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Research article

Equivalence of solutions for non-homogeneous $p(x)$-Laplace equations†

María Medina[1] **and Pablo Ochoa**[2,*]

[1] Departamento de Matemáticas, Universidad Autónoma de Madrid, Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain

[2] Facultad de Ingeniería, Universidad Nacional de Cuyo. CONICET. Parque Gral. San Martín 5500. Mendoza, Argentina

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* **Correspondence:** Email: pablo.ochoa@ingenieria.uncuyo.edu.ar.

Abstract: We establish the equivalence between weak and viscosity solutions for non-homogeneous $p(x)$-Laplace equations with a right-hand side term depending on the spatial variable, the unknown, and its gradient. We employ inf- and sup-convolution techniques to state that viscosity solutions are also weak solutions, and comparison principles to prove the converse. The new aspects of the $p(x)$-Laplacian compared to the constant case are the presence of log-terms and the lack of the invariance under translations.

Keywords: nonlinear elliptic equations; p(x)-Laplacian; viscosity solutions; weak solutions; comparison principle

To the memory of Ireneo Peral, mentor and friend. Te echamos de menos.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$. We will consider non-homogeneous $p(x)$-Laplace equations of the form

$$-\Delta_{p(x)}u = f(x, u, Du) \quad \text{in } \Omega, \tag{1.1}$$

where, given a function $p : \Omega \to (1, \infty)$, $-\Delta_{p(x)}$ is the $p(x)$-Laplace operator defined as

$$-\Delta_{p(x)}u := -\mathrm{div}\left(|Du|^{p(x)-2}Du\right). \tag{1.2}$$

For a smooth function φ with $D\varphi \neq 0$, we can expand the expression above and write

$$-\Delta_{p(x)}\varphi(x) = -|D\varphi|^{p(x)-2}\left(-\Delta\varphi + \frac{(p(x)-2)}{|D\varphi|^2}\Delta_\infty\varphi\right) - |D\varphi|^{p(x)-2}Dp(x)\cdot D\varphi \log|D\varphi|,$$

where

$$\Delta_\infty\varphi := D^2\varphi D\varphi \cdot D\varphi$$

is the ∞-Laplacian. Consequently, two fundamental differences exist between $-\Delta_{p(x)}$ and the p-Laplacian, with p constant: the fact that this operator is not invariant under translations in x and the presence of log-terms.

Recently, the study of partial differential equations with variable exponents has been motivated by the description of models in electrorheological and thermorheological fluids, image processing [3], or robotics. Moreover, classical references for existence and regularity of solution for $p(x)$-Laplacian Dirichlet problems are [7, 8, 11], among others.

In this work we are interested in analyzing the equivalence between weak and viscosity solutions (see Section 2.2 for the precise definitions of these notions) of the problem (1.1) under certain conditions on f. The relation among different types of solutions for different operators has been studied by several authors in the last decades. For linear problems, the equivalence between distributional and viscosity solutions was obtained by Ishii in [13]. Later on, for the homogeneous p-Laplace operator (i.e., (1.1) with $f \equiv 0$), the equivalence between weak and viscosity solutions was first obtained in [16], and later on in [14] with a different proof. For a source term like the one in (1.1), depending on all the lower-order terms, this equivalence for the p-Laplace equation was given in [19], following some ideas from [14]. Similar studies have been recently made for non-local operators, see [1, 18].

In the case of the variable $p(x)$-Laplacian, the equivalence for the homogeneous equation was proved in [16]. Related results appear in [20] between solutions of homogeneous equations involving the *strong* and the *normalized* $p(x)$-Laplacian. Up to our knowledge, no results are available in the case $f \not\equiv 0$. Indeed, combining techniques from [16, 20] to deal with the operator, and from [19] to deal with the function f, the goal of this work is to prove the equivalence of weak and viscosity solutions for the general problem (1.1).

Indeed, let us assume from now on that the exponent p satisfies

$$p \in C^1(\overline{\Omega}), \quad 1 < p^- \leq p^+ < \infty, \quad \text{where} \quad p^- := \min_{x\in\overline{\Omega}} p(x),\ p^+ := \max_{x\in\overline{\Omega}} p(x). \tag{1.3}$$

Hence, our first result is the following:

Theorem 1.1. *Let p satisfy* (1.3)*. Assume that $f = f(x, t, \eta)$ is uniformly continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, non increasing in t, Lipschitz continuous in η, and satisfies the growth condition*

$$|f(x, t, \eta)| \leq \gamma(|t|)|\eta|^{p(x)-1} + \phi(x), \tag{1.4}$$

where $\gamma \geq 0$ is continuous and $\phi \in L^\infty_{loc}(\Omega)$. Thus, if u is a locally Lipschitz viscosity supersolution of (1.1) *then it is a weak supersolution of the problem.*

It is worth to point out that the regularity assumption on u derives from a technical restriction on p (see Remark 3.2).

The proof of Theorem 1.1 relies on the approximation by the so called inf-convolutions (see Section 2.3). Roughly speaking, we will regularize u by some functions u_ε, that will satisfy a related problem in weak sense, and we will pass to the limit here. This idea was first used in [14] for the constant p-Laplacian in the homogeneous case, and then in more general settings in [1, 19, 20].

The reverse statement, weak solutions being viscosity, is strongly connected with comparison arguments, and a new class of functions needs to be considered.

Definition 1.2. *Let u be a weak supersolution to* (1.1) *in $D \subseteq \Omega$. We say that (u, f) satisfies the comparison principle property* (CPP) *in D if for every weak subsolution v of* (1.1) *such that $u \geq v$ a.e. in ∂D we have $u \geq v$ a.e. in D.*

In particular we will see that, for those functions satisfying this property, weak solutions are indeed viscosity solutions.

Theorem 1.3. *Let p satisfy* (1.3). *Assume u is a continuous weak supersolution of* (1.1) *and $f = f(x, t, \eta)$ is continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and Lipschitz continuous in η. If (CPP) holds then u is a viscosity supersolution of* (1.1).

In Section 4, apart from this theorem, we prove a comparison principle for the general equation (1.1), which has interest in itself.

Applications of the equivalence between viscosity and weak solutions can be found in [15, 20] to removability of sets and Radó type theorems. Also, the equivalence has been recently used in free-boundary problems (see [2, 12]).

The paper is organized as follows: in Section 2 we give an introduction into the theory of Sobolev spaces with variable exponents, we introduce the notions of viscosity and weak solutions in this context, and the definition and main properties of the infimal convolutions. Section 3 is devoted to the proof of Theorem 1.1, that is, to see that viscosity solutions of (1.1) are also weak solutions. In Section 4 we prove Theorem 1.3 (that weak solutions are viscosity solutions) and a general comparison principle for Eq (1.1).

