscr{B}x_+ = \tfrac{1}{2}X_- \quad , \quad \mathscr{B}x_- = -\tfrac{1}{2}X_+ \; .$$

From these results we obtain $(\mathscr{B}\pm\mathscr{G})$, which has nontrivial unidimensional kernel. Of course, it verifies that

$$(\mathscr{B}\pm\mathscr{G})\,[x,y]_{\mathscr{B}} = [(\mathscr{B}\pm\mathscr{G})\,x, (\mathscr{B}\pm\mathscr{G})\,y] \, .$$

or, equivalently, one may check that $\mathscr{B}$ is an $\mathscr{O}$-operator of extension $\mathscr{G}$ with mass $-1$

$$[\mathscr{B}x,\mathscr{B}y] - \mathscr{B}\Pi_{\mathfrak{sl}_2^*}[\mathscr{B}x,y] - \mathscr{B}\Pi_{\mathfrak{sl}_2^*}[x,\mathscr{B}y] = -[\mathscr{G}x,\mathscr{G}y].$$

The Lie algebra structure defined by $\mathscr{B}$, namely $((\mathfrak{sl}_2^*)_{\mathscr{B}}, [,]_{\mathscr{B}})$, with

$$[x,y]_{\mathscr{B}} = \Pi_{\mathfrak{sl}_2^*}[\mathscr{B}x,y] - \Pi_{\mathfrak{sl}_2^*}[\mathscr{B}y,x]$$

gives rise to the Lie brackets (22).

The second double Lie algebra constructed as the *twilled extension* of $\mathfrak{sl}_2$ and $(\mathfrak{sl}_2^*)^{op}$, namely $(\mathfrak{sl}_2\oplus(\mathfrak{sl}_2^*)^{op})_{\mathscr{B}}$, has the Lie bracket

$$\begin{aligned}
&[(x,X),(y,Y)]'\\
&= -[x,y]_{\mathscr{B}} + \Pi_{\mathfrak{sl}_2^*}[X,y] - \Pi_{\mathfrak{sl}_2^*}[Y,x]\\
&\quad + [X,Y] + \mathscr{B}\Pi_{\mathfrak{sl}_2^*}[X,y] + [\mathscr{B}y,X] - \mathscr{B}\Pi_{\mathfrak{sl}_2^*}[Y,x] - [\mathscr{B}x,Y]
\end{aligned}$$

that gives rise to the non-vanishing crossed Lie brackets

$$\begin{aligned}
&[X_+,h]' = \tfrac{1}{2}X_+ - x_-, && [X_-,h]' = x_+ + \tfrac{1}{2}X_-\\
&[H,x_+]' = -2x_+, && [X_+,x_+]' = 2h - \tfrac{1}{2}H \quad .\\
&[H,x_-]' = 2x_-, && [X_-,x_-]' = -2h - \tfrac{1}{2}H
\end{aligned}$$

The orthogonal subspaces $\mathscr{E}^+$and $\mathscr{E}^-$ are spanned by the graph of $(\mathscr{B}\pm\mathscr{G})$ on the basis $\{h,x_+,x_-\}\subset\mathfrak{sl}_2^*$, so that

$$\mathscr{E}^+ = \overline{\left\{h + \frac{1}{4}H, x_+ + X_-, x_-\right\}}$$

$$\mathscr{E}^- = \overline{\left\{h - \frac{1}{4}H, x_+, x_- - X_+\right\}}$$

where the overbar is meant to indicate the linear span by each set of vectors. The fundamental Lie brackets in each subspace are

$$\left[h + \frac{1}{4}H, x_+ + X_-\right]' = -(x_+ + X_-)$$

$$\left[h + \frac{1}{4}H, x_-\right]' = x_-$$

$$[x_+ + X_-, x_-]' = -2\left(h + \frac{1}{4}H\right)$$

and

$$\left[h - \frac{1}{4}H, x_+\right]' = x_+$$

$$\left[h - \frac{1}{4}H, x_- - X_+\right]' = -(x_- - X_+)$$

$$[x_+, x_- - X_+]' = 2\left(h - \frac{1}{4}H\right)$$

showing that $\mathscr{E}^+$ and $\mathscr{E}^-$ are Lie subalgebras of $(\mathfrak{sl}_2^*)^{op}\oplus\mathfrak{sl}_2$, isomorphic to $\mathfrak{sl}_2$ and $(\mathfrak{sl}_2)^{op}$, respectively. Of course, the crossed brackets between $\mathscr{E}^+$ and $\mathscr{E}^-$ vanish, confirming that they are ideals.

In the ordered basis $\{h,x_+,x_-;H,X_+,X_-\}\subset\mathfrak{sl}_2^*\oplus\mathfrak{sl}_2$, the linear operators

$$\mathscr{E}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ \mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix},$$

$$\mathscr{J}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ \mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix}$$

have associated the $6\times 6$ matrices

$$\mathscr{E}=\begin{pmatrix}0&0&0&4&0&0\\0&1&0&0&0&2\\0&0&-1&0&2&0\\1/4&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\end{pmatrix},$$

$$\mathscr{J}=\begin{pmatrix}0&0&0&4&0&0\\0&1&0&0&0&2\\0&0&-1&0&2&0\\-1/4&0&0&0&0&0\\0&0&-1&0&1&0\\0&-1&0&0&0&-1\end{pmatrix}.$$

## V. FROM LIE ALGEBRAS ORTHOGONAL DIRECT SUM TO MANIN TRIPLES

In this section we will reverse the path followed in the previous section: departing from a Lie algebra direct sum with a split bilinear form making both the Lie subalgebras orthogonal, we construct a Manin triple containing them as Lie ideals.

### A. Lie algebras direct sum and Manin quasi-triples

Let $\mathfrak{g}$ be the quadratic Lie algebra direct sum of the $n$-dimensional Lie algebras $\mathrm{E}^+$ and $\mathrm{E}^-$, with a split bilinear form $(,)_{\mathfrak{g}}$ rendering $\mathrm{E}^+$ and $\mathrm{E}^-$ mutually orthogonal. Note that Lie bracket of $\mathfrak{g}$ is

$$\left[X^++X^-,Y^++Y^-\right]_{\mathfrak{g}}=\left[X^+,Y^+\right]_{\mathrm{E}^+}+\left[X^-,Y^-\right]_{\mathrm{E}^-}$$

implying that $\mathrm{E}^+,\mathrm{E}^-$ are Lie ideals in $\mathfrak{g}$.

