Gravitation, Relativistic Astrophysics and Cosmology
Second Argentinian-Brazilian Meeting, 2014
G. S. Vila, F. L. Vieyro and J. Fabris, eds.

Gravitational entropy of a Kerr black hole

Daniela Pérez[1], and Gustavo E. Romero [1,2]

[1]*Instituto Argentino de Radioastronomía, C.C.5, (1894) Villa Elisa, Bs. As., Argentina*
[2]*Facultad de Ciencias Astronómicas y Geofísicas, UNLP, Paseo del Bosque s/n CP (1900), La Plata, Bs. As., Argentina*

Abstract. The gravitational entropy of a Kerr black hole is calculated using a classical estimator based on the Bel-Robinson tensor, which has been recently proposed by Clifton, Ellis, and Tavakol. We prove that, in the frame we consider, Clifton et al.'s estimator does not reproduce the Bekenstein-Hawking entropy of a Kerr black hole.

1. Introduction

Black holes are among the simplest objects of the universe. They can be fully described by a small number of parameters: mass (M), angular momentum (J), and electric charge (Q). Wheeler seems to have been the first to notice that, if we are not to abandon the Second Law of Thermodynamics, material accreted by a black hole should not only transfer to the hole its mass, angular momentum, and electric charge, but its entropy as well. Bekenstein (1972, 1973) noticed that the properties of the area of the event horizon of a black hole resemble those of entropy and proposed the following relation:

$$S_{\rm BH} = \frac{A}{4\, l_{\rm P}^2}. \tag{1}$$

Here, $S_{\rm BH}$ is the entropy of the black hole, A is the area of the event horizon, and $l_{\rm Pl} = \sqrt{G\hbar c^{-3}}$ is the Planck length. A generalized second law of black hole thermodynamics was also derived by Bekenstein (1974). Bardeen, Carter, and Hawking (1973) formulated the four laws of black hole physics, which are similar to the four laws of thermodynamics.

Since black holes can be fully described in terms of the gravitational field, it seems reasonable to associate an entropy with the gravitational field itself. In absence of a theory of quantum gravity, a statistical measure of the gravitational entropy is not possible. Instead, approximations might be obtained using classical invariants of General Relativity, as first suggested by Penrose (1979).

Several authors have tried to implement Penrose's proposal. Recently, Clifton, Ellis and Tavakol (2013) offered a novel definition for the entropy of the gravitational field based on integrals over quantities constructed from the pure Weyl form of the Bel-Robinson tensor. In particular, they calculated the gravitational entropy for a Schwarzschild black hole, for a spatially flat Robertson-Walker geometry with scalar perturbations, and for the inhomogeneous Lemaître-Tolman-Bondi solution.

The main goal of the present work is to calculate the gravitational entropy of a Kerr black hole using Clifton et al.'s proposal, and test whether such estimator still reproduces the Benkenstein-Hawking entropy of a Kerr black hole.

Throughout this paper we use geometrized units $G = c = 1$.

2. Bel-Robinson estimator

Clifton, Ellis, and Tavakol (2013) defined the entropy of the gravitational field $S_{\rm grav}$ following these five requirements: 1) $S_{\rm grav} \geq 0$, 2) $S_{\rm grav} = 0 \Leftrightarrow C_{\rm abcd} = 0$, where $C_{\rm abcd}$ is the Weyl tensor, 3) $S_{\rm grav}$ gives a measure of the local anisotropy of the free gravitational field, 4) $S_{\rm grav}$ should be equal to the Bekenstein-Hawking entropy on the event horizon of a black hole, 5) $S_{\rm grav}$ should increase monotonically as structure forms in the universe.