2. Preliminaries

2.1. Variable exponent spaces

In this section we introduce basic definitions and preliminary results concerning the spaces of variable exponent and the related theory of differential equations. Let

$$C_+(\overline{\Omega}) := \left\{ p \in C(\overline{\Omega}) : p(x) > 1 \quad \text{for any} \quad x \in \overline{\Omega} \right\}$$

and

$$p^- := \min_{\overline{\Omega}} p(\cdot), \qquad p^+ := \max_{\overline{\Omega}} p(\cdot).$$

Assume that p belongs to $C_+(\overline{\Omega})$ and satisfies the following log-Hölder condition: there exists $C > 0$ so that

$$|p(x) - p(y)| \leq C \frac{1}{|\log |x - y||}, \quad \text{for all} \quad x, y \in \Omega, \ x \neq y. \tag{2.1}$$

We define the Lebesgue variable exponent space as

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$

and we denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Consider also the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

Then the following results, that can be found in [5], hold.

Theorem 2.1 (Hölder's inequality)**.** *The space* $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ *is a separable, uniform convex Banach space. Furthermore, if* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$*, then*

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

The next proposition states the relation between norms and integrals of $p(x)$-th power.

Proposition 2.2. *Let*

$$\rho(u) := \int_\Omega |u|^{p(x)}\, dx, \quad u \in L^{p(\cdot)}(\Omega),$$

be the convex modular. Then the following assertions hold:

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ *(resp.* $= 1, > 1$*) if and only if* $\rho(u) < 1$ *(resp.* $= 1, > 1$*);*

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$ *implies* $\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$*, and* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ *implies* $\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$*;*

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} \to 0$ *if and only if* $\rho(u) \to 0$*, and* $\|u\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *if and only if* $\rho(u) \to \infty$*.*

The following result allows us to relate the norms of different Lebesgue variable exponent spaces (see [6] for a proof).

Lemma 2.3. *Suppose that* $p, q \in C_+(\overline{\Omega})$*. Let* $f \in L^{q(\cdot)p(\cdot)}(\Omega)$*. Then*

(i) $\|f\|^{p^+}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \|f^{p(\cdot)}\|_{L^{q(\cdot)}(\Omega)} \le \|f\|^{p^-}_{L^{p(\cdot)q(\cdot)}(\Omega)}$ *if* $\|f\|_{L^{p(\cdot)q(\cdot)}(\Omega)} \le 1$*;*

(ii) $\|f\|^{p^-}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \|f^{p(\cdot)}\|_{L^{q(\cdot)}(\Omega)} \le \|f\|^{p^+}_{L^{p(\cdot)q(\cdot)}(\Omega)}$ *if* $\|f\|_{L^{p(\cdot)q(\cdot)}(\Omega)} \ge 1$*.*

Let us denote the distributional gradiente by Du. Then we can define the variable Sobolev space $W^{1,p(\cdot)}(\Omega)$ as

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) : |Du| \in L^{p(\cdot)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)},$$

and we denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Notice that, due to the log-Hölder condition (2.1), $C_0^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$. The following Embedding Theorem can be proved (see for instance [10]).

Theorem 2.4. *If $p^+ < n$, then*

$$0 < S(p(\cdot), q(\cdot), \Omega) := \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} \frac{\|Dv\|_{L^{p(\cdot)}(\Omega)}}{\|v\|_{L^{q(\cdot)}(\Omega)}},$$

for all

$$1 \leq q(\cdot) \leq p^*(\cdot) = \frac{np(\cdot)}{n - p(\cdot)}.$$

Remark 2.5. The $q(\cdot)$ exponent has to be uniformly subcritical, i.e., $\inf_\Omega(p^*(\cdot) - q(\cdot)) > 0$, to enssure that $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

Let $X = W_0^{1,p(\cdot)}(\Omega)$. Recalling Definition 1.2, the operator $-\Delta_{p(x)}$ can be seen as the weak derivative of the functional $J : X \to \mathbb{R}$,

$$J(u) := \int_\Omega \frac{1}{p(x)} |Du|^{p(x)}\, dx,$$

in the sense that if $L := J' : X \to X^*$ then

$$(L(u), v) = \int_\Omega |Du|^{p(x)-2} DuDv\, dx, \quad u, v \in X.$$

We also recall the following properties from [11].

Theorem 2.6. *Let $X = W_0^{1,p(\cdot)}(\Omega)$. Then:*

(i) $L : X \to X^*$ is continuous, bounded and strictly monotone;
(ii) L is a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to\infty} (L(u_n) - L(u), u_n - u) \leq 0$$

then $u_n \to u$ in X;
(iii) L is a homeomorphism.

2.2. Notions of solutions

Considering the variable Sobolev spaces defined before, we can already introduce the notion of weak solution.

Definition 2.7. *We say that $u \in W^{1,p(x)}(\Omega)$ is a weak supersolution of* (1.1) *if for any non-negative $\varphi \in C_0^\infty(\Omega)$ there holds*

$$\int_\Omega |Du|^{p(x)-2} Du \cdot D\varphi\, dx \geq \int_\Omega f(x, u, Du)\varphi\, dx.$$

Likewise, we say that $u \in W^{1,p(x)}(\Omega)$ is a weak subsolution of (1.1) *if*

$$\int_\Omega |Du|^{p(x)-2} Du \cdot D\varphi\, dx \leq \int_\Omega f(x, u, Du)\varphi\, dx$$

for any non-negative $\varphi \in C_0^\infty(\Omega)$.

Finally, $u \in W^{1,p(x)}(\Omega)$ is a weak solution to (1.1) *if it is a weak sub- and supersolution.*

Denote by S^n the set of symmetric $n \times n$ matrices. In order to introduce the concept of viscosity solution, let us recall the definition of jets.

Definition 2.8. *The superjet $J^{2,-}u(x)$ of a function $u : \Omega \to \mathbb{R}$ at $x \in \Omega$ is defined as the set of pairs $(\eta, X) \in (\mathbb{R}^n \setminus \{0\}) \times S^n$ satisfying*

$$u(y) \geq u(x) + \eta \cdot (y - x) + \frac{1}{2} X(y - x) \cdot (y - x) + o(|x - y|^2)$$

as $y \to x$. The closure of a superjet is denoted by $\overline{J}^{2,-}u(x)$ and it is defined as the set of pairs $(\eta, X) \in \mathbb{R}^n \times S^n$ for which there exists a sequence $(\eta_i, X_i) \in J^{2,-}u(x_i)$, with $x_i \in \Omega$ so that

$$(x_i, \eta_i, X_i) \to (x, \eta, X) \quad \text{as } i \to \infty.$$

The subjet $J^{2,+}u(x)$ and its closure $\overline{J}^{2,+}u(x)$ are defined in a similar fashion.

Observe that the operator can be written as

$$\Delta_{p(x)}\varphi(x) = \operatorname{tr}\left(A(x, D\varphi(x))D^2\varphi(x)\right) + B(x, D\varphi(x)),$$

where

$$A(x, \xi) := |\xi|^{p(x)-2}\left(I + (p(x) - 2)\frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\right),$$

and

$$B(x, \xi) := |\xi|^{p(x)-2} \log |\xi| \xi \cdot Dp(x).$$

We can now precise the notion of viscosity solution.