We also assume that an antiisomorphism $\varphi:\mathrm{E}^+\longrightarrow\mathrm{E}^-$ such that $\varphi^\top=-\varphi^{-1}$ is given. As we saw in section II, there exist a couple of symmetric linear operators $\mathscr{E},\mathscr{J}:\mathfrak{g}\longrightarrow\mathfrak{g}$ with

$$\mathscr{E}^2=\mathscr{I}\quad,\quad\mathscr{J}^2=-\mathscr{I}\quad,\quad\mathscr{E}\mathscr{J}+\mathscr{J}\mathscr{E}=0$$

such that $\mathrm{E}^+,\mathrm{E}^-$ are the $n$-dimensional orthogonal eigenspaces of the product structure $\mathscr{E}$. In the direct sum decomposition $\mathfrak{g}=\mathrm{E}^+\oplus\mathrm{E}^-$, the block-matrices representing $\mathscr{E}$ and $\mathscr{J}$ are that given in eq. (2).

The eigenspaces of $\mathscr{F}=\mathscr{J}\mathscr{E}$, namely $\mathrm{F}_+$ and $\mathrm{F}_-$ with

$$\mathrm{F}_\pm=\left\{X^+\pm\varphi\left(X^+\right)/X^+\in\mathrm{E}^+\right\},$$

give rise to the Lagrangian splitting $\mathfrak{g}=\mathrm{F}_+\oplus\mathrm{F}_-$, with the Lie algebra brackets

$$\left[X^++\varphi\left(X^+\right),Y^+-\varphi\left(Y^+\right)\right]=\left[X^+,Y^+\right]+\varphi\left(\left[X^+,Y^+\right]\right)\tag{23}$$

$$\left[X^+\pm\varphi\left(X^+\right),Y^+\pm\varphi\left(Y^+\right)\right]=\left[X^+,Y^+\right]-\varphi\left(\left[X^+,Y^+\right]\right)$$

turning $(\mathfrak{g},\mathrm{F}_+,\mathrm{F}_-)$ in a *Manin quasi-triple* [23], where $\mathrm{F}_-$ is a Lie subalgebra while $\mathrm{F}_+$ is a $\mathrm{F}_-$-module,

$$[\mathrm{F}_-,\mathrm{F}_-]\subset\mathrm{F}_-\quad,\quad[\mathrm{F}_+,\mathrm{F}_-]\subset\mathrm{F}_+\quad,\quad[\mathrm{F}_+,\mathrm{F}_+]\subset\mathrm{F}_-\ .$$

In the case where $\mathfrak{g}$ is a semisimple Lie algebra, the pair $(\mathrm{F}_-,\mathrm{F}_+)$ is a Cartan decomposition of $\mathfrak{g}$.

### B. Generalized metrics and $\mathscr{O}$-operators

Consider now the linear bijection $\mathscr{G}:\mathrm{F}_+\longrightarrow\mathrm{F}_-$ defined in (3), and we denote $X_+=X^++\varphi\left(X^+\right)\in\mathrm{F}_+$. The original orthogonal subspaces can be expressed as

$$\mathrm{E}^\pm=\left\{X_+\pm\mathscr{G}\left(X_+\right)/X_+=X^++\varphi\left(X^+\right)\in\mathrm{F}_+\right\}.$$

Therefore, in the Lagrangian splitting $\mathfrak{g}=\mathrm{F}_+\oplus\mathrm{F}_-$, the operators $\mathscr{E},\mathscr{J},\mathscr{F}$ are represented by the block matrices

$$\mathscr{E}=\begin{pmatrix}0&\mathscr{G}^{-1}\\ \mathscr{G}&0\end{pmatrix}\quad,\quad\mathscr{J}=\begin{pmatrix}0&-\mathscr{G}^{-1}\\ \mathscr{G}&0\end{pmatrix}.$$

$$\mathscr{F}=\begin{pmatrix}I&0\\0&-I\end{pmatrix}$$

In section II, we saw that a wide family of orthogonal subspaces is obtained by applying the gauge transformation (4). They are parametrized by $\mathscr{B}\in\mathrm{Skew}\left(\mathrm{F}_+,\mathrm{F}_-\right)$, namely

$$\mathrm{E}^\pm_{\mathscr{B}}=\mathrm{graph}\left(\mathscr{B}\pm\mathscr{G}\right)=\left\{X_++\left(\mathscr{B}\pm\mathscr{G}\right)X_+/X_+\in\mathrm{F}_+\right\},$$

such that $\mathfrak{g}=\mathrm{E}^+_{\mathscr{B}}\oplus\mathrm{E}^-_{\mathscr{B}}$. There is an complex product structure $(\mathscr{E}_{\mathscr{B}},\mathscr{J}_{\mathscr{B}})$ associated with these orthogonal subspaces that, in the direct sum decomposition $\mathfrak{g}=\mathrm{F}_+\oplus\mathrm{F}_-$, are

$$\mathscr{E}_{\mathscr{B}}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ \mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix},$$

$$\mathscr{J}_{\mathscr{B}}=\begin{pmatrix}-\mathscr{G}^{-1}\mathscr{B} & \mathscr{G}^{-1}\\ -\mathscr{G}-\mathscr{B}\mathscr{G}^{-1}\mathscr{B} & \mathscr{B}\mathscr{G}^{-1}\end{pmatrix}.$$

Following analogous steps as in subsection IV C, we intend to introduce a new Lie algebra structure in such a way that $\mathrm{E}^+_{\mathscr{B}}$ and $\mathrm{E}^-_{\mathscr{B}}$ become Lie subalgebras. This is attained by asking for $\mathscr{B}$ to be an $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $-1$ and considering the vector subspace $\mathrm{F}_+$ as representation space of $\mathrm{F}_-$ under the bilinear map $\sigma:\mathrm{F}_-\longrightarrow\mathrm{End}\left(\mathrm{F}_+\right)$ defined as

$$\sigma_{X_-}Y_+=\left[X_-,Y_+\right].\tag{24}$$

**Proposition:** $\mathscr{G}:\mathrm{F}_+\longrightarrow\mathrm{F}_-$ *is* $\mathrm{F}_-$*-invariant and antisymmetric, namely*

$$\mathscr{G}\left(\sigma_{X_-}Y_+\right)=[X_-,\mathscr{G}Y_+]\quad,\quad \sigma_{\mathscr{G}X_+}Y_++\sigma_{\mathscr{G}Y_+}X_+=0\,.$$

