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Four point functions in the $SL(2,R)$ WZW model

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Abstract

We consider winding conserving four point functions in the $SL(2,R)$ WZW model for states in arbitrary spectral flow sectors. We compute the leading order contribution to the expansion of the amplitudes in powers of the cross ratio of the four points on the worldsheet, both in the m- and x-basis, with at least one state in the spectral flow image of the highest weight discrete representation. We also perform certain consistency check on the winding conserving three point functions.

The $SL(2,R)$ WZW model describes strings moving in AdS_3 and has important applications to gravity and black hole physics in two and three dimensions. It is also an interesting subject by itself, as a step beyond the well known rational conformal field theories. Actually, unlike string propagation in compact target spaces, where the spectrum is in general discrete and the model can then be studied using algebraic methods, the analysis of the worldsheet theory in the non-compact AdS_3 background requires the use of more intricate analytic techniques.

String theory on AdS_3 contains different sectors characterized by an integer number w, the spectral flow parameter or winding number [1]. The *short* string sectors correspond to maps from the worldsheet to a compact region in AdS_3 and the states in this sector belong to discrete representations of $SL(2,R)$ with spin $j \in \mathbf{R}$ and unitarity bound $\frac{1}{2} < j < (k-1)/2$. Other sectors contain *long* strings at infinity, near the boundary of spacetime, described by continuous representations of $SL(2,R)$ with spin $j = \frac{1}{2} + is$, $s \in \mathbf{R}$. Several correlation functions have been computed in various sectors [2,3]. In particular, four point functions of $w = 0$ states were computed in [2] analytically continuing previous results in the $SL(2,C)/SU(2)$ coset model [4] which corresponds to the Euclidean H_3 background. In this Letter we consider four point functions of states in arbitrary w sectors, a crucial ingredient to determine the consistency of the theory through factorization.

Different basis have been used in the literature to compute correlation functions in this theory. Vertex operators and expectation values for the spectral flow representations were constructed in [1,2] in the m-basis, where the generators $(J_0^3, \bar{J}_0^3)$ of the global $SL(2,C)$ isometry are diagonalized. The m-basis has the advantage that all values of w can be treated simultaneously. In particular, all winding conserving N-point functions have the same coefficient in this basis, for a given N, and they differ only in the worldsheet coordinate dependence [2,5] which reflects the change in the conformal weight of the states

$$\Delta(j) \to \Delta^w(j) = \Delta(j) - wm - \frac{k}{4}w^2, \tag{1}$$

where $\Delta(j) = -\frac{j(j-1)}{k-2}$ is the dimension of the unflowed operators.

Alternatively, the x-basis refers to the $SL(2,R)$ isospin parameter which can be interpreted as the coordinate of the boundary in the context of the AdS/CFT correspondence. The operators $\Phi_j(x,\bar{x})$ in the x-basis and $\Phi_{j;m,\bar{m}}(z,\bar{z})$ in the m-basis are related by

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the following transformation

$$\Phi_{j;m,\bar{m}}(z,\bar{z}) = \int \frac{d^2x}{|x|^2} x^{j-m} \bar{x}^{j-\bar{m}} \Phi_j(x,\bar{x};z,\bar{z}). \tag{2}$$

Finally, the μ-basis was found convenient to relate correlation functions in Liouville and $SL(2,C)/SU(2)$ models [5,6].

In this Letter we extend the results for the four point function of unflowed states in $SL(2,R)$ given in [2] to the case of winding conserving four point functions for states in arbitrary spectral flow sectors. This is accomplished by transforming the x-basis expression found in [4] to the m-basis in order to exploit the fact that the coefficient of all winding conserving correlators is the same (for a given number of external states).[1] In order to simplify the calculations we consider first the four point function in which one of the original unflowed operators is a highest weight and then analyze the more general case in which it is replaced by a global $SL(2,R)$ descendant. Finally we transform the result back to the x-basis.

Actually the explicit expression computed in [4] and further analyzed in [2] corresponds to the leading order in the expansion of the four point function in powers of the cross ratio of the worldsheet coordinates. It was pointed out in [4] that higher orders in the expansion may be determined iteratively once the lowest order is fixed as the initial condition. We will further discuss this topic for four point functions involving spectral flowed states.

Let us start recalling the result for the four point function of unflowed states originally computed for the $SL(2,C)/SU(2)$ model in [4] and later analytically continued to $SL(2,R)$ in [2], namely

$$\begin{aligned}
&\langle \Phi_{j_1}(x_1,z_1)\Phi_{j_2}(x_2,z_2)\Phi_{j_3}(x_3,z_3)\Phi_{j_4}(x_4,z_4)\rangle \\
&\quad = \int dj\, C(j_1,j_2,j)B(j)^{-1}C(j,j_3,j_4)\mathcal{F}(z,x)\bar{\mathcal{F}}(\bar{z},\bar{x})|x_{43}|^{2(j_1+j_2-j_3-j_4)}|x_{42}|^{-4j_2}|x_{41}|^{2(j_2+j_3-j_1-j_4)}|x_{31}|^{2(j_4-j_1-j_2-j_3)} \\
&\qquad \times |z_{43}|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)},
\end{aligned} \tag{3}$$

where the integral is over $j=\frac{1}{2}+is$ with s a positive real number. Here B and C are the coefficients corresponding to the two and three point functions of unflowed operators respectively (see [2] for the explicit expression in our conventions), and $\mathcal{F}$ is a function of the cross ratios $z=\frac{z_{21}z_{43}}{z_{31}z_{42}}$, $x=\frac{x_{21}x_{43}}{x_{31}x_{42}}$, with a similar expression for the antiholomorphic part. The function $\mathcal{F}$ is obtained by requiring (3) to be a solution of the Knizhnik–Zamolodchikov (KZ) equation [7,8].

Expanding $\mathcal{F}$ in powers of z as follows

$$\mathcal{F}(z,x) = z^{\Delta_j-\Delta_1-\Delta_2} x^{j-j_1-j_2} \sum_{n=0}^{\infty} f_n(x) z^n, \tag{4}$$

the lowest order f_0 is determined to be a solution of the standard hypergeometric equation thus giving two linearly independent solutions

$${}_2F_1(j-j_1+j_2, j+j_3-j_4, 2j; x), \quad \text{or} \quad x^{1-2j}{}_2F_1(1-j-j_1+j_2, 1-j+j_3-j_4, 2-2j; x).$$

Taking into account both the holomorphic and antiholomorphic parts, the unique monodromy invariant combination is of the form [2]

$$\begin{aligned}
\left|\mathcal{F}(z,x)\right|^2 = |z|^{2(\Delta_j-\Delta_1-\Delta_2)}|x|^{2(j-j_1-j_2)}\big\{&\left|{}_2F_1(j-j_1+j_2, j+j_3-j_4, 2j; x)\right|^2 \\
&+ \lambda\left|x^{1-2j}{}_2F_1(1-j-j_1+j_2, 1-j+j_3-j_4, 2-2j; x)\right|^2\big\} + \cdots,
\end{aligned} \tag{5}$$

where the ellipses denote higher orders in z and

$$\lambda = -\frac{\gamma(2j)^2\gamma(-j_1+j_2-j+1)\gamma(j_3-j_4-j+1)}{(1-2j)^2\gamma(-j_1+j_2+j)\gamma(j_3-j_4+j)},$$

with $\gamma(a)\equiv\frac{\Gamma(a)}{\Gamma(1-a)}$. Higher orders in (4) are determined iteratively by the KZ equation starting from f_0 as the initial condition [4].