Definition 2.9. *A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of* (1.1) *if for any $(\eta, X) \in J^{2,-}u(x)$ there holds*

$$-tr\,(A(x, X)) - B(x, \eta) \geq f(x, u(x), \eta).$$

Similarly, an upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (1.1) *if*

$$-tr\,(A(x, X)) - B(x, \eta) \leq f(x, u(x), \eta),$$

for all $(\eta, X) \in J^{2,+}u(x)$. Finally, a viscosity solution is a continuous function which is a viscosity sub- and a supersolution.

Observe that we do not require anything at jets of the form $(0, X)$ or if $J^{2,-}u(x) = \emptyset$. Moreover, the above definition of viscosity supersolution is equivalently given if we replace the superjet by its closure or if we take for jets (η, X) pairs of the form $(D\varphi(x), D^2\varphi(x)) \in (\mathbb{R}^n \setminus \{0\}) \times S^n$, where φ is smooth and touches u from below at x.

2.3. Infimal convolutions

A standard smoothing operator in the theory of viscosity solutions is the infimal convolution.

Definition 2.10. *Given $\varepsilon > 0$ and $q \geq 2$ we define the infimal convolution of a function $u\colon \Omega \to \mathbb{R}$ as*

$$u_\varepsilon(x) := \inf_{y\in\Omega}\left(u(y) + \frac{|x-y|^q}{q\varepsilon^{q-1}}\right), \qquad x \in \Omega.$$

The infimal convolution will be one of the main tools to prove that viscosity solutions are weak solutions. For the next result see for instance [14, 20] and the references therein.

Lemma 2.11. *Let u be a bounded and lower semicontinuous function in Ω. Then:*

(i) There exists $r(\varepsilon) > 0$ such that

$$u_\varepsilon(x) = \inf_{y\in B_{r(\varepsilon)}(x)}\left(u(y) + \frac{|x-y|^q}{q\varepsilon^{q-1}}\right),$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(ii) The sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is increasing as $\varepsilon \to 0$ and $u_\varepsilon \to u$ pointwise in Ω.

(iii) u_ε is locally Lipschitz and twice differentiable a.e. Actually, for almost every $x, y \in \Omega$,

$$u_\varepsilon(y) = u_\varepsilon(x) + Du_\varepsilon(x)\cdot(x-y) + \frac{1}{2}D^2u_\varepsilon(x)(x-y)^2 + o(|x-y|^2).$$

(iv) u_ε is semiconcave, that is, there exists a constant $C = C(q, \varepsilon, osc(u)) > 0$ such that the function $x \mapsto u_\varepsilon(x) - C|x|^2$ is concave. In particular

$$D^2u_\varepsilon(x) \leq 2CI, \quad a.e.\ \ x \in \Omega,$$

where I is the identity matrix.

(v) The set $Y_\varepsilon(x) := \left\{y \in B_{r(\varepsilon)}(x)\colon u_\varepsilon(x) = u(y) + \frac{|x-y|^q}{q\varepsilon^{q-1}}\right\}$ is non empty and closed for every $x \in \Omega$.

(vi) If $x \in \Omega_{r(\varepsilon)} := \{x \in \Omega :\ dist\,(x, \partial\Omega) > r(\varepsilon)\}$, then there exists $x_\varepsilon \in B_{r(\varepsilon)}$ such that

$$u_\varepsilon(x) = u(x_\varepsilon) + \frac{|x-x_\varepsilon|^q}{q\varepsilon^{q-1}}.$$

(vii) If $(\eta, X) \in J^{2,-}u_\varepsilon(x)$ with $x \in \Omega_{r(\varepsilon)}$, then

$$\eta = \frac{(x-x_\varepsilon)}{\varepsilon^{q-1}}|x-x_\varepsilon|^{q-2} \quad and \quad X \leq \frac{q-1}{\varepsilon}|\eta|^{\frac{q-2}{q-1}}I.$$

Remark 2.12. For later purposes (see the proof of Lemma 3.3) we will choose

$$q \geq 2 \quad \text{such that} \quad p^- - 2 + \frac{q-2}{q-1} \geq 0. \tag{2.2}$$

3. Viscosity solutions are weak solutions: proof of Theorem 1.1

Let us consider the inf-convolution u_ε given by Definition 2.10. We can summarize the strategy to prove Theorem 1.1 in several steps. Assuming that u is a viscosity supersolution, we will identify what problem is satisfied by u_ε in a pointwise sense, and later on in a weak sense. We will finish from here by passing to the limit in ε, obtaining the weak problem satisfied by u.

We thus start by identifying the problem fulfilled by u_ε.

Lemma 3.1. *Assume p satisfies* (1.3). *Let $u : \Omega \to \mathbb{R}$ locally Lipschitz, and let $f = f(x,t,\eta)$ be continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and non increasing in t. If u is a viscosity supersolution of* (1.1) *then*

$$\Delta_{p(x)} u_\varepsilon(x) \geq f_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x)) + E(\varepsilon) \qquad \text{a.e. in } \Omega_{r(\varepsilon)}, \tag{3.1}$$

where

$$f_\varepsilon(x,s,\eta) := \inf_{y \in B_{r(\varepsilon)}(x)} f(y,s,\eta),$$

and $E(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Here $E(\varepsilon)$ depends only on p, q and ε.

Notice that, differently from the constant case (see [19, Lemma 3.3]), when we identify the problem satisfied by u_ε in a pointwise sense, an error term $E(\varepsilon)$ arises. We expect it that to disappear when passing to the limit in the final step. To prove this lemma we borrow some computations from [17, proof of Proposition 6.1], where they are used to prove a comparison-type result.

Proof. Fix $x \in \Omega_{r(\varepsilon)}$ and let $(\eta, Z) \in J^{2,-} u_\varepsilon(x)$, with $\eta \neq 0$. Then, by Lemma 2.11, there is $x_\varepsilon \in B_{r(\varepsilon)}(x)$ such that

$$u_\varepsilon(x) = u(x_\varepsilon) + \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}} \quad \text{and} \quad \eta = \frac{(x_\varepsilon - x)}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}. \tag{3.2}$$