**Proof**: Recall that $\mathscr{G}:\mathrm{F}_+\longrightarrow\mathrm{F}_-$ is defined as in (3), then using the Lie brackets (23), it is immediate to see that

$$\mathscr{G}\left(\sigma_{X_-}Y_+\right)=[X_-,\mathscr{G}Y_+]\,.$$

For the antisymmetry consider

$$\mathscr{G}\left(\sigma_{\mathscr{G}X_+}Y_++\sigma_{\mathscr{G}Y_+}X_+\right)=\mathscr{G}\left([\mathscr{G}X_+,Y_+]+[\mathscr{G}Y_+,X_+]\right)$$

and, by the previous result, the rhs vanish.■

Next we seek a linear operator $\mathscr{B}:\mathrm{F}_+\longrightarrow\mathrm{F}_-$ fulfilling the equation.

$$[\mathscr{B}X_+,\mathscr{B}Y_+]-\mathscr{B}\left(\sigma_{\mathscr{B}X_+}Y_+-\sigma_{\mathscr{B}Y_+}X_+\right)=-[\mathscr{G}X_+,\mathscr{G}Y_+]\,,$$

so $\mathscr{B}$ is an $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $-1$. In turn, this implies that

1. *The bracket*

$$[X_+,Y_+]_{\mathscr{B}}=[\mathscr{B}X_+,Y_+]-[\mathscr{B}Y_+,X_+]$$

*defines a Lie algebra on* $\mathrm{F}_+$,

2. $(\mathscr{B}\pm\mathscr{G}):\mathrm{F}_+\longrightarrow\mathrm{F}_-$ *is a Lie algebra homomorphism from* $(\mathrm{F}_+,[,]_{\mathscr{B}})$ *to* $(\mathrm{F}_-,[,])$*, namely*

$$(\mathscr{B}\pm\mathscr{G})[X_+,Y_+]_{\mathscr{B}}=[(\mathscr{B}\pm\mathscr{G})X_+,(\mathscr{B}\pm\mathscr{G})Y_+]$$

$\forall X_+,Y_+\in\mathrm{F}_+$.

Let us denote the Lie algebra $(\mathrm{F}_+,[,]_{\mathscr{B}})$ as $\mathrm{F}_+^{\mathscr{B}}$. There is a one-to-one correspondence between solutions to the $\mathscr{O}$-operator condition (14) on $\mathrm{F}_-$ and solutions of the modified classical Yang-Baxter equation on $\mathrm{E}^+$. Recalling that $\mathrm{F}_\pm=\mathrm{graph}(\pm\varphi)$ we may think of $\mathscr{B}:\mathrm{F}_+\to\mathrm{F}_-$ as realized by some linear operator $\theta:\mathrm{E}^+\longrightarrow\mathrm{E}^+$ in such a way that

$$\mathscr{B}\left(X^++\varphi\left(X^+\right)\right)=\theta X^+-\varphi\left(\theta X^+\right).$$

**Theorem:** $\mathscr{B}:\mathrm{F}_+\to\mathrm{F}_-$ *is an extended* $\mathscr{O}$*-operator with extension* $\mathscr{G}$ *of mass* $\kappa=-1$ *if and only if* $\theta:\mathrm{E}^+\longrightarrow\mathrm{E}^+$ *is a skew-symmetric solution to the modified classical Yang-Baxter equation.*

**Proof:** From the eq. (14) written in terms of $X_+=X^++\varphi(X^+)$ and $Y_+=Y^++\varphi(Y^+)$ and having in mind the explicit form of the left action $\sigma_{\mathscr{B}X_+}Y_+$,

$$\sigma_{(\theta X^+-\varphi(\theta X^+))}\left(Y^++\varphi\left(Y^+\right)\right)=\left[\theta X^+,Y^+\right]+\varphi\left[\theta X^+,Y^+\right]$$

and the same for $\sigma_{\mathscr{B}Y_+}X_+$, one gets

$$\begin{aligned}&\left[\theta X^+,\theta Y^+\right]-\varphi\left(\left[\theta X^+,\theta Y^+\right]\right)\\&-\mathscr{B}\left(\left(\left[\theta X^+,Y^+\right],\varphi\left[\theta X^+,Y^+\right]\right)\right)\\&-\mathscr{B}\left(\left(\left[\theta Y^+,X^+\right],\varphi\left[\theta Y^+,X^+\right]\right)\right)\\&=-\left[\left(X^+-\varphi\left(X^+\right)\right),\left(Y^+-\varphi\left(Y^+\right)\right)\right]\end{aligned}$$

Applying again the action of $\mathscr{B}$ in terms of $\theta$, it turns in

$$\begin{aligned}&\left[\theta X^+,\theta Y^+\right]-\theta\left(\left[\theta X^+,Y^+\right]-\left[\theta Y^+,X^+\right]\right)+\left[X^+,Y^+\right]=\\&\varphi\left(\left[\theta X^+,\theta Y^+\right]-\theta\left(\left[\theta X^+,Y^+\right]-\left[\theta Y^+,X^+\right]\right)+\left[X^+,Y^+\right]\right)\end{aligned}$$

which holds if and only if

$$\left[\theta X^+,\theta Y^+\right]-\theta\left[\theta X^+,Y^+\right]-\theta\left[X^+,\theta Y^+\right]=-\left[X^+,Y^+\right]$$

so $\theta:\mathrm{E}^+\longrightarrow\mathrm{E}^+$ is a solution of the modified classical Yang-Baxter equation on $\mathrm{E}^+$.

On the other side, because $\mathscr{B}$ is skew-symmetric relative to the bilinear form $(,)_{\mathfrak{g}}$, $\theta$ must be also skew-symmetric.■

### C. Twilled extension of $\mathrm{F}_+^{\mathscr{B}},\mathrm{F}_-$

So far, we have built a Lagrangian decomposition $\mathfrak{g}=\mathrm{F}_+\oplus\mathrm{F}_-$ of a Lie algebra direct sum $\mathfrak{g}=\mathrm{E}^+\oplus\mathrm{E}^-$, where $\mathrm{F}_-$ is a Lie subalgebra. Then, by introducing the generalized metric $\mathscr{G}\pm\mathscr{B}$ and regarding it as an $\mathscr{O}$-operator, we obtained a Lie algebra structure on $\mathrm{F}_+$, denoted as $\mathrm{F}_+^{\mathscr{B}}$, which is homomorphic to $\mathrm{F}_-$. Next, by constructing a *twilled extension* out of the Lie algebras $\left(\mathrm{F}_+^{\mathscr{B}}\right)^{op},\mathrm{F}_-$[9], we shall get a new Lie algebra structure on $\mathfrak{g}$ having $\left(\mathrm{F}_+^{\mathscr{B}}\right)^{op}$ and $\mathrm{F}_-$ as Lie subalgebras.