In particular, Clifton and coworkers constructed a definition of $S_{\rm grav}$ in analogy with the fundamental law of thermodynamics:

$$T_{\rm grav} dS_{\rm grav} = dU_{\rm grav} + p_{\rm grav} dV. \tag{2}$$

Here, $T_{\rm grav}$, $S_{\rm grav}$, $U_{\rm grav}$ and $p_{\rm grav}$ stand for the effective temperature, entropy, internal energy, and isotropic pressure of the free gravitational field respectively, whereas V is the spatial volume. Expressions for the effective energy density $\rho_{\rm grav}$ and pressure $p_{\rm grav}$ are derived from the Bel-Robinson tensor, which for Coulomb-like gravitational fields, such as black hole spacetimes, take the form:

$$8\pi\rho_{\rm grav} = 2\alpha\sqrt{\frac{2\mathcal{W}}{3}}, \tag{3}$$

$$p_{\rm grav} = 0, \tag{4}$$

where α is a constant and $\mathcal{W}$ is the "super-energy density":

$$\mathcal{W} = \frac{1}{4}\left(E_a{}^b E^a{}_b + H_a{}^b H^a{}_b\right), \tag{5}$$

and E_{ab} and H_{ab} denote the electric and magnetic part of the Weyl tensor, respectively.

The temperature of the gravitational field is defined as a local quantity that reproduces the Hawking (1974, 1975), Unruh (1976), and de Sitter temperatures (Gibbons and Hawking, 1977) in the appropriate limits (Clifton at al., 2013). It has the following expression:

$$T_{\rm grav} = \frac{\left|\dot{u}_a z^a + H + \sigma_{ab} z^a z^b\right|}{2\pi}. \tag{6}$$

Here u_a is a timelike unit vector, z^a is a spacelike unit vector aligned with the Weyl principal tetrad, $H = \Theta/3$ being $\Theta = \nabla_a u^a$ the expansion scalar and $\sigma_{ab} = \nabla_{(a} u_{b)} + a_{(a} u_{b)} - 1/3\,\Theta h_{ab}$ is the shear tensor; h_{ab} is the projection tensor $h_{ab} = g_{ab} - (u_c u^c)\, u_a u_b$.

Clifton and coworkers calculated the gravitational entropy of a Schwarzschild black hole, recovering the Bekenstein-Hawking entropy on the event horizon of the hole. In the following section we extend their calculations to a Kerr black hole and analyze whether such estimator still represents a good classical measure of the entropy of the gravitational field.

3. Bel-Robinson estimator for Kerr black holes

The line element of the Kerr spacetime in oblate spheroidal coordinates (t, r, θ, ϕ) takes the form (Doran, 2000):

$$\begin{aligned} d\tau^2 &= -\left(1-\frac{2Mr}{\rho^2}\right)dt^2+\frac{\rho^2}{r^2+a^2}dr^2-2\sqrt{\frac{2Mr}{r^2+a^2}}dt\,dr-4aMr\frac{\sin^2\theta}{\rho^2}dt\,d\phi \\ &+ 2a\sqrt{\frac{2Mr}{r^2+a^2}}\sin^2\theta d\phi dr+\rho^2 d\theta^2+\left[\left(r^2+a^2\right)+2Mra^2\frac{\sin^2\theta}{\rho^2}\right]\sin\theta^2 d\phi^2, \end{aligned}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$. The constant M represents the mass of the black hole and a its angular momentum.

Clifton et al.'s proposal is frame-dependent. We make the simplest choice for a Kerr spacetime (see below). Specifically, we adopt the following unit vectors u^a and z^a:

$$u^a = \left(\frac{r^2+a^2}{\sqrt{-\Delta}\,\sqrt{r^2+u^2}}, 0, 0, \frac{a}{\sqrt{r^2+u^2}\,\sqrt{-\Delta}}\right), \tag{7}$$

$$z^a = \left(\frac{\sqrt{r^2+a^2}\,\sqrt{2Mr}}{\sqrt{-\Delta}\,\sqrt{r^2+u^2}}, \frac{\sqrt{-\Delta}}{\sqrt{r^2+u^2}}, 0, \frac{a\,\sqrt{2Mr}}{\sqrt{-\Delta}\,\sqrt{r^2+u^2}\,\sqrt{r^2+a^2}}\right), \tag{8}$$