Now we perform the transformation of (3) to the m-basis with the solution (5). Let us consider first the case in which one of the operators in the four point function is a highest weight,[2] say Φ_{j_1}, and look at the contribution of the first term in the r.h.s. of (5) (the second term will be considered later). The x_i dependence is given by

$$\left|K(x_i,j_i,j)\right|^2 = \left|x_{43}^{j_1+j_2-j_3-j_4}x_{42}^{-2j_2}x_{41}^{j_2+j_3-j_1-j_4}x_{31}^{j_4-j_1-j_2-j_3}x^{j-j_1-j_2}{}_2F_1(a,b,c;x)\right|^2, \tag{6}$$

where $a=j-j_1+j_2$, $b=j+j_3-j_4$ and $c=2j$.

[1] This fact is shown in Appendix B for the winding conserving three point function with states in the sectors $w=+1,-1$, and 0.

[2] The more general case where the highest weight is replaced by a global $SL(2,R)$ descendant is analyzed below.

We have to evaluate the residue of the pole at $j_1 = -m_1 = -\bar{m}_1$ in the x_1 integral transform of (6). This is obtained applying the following operator

$$\int d^2x_2\, d^2x_3\, d^2x_4\, x_2^{j_2-m_2-1}\bar{x}_2^{j_2-\bar{m}_2-1}x_3^{j_3-m_3-1}\bar{x}_3^{j_3-\bar{m}_3-1}x_4^{j_4-m_4-1}\bar{x}_4^{j_4-\bar{m}_4-1}\lim_{x_1,\bar{x}_1\to\infty} x_1^{2j_1}\bar{x}_1^{2j_1},$$

where we are using (2).

Noticing that

$$\lim_{x_1\to\infty}\left[x_1^{2j_1}K(x_i,j_i,j)\right] = x_4^{j_1-j_2-j_3-j_4}\left(1-\frac{x_3}{x_4}\right)^{j-j_3-j_4}\left(1-\frac{x_2}{x_4}\right)^{j_1-j_2-j}{}_2F_1\left(a,b,c;\frac{1-\frac{x_3}{x_4}}{1-\frac{x_2}{x_4}}\right),$$

and performing the change of variables $x_2 \to x_2/x_4$, $x_3 \to x_3/x_4$, we may write

$$\begin{aligned}&\lim_{x_1,\bar{x}_1\to\infty}\int d^2x_2\, d^2x_3\, d^2x_4\, x_2^{j_2-m_2-1}\bar{x}_2^{j_2-\bar{m}_2-1}x_3^{j_3-m_3-1}\bar{x}_3^{j_3-\bar{m}_3-1}x_4^{j_4-m_4-1}\bar{x}_4^{j_4-\bar{m}_4-1}\left|x_1^{2j_1}K(x_i,j_i,j)\right|^2\\&\quad\sim V_{\mathrm{conf}}\delta^2(m_1+m_2+m_3+m_4)\Omega(j,j_i,m_i,\bar{m}_i),\end{aligned}\tag{7}$$

where

$$\begin{aligned}\Omega(j,j_i,m_i,\bar{m}_i) \equiv{}& \int d^2x_2\, d^2x_3\, x_2^{j_2-m_2-1}\bar{x}_2^{j_2-\bar{m}_2-1}x_3^{j_3-m_3-1}\bar{x}_3^{j_3-\bar{m}_3-1}|1-x_2|^{2(j_1-j_2-j)}|1-x_3|^{2(j-j_3-j_4)}\\&\times{}_2F_1\left(a,b,c;\frac{1-x_3}{1-x_2}\right){}_2F_1\left(a,b,c;\frac{1-\bar{x}_3}{1-\bar{x}_2}\right),\end{aligned}\tag{8}$$

and the δ-function comes from the x_4 integral. The factor V_{conf} is the volume of the conformal group of S^2, and it arises from the fact that we are looking at the residue of the pole at $j_1 = -m_1 = -\bar{m}_1$ (see [2]).

In order to compute Ω, we begin by considering the following integral

$$I \equiv \int d^2y\, y^p\bar{y}^q(1-y)^r(1-\bar{y})^s{}_2F_1\big(a,b,c;t(1-y)\big){}_2F_1\big(a,b,c;\bar{t}(1-\bar{y})\big).$$

We approach it by first redefining variables and integration contours.[3] Let us define y_1 and y_2 through $y = y_1 + iy_2$, $\bar{y} = y_1 - iy_2$, and then perform the scaling $y_2 \to -ie^{2i\epsilon}y_2$, where ϵ is a small positive number. Thus we may write

$$\begin{aligned}I \sim{}& \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2\, (y_1-y_2+2i\epsilon y_2)^p(y_1+y_2-2i\epsilon y_2)^q(1-y_1+y_2-2i\epsilon y_2)^r\\&\times(1-y_1-y_2+2i\epsilon y_2)^s{}_2F_1\big(a,b,c;t(1-y_1+y_2-2i\epsilon y_2)\big){}_2F_1\big(a,b,c;\bar{t}(1-y_1-y_2+2i\epsilon y_2)\big).\end{aligned}$$

It is convenient to introduce $y_\pm \equiv y_1 \pm y_2$ so that the integral can be rewritten as

$$\begin{aligned}I \sim{}& \int_{-\infty}^{\infty} dy_+ \int_{-\infty}^{\infty} dy_-\left[y_+-i\epsilon(y_+-y_-)\right]^q\left[1-y_++i\epsilon(y_+-y_-)\right]^s{}_2F_1\big(a,b,c;\bar{t}\left[1-y_++i\epsilon(y_+-y_-)\right]\big)\\&\times\left[y_-+i\epsilon(y_+-y_-)\right]^p\left[1-y_--i\epsilon(y_+-y_-)\right]^r{}_2F_1\big(a,b,c;t\left[1-y_--i\epsilon(y_+-y_-)\right]\big).\end{aligned}\tag{9}$$

Here the ϵ terms define the way we go around the singular points (see [9] for details). We decompose the integration interval for y_+ into $(-\infty,0)\cup(0,1)\cup(1,\infty)$. The only non-vanishing contribution comes from the interval $y_+ \in (0,1)$, since the integration contours for y_- can be deformed to infinity in the other two intervals, and we assume that the integrals are convergent. For the non-vanishing contribution, we deform the integration contour to y_- in $(1,\infty)$.

In this way the m and $\bar{m}$ contributions are factorized as follows

$$I = \sin(\pi r)\int_0^1 dy_+\, y_+^q(1-y_+)^s{}_2F_1\big(a,b,c;\bar{t}(1-y_+)\big)\int_1^{\infty} dy_-\, y_-^p(1-y_-)^r{}_2F_1\big(a,b,c;t(1-y_-)\big).$$

Now changing variable $y_- \to 1/y_-$ we arrive at

$$I = (-1)^r\sin(\pi r)\int_0^1 dy_+\, y_+^q(1-y_+)^s{}_2F_1\big(a,b,c;\bar{t}(1-y_+)\big)\int_0^1 dy_-\, y_-^{-p-r-2}(1-y_-)^r{}_2F_1\left(a,b,c;t\frac{y_--1}{y_-}\right).\tag{10}$$

[3] This is similar to the computation of the integral $\int d^2y\,|y^\alpha(1-y)^\beta|^2$ in [9].