Let $\varphi \in C^2(\mathbb{R}^n)$ such that φ touches u_ε from below at x and

$$D\varphi(x) = \eta, \quad D^2\varphi(x) = Z.$$

Then, by definition of u_ε,

$$u(y) - \varphi(z) + \frac{|y-z|^q}{q\varepsilon^{q-1}} \geq u_\varepsilon(z) - \varphi(z) \geq 0, \tag{3.3}$$

for all $y, z \in \Omega_{r(\varepsilon)}$. Since by (3.2) we have

$$u(x_\varepsilon) = \varphi(x) - \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}},$$

it follows from (3.3) that

$$u(y) - \varphi(z) + \frac{|y-z|^q}{q\varepsilon^{q-1}}$$

has a minimum at (x_ε, x). Thus,

$$-u(y) + \varphi(z) - \frac{|y-z|^q}{q\varepsilon^{q-1}}$$

attains its maximum over $\Omega_{r(\varepsilon)} \times \Omega_{r(\varepsilon)}$ at (x_ε, x). Let us consider

$$\Phi(y,z) := \frac{|y-z|^q}{q\varepsilon^{q-1}}.$$

By the Maximum Principle for semicontinuous functions (see [4, Theorem 3.2]), there exist symmetric matrices (Y, Z) such that

$$(\eta, Y) \in \overline{J}^{2,-} u(x_\varepsilon), \quad (\eta, Z) \in \overline{J}^{2,+} \varphi(x),$$

and

$$\begin{pmatrix} -Y & 0 \\ 0 & Z \end{pmatrix} \leq D^2\Phi(x_\varepsilon, x) + \varepsilon^{q-1} \left(D^2\Phi(x_\varepsilon, x)\right)^2, \tag{3.4}$$

with

$$\begin{aligned} D^2\Phi(x_\varepsilon, x) = \varepsilon^{1-q}|x_\varepsilon - x|^{q-4} \Bigg[& |x_\varepsilon - x|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\ & +(q-2)\begin{pmatrix} (x_\varepsilon - x) \otimes (x_\varepsilon - x) & -(x_\varepsilon - x) \otimes (x_\varepsilon - x) \\ -(x_\varepsilon - x) \otimes (x_\varepsilon - x) & (x_\varepsilon - x) \otimes (x_\varepsilon - x) \end{pmatrix} \Bigg]. \end{aligned}$$

Inequality (3.4) implies that, for any $\xi, \eta \in \mathbb{R}^n$,

$$Z\xi \cdot \xi - Y\eta \cdot \eta \leq \varepsilon^{1-q} \left[(q-1)|x_\varepsilon - x|^{q-2} + 2(q-1)^2|x_\varepsilon - x|^{2(q-2)}\right] |\eta - \xi|^2. \tag{3.5}$$

By the equivalence of the definition of viscosity solutions between tests functions and the closure of jets for continuous operators (recall $\eta \neq 0$), we deduce

$$\begin{aligned} f(x_\varepsilon, u(x_\varepsilon), \eta) &\leq -\mathrm{tr}\,(A(x_\varepsilon, \eta)Y) - B(x_\varepsilon, \eta) \\ &= \mathrm{tr}\,(A(x, \eta)Z) - \mathrm{tr}\,(A(x_\varepsilon, \eta)Y) - \mathrm{tr}\,(A(x, \eta)Z) + B(x, \eta) - B(x_\varepsilon, \eta) - B(x, \eta). \end{aligned} \tag{3.6}$$

Observe that since $\eta \neq 0$, $A(\cdot, \eta)$ is symmetric and positive definite, and hence the square root $A(x, \eta)^{1/2}$ exists and is symmetric. We define

$$A(x)^{1/2} := A(x, \eta)^{1/2} \quad \text{and} \quad A(x_\varepsilon)^{1/2} := A(x_\varepsilon, \eta)^{1/2}.$$

Now,

$$\mathrm{tr}\,(A(x, \eta)Z) = \mathrm{tr}\left(A(x)^{1/2}A(x)^{1/2}Z\right) = \sum_{k=1}^{n} ZA_k(x)^{1/2} \cdot A_k(x)^{1/2}, \tag{3.7}$$

where $A_k(\cdot)^{1/2}$ is the k-th column of $A(\cdot)^{1/2}$. Hence, (3.5), (3.6), and (3.7) give

$$\begin{aligned} f(x_\varepsilon, u(x_\varepsilon), \eta) \leq & \sum_{k=1}^{n} ZA_k(x)^{1/2} \cdot A_k(x)^{1/2} - \sum_{k=1}^{n} YA_k(x_\varepsilon)^{1/2} \cdot A_k(x_\varepsilon)^{1/2} - \mathrm{tr}\,(A(x, \eta)Z) \\ & + B(x, \eta) - B(x_\varepsilon, \eta) - B(x, \eta) \\ \leq & C\varepsilon^{1-q}|x_\varepsilon - x|^{q-2}\|A(x)^{1/2} - A(x_\varepsilon)^{1/2}\|_2^2 + B(x, \eta) - B(x_\varepsilon, \eta) \\ & - \mathrm{tr}\,(A(x, \eta)Z) - B(x, \eta). \end{aligned} \tag{3.8}$$

Proceeding as in [17, proof of Proposition 6.1], it can be seen that

$$\|A(x)^{1/2} - A(x_\varepsilon)^{1/2}\|_2^2 \leq \frac{\|A(x) - A(x_\varepsilon)\|_2^2}{(\lambda_{min}(A(x)) + \lambda_{min}(A(x_\varepsilon)))^2}. \tag{3.9}$$

and, using that $p \in C^1$,

$$B(x,\eta) - B(x_\varepsilon,\eta) \le |\eta|^{p(x)-1} |\log|\eta|| |Dp(x) - Dp(x_\varepsilon)| + C|\eta|^{s-1} \log^2|\eta| |p(x) - p(x_\varepsilon)|. \tag{3.10}$$

for some s in the interval connecting $p(x)$ and $p(x_\varepsilon)$. Furthermore,

$$\|A(x,\eta) - A(x_\varepsilon,\eta)\|_2 \le C\left((p^+ + 1)|\log|\eta|||\eta|^{s-2} + |\eta|^{p(x_\varepsilon)-2}\right)|x - x_\varepsilon|, \tag{3.11}$$

and

$$\lambda_{min}\left(A(x)^{1/2}\right) = \left(\min_{|\xi|=1} A(x,\eta)\xi\cdot\xi\right)^{1/2} \ge \min\left\{1, \sqrt{p(x)-1}\right\}|\eta|^{\frac{p(x)-2}{2}}. \tag{3.12}$$

Thus, combining (3.8), (3.9), (3.10), (3.11) and (3.12), we deduce

$$\begin{aligned} f(x_\varepsilon, u(x_\varepsilon), \eta) &\le C|\eta|^{p(x)-1}|\log|\eta|||x - x_\varepsilon| + C|\eta|^{s-1}\log^2|\eta||x - x_\varepsilon| \\ &\quad + C\varepsilon^{1-q} \frac{\left(|\log|\eta|||\eta|^{s-2} + |\eta|^{p(x_\varepsilon)-2}\right)^2}{\min\{1, p^- - 1\}\left[|\eta|^{\frac{p(x)-2}{2}} + |\eta|^{\frac{p(x_\varepsilon)-2}{2}}\right]^2}|x - x_\varepsilon|^q \\ &\quad - \operatorname{tr}(A(x,\eta)Z) - B(x,\eta). \end{aligned} \tag{3.13}$$