Together with map $\sigma:\mathrm{F}_-\longrightarrow\mathrm{End}(\mathrm{F}_+)$ given in eq. (24) and following the results of subsection IV D, in particular eq. (16), we introduced the map $\rho:\left(\mathrm{F}_+^{\mathscr{B}}\right)^{op}\longrightarrow\mathrm{End}(\mathrm{F}_-)$ defined as

$$\rho_{X_+}Y_-=\mathscr{B}[X_+,Y_-]-[\mathscr{B}X_+,Y_-]\tag{25}$$

It is a 1-cocycle making the pair on maps $\sigma$,$\rho$ to fulfill the constraints (15), so we get a twilled extension $\mathfrak{g}_{\mathscr{B}}=\left(\mathrm{F}_+^{\mathscr{B}}\right)^{op}\oplus\mathrm{F}_-$ of $\left(\mathrm{F}_+^{\mathscr{B}}\right)^{op}$ and $\mathrm{F}_-$ with Lie bracket

$$\begin{aligned}&[X_++X_-,Y_++Y_-]\\&=-[X_+,Y_+]_{\mathscr{B}}+[X_-,Y_+]-[Y_-,X_+]\\&\quad\oplus[X_-,Y_-]-[\mathscr{B}X_+,Y_-]-\mathscr{B}[Y_-,X_+]\\&\quad+[\mathscr{B}Y_+,X_-]+\mathscr{B}[X_-,Y_+]\,.\end{aligned}$$

**Proposition:** $\left(\mathfrak{g}_{\mathscr{B}},\left(\mathrm{F}_+^{\mathscr{B}}\right)^{op},\mathrm{F}_-\right)$ *equipped with the bilinear form* $(,)_{\mathfrak{g}}$ *is a Manin triple.*

**Proof:** It just remains to prove that the bilinear form $(,)_{\mathfrak{g}}$ is invariant under the adjoint action of $\mathfrak{g}_{\mathscr{B}}$. Let $(X_++X_-)$, $(Y_++Y_-)$, $(Z_++Z_-)\in\mathfrak{g}_{\mathscr{B}}$ then

$$\begin{aligned}&\left([X_++X_-,Y_++Y_-]_{\mathfrak{g}_{\mathscr{B}}},Z_++Z_-\right)_{\mathfrak{g}}\\&=(Y_+,[\mathscr{B}X_+,Z_-]-\mathscr{B}[X_+,Z_-]-[X_-,Z_-])_{\mathfrak{g}}\\&\quad+(Y_+,[X_-,\mathscr{B}Z_+]+\mathscr{B}[Z_+,X_-])_{\mathfrak{g}}\\&\quad+(Y_-,[Z_-,X_+]-[X_-,Z_+]-[\mathscr{B}Z_+,X_+]+[\mathscr{B}X_+,Z_+])_{\mathfrak{g}}\\&=-(Y_-,-[\mathscr{B}X_+,Z_+]-[X_+,\mathscr{B}Z_+,]+[X_-,Z_+]-[Z_-,X_+])_{\mathfrak{g}}\\&\quad-(Y_+,[X_-,Z_-]+\mathscr{B}[X_-,Z_+]+[\mathscr{B}Z_+,X_-])_{\mathfrak{g}}\\&\quad+(Y_+,\mathscr{B}[Z_-,X_+]+[\mathscr{B}X_+,Z_-])_{\mathfrak{g}}\\&=-\left((Y_+,Y_-),[(X_+,X_-),(Z_+,Z_-)]_{\mathfrak{g}_{\mathscr{B}}}\right)_{\mathfrak{g}}\end{aligned}$$

as expected. Thus it shows that $\left(\mathfrak{g}_{\mathscr{B}},\left(\mathrm{F}_{+}^{\mathscr{B}}\right)^{op},\mathrm{F}_{-}\right)$ with the nondegenerate invariant symmetric bilinear form $(,)_{\mathfrak{g}}$ is a Manin triple.■

This Manin triple $\left(\mathfrak{g}_{\mathscr{B}},\left(\mathrm{F}_{+}^{\mathscr{B}}\right)^{op},\mathrm{F}_{-}\right)$ also admit an orthogonal splitting in Lie ideals $\mathfrak{g}_{\mathscr{B}}=\mathrm{E}_{\mathscr{B}}^{+}\oplus\mathrm{E}_{\mathscr{B}}^{-}$, with $\mathrm{E}_{\mathscr{B}}^{\pm}=\{X_{+}+(\mathscr{B}\pm\mathscr{G})X_{+}/X_{+}\in\mathrm{F}_{+}\}$. The Lie brackets in these subalgebras are

$$
\begin{aligned}
&[(X_{+},(\mathscr{B}\pm\mathscr{G})X_{+}),(Y_{+},(\mathscr{B}\pm\mathscr{G})Y_{+})]_{\mathfrak{g}_{\mathscr{B}}}\\
&=\pm2\left(\mathscr{G}^{-1}[\mathscr{G}X_{+},\mathscr{G}Y_{+}]+(\mathscr{B}\pm\mathscr{G})\mathscr{G}^{-1}[\mathscr{G}X_{+},\mathscr{G}Y_{+}]\right)
\end{aligned}
$$

and

$$
[(X_{+},(\mathscr{B}+\mathscr{G})X_{+}),(Y_{+},(\mathscr{B}-\mathscr{G})Y_{+})]_{\mathfrak{g}_{\mathscr{B}}}=0.
$$

### D. Example: Anti-isomorphic Lie algebras

Let $\mathrm{E}^{+}$ and $\mathrm{E}^{-}$ be a couple of Lie algebras connected by an antiisomorphism $\varphi:\mathrm{E}^{+}\longrightarrow\mathrm{E}^{-}$, and $\mathrm{E}^{-}$ equipped with an invariant, symmetric nondegenerate bilinear form $(,)_{\mathrm{E}^{-}}$. Then a symmetric nondegenerate bilinear form on $\mathrm{E}^{+}$ can be defined as

$$
\left(X^{+},Y^{+}\right)_{\mathrm{E}^{+}}=-\left(\varphi\left(X^{+}\right),\varphi\left(Y^{+}\right)\right)_{\mathrm{E}^{-}}
$$

which is also $\mathrm{E}^{+}$-invariant and implies $\varphi^{\top}=-\varphi^{-1}$.