Consider now the integral in (8). Using twice the result above we find

$$\Omega(j, j_i, m_i, \bar{m}_i) = \Omega_-(j, j_i, m_i)\Omega_+(j, j_i, \bar{m}_i), \tag{11}$$

where

$$\begin{aligned}\Omega_-(j, j_i, m_i) = {}& (-1)^{j_1-j_2-j_3-j_4} \sin\bigl[\pi(j_1 - j_2 - j)\bigr] \sin\bigl[\pi(j - j_3 - j_4)\bigr] \\ & \times \int_0^1 du_- \int_0^1 dv_-\, u_-^{j_4-j+m_3-1}(1-u_-)^{j-j_3-j_4} v_-^{j-j_1+m_2-1}(1-v_-)^{j_1-j_2-j}\, {}_2F_1\biggl(a, b, c; \frac{v_-}{u_-}\frac{1-u_-}{1-v_-}\biggr), \\ \Omega_+(j, j_i, \bar{m}_i) = {}& \int_0^1 du_+ \int_0^1 dv_+\, u_+^{j-j_3-j_4}(1-u_+)^{j_3-\bar{m}_3-1} v_+^{j_1-j_2-j}(1-v_+)^{j_2-\bar{m}_2-1}\, {}_2F_1\biggl(a, b, c; \frac{u_+}{v_+}\biggr),\end{aligned} \tag{12}$$

and in the last integral we have performed the additional change of variables $u_+ \to 1 - u_+$, $v_+ \to 1 - v_+$.

Now notice that the roles of y_+ and y_- can be exchanged in the manipulations leading from (9) to (10). In this case one would arrive at an equivalent expression

$$I = (-1)^s \sin(\pi s) \int_0^1 dy_+\, y_+^{-q-s-2}(1-y_+)^s\, {}_2F_1\biggl(a, b, c; \bar{t}\frac{y_+ - 1}{y_+}\biggr) \int_0^1 dy_-\, y_-^p (1-y_-)^r\, {}_2F_1\bigl(a, b, c; t(1-y_-)\bigr),$$

which leads to

$$\Omega_-(j, j_i, m_i) = \int_0^1 du_- \int_0^1 dv_-\, u_-^{j-j_3-j_4}(1-u_-)^{j_3-m_3-1} v_-^{j_1-j_2-j}(1-v_-)^{j_2-m_2-1}\, {}_2F_1\biggl(a, b, c; \frac{u_-}{v_-}\biggr). \tag{13}$$

Comparing Eqs. (12) and (13) we are now able to write Ω in a form which is *manifestly* symmetric with respect to m_i and $\bar{m}_i$, namely

$$\begin{aligned}\Omega = {}& \int_0^1 du_- \int_0^1 dv_-\, u_-^{j-j_3-j_4}(1-u_-)^{j_3-m_3-1} v_-^{j_1-j_2-j}(1-v_-)^{j_2-m_2-1}\, {}_2F_1\biggl(a, b, c; \frac{u_-}{v_-}\biggr) \\ & \times \int_0^1 du_+ \int_0^1 dv_+\, u_+^{j-j_3-j_4}(1-u_+)^{j_3-\bar{m}_3-1} v_+^{j_1-j_2-j}(1-v_+)^{j_2-\bar{m}_2-1}\, {}_2F_1\biggl(a, b, c; \frac{u_+}{v_+}\biggr).\end{aligned} \tag{14}$$

Therefore, the problem of computing Ω reduces to that of solving the following integral

$$\Sigma = \int_0^1 du \int_0^1 dv\, u^\alpha (1-u)^\beta v^\mu (1-v)^\nu\, {}_2F_1\biggl(a, b, c; \frac{u}{v}\biggr).$$

Now using (A.1) (see Appendix A) we may write

$$\Sigma = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \int_0^1 dv\, v^\mu (1-v)^\nu\, {}_3F_2\biggl(a, b, \alpha+1; c, \alpha+\beta+2; \frac{1}{v}\biggr),$$

and with the help of (A.2) and (A.3) (see Appendix A) we arrive at

$$\begin{aligned}\Sigma = {}& \Gamma(\alpha+1)\Gamma(\beta+1)\biggl(\Lambda\begin{bmatrix} a, a-c+1, a-\alpha-\beta-1, a+\mu+1 \\ a-b+1, a-\alpha, a+\mu+\nu+2 \end{bmatrix} + \Lambda\begin{bmatrix} b, b-c+1, b-\alpha-\beta-1, b+\mu+1 \\ b-a+1, b-\alpha, b+\mu+\nu+2 \end{bmatrix} \\ & + \Lambda\begin{bmatrix} \alpha+1, \alpha-c+2, -\beta, \alpha+\mu+2 \\ \alpha-a+2, \alpha-b+2, \alpha+\mu+\nu+3 \end{bmatrix}\biggr),\end{aligned} \tag{15}$$

where we have defined

$$\Lambda\begin{bmatrix} \rho_1, \rho_2, \rho_3, \rho_4 \\ \sigma_1, \sigma_2, \sigma_3 \end{bmatrix} \equiv (-1)^{\rho_1} \frac{\Gamma(1-\sigma_1)\Gamma(1-\sigma_2)\Gamma(1+\rho_1-\rho_2)\Gamma(\rho_4)\Gamma(\sigma_3-\rho_4)}{\Gamma(1-\rho_2)\Gamma(1-\rho_3)\Gamma(1+\rho_1-\sigma_1)\Gamma(1+\rho_1-\sigma_2)\Gamma(\sigma_3)} \times {}_4F_3(\rho_1, \rho_2, \rho_3, \rho_4; \sigma_1, \sigma_2, \sigma_3; 1).$$