By Lemma 2.11 ((i) and (vi)), and since u is locally Lipschitz,

$$|x - x_\varepsilon|^q \le q\varepsilon^{q-1}|u_\varepsilon(x) - u(x_\varepsilon)| \le q\varepsilon^{q-1}|u(x) - u(x_\varepsilon)| \le C\varepsilon^{q-1}|x - x_\varepsilon|. \tag{3.14}$$

Hence,

$$|\eta| = q\varepsilon^{1-q}|x - x_\varepsilon|^{q-1} \le C. \tag{3.15}$$

Consequently, the terms

$$|\eta|^{p(x)-1}|\log|\eta|| \quad \text{and} \quad |\eta|^{s-1}\log^2|\eta|$$

remain bounded, and so the first two terms in (3.13) tend to 0 as $\varepsilon \to 0^+$. For the third term, we obtain by (3.15) that

$$\begin{aligned} \varepsilon^{1-q}\left(\frac{|\log|\eta||\eta|^{s-2}|}{|\eta|^{\frac{p(x)-2}{2}} + |\eta|^{\frac{p(x_\varepsilon)-2}{2}}}\right)^2 |x - x_\varepsilon|^q &\le C\log^2|\eta||\eta|^{2s-p(x)-2}|\eta||x - x_\varepsilon| \\ &\le C\varepsilon\log^2|\eta||\eta|^{2s-p(x)-1}. \end{aligned}$$

Since $2s - p(x) - 1 \to p(x) - 1 \ge p^- - 1 > 0$, the term $\log^2|\eta||\eta|^{2s-p(x)-1}$ is uniformly bounded for ε sufficiently small and thus

$$\varepsilon^{1-q}\left(\frac{|\log|\eta||\eta|^{s-2}|}{|\eta|^{\frac{p(x)-2}{2}} + |\eta|^{\frac{p(x_\varepsilon)-2}{2}}}\right)^2 |x - x_\varepsilon|^q \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Regarding the term

$$\varepsilon^{1-q}\frac{\left(|\eta|^{p(x_\varepsilon)-2}\right)^2}{\min\{1,p^- -1\}\left[|\eta|^{\dfrac{p(x)-2}{2}}+|\eta|^{\dfrac{p(x_\varepsilon)-2}{2}}\right]^2}|x-x_\varepsilon|^q,$$

by (3.15) it may be bounded by

$$\varepsilon^{1-q}|\eta|^{p(x_\varepsilon)-2}|x-x_\varepsilon|^q = q^{-1}|\eta|^{p(x_\varepsilon)-1}|x-x_\varepsilon| \to 0$$

as $\varepsilon \to 0^+$. Therefore, we get from (3.13) that

$$f(x_\varepsilon, u_\varepsilon(x), \eta) = f(x_\varepsilon, u(x_\varepsilon), \eta) \le -\mathrm{tr}\,(A(x,\eta)Z) - B(x,\eta) + E(\varepsilon),$$

with $E(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, u_ε is a viscosity supersolution of (3.1) and, since it is twice differentiable almost everywhere, (3.1) holds a.e. in $\Omega_{r(\varepsilon)}$. □

Remark 3.2. Observe that the Lipschitz condition on u was used to prove (3.14). If u is merely uniformly continuous, there exists a modulus of continuity ω so that $|u(x) - u(y)| \le \omega(x - y)$ for all $x, y \in \Omega$. Hence, by Lemma 2.11 ((i) and (vi)), we get

$$|x_\varepsilon - x|^q \le C\varepsilon^{q-1}\omega(r(\varepsilon))$$

and so

$$|\eta| \le C\varepsilon^{1-q}|x_\varepsilon - x|^{q-1} \le C\varepsilon^{-(q-1)/q}.$$

Consequently, the converges in (3.13) may not hold.

Let us pass now from the pointwise formulation in Lemma 3.1 to a weak inequality. We will use in the proof some computations from the proof of [20, Lemma 5.5].

Lemma 3.3. *Assume $f = f(x,t,\eta)$ uniformly continuous in $\Omega\times\mathbb{R}\times\mathbb{R}^n$ and Lipschitz continuous in η, satisfying* (1.4)*. Suppose in addition that $f(x,r,0) = 0$ for all $(x,r) \in \Omega\times\mathbb{R}$. If u is a locally Lipschitz viscosity solution of* (1.1)*, then for any non-negative $\varphi \in C_0^\infty(\Omega)$ there holds*

$$\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\cdot D\varphi\,dx \ge \int_{\Omega_{r(\varepsilon)}} f_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi\,dx + E(\varepsilon)\int_{\Omega_{r(\varepsilon)}\setminus\{Du_\varepsilon=0\}}\varphi\,dx,$$

for all $\varepsilon > 0$ small enough.

Proof. Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \gneq 0$. Let ε be small enough so that $\varphi \in C_0^\infty(\Omega_{r(\varepsilon)})$. Since u_ε is semi-concave, there is a constant $C(q,\varepsilon,u) > 0$ so that

$$\phi(x) := u_\varepsilon(x) - C(q,\varepsilon,u)|x|^2$$

is concave in $\Omega_{r(\varepsilon)}$. Hence, by mollification, there is a sequence of smooth concave functions ϕ_j so that

$$(\phi_j, D\phi_j, D^2\phi_j) \to (\phi, D\phi, D^2\phi) \quad a.e.\ \Omega_{r(\varepsilon)}.$$

Define

$$u_{\varepsilon,j}(x) := \phi_j(x) + C(q,\varepsilon,u)|x|^2.$$

Given $\delta > 0$, by integration by parts we obtain

$$\int_{\Omega_{r(\varepsilon)}} -\mathrm{div}\left[\left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p(x)-2}{2}} Du_{\varepsilon,j}\right]\varphi\,dx = \int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p(x)-2}{2}} Du_{\varepsilon,j}\cdot D\varphi\,dx. \tag{3.16}$$

Observe that, since u_ε is locally Lipschitz, there exists a constant $M > 0$, independent of j, so that

$$\sup_j \|Du_{\varepsilon,j}\|_{L^\infty(supp\,\varphi)}, \sup_j \|Dp_j\|_{L^\infty(supp\,\varphi)} \le M. \tag{3.17}$$

Hence, by the Dominated Convergence Theorem, the right-hand side of (3.16) converges, as $j \to \infty$, to

$$\int_{\Omega_{r(\varepsilon)}} (\delta + |Du_\varepsilon|^2)^{\frac{p(x)-2}{2}} Du_\varepsilon \cdot D\varphi\,dx.$$

