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On the zeros of certain polynomials and entire functions

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Abstract

Entire functions whose coefficients are polynomials having real and negative roots are built from Touchard polynomials. The particular set of polynomials

$$\sum_{j=0}^{n} \binom{n}{j} (z+2j-2n+1)(z+2j-2n+3)\cdots(z+2j-1), \quad (1 \leq n)$$

is shown to have purely complex roots, where we show a connection of these polynomials with certain approximations of the Riemann's zeta function. Also a certain class of Fourier transforms is shown to have only real roots.

Keywords: *Zeros. Polynomials. Entire functions. Riemann's zeta function.*

Subjclass [2010]: *26C10, 30C15.*

1. Introduction and results

The aim of this note is to prove some results about the zeros of certain polynomials and Fourier transforms. We add an appendix showing the connection of some of our results to certain approximations in number theory. Each section is independent to read.

The location of roots of polynomials and entire functions is an old subject. Classical texts are [9], [12]. The reader may consult [6] for an abundant bibliography up to 2011 on zeros of Fourier transforms.

We recall that an entire function $f(z)$ is said to be in the Laguerre-Pólya class if it is the uniform limit on compact sets of polynomials with real roots. This is known [6] to be equivalent to the fact that there exists β, α, α_n real, $0 \leq \alpha$ with $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and

$$f(z) = e^{-\alpha z^2 + \beta z} \prod_{n=1}^{\infty} (1 + z\alpha_n)\, e^{-z\alpha_n}.$$

We will use Hurwtiz theorem in its simplest form: *if* $f_n(z), f(z)$ are entire functions and $f_n(z) \to f(z)$ uniformly on compact sets of the complex plane where the roots of $f_n(z)$ are real then either $f(z)$ is identically zero or it has only real roots.

2. Use of Touchard polynomials.

Theorem 1. *i) Assume that $f(z)$ is in the Laguerre-Pólya class and that*

$$exp\left(x\left\{e^{\{e^t-1\}}-1\right\}\right) = 1 + x\sum_{n=1}^{\infty} p_{n-1}(x)t^n,$$

$$f\left(x\left\{e^{\{e^t-1\}}-1\right\}\right) = \sum_{n=0}^{\infty} q_n^f(x)t^n.$$

Then $p_{n-1}(x)$, $n = 1, 2, \ldots,$ is a polynomial with positive coefficients of $n-1$ degree with only real negative roots.

Also the polynomials $q_n^f(x)$, $n = 0, 1, 2, \ldots,$ have only real roots.

ii) Set

$$exp\left(x\left\{e^{\left\{e^{\left(e^t-1\right)}-1\right\}}-1\right\}\right) = 1 + x\sum_{n=1}^{\infty} p_{n-1}^*(x)t^n,$$

Then $p_{n-1}^*(x)$, $n = 1, 2, \ldots,$ is a polynomial with positive coefficients of $n-1$ degree with only real negative roots.

Proof. *i*) The Touchard polynomials $T_n(x)$ have only real roots. These roots are all negative except a simple root at $x = 0$. They can be defined using the generating function (see [8])

$$\sum_{n=0}^{\infty} \frac{T_n(x)}{n!} t^n = exp\left\{x(e^t - 1)\right\}.$$

Next observe that by Laguerre's theorem ([12] (II), problem 67, part V) and Hurwitz theorem if a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ has only real roots then

$$a_0 T_m(0) + a_1 T_m(1)x + \cdots + a_k T_m(k)x^k = \frac{m!}{2\pi i} \int_\gamma \frac{\sum_{n=0}^k a_n x^n exp\left\{n(e^t - 1)\right\}}{t^{m+1}} dt, \tag{2.1}$$

has real roots. Here the equality follows from Cauchy's formula where γ is a curve enclosing the origin.

If fact from careful examination of the proof given in [12] if a polynomial $p(z)$ has real and negative roots then (2.1) has only real and non positive roots (or more simply observe that $a_i > 0, T_m(i) > 0$ if $i = 1, 2, \ldots,$ and $T_m(0) = 0$).

Now taking $p(x) = (1 + x/k)^k$ yields that $\frac{m!}{2\pi i} \int_\gamma \left(1 + \frac{xe^{e^t - 1}}{k}\right)^k \frac{1}{t^{m+1}} dt$ has only real and non positive roots. Letting $k \to \infty$ in this last formula and writing

$$exp(xe^{e^t - 1}) = \sum_{n=0}^{\infty} q_n(x) t^n,$$

yields that the function

$$q_m(x) = \frac{1}{2\pi i} \int_\gamma exp(xe^{e^t - 1}) \frac{1}{t^{m+1}} dt,$$

has only real and non positive roots.

Finally observe that $e^{e^t - 1} - 1 = c_1 t + c_2 t^2 + \ldots$ with $c_i > 0$ for all i. This implies that

$$exp\left(x\left\{e^{e^t - 1} - 1\right\}\right) = 1 + x \sum_{n=1}^{\infty} p_{n-1}(x) t^n,$$

where $p_{n-1}(x)$ is a polynomial of $n - 1$ degree with real and negative roots (observe that $e^x x p_{n-1}(x) = q_n(x)$).

If $f(z)$ is in the Laguerre-Pólya class then it can be approximated on compact sets by polynomials $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ and the proof is exactly the same.
ii) Again by Laguerre's theorem if a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ has real and negative roots then similarly as in (*i*)

$$a_1p_{m-1}(1)x + a_2 2p_{m-1}(2)x^2 + \cdots = \frac{1}{2\pi i}\int_\gamma \frac{\sum_{n=0}^{k} a_nx^n exp\left(n\left\{e^{e^t-1}-1\right\}\right)}{t^{m+1}}dt,$$

has real and non positive roots.