Finally, replacing (15) in (14) we find the explicit form of Ω as follows

$$\begin{aligned}\Omega(j, j_i, m_i, \bar{m}_i) = {}& \Gamma(-j_3 - j_4 + j + 1)^2 \Gamma(j_3 - m_3)\Gamma(j_3 - \bar{m}_3) \\ &\times \left(\Lambda\begin{bmatrix} -j_1 + j_2 + j, -j_1 + j_2 - j + 1, -j_1 + j_2 + j_4 + m_3, 1 \\ -j_1 + j_2 - j_3 + j_4 + 1, -j_1 + j_2 + j_3 + j_4, j_2 - m_2 + 1 \end{bmatrix} \right. \\ &+ \Lambda\begin{bmatrix} j_3 - j_4 + j, j_3 - j_4 - j + 1, j_3 + m_3, j_1 - j_2 + j_3 - j_4 + 1 \\ j_1 - j_2 + j_3 - j_4 + 1, 2j_3, j_1 + j_3 - j_4 - m_2 + 1 \end{bmatrix} \\ &\left. + \Lambda\begin{bmatrix} -j_3 - j_4 + j + 1, -j_3 - j_4 - j + 2, -j_3 + m_3 + 1, j_1 - j_2 - j_3 - j_4 + 2 \\ j_1 - j_2 - j_3 - j_4 + 2, -2j_3 + 2, j_1 - j_3 - j_4 - m_2 + 2 \end{bmatrix} \right) \\ &\times \left(\Lambda\begin{bmatrix} -j_1 + j_2 + j, -j_1 + j_2 - j + 1, -j_1 + j_2 + j_4 + \bar{m}_3, 1 \\ -j_1 + j_2 - j_3 + j_4 + 1, -j_1 + j_2 + j_3 + j_4, j_2 - \bar{m}_2 + 1 \end{bmatrix} \right. \\ &+ \Lambda\begin{bmatrix} j_3 - j_4 + j, j_3 - j_4 - j + 1, j_3 + \bar{m}_3, j_1 - j_2 + j_3 - j_4 + 1 \\ j_1 - j_2 + j_3 - j_4 + 1, 2j_3, j_1 + j_3 - j_4 - \bar{m}_2 + 1 \end{bmatrix} \\ &\left. + \Lambda\begin{bmatrix} -j_3 - j_4 + j + 1, -j_3 - j_4 - j + 2, -j_3 + \bar{m}_3 + 1, j_1 - j_2 - j_3 - j_4 + 2 \\ j_1 - j_2 - j_3 - j_4 + 2, -2j_3 + 2, j_1 - j_3 - j_4 - \bar{m}_2 + 2 \end{bmatrix} \right). \end{aligned} \tag{16}$$

We still have to take into account the second term in the r.h.s. of (5). However, as pointed out in [2], this can be obtained from the first term by just replacing $j \to 1 - j$. Therefore the explicit form of the four point function of unflowed operators in the m-basis is the following

$$\begin{aligned}&\big\langle \Phi_{j_1;-j_1,-j_1}(z_1)\Phi_{j_2;m_2,\bar{m}_2}(z_2)\Phi_{j_3;m_3,\bar{m}_3}(z_3)\Phi_{j_4;m_4,\bar{m}_4}(z_4)\big\rangle \\ &\quad\sim V_{\mathrm{conf}}\delta^2(m_1 + m_2 + m_3 + m_4)|z_{43}|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)} \\ &\qquad\times \int dj\, C(j_1, j_2, j)B(j)^{-1}C(j, j_3, j_4)\big[\Omega(j, j_i, m_i, \bar{m}_i) + \lambda\Omega(1 - j, j_i, m_i, \bar{m}_i)\big]|z|^{2(\Delta_j-\Delta_1-\Delta_2)} + \cdots, \end{aligned} \tag{17}$$

where the ellipses denote higher orders in z.

We would now like to discuss the more general case in which the highest weight operator $\Phi_{j_1;-j_1,-j_1}$ is replaced by a global $SL(2,R)$ descendant, namely, we will consider the extension to $m_1 = -j_1 - n_1$, $\bar{m}_1 = -j_1 - \bar{n}_1$ with $n_1, \bar{n}_1 = 0, 1, \ldots$, which corresponds to an operator in the highest weight principal discrete representation $\mathcal{D}^-_{j_1}$ (see [1] for notations and conventions).

The procedure is analogous to that followed in [10] in the case of the three point function, and it involves acting on the correlator (17) with the lowering operator J^+ and making use of the Baker–Campbell–Hausdorff formula (see [10] for details). In the case of the three point function, the result can be expressed in terms of a sum over one running index.[4] In our case, however, there are sums over two holomorphic and two antiholomorphic indices, since we are dealing with a four point function and $m_i - \bar{m}_i \in \mathbf{Z}$. In fact, performing these operations on (17), the following generalization is obtained

$$\begin{aligned}&\big\langle \Phi_{j_1;-j_1-n_1,-j_1-\bar{n}_1}(z_1)\Phi_{j_2;m_2,\bar{m}_2}(z_2)\Phi_{j_3;m_3,\bar{m}_3}(z_3)\Phi_{j_4;m_4,\bar{m}_4}(z_4)\big\rangle \\ &\quad= (-1)^{n_1+\bar{n}_1}\frac{\Gamma(2j_1)^2}{\Gamma(j_1 - m_1)\Gamma(j_1 - \bar{m}_1)}\sum_{n_2,n_3=0}^{n_1}\sum_{\bar{n}_2,\bar{n}_3=0}^{\bar{n}_1}\mathcal{G}_{n_2,n_3}(j_i, m_i)\mathcal{G}_{\bar{n}_2,\bar{n}_3}(j_i, \bar{m}_i) \\ &\qquad\times \big\langle \Phi_{j_1;-j_1,-j_1}(z_1)\Phi_{j_2;m_2-n_2,\bar{m}_2-\bar{n}_2}(z_2)\Phi_{j_3;m_3-n_3,\bar{m}_3-\bar{n}_3}(z_3)\Phi_{j_4;m_4-n_4,\bar{m}_4-\bar{n}_4}(z_4)\big\rangle, \end{aligned} \tag{18}$$

where $n_i \in \mathbf{Z}$, $n_1 = n_2 + n_3 + n_4$, $\bar{n}_1 = \bar{n}_2 + \bar{n}_3 + \bar{n}_4$, and we have defined

$$\begin{aligned}\mathcal{G}_{n_2,n_3}(j_i, m_i) \equiv {}& \frac{1}{\Gamma(n_2 + 1)\Gamma(n_3 + 1)}\frac{\Gamma(-j_1 - m_1 + 1)}{\Gamma(-j_1 - m_1 - n_2 - n_3 + 1)} \\ &\times \frac{\Gamma(j_2 - m_2 + n_2)\Gamma(j_3 - m_3 + n_3)\Gamma(j_4 - j_1 - m_4 - m_1 - n_2 - n_3)}{\Gamma(j_2 - m_2)\Gamma(j_3 - m_3)\Gamma(j_4 - m_4)}. \end{aligned}$$

The correlator in the r.h.s. of (18) can be obtained by performing the replacements $m_i \to m_i - n_i$, $\bar{m}_i \to \bar{m}_i - \bar{n}_i$ $(i = 2, 3, 4)$ in (17). In particular, note that this transforms

$$\begin{aligned}\delta^2(m_1 + m_2 + m_3 + m_4) &\to \delta^2\big(m_1 + (m_2 - n_2) + (m_3 - n_3) + (m_4 + n_2 + n_3 - n_1)\big) \\ &= \delta^2\big((m_1 - n_1) + m_2 + m_3 + m_4\big),\end{aligned}$$

as expected for the correlator in the l.h.s. of (18) since m_1 is lowered to $m_1 - n_1$.

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[4] See [10] for $m_i = \bar{m}_i$ and [11] for the more general case of the three point function with $m_i - \bar{m}_i \in \mathbf{Z}$.