Let us treat now the left-hand side of (3.16). Observe that

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} -\mathrm{div}\left[\left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} Du_{\varepsilon,j}\right]\varphi\,dx\\
&= -\int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} \left(\Delta u_{\varepsilon,j} + \frac{p_j(x)-2}{\delta + |Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right)\varphi\,dx\\
&\quad - \frac{1}{2}\int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} \log(\delta + |Du_{\varepsilon,j}|^2) Du_{\varepsilon,j}\cdot Dp_j\varphi\,dx\\
&=: I_1 + I_2.
\end{aligned} \tag{3.18}$$

By (3.17) and the Dominated Convergence Theorem we obtain that, when $j \to \infty$,

$$I_2 \to -\frac{1}{2}\int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \log(\delta + |Du_\varepsilon|^2) Du_\varepsilon \cdot Dp\,\varphi\,dx.$$

For I_1 we will use Fatou's Lemma. Observe that by concavity of ϕ_j,

$$D^2 u_{\varepsilon,j} \le C(q,\varepsilon,u)I.$$

Hence, the integrand in I_1 is bounded from below by a constant independent of j if $Du_{\varepsilon,j} = 0$. On the other hand, if $Du_{\varepsilon,j} \neq 0$, it can be checked (see [20, Lemma 5.5]) that

$$\left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} \left(\Delta u_{\varepsilon,j} + \frac{p_j(x)-2}{\delta + |Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right) \le C(\varepsilon, q, u, M, \delta).$$

Taking $\liminf$ as $j \to \infty$ in (3.18), we obtain

$$\begin{aligned}
&-\int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \left(\Delta u_\varepsilon + \frac{p(x)-2}{\delta + |Du_\varepsilon|^2}\Delta_\infty u_\varepsilon\right)\varphi\,dx\\
&\qquad - \frac{1}{2}\int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \log(\delta + |Du_\varepsilon|^2) Du_\varepsilon \cdot Dp\,\varphi\,dx\\
&\le \int_{\Omega_{r(\varepsilon)}} (\delta + |Du_\varepsilon|^2)^{\frac{p(x)-2}{2}} Du_\varepsilon \cdot D\varphi\,dx.
\end{aligned} \tag{3.19}$$

By the Dominated Convergence Theorem, as $\delta \to 0$ we have

$$\int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \log(\delta + |Du_\varepsilon|^2) Du_\varepsilon \cdot Dp\, \varphi\, dx \to 2 \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} \log |Du_\varepsilon| Du_\varepsilon \cdot Dp\, \varphi\, dx \tag{3.20}$$

and

$$\int_{\Omega_{r(\varepsilon)}} (\delta + |Du_\varepsilon|^2)^{p(x)-2} Du_\varepsilon \cdot D\varphi\, dx \to \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot D\varphi\, dx. \tag{3.21}$$

Moreover, using (2.2) and proceeding as in the proof of [20, Lemma 5.5], we can apply Fatou's lemma in the integral

$$\int_{\Omega_{r(\varepsilon)} \setminus \{Du_\varepsilon = 0\}} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \left(\Delta u_\varepsilon + \frac{p(x)-2}{\delta + |Du_\varepsilon|^2} \Delta_\infty u_\varepsilon\right) dx.$$

Thus, from (3.19)–(3.21), and Lemma 3.1 we conclude that

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot D\varphi\, dx \\
&\quad \geq \liminf_{\delta \to 0} \int_{\Omega_{r(\varepsilon)} \setminus \{Du_\varepsilon = 0\}} -\left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \left(\Delta u_\varepsilon + \frac{p(x)-2}{\delta + |Du_\varepsilon|^2} \Delta_\infty u_\varepsilon\right) \varphi\, dx \\
&\quad + \liminf_{\delta \to 0} \int_{\Omega_{r(\varepsilon)} \setminus \{Du_\varepsilon = 0\}} -\frac{1}{2} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \log(\delta + |Du_\varepsilon|^2) Du_\varepsilon \cdot Dp\, \varphi\, dx \\
&\quad \geq -\int_{\Omega_{r(\varepsilon)} \setminus \{Du_\varepsilon = 0\}} \Delta_{p(x)} u_\varepsilon \varphi\, dx \\
&\quad \geq \int_{\Omega_{r(\varepsilon)} \setminus \{Du_\varepsilon = 0\}} f_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \varphi\, dx + E(\varepsilon) \int_{\Omega_{r(\varepsilon)} \setminus \{Du_\varepsilon = 0\}} |Du_\varepsilon|^{\max\{p(x)-2, 0\}} \varphi\, dx \\
&\quad = \int_{\Omega_{r(\varepsilon)}} f_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \varphi\, dx + E(\varepsilon) \int_{\Omega_{r(\varepsilon)}} \varphi\, dx.
\end{aligned}$$

□

The proofs of the following lemmas follow the strategy in [20, Lemma 5.6, Lemma 5.7] for the homogeneous case, so we will only highlight the differences coming from the non-homogeneous term.

Lemma 3.4. *Under the assumptions of Lemma 3.3, $u \in W^{1,p(x)}_{loc}(\Omega)$ and, for each $\Omega' \Subset \Omega$, we have that, up to a subsequence, $u_\varepsilon \to u$ weakly in $W^{1,p(x)}(\Omega')$ as $\varepsilon \to 0$.*

Proof. Let $\Omega' \Subset \Omega$ and let $\xi \in C_0^\infty(\Omega)$ be such that $0 \leq \xi \leq 1$ and $\xi = 1$ in $\overline{\Omega'}$. Assume that

$$K := supp\, \xi \subset \Omega_{r(\varepsilon)},$$

and define

$$\varphi := (L - u_\varepsilon)\xi^{p^+}, \quad \text{with } L := \sup_{\varepsilon, \Omega'} |u_\varepsilon(x)|.$$

By Lemma 3.3, we have

$$\begin{aligned}
\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)} \xi^{p^+} dx &\leq \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-1} \xi^{p^+-1} (L - u_\varepsilon) p^+ |D\xi|\, dx \\
&+ \int_{\Omega_{r(\varepsilon)}} |f_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \varphi\, dx + |E(\varepsilon)| \int_{\Omega_{r(\varepsilon)}} \varphi\, dx.
\end{aligned} \tag{3.22}$$

By using Young's inequality it can be seen that

$$\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-1}\xi^{p^+-1}(L-u_\varepsilon)p^+|D\xi|\,dx \le \delta\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)}\xi^{p^+}dx + C(\delta,p,,L,D\xi), \tag{3.23}$$

and, using (1.4),

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} |f_\varepsilon(x,u_\varepsilon,Du_\varepsilon)|\varphi\,dx \\
&\qquad \le \gamma_\infty\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-1}\xi^{p^+-1}(L-u_\varepsilon)\xi\,dx + \int_{\Omega_{r(\varepsilon)}} \phi(x)(L-u_\varepsilon)\xi^{p^+}\,dx \\
&\qquad \le \gamma_\infty\delta\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)}\xi^{p^+} + \gamma_\infty\int_{\Omega_{r(\varepsilon)}}\left(\frac{2}{\delta}Lp^+\right)^{p(x)}dx + C(\phi,L,\Omega) \\
&\qquad \le \gamma_\infty\delta\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)}\xi^{p^+} + C(\phi,\delta,p,L,\gamma,\Omega).
\end{aligned} \tag{3.24}$$