Consider the Lie algebra direct sum $\mathfrak{g}=\mathrm{E}^{+}\oplus\mathrm{E}^{-}$ equipped with the invariant non-degenerate symmetric bilinear form $(,)_{\mathfrak{g}}:\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathbb{R}$ defined as

$$
\left(\left(X^{+},X^{-}\right),\left(Y^{+},Y^{-}\right)\right)_{\mathfrak{g}}=\left(X^{+},Y^{+}\right)_{\mathrm{E}^{+}}+\left(X^{-},Y^{-}\right)_{\mathrm{E}^{-}}
$$

making $\mathrm{E}^{+}$ and $\mathrm{E}^{-}$ mutually orthogonal subspaces of $\mathfrak{g}$.

The graph of $\pm\varphi$ provides a Lagrangian splitting $\mathfrak{g}=\mathrm{F}_{+}\oplus\mathrm{F}_{-}$ where

$$
\mathrm{F}_{\pm}=\left\{\left(X^{+},\pm\varphi\left(X^{+}\right)\right)/X^{+}\in\mathrm{E}^{+}\right\}.
$$

so the invertible linear map $\mathscr{G}:\mathrm{F}_{+}\longrightarrow\mathrm{F}_{-}$ introduced in eq. (3) is defined as

$$
\mathscr{G}\left(X^{+},\varphi\left(X^{+}\right)\right)=\left(X^{+},-\varphi\left(X^{+}\right)\right)
$$

For instance, consider the Lie algebra $\mathrm{E}^{+}=\mathfrak{h}$ equipped with a symmetric, non-degenerate, ad-invariant bilinear form $(,)_{\mathfrak{h}}$, and $\mathrm{E}^{-}=\mathfrak{h}^{op}$, with the antihomomorphism $\varphi:\mathfrak{h}\longrightarrow\mathfrak{h}^{op}$ defined as $\varphi(X)=X$ for $X\in\mathfrak{h}$, then $[\varphi(X),\varphi(Y)]_{\mathfrak{h}^{op}}=-[X,Y]=-\varphi([X,Y])$. Define the Lie algebra direct sum $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^{op}$ with the Lie bracket

$$
\left[\left(X,X'\right),\left(Y,Y'\right)\right]_{\mathfrak{g}}=\left([X,Y],-\left[X',Y'\right]\right)
$$

and the bilinear form

$$
\left(\left(X,X'\right),\left(Y,Y'\right)\right)_{\mathfrak{g}}=(X,Y)_{\mathfrak{h}}-\left(X',Y'\right)_{\mathfrak{h}}
$$

which is invariant and makes $\mathfrak{h}$ and $\mathfrak{h}^{op}$ orthogonal Lie subalgebras of $\mathfrak{g}$. The subspaces

$$
\mathfrak{h}_{\pm}=\{(X,\pm X)/X\in\mathfrak{h}\}=\operatorname{graph}(\pm\varphi)
$$

are Lagrangian and $\mathfrak{g}=\mathfrak{h}_{+}\oplus\mathfrak{h}_{-}$, and the Lie brackets in $\mathfrak{h}_{\pm}$ are

$$
\begin{aligned}
\left[(X,\pm X),(Y,\pm Y)\right]_{\mathfrak{g}}&=([X,Y],-[X,Y])\in\mathfrak{h}_{-}\\
\left[(X,+X),(Y,-Y)\right]_{\mathfrak{g}}&=([X,Y],+[X,Y])\in\mathfrak{h}_{+}
\end{aligned}
$$

Now, consider the linear map $\mathscr{G}:\mathfrak{h}_{+}\longrightarrow\mathfrak{h}_{-}$ is

$$
\mathscr{G}(X,+X)=(X,-X)
$$

for $X\in\mathfrak{h}$, which is $\mathfrak{h}_{-}$-invariant:

$$
\begin{aligned}
&\mathscr{G}([(X,-X),(Y,+Y)])\\
&=([X,Y],-[X,Y])\\
&=[(X,-X),\mathscr{G}(Y,+Y)]
\end{aligned}
$$

and antisymmetric. Together with $\mathscr{G}$, we introduce a linear operator $\mathscr{B}:\mathfrak{h}_{+}\longrightarrow\mathfrak{h}_{-}$ satisfying the $\mathscr{O}$-operator condition (12), which is realized by some linear operator $\theta:\mathfrak{h}\longrightarrow\mathfrak{h}$ such that

$$
\mathscr{B}(X,+X)=(\theta X,-\theta X)
$$

provided $\theta$ is a solution of the modified classical Yang-Baxter equation on $\mathfrak{h}$. With these operator we get a new Lie algebra structure on $\mathfrak{h}_{+}$ with Lie bracket

$$
[(X,+X),(Y,+Y)]_{\mathscr{B}}=\left([X,Y]_{\theta},[X,Y]_{\theta}\right).
$$

where, for $X,Y\in\mathfrak{h}$

$$
[X,Y]_{\theta}=[\theta X,Y]-[\theta Y,X]
$$

We denote $(\mathfrak{h}_{+},[,]_{\theta})$ as $\mathfrak{h}_{+}^{\theta}$.

The next step is to construct the twilled action, so we need the map $\sigma:\mathfrak{h}_{-}\longrightarrow\operatorname{End}(\mathfrak{h}_{+})$ (24) that is defined here as

$$
\sigma_{(X,-X)}(Y,+Y)=[(X,-X),(Y,+Y)]=[X,Y]+[X,Y]\in\mathfrak{h}_{+}
$$

and the map $\rho:\mathfrak{h}_{+}\longrightarrow\operatorname{End}(\mathfrak{h}_{-})$ (25) which turns in

$$
\rho_{(X,+X)}(Y,-Y)=(\theta[X,Y]-[\theta X,Y],-\theta[X,Y]-[\theta X,Y])
$$

Thus the twilled extension $\mathfrak{g}_{\mathscr{B}}=\left(\mathfrak{h}_{+}^{\theta}\right)^{op}\oplus\mathfrak{h}_{-}$ of $\left(\mathfrak{h}_{+}^{\theta}\right)^{op}$ and $\mathfrak{h}_{-}$ is defined by the Lie bracket

$$
\begin{aligned}
&[(X,+X)+(X',-X'),(Y,+Y)+(Y',-Y')]\\
&=-[(X,+X),(Y,+Y)]_{\mathscr{B}}+[(X',-X'),(Y,+Y)]\\
&-[(Y',-Y'),(X,+X)]+[(X',-X'),(Y',-Y')]\\
&\quad-[\mathscr{B}(X,+X),(Y',-Y')]-\mathscr{B}[(Y',-Y'),(X,+X)]\\
&\quad+[\mathscr{B}(Y,+Y),(X',-X')]+\mathscr{B}[(X',-X'),(Y,+Y)].
\end{aligned}
$$