Now plugging (17) into (18) we get

$$\begin{aligned}
&\big\langle \Phi_{j_1;-j_1-n_1,-j_1-\bar n_1}(z_1)\Phi_{j_2;m_2,\bar m_2}(z_2)\Phi_{j_3;m_3,\bar m_3}(z_3)\Phi_{j_4;m_4,\bar m_4}(z_4)\big\rangle\\
&\quad\sim V_{\text{conf}}\delta^2\big((m_1-n_1)+m_2+m_3+m_4\big)\frac{\Gamma(2j_1)^2}{\Gamma(j_1-m_1)\Gamma(j_1-\bar m_1)}\\
&\qquad\times |z_{43}|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}|z_{42}|^{-4\Delta_2}|z_{41}|^{2(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}\\
&\qquad\times\sum_{n_2,n_3=0}^{n_1}\sum_{\bar n_2,\bar n_3=0}^{\bar n_1}\mathcal{G}_{n_2,n_3}(j_i,m_i)\mathcal{G}_{\bar n_2,\bar n_3}(j_i,\bar m_i)\int dj\,C(j_1,j_2,j)B(j)^{-1}C(j,j_3,j_4)\\
&\qquad\times\big[\Omega(j,j_i,m_2-n_2,m_3-n_3,\bar m_2-\bar n_2,\bar m_3-\bar n_3)\\
&\qquad+\lambda\Omega(1-j,j_i,m_2-n_2,m_3-n_3,\bar m_2-\bar n_2,\bar m_3-\bar n_3)\big]|z|^{2(\Delta_j-\Delta_1-\Delta_2)}+\cdots,
\end{aligned}\tag{19}$$

where the ellipses denote higher orders in z.

A comment is in order. In the case of the three point function, the sum over one running index considered in [10] was extended to ∞ and then identified with the generalized hypergeometric function ${}_3F_2$. However, in the present case we have not been able to reduce the sums in (19) to any elementary function, due to the much involved nature of the coefficient (16) (it is even possible that such reduction is not at all practicable).

Since the results (17) and (19) are written in the m-basis, we may now proceed to perform the spectral flow operation following the prescription (1) in order to obtain the winding conserving four point functions for states in arbitrary spectral flow sectors. For instance, applying the spectral flow operation to (17) gives

$$\begin{aligned}
&\big\langle \Phi^{w_1,j_1=-m_1=-\bar m_1}_{J_1,M_1;\bar J_1,\bar M_1}(z_1)\Phi^{w_2,j_2}_{J_2,M_2;\bar J_2,\bar M_2}(z_2)\Phi^{w_3,j_3}_{J_3,M_3;\bar J_3,\bar M_3}(z_3)\Phi^{w_4,j_4}_{J_4,M_4;\bar J_4,\bar M_4}(z_4)\big\rangle\\
&\quad\sim V_{\text{conf}}\delta^2(m_1+m_2+m_3+m_4)z_{43}^{\Delta_1^{w_1}+\Delta_2^{w_2}-\Delta_3^{w_3}-\Delta_4^{w_4}}\bar z_{43}^{\bar\Delta_1^{w_1}+\bar\Delta_2^{w_2}-\bar\Delta_3^{w_3}-\bar\Delta_4^{w_4}}z_{42}^{-2\Delta_2^{w_2}}\bar z_{42}^{-2\bar\Delta_2^{w_2}}z_{41}^{\Delta_2^{w_2}+\Delta_3^{w_3}-\Delta_1^{w_1}-\Delta_4^{w_4}}\\
&\qquad\times\bar z_{41}^{\bar\Delta_2^{w_2}+\bar\Delta_3^{w_3}-\bar\Delta_1^{w_1}-\bar\Delta_4^{w_4}}z_{31}^{\Delta_4^{w_4}-\Delta_1^{w_1}-\Delta_2^{w_2}-\Delta_3^{w_3}}\bar z_{31}^{\bar\Delta_4^{w_4}-\bar\Delta_1^{w_1}-\bar\Delta_2^{w_2}-\bar\Delta_3^{w_3}}\\
&\qquad\times\int dj\,C(j_1,j_2,j)B(j)^{-1}C(j,j_3,j_4)z^{\Delta_j^{w}-\Delta_1^{w_1}-\Delta_2^{w_2}}\bar z^{\bar\Delta_j^{w}-\bar\Delta_1^{w_1}-\bar\Delta_2^{w_2}}\big[\Omega(j,j_i,m_i,\bar m_i)+\lambda\Omega(1-j,j_i,m_i,\bar m_i)\big]+\cdots,
\end{aligned}\tag{20}$$

where $J_i=|M_i|=|m_i+\frac{k}{2}w_i|$, $\bar J_i=|\bar M_i|=|\bar m_i+\frac{k}{2}w_i|$ are the spacetime conformal weights of the spectral flowed operators and they should be distinguished from the j_i of the original unflowed states. Recall that this is a winding conserving correlator with states in arbitrary spectral flow sectors up to the requirement $\sum_{i=1}^4 w_i=0$.

Similarly, applying the spectral flow operation to (19) one gets

$$\begin{aligned}
&\big\langle \Phi^{w_1,j_1=-m_1-n_1=-\bar m_1-\bar n_1}_{J_1,M_1;\bar J_1,\bar M_1}(z_1)\Phi^{w_2,j_2}_{J_2,M_2;\bar J_2,\bar M_2}(z_2)\Phi^{w_3,j_3}_{J_3,M_3;\bar J_3,\bar M_3}(z_3)\Phi^{w_4,j_4}_{J_4,M_4;\bar J_4,\bar M_4}(z_4)\big\rangle\\
&\quad\sim V_{\text{conf}}\delta^2\big((m_1-n_1)+m_2+m_3+m_4\big)\frac{\Gamma(2j_1)^2}{\Gamma(j_1-m_1)\Gamma(j_1-\bar m_1)}\\
&\qquad\times z_{43}^{\Delta_1^{w_1}(n_1)+\Delta_2^{w_2}-\Delta_3^{w_3}-\Delta_4^{w_4}}\bar z_{43}^{\bar\Delta_1^{w_1}(\bar n_1)+\bar\Delta_2^{w_2}-\bar\Delta_3^{w_3}-\bar\Delta_4^{w_4}}z_{42}^{-2\Delta_2^{w_2}}\bar z_{42}^{-2\bar\Delta_2^{w_2}}\\
&\qquad\times z_{41}^{\Delta_2^{w_2}+\Delta_3^{w_3}-\Delta_1^{w_1}(n_1)-\Delta_4^{w_4}}\bar z_{41}^{\bar\Delta_2^{w_2}+\bar\Delta_3^{w_3}-\bar\Delta_1^{w_1}(\bar n_1)-\bar\Delta_4^{w_4}}z_{31}^{\Delta_4^{w_4}-\Delta_1^{w_1}(n_1)-\Delta_2^{w_2}-\Delta_3^{w_3}}\bar z_{31}^{\bar\Delta_4^{w_4}-\bar\Delta_1^{w_1}(\bar n_1)-\bar\Delta_2^{w_2}-\bar\Delta_3^{w_3}}\\
&\qquad\times\sum_{n_2,n_3=0}^{n_1}\sum_{\bar n_2,\bar n_3=0}^{\bar n_1}\mathcal{G}_{n_2,n_3}(j_i,m_i)\mathcal{G}_{\bar n_2,\bar n_3}(j_i,\bar m_i)\int dj\,C(j_1,j_2,j)B(j)^{-1}C(j,j_3,j_4)\\
&\qquad\times\big[\Omega(j,j_i,m_2-n_2,m_3-n_3,\bar m_2-\bar n_2,\bar m_3-\bar n_3)+\lambda\Omega(1-j,j_i,m_2-n_2,m_3-n_3,\bar m_2-\bar n_2,\bar m_3-\bar n_3)\big]\\
&\qquad\times z^{\Delta_j^{w}-\Delta_1^{w_1}(n_1)-\Delta_2^{w_2}}\bar z^{\bar\Delta_j^{w}-\bar\Delta_1^{w_1}(\bar n_1)-\bar\Delta_2^{w_2}}+\cdots,
\end{aligned}\tag{21}$$

where $\Delta_1^{w_1}(n_1)=\Delta_1^{w_1}+w_1n_1$ and $\bar\Delta_1^{w_1}(\bar n_1)=\bar\Delta_1^{w_1}+w_1\bar n_1$.