where $u = a\cos\theta$ and $\Delta = r^2 + a^2 - 2Mr$. In the region interior to the outer event horizon $u^a u_a = 1$ and $z^a z_a = -1$. The vector z^a is chosen to be orthogonal to the hypersurfaces of constant time t. Because of the four-fold coordinate degrees of freedom inherent to General Relativity, there is not a unique foliation of spacetime into a family of nonintersecting spacelike 3-surfaces Σ. For the Kerr spacetime metric given by Eq. (7), we have checked that the unit vectors u^a and z^a satisfy all conditions for the calculation of the gravitational entropy as stated by Clifton et al. (2013).

We now proceed to the calculation of the gravitational energy density and temperature according to Eqs. (3) and (6), respectively.

The gravitational energy density takes the form:

$$\rho_{\rm grav} = \frac{\alpha}{4\pi}\frac{M}{(r^2+u^2)^{3/2}}. \tag{9}$$

In Figures 1 and 2,we show plots of $\rho_{\rm grav}$ as a function of the radial coordinate for $\theta = \pi/2$ and $\theta = \pi/4$, respectively. The gravitational energy density is everywhere well-defined and positive, except towards the ring singularity, as expected.

Figure 1. Plot of $\rho_{\rm grav}$ as a function of the radial coordinate for $a = 0.8$ and $\theta = \pi/2$.

Figure 2. Plot of $\rho_{\rm grav}$ as a function of the radial coordinate for $a = 0.8$ and $\theta = \pi/4$.

We obtain the following expression for the temperature of the gravitational field:

$$T_{\rm grav} = \frac{\left|-ra^2 - Mu^2 + ru^2 + Mr^2\right|}{2\pi\ (r^2+u^2)^{3/2}\sqrt{-\Delta}}, \tag{10}$$

where the absolute value brackets were added to avoid negative or complex values. A 3-dimensional plot of $T_{\rm grav}$ as a function of the radial and angular coordinate for $a = 0.8$ is shown in Figure 3. We see that $T_{\rm grav}$ is everywhere well-defined except towards the inner and outer horizons.

As explained by Clifton et al. (2013), a small change in the gravitational entropy density of a black hole occurs when a small amount of mass is added:

$$\delta s_{\rm grav} = \frac{\delta \left(\rho_{\rm grav} v\right)}{T_{\rm grav}}. \tag{11}$$

In the expression above the element of volume is $v = z^a \eta_{abcd} dx^b dx^c dx^d$, where $\eta_{abcd} = \eta_{[abcd]}$, $\eta_{0123} = \sqrt{|g_{ab}|}$. For the our particular coordinate choice:

$$v = \frac{\sqrt{2Mr}\left(r^2 + a^2\right)^{1/2}\left(r^2 + u^2\right)^{1/2}}{a\,\sqrt{-\Delta}} d\phi\; du\; dr. \tag{12}$$

Figure 3. Plot of $T_{\rm grav}$ as a function of the coordinates r and θ for $a = 0.8$.

Figure 4. Plot of $S_{\rm BR}$ and $S_{\rm BH}$ as a function of the angular momentum a.

We now proceed to calculate the gravitational entropy $S_{\rm grav}$ by performing the integration of Eq. (11) over the volume V enclosed by the outer event horizon on a hypersurface of constant t, for a fixed value of a:

$$S_{\rm grav} = \int_{\rm V} \frac{\rho_{\rm grav} v}{T_{\rm grav}}, \tag{13}$$

in order to test whether the Bel-Robinson proposal in the choosen frame reproduces the Bekenstein-Hawking entropy of a Kerr black hole. We notice, however, that independently of the coordinate choice, the region inside the inner horizon is not time-orientable since the region is chronology-violating (Visser, 1996). The contribution to the gravitational entropy should come from the region between the inner and outer horizons.