The orthogonal splitting $\mathfrak{g}_{\theta}=\mathfrak{h}_{\theta}^{+}\oplus\mathfrak{h}_{\theta}^{-}$, is defined by the graph of $(\mathscr{B}\pm\mathscr{G})$ namely

$$
\begin{aligned}
\mathfrak{h}_{\theta}^{+}&=\{(X,+X)+(\theta X+X,-(\theta X+X))/X\in\mathfrak{h}\}\\
\mathfrak{h}_{\theta}^{-}&=\{(X,+X)+(\theta X-X,-(\theta X-X))/X\in\mathfrak{h}\}.
\end{aligned}
$$

Let us introduce the map $\Theta^{\pm}:\mathfrak{h}\longrightarrow\mathfrak{h}_{\theta}^{\pm}$ such that $\Theta^{\pm}(X)=(X,+X)+(\theta X\pm X,-(\theta X+X))$, then the Lie bracket in $\mathfrak{h}_{\theta}^{\pm}$ is

$$
\left[\Theta^{\pm}(X),\Theta^{\pm}(Y)\right]=\pm2\left(\Theta^{\pm}([X,Y])\right)
$$

showing that $\mathfrak{h}_\theta^\pm$ is a Lie subalgebra and that the linear map $\frac{1}{2}\Theta^\pm : \mathfrak{h} \longrightarrow \mathfrak{h}_\theta^\pm$ is a Lie algebra isomorphism. Of course, the crossed bracket is

$$\left[\Theta^+(X), \Theta^-(Y)\right] = (0,0)$$

verifying that $\mathfrak{h}_\theta^\pm$ is an ideal.

### 1. *Example:* $\mathfrak{sl}_2(\mathbb{C})$

The above results can be applied to the example of subsection IV G 1: Define

$$\begin{aligned} a_1 &= 2\left(h - \tfrac{1}{4}H\right) \ , \ b_1 = 2\left(h + \tfrac{1}{4}H\right) \\ a_2 &= x_+ \qquad\qquad , \ b_2 = (x_+ + X_-) \\ a_3 &= (x_- - X_+) \ \ , \ b_3 = x_- \end{aligned}$$

then consider the Lie algebras $\mathfrak{h}^- = \mathfrak{sl}_2$ and $\mathfrak{h}^+ = (\mathfrak{sl}_2)^{op}$ generated by the basis $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, respectively, with the Lie brackets

$$[a_1, a_2] = 2a_2 \quad [a_1, a_3] = -2a_3 \quad [a_2, a_3] = a_1$$

and

$$[b_1, b_2] = -2b_2 \quad [b_1, b_3] = 2b_3 \quad [b_2, b_3] = -b_1 \ .$$

Let $\mathfrak{h}^-$ be equipped with a symmetric nondegenerate invariant bilinear form $(,)_{\mathfrak{h}^-}$ defined by the matrix

$$(a_i, a_j)_{\mathfrak{h}^-} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

which is symmetric, nondegenerate and invariant. We use it to equip the Lie algebra direct sum $\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ with the bilinear form

$$\left(X^+ + X^-, Y^+ + Y^-\right)_{\mathfrak{g}} = \left(X^-, Y^-\right)_{\mathfrak{h}^-} - \left(\varphi\left(X^+\right), \varphi\left(Y^+\right)\right)_{\mathfrak{h}^-}$$

making $\mathfrak{h}^+$ and $\mathfrak{h}^-$ mutually orthogonal.

The linear map $\varphi : \mathfrak{h}^+ \longrightarrow \mathfrak{h}^-$ defined as $\varphi(b_i) = a_i$ is an antiisomorphism of Lie algebras and the graph of $\pm\varphi$ is

$$\mathfrak{h}_\pm = \mathrm{graph}(\pm\varphi) = \overline{\{b_1 \pm a_1, b_2 \pm a_2, b_3 \pm a_3\}}$$

Then

$$\begin{aligned} &(b_i \pm \varphi(b_i), b_j \pm \varphi(b_j))_{\mathfrak{g}} \\ &= (\varphi(b_i), \varphi(b_j))_{\mathfrak{h}^-} - (\varphi(b_i), \varphi(b_j))_{\mathfrak{h}^-} \\ &= 0 \end{aligned}$$

so that $\mathfrak{h}_\pm$ is a Lagrangian subspace.

Let us denote $e^i_\pm = b_i \pm \varphi(b_i) = b_i \pm a_i$, then the non-vanishing Lie brackets in the Lie algebra direct sum are

$$\begin{array}{lll} \left[e^1_+, e^2_+\right] = -2e^2_- \ , & \left[e^1_+, e^3_+\right] = 2e^3_- \ , & \left[e^2_+, e^3_+\right] = -e^1_- \\ \left[e^1_-, e^2_-\right] = -2e^2_- \ , & \left[e^1_-, e^3_-\right] = 2e^3_- \ , & \left[e^2_-, e^3_-\right] = -e^1_- \\ \left[e^1_+, e^2_-\right] = -2e^2_+ \ , & \left[e^1_+, e^3_-\right] = 2e^3_+ \ , & \left[e^2_+, e^1_-\right] = 2e^2_+ \\ \left[e^2_+, e^3_-\right] = -e^1_+ \ , & \left[e^3_+, e^1_-\right] = -2e^3_+ \ , & \left[e^3_+, e^2_-\right] = e^1_+ \end{array} .$$

Next we introduce the map $\mathscr{G} : \mathfrak{h}_+ \longrightarrow \mathfrak{h}_-$ defined as

$$\mathscr{G} e^i_+ = e^i_- \quad i = 1,2,3$$

which is $\mathfrak{h}_-$-invariant:

$$\mathscr{G}\left[e^i_-, e^j_+\right] = \left[e^i_-, \mathscr{G} e^j_+\right],$$

and antisymmetric. Also, we introduce a linear operator $\mathscr{B} : \mathfrak{h}_+ \longrightarrow \mathfrak{h}_-$ satisfying the $\mathscr{O}$-operator condition (12), realized through a solution $\theta : \mathfrak{h}^+ \longrightarrow \mathfrak{h}^+$ of the modified classical Yang-Baxter equation on $\mathfrak{h}^+$

$$\mathscr{B} e^i_+ = \mathscr{B}(b_i + a_i) = \theta b_i - \varphi(\theta(b_i)) .$$