Having completed the analysis in the m-basis, our aim is to transform the four point functions (20) and (21) back to the x-basis. We follow a procedure analogous to that considered in [2] in the case of two and three point functions. It was shown that it is not necessary to compute the most general expression since the x-basis correlators are the pole residue of the m-basis results at $J_i=M_i$, $\bar J_i=\bar M_i$, for a given spectral flowed state.[5] Similarly as in the case of the two point functions, the pole here is in the

[5] Actually, this procedure was applied in [2] to the two point function and the winding violating three point function. However, it may be shown that it also gives the correct result for winding conserving three point functions comparing expressions in [3] and [11] (see Appendix B).

divergent factor V_{conf}, and (20) can then be interpreted as resulting from an x-basis expression of the form

$$\begin{aligned}
&\left\langle \Phi^{|w_1|,j_1}_{J_1,\bar J_1}(x_1,z_1)\Phi^{|w_2|,j_2}_{J_2,\bar J_2}(x_2,z_2)\Phi^{|w_3|,j_3}_{J_3,\bar J_3}(x_3,z_3)\Phi^{|w_4|,j_4}_{J_4,\bar J_4}(x_4,z_4)\right\rangle \\
&\sim x_{43}^{J_1+J_2-J_3-J_4}\bar x_{43}^{\bar J_1+\bar J_2-\bar J_3-\bar J_4}x_{42}^{-2J_2}\bar x_{42}^{-2\bar J_2}x_{41}^{J_2+J_3-J_1-J_4}\bar x_{41}^{\bar J_2+\bar J_3-\bar J_1-\bar J_4}x_{31}^{J_4-J_1-J_2-J_3}\bar x_{31}^{\bar J_4-\bar J_1-\bar J_2-\bar J_3} \\
&\quad\times z_{43}^{\Delta_1^{|w_1|}+\Delta_2^{|w_2|}-\Delta_3^{|w_3|}-\Delta_4^{|w_4|}}\bar z_{43}^{\bar\Delta_1^{|w_1|}+\bar\Delta_2^{|w_2|}-\bar\Delta_3^{|w_3|}-\bar\Delta_4^{|w_4|}}z_{42}^{-2\Delta_2^{|w_2|}}\bar z_{42}^{-2\bar\Delta_2^{|w_2|}} \\
&\quad\times z_{41}^{\Delta_2^{|w_2|}+\Delta_3^{|w_3|}-\Delta_1^{|w_1|}-\Delta_4^{|w_4|}}\bar z_{41}^{\bar\Delta_2^{|w_2|}+\bar\Delta_3^{|w_3|}-\bar\Delta_1^{|w_1|}-\bar\Delta_4^{|w_4|}}z_{31}^{\Delta_4^{|w_4|}-\Delta_1^{|w_1|}-\Delta_2^{|w_2|}-\Delta_3^{|w_3|}}\bar z_{31}^{\bar\Delta_4^{|w_4|}-\bar\Delta_1^{|w_1|}-\bar\Delta_2^{|w_2|}-\bar\Delta_3^{|w_3|}} \\
&\quad\times\int dj\, C(j_1,j_2,j)B(j)^{-1}C(j,j_3,j_4)\left[\Omega(j,j_i,m_i,\bar m_i)+\lambda\Omega(1-j,j_i,m_i,\bar m_i)\right] \\
&\quad\times z^{\Delta_j^{|w|}-\Delta_1^{|w_1|}-\Delta_2^{|w_2|}}\bar z^{\bar\Delta_j^{|w|}-\bar\Delta_1^{|w_1|}-\bar\Delta_2^{|w_2|}}x^{j-J_1-J_2}\bar x^{j-\bar J_1-\bar J_2}\left\{\left|{}_2F_1(j-J_1+J_2,j+J_3-J_4,2j;x)\right|^2\right. \\
&\quad\left.+\hat\lambda\left|x^{1-2j}{}_2F_1(1-j-J_1+J_2,1-j+J_3-J_4,2-2j;x)\right|^2\right\}+\cdots,
\end{aligned}\tag{22}$$

where the ellipses denote higher order terms in z. We have replaced $w_i\to|w_i|$ because in the x-basis the operators are labeled with positive winding number [2] and

$$\hat\lambda=-\frac{\gamma(2j)^2\gamma(-J_1+J_2-j+1)\gamma(J_3-J_4-j+1)}{(1-2j)^2\gamma(-J_1+J_2+j)\gamma(J_3-J_4+j)}.\tag{23}$$

In order to determine the x dependence of (22) we have used that the Ward identities satisfied by correlators involving either unflowed or spectral flowed fields in the x-basis are the same up to the replacements $\Delta_i\to\Delta_i^w$, $j_i\to J_i$ for the spectral flowed fields [3]. In addition, although the modified KZ and null vector equations to be satisfied by correlators involving spectral flowed fields in the x-basis were shown in [3] to be iterative in the spins J_i, and their forms differ from the usual KZ and null vector equations for the unflowed case, at the lowest order in z that we are considering here the modified KZ equation actually reduces to that of the unflowed case with the replacements $j_i\to J_i$, and the iterative terms do not contribute. Therefore we expect that the four point functions have the same dependence on the coordinates and anharmonic ratios as (3), (5), with the replacements mentioned above.

Similarly, the same procedure can be followed in order to transform (21) back to the x-basis and we get