Integral (13) can explicitly be written as:

$$S_{\rm grav} = \beta \int_{r_-}^{r_+} \int_0^{\pi} \frac{r^{1/2}\ \left(r^2 + a^2\right)^{1/2}\ \left(r^2 + a^2 \cos^2\theta\right)^{1/2} \sin\theta\; d\theta\; dr}{|f(r,\theta)|}, \tag{14}$$

where $\beta = 2^{1/2}\pi\alpha\ M^{3/2}$. In the latter equation we have already integrated over the azimuthal coordinate ϕ. The function $f(r,\theta)$ is defined as:

$$f(r,\theta) \equiv -ra^2 + Mr^2 + a^2 \cos^2\theta \left(-M + r\right). \tag{15}$$

The domain of integration of Eq. (14) is:

$$T = \left\{ (r,\theta) \in \Re^2 /\ r_- \leq r \leq r_+ \ \wedge\ 0 \leq \theta \leq \pi \right\}. \tag{16}$$

We divide T into two subregions denoted D and G respectively, such that $T = D \cup G$,

$$D = \left\{ (r, \theta) \in \Re^2 / \ r_- \leq r \leq r_* \ \wedge \ 0 \leq \theta \leq \pi \right\}, \tag{17}$$

where r_* is the solution of the equation $f(r_*, 0) = 0$, and $G = T - D$.

Given the definitions above, Eq. (14) can be written as $S_{\rm grav} = \beta \left(S^{\rm D}_{\rm grav} + S^{\rm G}_{\rm grav} \right)$ where,

$$S^{\rm D}_{\rm grav} = \int\int_D \frac{r^{1/2} \ \left(r^2 + a^2\right)^{1/2} \ \left(r^2 + a^2 \cos^2\theta\right)^{1/2} \sin\theta \ d\theta \ dr}{|f(r, \theta)|}, \tag{18}$$

$$S^{\rm G}_{\rm grav} = \int\int_G \frac{r^{1/2} \ \left(r^2 + a^2\right)^{1/2} \ \left(r^2 + a^2 \cos^2\theta\right)^{1/2} \sin\theta \ d\theta \ dr}{|f(r, \theta)|}. \tag{19}$$

The integral given by Eq. (18) is an improper divergent integral; in particular it tends to infinity for those values of r and θ such that $f(r, \theta) = 0$. Contrary, integral (19) is well defined for $a \in (0, 1)$, and can be integrated numerically.

We show in Figure 4 the result of the numerical integration of $S^{\rm G}_{\rm grav}$ (see Eq. 19). We also plot the Bekenstein-Hawking entropy, denoted $S_{\rm BH}$, as a function of the angular momentum of the hole. It is clear that $S_{\rm BR}$ does not reproduce the Bekenstein-Hawking entropy of a black hole. We conclude that even in the domain of integration G where the entropy is well defined, it is not a good approximation to the Bekenstein-Hawking entropy, at least for the current coordinate choice. We do not discard that for a different choice of vectors u^a and z^a, the Bel-Robinson proposal may coincide with Bekenstein-Hawking result. However, the fact that the innermost region of the Kerr spacetime is not folliable and time-orientable suggests that our result might be general.

4. Final remarks

We have computed the gravitational energy density, temperature, and gravitational entropy of a Kerr black hole according to the Bel-Robinson estimator. The calculations were performed using a pair of vectors u_a and z^a spacelike and timelike, respectively, that determine a Weyl principal tetrad. The choice of such vectors, however, is not unique, thus being $\rho_{\rm grav}$ and $T_{\rm grav}$ frame dependent quantities. Under the simplest coordinate choice, we proved that the gravitational entropy is not well defined.

The first requirement that a reliable classical estimator of the gravitational entropy needs to fulfill is that it should be well-behaved in all types of horizons where quantum field calculations can be used as an independent probe of the entropy. Only when a complete match be obtained, the classical estimators can be used to evaluate other families of spacetimes with some confidence.

Acknowledgments. This work is supported by PICT 2012-00878, Préstamo BID (ANPCyT). We thank Santiago E. Perez Bergliaffa for useful advice.

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