We use the quasitriangular factorizable solution of the modified classical Yang-Baxter equation $2r \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ introduced in eq. (21) and define $\theta(b_i) = -(b_i \otimes I, r^-)$ from $r_- = b_2 \otimes b_3 - b_3 \otimes b_2$ so that

$$\theta(b_1) = 0 \ , \ \theta(b_2) = b_2 \ , \ \theta(b_3) = -b_3 \ .$$

and the map $\mathscr{B}$ on the basis $\left\{e^1_+, e^2_+, e^3_+\right\} \subset \mathfrak{h}_+$ is

$$\mathscr{B} e^1_+ = 0 \ , \ \mathscr{B} e^2_+ = e^2_- \ , \ \mathscr{B} e^3_+ = -e^3_- \ .$$

From it we calculate the new Lie bracket on $\mathfrak{h}_+$, namely $[X,Y]_{\mathscr{B}} = [\mathscr{B}X, Y] - [\mathscr{B}Y, X]$, which turns out to be

$$\left[e^1_+, e^2_+\right]_{\mathscr{B}} = -2e^2_+ \ , \ \left[e^1_+, e^3_+\right]_{\mathscr{B}} = -2e^3_+ \ , \ \left[e^2_+, e^3_+\right]_{\mathscr{B}} = 0 \tag{26}$$

and we denote it as $\mathfrak{h}_+^{\mathscr{B}}$, which can be identified as an $\mathfrak{sl}_2^*$ Lie algebra.

The next step is to construct the twilled action, so we need the bilinear map $\sigma : \mathfrak{h}_- \longrightarrow \mathrm{End}(\mathfrak{h}_+)$ (24) and $\rho : \mathfrak{h}_+ \longrightarrow \mathrm{End}(\mathfrak{h}_-)$ (25), whose non-vanishing actions are

$$\begin{array}{ll} \sigma_{e^1_-} e^2_+ = -2e^2_+ \ , & \sigma_{e^1_-} e^3_+ = 2e^3_+ \\ \sigma_{e^2_-} e^1_+ = 2e^2_+ \ , & \sigma_{e^2_-} e^3_+ = -e^1_+ \\ \sigma_{e^3_-} e^1_+ = -2e^3_+ \ , & \sigma_{e^3_-} e^2_+ = e^1_+ \end{array}$$

and

$$\rho\left(e^1_+, e^2_-\right) = -2e^2_- \ , \ \rho\left(e^1_+, e^3_-\right) = -2e^3_- \ .$$

The twilled extension $\mathfrak{g}_{\mathscr{B}} = \left(\mathfrak{h}_+^{\mathscr{B}}\right)^{op} \oplus \mathfrak{h}_-$ of $\left(\mathfrak{h}_+^{\mathscr{B}}\right)^{op}$ and $\mathfrak{h}_-$ is then defined by the Lie bracket

$$\begin{aligned}&[X_+ + X_-, Y_+ + Y_-] \\ &= -[X_+, Y_+]_{\mathscr{B}} + [X_-, Y_+] - [Y_-, X_+] \\ &\quad + [X_-, Y_-] - [\mathscr{B}X_+, Y_-] - \mathscr{B}[Y_-, X_+] \\ &\quad + \mathscr{B}[X_-, Y_+] + [\mathscr{B}Y_+, X_-]\end{aligned} ,$$

giving rise to the nonvanishing crossed Lie brackets

$$\begin{aligned}&\left[e_+^1, e_-^2\right]_{\mathscr{B}} = -2e_+^2 - 2e_-^2 \quad , \quad \left[e_+^1, e_-^3\right]_{\mathscr{B}} = 2e_+^3 - 2e_-^3 \\ &\left[e_+^2, e_-^1\right]_{\mathscr{B}} = 2e_+^2 \quad , \quad \left[e_+^2, e_-^3\right]_{\mathscr{B}} = -e_+^1 + e_-^1 \\ &\left[e_+^3, e_-^1\right]_{\mathscr{B}} = -2e_+^3 \quad , \quad \left[e_+^3, e_-^2\right]_{\mathscr{B}} = e_+^1 + e_-^1\end{aligned}$$

that is completed with

$$\left[e_-^1, e_-^2\right] = -2e_-^2 \quad , \quad \left[e_-^1, e_-^3\right] = 2e_-^3 \quad , \quad \left[e_-^2, e_-^3\right] = -e_-^1$$

and (26). It coincides with the crossed brackets of the *twilled extension* of $\mathfrak{sl}_2$ and $(\mathfrak{sl}_2^*)^{op}$ namely $\left(\mathfrak{sl}_2 \oplus (\mathfrak{sl}_2^*)^{op}\right)_{\mathscr{B}}$, obtained in subsection IV G 1, after the substitutions $e_-^1 = H$, $e_-^2 = X_-$, $e_-^3 = X_+$, $e_+^1 = 4h$, $e_+^2 = 2x_+$ and $e_+^3 = 2x_-$.

The orthogonal subspaces

$$\mathrm{E}_{\mathscr{B}}^{\pm} = \overline{\left\{e_+^i + (\mathscr{B} \pm \mathscr{G})\, e_+^i / i = 1,2,3\right\}}$$

are spanned by the sets $\left\{e_+^1 + e_-^1, e_+^2 + 2e_-^2, e_+^3\right\}$ and $\left\{e_+^1 - e_-^1, e_+^2, e_+^3 - 2e_-^3\right\}$, and one may easily verify that they are Lie ideals in $\left(\mathfrak{sl}_2 \oplus (\mathfrak{sl}_2^*)^{op}\right)_{\mathscr{B}}$.

## VI. CONCLUSIONS

We have studied some algebraic aspects of quadratic vector spaces with Lagrangian and orthogonal splittings associated with complex product structures. This algebraic structure comes to complete the role of the involutive operator $\mathscr{E}$ which is well known in various settings such as string theory, Poisson Lie T-duality and generalized Kähler geometry, by associating a complex structure $\mathscr{J}$ constructed from $\mathscr{E}$. In addition, this structure can be tied to a generalized metric on one of the Lagrangian components, and this is the origin of the physical interest on it, since it arise as twisting of a genuine metric. When this metric is encoded in the operator $\mathscr{E}$, the duality idea becomes more apparent because a metric in the Lagrangian complement, related to the former, is introduced.