$$\begin{aligned}
&\left\langle \Phi^{|w_1|,j_1}_{J_1(n_1),\bar J_1(\bar n_1)}(x_1,z_1)\Phi^{|w_2|,j_2}_{J_2,\bar J_2}(x_2,z_2)\Phi^{|w_3|,j_3}_{J_3,\bar J_3}(x_3,z_3)\Phi^{|w_4|,j_4}_{J_4,\bar J_4}(x_4,z_4)\right\rangle \\
&\sim\frac{\Gamma(2j_1)^2}{\Gamma(j_1-m_1)\Gamma(j_1-\bar m_1)}x_{43}^{J_1(n_1)+J_2-J_3-J_4}\bar x_{43}^{\bar J_1(\bar n_1)+\bar J_2-\bar J_3-\bar J_4}x_{42}^{-2J_2}\bar x_{42}^{-2\bar J_2} \\
&\quad\times x_{41}^{J_2+J_3-J_1(n_1)-J_4}\bar x_{41}^{\bar J_2+\bar J_3-\bar J_1(\bar n_1)-\bar J_4}x_{31}^{J_4-J_1(n_1)-J_2-J_3}\bar x_{31}^{\bar J_4-\bar J_1(\bar n_1)-\bar J_2-\bar J_3} \\
&\quad\times z_{43}^{\Delta_1^{|w_1|}(n_1)+\Delta_2^{|w_2|}-\Delta_3^{|w_3|}-\Delta_4^{|w_4|}}\bar z_{43}^{\bar\Delta_1^{|w_1|}(\bar n_1)+\bar\Delta_2^{|w_2|}-\bar\Delta_3^{|w_3|}-\bar\Delta_4^{|w_4|}}z_{42}^{-2\Delta_2^{|w_2|}}\bar z_{42}^{-2\bar\Delta_2^{|w_2|}} \\
&\quad\times z_{41}^{\Delta_2^{|w_2|}+\Delta_3^{|w_3|}-\Delta_1^{|w_1|}(n_1)-\Delta_4^{|w_4|}}\bar z_{41}^{\bar\Delta_2^{|w_2|}+\bar\Delta_3^{|w_3|}-\bar\Delta_1^{|w_1|}(\bar n_1)-\bar\Delta_4^{|w_4|}} \\
&\quad\times z_{31}^{\Delta_4^{|w_4|}-\Delta_1^{|w_1|}(n_1)-\Delta_2^{|w_2|}-\Delta_3^{|w_3|}}\bar z_{31}^{\bar\Delta_4^{|w_4|}-\bar\Delta_1^{|w_1|}(\bar n_1)-\bar\Delta_2^{|w_2|}-\bar\Delta_3^{|w_3|}} \\
&\quad\times\sum_{n_2,n_3=0}^{n_1}\sum_{\bar n_2,\bar n_3=0}^{\bar n_1}\mathcal G_{n_2,n_3}(j_i,m_i)\mathcal G_{\bar n_2,\bar n_3}(j_i,\bar m_i)\int dj\, C(j_1,j_2,j)B(j)^{-1}C(j,j_3,j_4) \\
&\quad\times\left[\Omega(j,j_i,m_2-n_2,m_3-n_3,\bar m_2-\bar n_2,\bar m_3-\bar n_3)+\lambda\Omega(1-j,j_i,m_2-n_2,m_3-n_3,\bar m_2-\bar n_2,\bar m_3-\bar n_3)\right] \\
&\quad\times z^{\Delta_j^{|w|}-\Delta_1^{|w_1|}(n_1)-\Delta_2^{|w_2|}}\bar z^{\bar\Delta_j^{|w|}-\bar\Delta_1^{|w_1|}(\bar n_1)-\bar\Delta_2^{|w_2|}}x^{j-J_1(n_1)-J_2}\bar x^{j-\bar J_1(\bar n_1)-\bar J_2} \\
&\quad\times\left\{\left|{}_2F_1(j-J_1(n_1)+J_2,j+J_3-J_4,2j;x)\right|^2\right. \\
&\quad\left.+\hat\lambda(n_1)\left|x^{1-2j}{}_2F_1(1-j-J_1(n_1)+J_2,1-j+J_3-J_4,2-2j;x)\right|^2\right\}+\cdots,
\end{aligned}\tag{24}$$

where $J_1(n_1)=|-j_1-n_1+\frac{k}{2}w_1|$.

Therefore we have extended the result (3), (5) for the four point function of unflowed operators in the $SL(2,R)$ WZW model, to all winding conserving four point functions for states in arbitrary spectral flow sectors. We have obtained results both in the m- and the x-basis, with the simplifying assumption that at least one operator is in the spectral flow image of $\mathcal D_j^-$.

Notice that, while the m-basis expressions (20) and (21) remain valid for all winding conserving four point functions, including in particular the case in which all the external operators are unflowed, the results (22) and (24) in the x-basis do not hold when all the external states are unflowed. This is consistent with the fact that in the m-basis, all N-point functions are the same, for a given N, up to some free boson correlator [1,2,5] which only modifies the z_i dependence. On the other hand, the procedure we have followed to transform from the m- to the x-basis, by evaluating the pole residue at $J = M$, $\bar{J} = \bar{M}$, requires at least one spectral flowed state in the correlator.[6] Consequently (22) and (24) result in this case, whereas (3), (5) hold for four unflowed operators.

We would like to comment on the higher order contributions to the expansion in z. As in the case of the four point function of unflowed operators [4], we expect them to be determined once the lowest order is given as the initial condition. This works in two different but equivalent ways. The first one is that higher orders in the spectral flowed case are fixed by first iteratively determining the higher orders in (4) starting from (5), and then performing the spectral flow operation to the result, in a similar fashion as we have done here to the lowest order contribution. Alternatively one may determine the higher order contributions using modified KZ equations for amplitudes involving spectral flowed states, starting from (20)–(24) as the initial conditions. In the m- and μ-basis, such modified KZ equations were discussed in [5]. In the x-basis, modified KZ equations were computed in [3] and the determination of the higher order contributions was also discussed in the one-unit violating case.

Finally, one important application of our results would be to investigate the structure of the factorization of (20)–(24) in order to establish the consistency of string theory on AdS_3 and verify the winding violation pattern suggested in [2]. We hope to tackle this problem in the future.

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Appendix A. Useful formulae

The following identities for the hypergeometric functions can be found e.g. in [12]

$$\int_0^1 du\, u^{\alpha}(1-u)^{\beta}\, {}_2F_1(a,b,c;\lambda u) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\, {}_3F_2(a,b,\alpha+1;c,\alpha+\beta+2;\lambda). \tag{A.1}$$

$$\int_0^1 du\, u^{\alpha}(1-u)^{\beta}\, {}_3F_2(a,b,c;d,e;u) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\, {}_4F_3(a,b,c,\alpha+1;d,e,\alpha+\beta+2;1). \tag{A.2}$$

$$\begin{aligned} &{}_3F_2(a,b,c;d,e;u) \\ &\quad= \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)}\left[\frac{\Gamma(a)\Gamma(b-a)\Gamma(c-a)}{\Gamma(d-a)\Gamma(e-a)}(-u)^{-a}\, {}_3F_2\left(a-d+1,a-e+1,a;a-b+1,a-c+1;\frac{1}{u}\right)\right. \\ &\qquad+ \frac{\Gamma(b)\Gamma(a-b)\Gamma(c-b)}{\Gamma(d-b)\Gamma(e-b)}(-u)^{-b}\, {}_3F_2\left(b-d+1,b-e+1,b;b-a+1,b-c+1;\frac{1}{u}\right) \\ &\qquad\left.+ \frac{\Gamma(c)\Gamma(a-c)\Gamma(b-c)}{\Gamma(d-c)\Gamma(e-c)}(-u)^{-c}\, {}_3F_2\left(c-d+1,c-e+1,c;c-a+1,c-b+1;\frac{1}{u}\right)\right]. \end{aligned} \tag{A.3}$$

Appendix B. Three point functions

In this appendix we transform the three point function including two $w=1$ operators, computed in [3], from the x-basis to the m-basis. We verify that the result equals the three point function for unflowed operators in the m-basis computed in [11]. This may be considered a check not only on the expressions in [3,11], but also on the claim that the coefficient of all winding conserving correlators is the same in the m-basis (for a given number of external states) [1,2,5].