Finally, it is easy to check that

$$|E(\varepsilon)|\int_{\Omega_{r(\varepsilon)}} \varphi\,dx \le |E(\varepsilon)|C(L,p,\Omega). \tag{3.25}$$

Combining (3.22)–(3.25) and recalling that $\xi = 1$ in Ω', we obtain the uniform boundedness of Du_ε in $L^{p(x)}(\Omega')$. Therefore, up to a subsequence, $u_\varepsilon \to u$ weakly in $W^{1,p(x)}(\Omega')$ as $\varepsilon \to 0$. □

Lemma 3.5. *Under the assumptions of Lemma 3.3, for each $\Omega' \Subset \Omega$, we have that, up to a subsequence, $u_\varepsilon \to u$ in $W^{1,p(x)}(\Omega')$ as $\varepsilon \to 0$.*

Proof. Let $\Omega' \Subset \Omega$ and let $\xi \in C_0^\infty(\Omega)$ be such that $0 \le \xi \le 1$ and $\xi = 1$ in $\overline{\Omega'}$. Consider the test function

$$\varphi := (u-u_\varepsilon)\xi,$$

and choose ε small enough so that $K := \operatorname{supp}\varphi \subset \Omega_{r(\varepsilon)}$. Observe that $\varphi \in W^{1,p(x)}(\Omega)$ and has compact support. By Lemma 3.3 and (1.4), we have

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} \left(|Du|^{p(x)-2}Du - |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du-Du_\varepsilon)\xi\,dx \\
&\quad \le \int_{\Omega_{r(\varepsilon)}} |Du|^{p(x)-2}Du\cdot(Du-Du_\varepsilon)\xi\,dx + \int_{\Omega_{r(\varepsilon)}} |f_\varepsilon(x,u_\varepsilon,Du_\varepsilon)|(u-u_\varepsilon)\xi\,dx \\
&\qquad + |E(\varepsilon)|\int_{\Omega_{r(\varepsilon)}} (u-u_\varepsilon)\xi\,dx + \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2}Du_\varepsilon\cdot D\xi(u-u_\varepsilon)dx \\
&\quad \le \int_{\Omega_{r(\varepsilon)}} |Du|^{p(x)-2}Du\cdot(Du-Du_\varepsilon)\xi\,dx \\
&\qquad + \|u-u_\varepsilon\|_{L^\infty(K)}\left(\gamma_\infty\int_K |Du_\varepsilon|^{p(x)-1}dx + \int_K \phi(x)\,dx\right) \\
&\qquad + \|u-u_\varepsilon\|_{L^\infty(K)}\left(|E(\varepsilon)||K| + \int_K |Du_\varepsilon|^{p(x)-1}D\xi dx\right).
\end{aligned} \tag{3.26}$$

Since $u_\varepsilon \to u$ locally uniformly and, by Lemma 3.4, $u_\varepsilon \to u$ weakly in $W^{1,p(x)}(K)$, the right-hand side of (3.26) converges to 0, up to a subsequence, when $\varepsilon \to 0$. By Theorem 2.6, the operator $L : W^{1,p(x)}(\Omega') \to [W^{1,p(x)}(\Omega')]^*$ given by

$$\langle L(v), w\rangle = \int_{\Omega'} |Dv|^{p(x)-2} Dv \cdot Dw\, dx, \quad v, w \in W^{1,p(x)}(\Omega')$$

is a mapping of type (S_+). Then, it follows from (3.26) that $u_\varepsilon \to u$ strongly in $W^{1,p(x)}(\Omega')$ as $\varepsilon \to 0$. □

We can already prove that viscosity solutions of (1.1) are also weak solutions.

Proof of Theorem 1.1. Let $\varphi \in C_0^\infty(\Omega)$ and take $\Omega' \Subset \Omega$ such that

$$\operatorname{supp} \varphi \subset \Omega'.$$

Let us fix $\varepsilon_0 > 0$ such that $0 < \varepsilon < \varepsilon_0$ implies

$$\Omega' \subset \Omega_{r(\varepsilon)}.$$

In view of Lemma 3.3, to prove the theorem, it will be enough to show the following convergences:

(I)
$$\lim_{\varepsilon\to 0^+} \int_{\Omega'} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot D\varphi\, dx = \int_{\Omega'} |Du|^{p(x)-2} Du \cdot D\varphi\, dx$$

(II)
$$\lim_{\varepsilon\to 0^+} \int_{\Omega_{r(\varepsilon)}} f_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\varphi\, dx = \int_{\Omega'} f(x, u, Du)\varphi\, dx$$

(III)
$$\lim_{\varepsilon\to 0^+} E(\varepsilon) \int_{\Omega_{r(\varepsilon)}} \varphi\, dx = 0.$$

Proceeding exactly as in the proof of [20, Theorem 5.8], it can be seen that (I) holds, and (III) follows in a straightforward way.

Let us prove (II). Let ε, φ and Ω' as above. By the uniform continuity of f, for every $\rho > 0$, there exists $\delta > 0$ such that

$$|f(x, u_\varepsilon(x), Du_\varepsilon(x)) - f(y, u_\varepsilon(x), Du_\varepsilon(x))| \le \rho, \quad y \in B_\delta(x).$$

Choose $\varepsilon_0 > 0$ so that $r(\varepsilon) < \delta$ for every $\varepsilon < \varepsilon_0$. Thus, from the previous inequality we get

$$f(x, u_\varepsilon(x), Du_\varepsilon(x)) < \rho + f(y, u_\varepsilon(x), Du_\varepsilon(x)),$$

for every $x \in \Omega'$ and $y \in B_{r(\varepsilon)}(x)$. In particular,

$$f(x, u_\varepsilon(x), Du_\varepsilon(x)) < \rho + f_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x)),$$

and therefore

$$0 \le |f(x, u_\varepsilon(x), Du_\varepsilon(x)) - f_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x))| < \rho.$$

Hence,

$$\int_{\Omega'} |f(x, u_\varepsilon, Du_\varepsilon) - f_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\varphi \, dx \le \rho \|\varphi\|_{L^\infty(\Omega')}|\Omega'|, \tag{3.27}$$

for ρ arbitrarily small. Since $\|u_\varepsilon\|_{L^\infty(\Omega')} \le \|u\|_{L^\infty(\Omega')}$ for all ε, it follows

$$\max_{[-\|u_\varepsilon\|_{L^\infty}, \|u_\varepsilon\|_{L^\infty}]} |\gamma(t)| \le \max_{[-\|u\|_{L^\infty}, \|u\|_{L^\infty}]} |\gamma(t)|,$$

and then by (1.4) we have

$$|f(x, u_\varepsilon, Du)| \le C|Du|^{p(x)-1} + \phi(x) \in L^{p'(x)}(\Omega') \subset L^1(\Omega')$$

for a constant C independent of ε. Then, by the Lebesgue Convergence Theorem,

$$\lim_{\varepsilon \to 0} \int_{\Omega'} f(x, u_\varepsilon, Du)\varphi \, dx = \int_{\Omega'} f(x, u, Du)\varphi \, dx. \tag{3.28}$$