When dealing Lie bialgebras, we have shown that starting from a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ with a generalized metric $\mathscr{G} + \mathscr{B} : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_-$, that gives rise to an orthogonal splitting $\mathfrak{g} = \mathscr{E}^+ \oplus \mathscr{E}^-$, we can build the Manin triple $\left(\mathfrak{g}, \left(\mathfrak{g}_+^{\mathscr{B}}\right)^{op}, \mathfrak{g}_-\right)$ where $\mathscr{E}^+$ and $\mathscr{E}^-$ become anti-isomorphic Lie ideals provided $\mathscr{B}$ is an $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $-1$. This can be interpreted as $\mathfrak{g} = \mathscr{E}^+ \oplus \mathscr{E}^-$ is a decoupling of the Manin triple because there are no non trivial dressing action. Conversely, starting from a quadratic Lie algebra direct sum of a pair of orthogonal Lie algebras $\mathrm{E}^+$ and $\mathrm{E}^-$ we built a quasi-Manin triple $(\mathfrak{g}, \mathrm{F}_+, \mathrm{F}_-)$ with a metric $\mathscr{G}$ on the $\mathrm{F}_-$-module $\mathrm{F}_+$. Then, after introducing a twisting $\mathscr{B} : \mathrm{F}_+ \longrightarrow \mathrm{F}_-$, a Lie algebra structure $\mathrm{F}_+^{\mathscr{B}} = (\mathrm{F}_+, [,]_{\mathscr{B}})$ is defined on the vector space $\mathrm{F}_+$ by taking $\mathscr{B}$ as an $\mathscr{O}$-operator with extension $\mathscr{G}$ of mass $-1$ and, in turn, we get a Main triple $\left(\mathfrak{g}_{\mathscr{B}}, (\mathrm{F}_+^{\mathscr{B}})^{op}, \mathrm{F}_-\right)$ where the orhogonal subspaces $\mathrm{E}^+$ and $\mathrm{E}^-$ are Lie ideals.

In both cases, the $\mathscr{O}$-operator is a masked version of a quasitriangular factorizable solution of the classical Yang-Baxter equation so the key behind these constructions is the fact that if $\mathfrak{k}$ is a quasitriangular factorizable Lie bialgebra then its double $\mathfrak{k} \oplus (\mathfrak{k}^*)^{op}$ admits an orthogonal splitting in anti-isomorphic Lie ideals, as shown in subsection 4.7. However, the approach through generalized metrics and $\mathscr{O}$-operators allows making a direct contact with some subjects with geometric contents as those described at the Introduction and in the end of Section 1.

From the point of view of possible application in some Theoretical Physics problems, it is interesting to see this procedure as a way to obtain the *decoupled modes* inside a Manin triple, since there are no nontrivial dressing actions in the orthogonal Lie ideals splitting.

[1] C. Klimcik and P. Severa, "Poisson-Lie T-duality and loop groups of Drinfeld doubles," Phys. Lett. B **351**, 455–462 (1995).
[2] C. Klimcik and P. Severa, "Dual non-Abelian duality and the Drinfeld double," Phys. Lett. B **372**, 65–71 (1996).
[3] C. Klimcik, "Poisson-Lie T-duality," (1996) pp. 116–121.
[4] C. Klimcik and P. Severa, "T-Duality and the Moment Map," in *Quantum Fields and Quantum Space Time*, NATO ASI, Vol. 364 (Springer, Boston, MA, 1997) pp. 323–329.
[5] G. Aldazabal, D. Marqués, and C. Núnez, "Double field theory: a pedagogical review," Class. Quantum Gravity **30**, 163001 (2013).
[6] A. Andrada and S. Salamon, "Complex product structure on Lie algebras," Forum Math. **17**, 261–295 (2005).
[7] P. Ševera and A. Weinstein, "Poisson Geometry with a 3-Form Background," Progress of Theoretical Physics Supplement **144**, 145–154 (2001).
[8] C. Bai, L. Guo, and X. Ni, "Nonabelian Generalized Lax pairs, the classical Yang-Baxter equation and postLie algebras," Commun. Math. Phys. **297**, 553–596 (2010).
[9] I. Kosmann-Schwarzbach and F. Magri, "Poisson-Lie groups and complete integrability. i. Drinfeld bigebras, dual extensions and their canonical representations," Annales de l'I.H.P. Physique théorique **49**, 433–460 (1988).
[10] E. Meinrenken, *Clifford Algebras and Lie Theory* (Springer-Verlag, Berlin Heidelberg, 2013).
[11] P. Libermann, "Sur les structures presque paracomplexes," C. R. Acad. Sci. Paris **234**, 2517–2519 (1952).
[12] J. Scherk and J. Schwarz, "How to get masses from extra dimensions," Nuc. Phys. B **153**, 61–88 (1979).
[13] J. Maharana and J. H. Schwarz, "Noncompact symmetries in string theory," Nuclear Physics B **390**, 3–32 (1993).
[14] C. Hull and B. Zwiebach, "Double field theory," J. High Energ. Phys **2009**, 099–099 (2009).
[15] O. Hohm, C. Hull, and B. Zwiebach, "Generalized metric formulation of double field theory," J. High Energ. Phys **2010**, 099–099 (2009).
[16] M. Gualtieri, *Generalized Complex Geometry*, Ph.D. thesis, Oxford University (2003).
[17] J.-H. Lu and A. Weinstein, "Poisson Lie groups, dressing transformations and Bruhat decompositions," J. Diff. Geom. **31**, 501–526 (1990).
[18] V. G. Drinfeld, "Hamiltonian structures on Lie groups, Lie algebras and the geometric meaning of the classical Yang-Baxter equation," Soviet Math. Dokl. **27**, 68–71 (1983).
[19] M. A. Semenov-Tian-Shansky, "What is a classical R-matrix?" Funct. Anal. Appl. **17**, 259–273 (1983).
[20] M. Bordemann, "Generalized Lax pairs, the modified classical Yang-Baxter equation, and affine geometry of Lie groups," Commun. Math. Phys. **135**, 201–216 (1990).

[21]C. Bai, L. Guo, and X. Ni, “Generalizations of the classical Yang-Baxter equation and O-operators,” J. Math. Phys. **52**, 063515 (2011).
[22]S. Majid, *Foundations of Quantum Groups* (Cambridge Univerity Press, Cambridge, 1995).
[23]A. Y. Alekseev and I. Kosmann-Schwarzbach, “Manin pairs and momentum maps,” J. Diff. Geom. **56**, 133–165 (2000).