The three point function including two $w=1$ operators in the x-basis is [3]

$$\begin{aligned} &\left\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\Phi_{j_3}(x_3,z_3)\right\rangle \\ &\quad\sim \frac{1}{V_{\text{conf}}^2} B(j_1)B(j_2)C\left(\frac{k}{2}-j_1,\frac{k}{2}-j_2,j_3\right)W(j_1,j_2,j_3,m_1,m_2,\bar{m}_1,\bar{m}_2) \\ &\qquad\times x_{12}^{j_3-J_1-J_2}\bar{x}_{12}^{j_3-\bar{J}_1-\bar{J}_2}x_{13}^{J_2-J_1-j_3}\bar{x}_{13}^{\bar{J}_2-\bar{J}_1-j_3}x_{23}^{J_1-J_2-j_3}\bar{x}_{23}^{\bar{J}_1-\bar{J}_2-j_3} \\ &\qquad\times z_{12}^{\Delta_3-\Delta_1^{w=1}-\Delta_2^{w=1}}\bar{z}_{12}^{\Delta_3-\bar{\Delta}_1^{w=1}-\bar{\Delta}_2^{w=1}}z_{13}^{\Delta_2^{w=1}-\Delta_1^{w=1}-\Delta_3}\bar{z}_{13}^{\bar{\Delta}_2^{w=1}-\bar{\Delta}_1^{w=1}-\Delta_3}z_{23}^{\Delta_1^{w=1}-\Delta_2^{w=1}-\Delta_3}\bar{z}_{23}^{\bar{\Delta}_1^{w=1}-\bar{\Delta}_2^{w=1}-\Delta_3}, \end{aligned} \tag{B.1}$$

[6] In fact there should be at least two spectral flowed operators in the winding conserving case.

where

$$W(j_1, j_2, j_3, m_1, m_2, \bar{m}_1, \bar{m}_2) = \int d^2u\, d^2v\, u^{j_1-m_1-1}\bar{u}^{j_1-\bar{m}_1-1} v^{j_2-m_2-1}\bar{v}^{j_2-\bar{m}_2-1} \times |1-u|^{2(j_2-j_1-j_3)}|1-v|^{2(j_1-j_2-j_3)}|u-v|^{2(j_3-j_1-j_2)},$$

was computed in [13], but the explicit result is not relevant for our purposes here.

In order to transform this expression to the m-basis, we use (2) and evaluate the residue of the pole at $J_2 = -M_2$, $\bar{J}_2 = -\bar{M}_2$ in the x_2 integral, i.e. we perform the following operation to (B.1)

$$\int d^2x_1\, d^2x_3\, x_1^{J_1-M_1-1}\bar{x}_1^{\bar{J}_1-\bar{M}_1-1} x_3^{j_3-m_3-1}\bar{x}_3^{j_3-\bar{m}_3-1} \lim_{x_2,\bar{x}_2\to\infty} x_2^{2J_2}\bar{x}_2^{2\bar{J}_2}.$$

Thus we find that the x-dependent part in (B.1) contributes the following factor

$$V_{\text{conf}}\delta^2(M_1 + M_2 + m_3)\frac{\Gamma(j_3 - m_3)\Gamma(J_2 - J_1 - j_3 + 1)\Gamma(\bar{J}_1 - \bar{J}_2 + \bar{m}_3)}{\Gamma(-j_3 + \bar{m}_3 + 1)\Gamma(\bar{J}_1 - \bar{J}_2 + j_3)\Gamma(J_2 - J_1 - m_3 + 1)}, \tag{B.2}$$

up to some k-dependent (j independent) coefficient. Considering that $J_1 = M_1 = m_1 + k/2$, $J_2 = -M_2 = -m_2 + k/2$, $M_1 + M_2 + m_3 = 0$ and $m_3 - \bar{m}_3 \in \mathbf{Z}$, this can be reduced to

$$V_{\text{conf}}^2\delta^2(m_1 + m_2 + m_3). \tag{B.3}$$

Therefore, recalling the identity [3]

$$B(j_1)B(j_2)C\left(\frac{k}{2} - j_1, \frac{k}{2} - j_2, j_3\right) \sim C(j_1, j_2, j_3), \tag{B.4}$$

we obtain the following expression for the winding conserving ($w_1 = -w_2 = 1$, $w_3 = 0$) three point function in the m-basis

$$\begin{aligned}&\left\langle \Phi^{w=1,j_1}_{J_1,M_1;\bar{J}_1,\bar{M}_1}(z_1)\Phi^{w=-1,j_2}_{J_2,M_2;\bar{J}_2,\bar{M}_2}(z_2)\Phi_{j_3;m_3,\bar{m}_3}(z_3)\right\rangle \\ &\quad\sim \delta^2(m_1 + m_2 + m_3)C(j_1, j_2, j_3)W(j_1, j_2, j_3, m_1, m_2, \bar{m}_1, \bar{m}_2) \\ &\qquad\times z_{12}^{\Delta_3-\Delta_1^{w=1}-\Delta_2^{w=-1}}\bar{z}_{12}^{\bar{\Delta}_3-\bar{\Delta}_1^{w=1}-\bar{\Delta}_2^{w=-1}} z_{13}^{\Delta_2^{w=-1}-\Delta_1^{w=1}-\Delta_3}\bar{z}_{13}^{\bar{\Delta}_2^{w=-1}-\bar{\Delta}_1^{w=1}-\Delta_3} z_{23}^{\Delta_1^{w=1}-\Delta_2^{w=-1}-\Delta_3}\bar{z}_{23}^{\bar{\Delta}_1^{w=1}-\bar{\Delta}_2^{w=-1}-\Delta_3}.\end{aligned} \tag{B.5}$$

This expression coincides with the three point function of unflowed operators computed in [11] up to the powers of z_{ij}, which have to be transformed according to (1).

References

[1] J. Maldacena, H. Ooguri, J. Math. Phys. 42 (2001) 2929.
[2] J. Maldacena, H. Ooguri, Phys. Rev. D 65 (2002) 106006.
[3] E. Herscovich, P. Minces, C. Núñez, JHEP 0606 (2006) 047.
[4] J. Teschner, Nucl. Phys. B 571 (2000) 555.
[5] S. Ribault, JHEP 0509 (2005) 045.
[6] S. Ribault, J. Teschner, JHEP 0506 (2005) 014.
[7] V.G. Knizhnik, A.B. Zamolodchikov, Nucl. Phys. B 247 (1984) 83.
[8] A.B. Zamolodchikov, V.A. Fateev, Sov. J. Nucl. Phys. 43 (1986) 657.
[9] V.S. Dotsenko, Lectures on Conformal Field Theory, in: Proceedings of RIMS Workshop, Conformal Field Theory and Solvable Lattice Models, Kyoto, 1986.
[10] K. Becker, M. Becker, Nucl. Phys. B 418 (1994) 206.
[11] Y. Satoh, Nucl. Phys. B 629 (2002) 188.
[12] Y. Luke, The Special Functions and Their Approximations, Academic Press, 1969.
[13] T. Fukuda, K. Hosomichi, JHEP 0109 (2001) 003.