Moreover, the convergence $Du_\varepsilon \to Du$ in $L^{p(x)}(\Omega')$ and the Lipschitz continuity of f with respect to the third variable imply

$$\int_{\Omega'} |f(x, u_\varepsilon, Du_\varepsilon) - f(x, u_\varepsilon, Du)|\varphi \, dx \le C \int_{\Omega'} |Du_\varepsilon - Du| \, dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.29}$$

Therefore, combining (3.27)–(3.29) we obtain (II). □

4. Weak solutions are viscosity solutions: proof of Theorem 1.3

To prove this implication we follow the strategy of [19]. As we said in the Introduction, the argument is strongly connected to the availability of comparison principles. After the proof of the theorem we will state and prove an example of comparison result that applies here.

Proof of Theorem 1.3. Let $u \in C(\Omega)$ be a weak supersolution to (1.1). To reach a contradiction, assume that u is not a viscosity supersolution. By assumption, there exist $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ so that $D\varphi(x_0) \neq 0$,

$$u(x_0) = \varphi(x_0), \quad u(x) > \varphi(x) \text{ for all } x \neq x_0, \tag{4.1}$$

and

$$-\Delta_{p(x_0)}\varphi(x_0) < f(x_0, u(x_0), D\varphi(x_0)). \tag{4.2}$$

Moreover, the mapping

$$x \to f(x, u(x), D\varphi(x))$$

is continuous in Ω, and (4.1) yields

$$-\Delta_{p(x)}\varphi(x) - f(x, u(x), D\varphi(x)) < 0, \quad \text{for all } x \in B_r(x_0),$$

for some $r > 0$ small eunogh. Hence, there exists $r_0 > 0$ so that $D\varphi(x) \neq 0$ for all $x \in B_{r_0}(x_0)$ and

$$-\Delta_{p(x)}\varphi(x) \le f(x, u(x), D\varphi(x)), \qquad x \in B_{r_0}(x_0). \tag{4.3}$$

Let

$$m := \min_{\partial B_{r_0}(x_0)} (u - \varphi)\,.$$

Then by (4.1), $m > 0$. Consider

$$\tilde{\varphi}(x) := \varphi(x) + m, \qquad x \in \Omega.$$

By (4.3), $\tilde{\varphi}$ is a weak subsolution to

$$-\Delta_{p(x)} v = \tilde{f}(x, Dv), \tag{4.4}$$

in $B_{r_0}(x_0)$, where $\tilde{f}(x,\eta) := f(x, u(x), \eta)$. Observe that $\tilde{f}$ is locally Lipschitz in $\Omega \times \mathbb{R}^n$. Moreover, in the weak sense, we have

$$-\Delta_{p(x)} u \geq f(x, u, Du) = \tilde{f}(x, Du),$$

which shows that u is a weak supersolution to (4.4). In addition, observe that $u \geq \tilde{\varphi}$ on $\partial B_{r_0}(x_0)$, and that (4.5) holds since $D\tilde{\varphi} \neq 0$ in $B_{r_0}(x_0)$. Thus, by the (CPP) we conclude that $u \geq \tilde{\varphi}$ in $B_{r_0}(x_0)$. This contradicts (4.1). □

4.1. Maximum principles

Some comparison principles may be found in the literature for $p(x)$-Laplace equations. See for instance [9, 17, 21]. Here, we provide a comparison principle for a Lipschitz right-hand side depending on all the lower terms.

Theorem 4.1. *Assume that $f = f(x, r, \eta)$ is locally Lipschitz in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Let $u, v \in C^1(\Omega)$ be weak sub- and supersolutions, respectively, of* (1.1) *such that*

$$|Du(x)| + |Dv(x)| > 0, \tag{4.5}$$

for a.e. x satisfying $p(x) > 2$. Then, there exists $\delta > 0$ such that for any domain B, $\overline{B} \subset \Omega$, with $|B| < \delta$ and $u \leq v$ on ∂B, there holds $u \leq v$ in B.

Proof. Take $(u - v)^+ \chi_B \in W_0^{1,p(x)}(\Omega)$ as a test function in (1.1) to get

$$\begin{aligned} \int_B \left(|Du|^{p(x)-2} Du - |Dv|^{p(x)-2} Dv \right) \cdot D(u-v)^+ \\ \leq \int_B \frac{f(x,u,Du) - f(x,v,Dv)}{u - v} [(u-v)^+]^2. \end{aligned} \tag{4.6}$$

Now, using the inequality

$$c(|\xi| + |\eta|)^{p(x)-2} |\xi - \eta|^2 \leq \left(|\xi|^{p(x)-2} \xi - |\eta|^{p(x)-2} \eta \right) \cdot (\xi - \eta),$$

the Lipschitz assumption on f and the boundedness of u and v in C^1, we get

$$\int_B (|Du| + |Dv|)^{p(x)-2} |D(u-v)^+|^2 \leq C(u,v) \left[\int_B [(u-v)^+]^2 + \int_B |Du - Dv| (u-v)^+ \right]. \tag{4.7}$$

By Poincaré and Hölder inequalities, and the assumption (4.5), we obtain

$$\begin{aligned}
C(u,v)&\left[\int_B[(u-v)^+]^2+\int_B|Du-Dv|(u-v)^+\right]\\
&\le C(B)C(u,v)\int_B|D(u-v)^+|^2 \quad (\text{here, } C(B)\to 0 \text{ as } |B|\to 0)\\
&= C(B)C(u,v)\int_B(|Du|+|Dv|)^{2-p(x)}(|Du|+|Dv|)^{p(x)-2}|D(u-v)^+|^2\\
&\le C(B)C(u,v)\int_B(|Du|+|Dv|)^{p(x)-2}|D(u-v)^+|^2.
\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\int_B(|Du|+|Dv|)^{p(x)-2}|D(u-v)^+|^2 \le C(B)C(u,v)\int_B(|Du|+|Dv|)^{p(x)-2}|D(u-v)^+|^2.$$

Hence, for $|B|$ small enough, we get $(u-v)^+=0$ in B and hence $u\le v$ in B. □

Remark 4.2. Keeping track of the proof of Theorem 1.3, it is easy to see that Theorem 4.1 allows us to prove that weak solutions are viscosity. Indeed, it is enough with choosing r_0 sufficiently small so that $|B_{r_0}(x_0)|<\delta$, with $\delta>0$ provided by Theorem 4.1.

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Conflict of interest

The authors declare no conflict of interest.

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