Now taking $p(x) = (1 + x/k)^k$, writing

$$exp\left(xe^{e^{e^t-1}-1}\right) = \sum_{n=0}^{\infty} r_n(x)t^n,$$

and making $k \to \infty$ yields that the function

$$r_m(x) = \frac{1}{2\pi i}\int_\gamma exp(xe^{e^t-1})\frac{1}{t^{m+1}}dt,$$

has only real and non positive roots. But then it can be proved in a similar way as before that

$$exp\left(x\left\{e^{e^{e^t-1}-1} - 1\right\}\right) = 1 + x\sum_{n=1}^{\infty} p^*_{n-1}(x)t^n,$$

where $p^*_{n-1}(x)$ is a polynomial with positive coefficients (and therefore with negative roots). The proof is complete. $\square$

Iterated exponentials have been studied by Bell [1] and also by Ramanujan ([2] see Chapter 4 and bibliography therein).

3. On the zeros of a class of polynomials

The main result of this section is the following theorem.

Theorem 2. Let $1 \le n$. The polynomials

$$\sum_{j=0}^{n}\binom{n}{j}(z+2j-2n+1)(z+2j-2n+3)\cdots(z+2j-1), \tag{3.1}$$

have all its roots on the line $z = it$, $t \in \mathbf{R}$.

Proof. Given a function $f(z)$, we define the operator $Tf(z) := f(z+1) + f(z-1)$ and let $g_m(z)$ be a polynomial of $m+1$ degree defined by

$$g_m(z) := (z-m)(z-m+2)\cdots(z+m-2)(z+m).$$

It is understood that $g_0(z) = z$.

Then $T^n g_{n-1}(z)$ is equal to (3.1).

The following lemma is the key to the proof.

Lemma 1. *Assume that*

$$f(z) := g_m(z)\prod_{j=1}^{r}(z^2+\Delta_j),$$

with m a non negative integer, $\Delta_j > 0$ and $0 \leq r$ (in case $r=0$ the last product is set equal to 1).

If $m=0$ or $m=1$ then $Tf(z)$ and $T^2 f(z)$ have all their roots on the line $z = it$.

If $m \geq 2$ then $T^2 f(z)$ is of the form

$$4g_{m-2}(z)\prod_{j=1}^{r+1}(z^2+\Delta_j^*)$$

with $\Delta_j^* > 0$.

In order to prove Lemma 1 we shall need the following lemma (compare with Lemma 1 in [4]).

Lemma 2. *Let $\Delta \geq 0$. If $\Re(z) > 0$ then*

$$|(z+1)^2+\Delta| > |(z-1)^2+\Delta|.$$

For $\Re(z) < 0$, the reverse inequality holds.

Proof. The lemma readily follows by observing that, for $z = x+iy$, $x \in \mathbf{R}$, $y \in \mathbf{R}$,

$$|(z+1)^2+\Delta|^2 - |(z-1)^2+\Delta|^2 = 8x(x^2+y^2+\Delta+1).$$

$\square$

We shall prove Lemma 1 for $m \geq 2$ (the proof for cases $m = 0, 1$ is implicitly contained in what follows).

We have that

$$Tf(z) = g_{m-1}(z)E(z),$$

where

$$E(z) = (z+m+1)\prod_{j=1}^{r}((z+1)^2+\Delta_j) + (z-m-1)\prod_{j=1}^{r}((z-1)^2+\Delta_j).$$

We will show that all the roots of $E(z)$ are on the line it, $t \in \mathbf{R}$. In fact, if $E(z) = 0$, then

$$|(z+m+1)\prod_{j=1}^{r}((z+1)^2+\Delta_j)| = |(z-m-1)\prod_{j=1}^{r}((z-1)^2+\Delta_j)|.$$

Thus, it cannot be $\Re(z) > 0$ since then $|(z+m+1)| > |(z-m-1)|$ and, by Lemma 2, the same happens to each term in the product. A similar reasoning for the case $\Re(z) < 0$ yields that all the roots of $E(z)$ are on the line it, $t \in \mathbf{R}$.

As $E(z)$ has degree $2r+1$, $E(0) = 0$, $E'(0) \neq 0$ and $E(\bar{z}) = \overline{E(z)}$, we have that $E(z) = 2z\prod_{j=1}^{r}(z^2+\Delta_j^{'})$ with $\Delta_j^{'} > 0$.

Applying T one more time, one gets

$$T^2 f(z) = g_{m-2}(z)E^*(z),$$

where

$$E^*(z) = (z-m)E(z-1) + (z+m)E(z+1).$$

Since $E^*(z) = 0$ implies

$$|(z+m)2(z+1)\prod_{j=1}^{r}((z+1)^2+\Delta_j^{'})| = |(z-m)2(z-1)\prod_{j=1}^{r}((z-1)^2+\Delta_j^{'})|,$$

then the same argument using Lemma 2 yields that all the roots of $E^*(z)$ are on the line it, $t \in \mathbf{R}$. As $E^*(z)$ has degree $2r+2$, $E^*(0) \neq 0$ and $E^*(\bar{z}) = \overline{E^*(z)}$, then $E^*(z) = 4\prod_{j=1}^{r+1}(z^2+\Delta_j^*)$ with $\Delta_j^* > 0$. This ends the proof of the Lemma 1.

The theorem follows using Lemma 1 repeatedly with $f(z) = g_{n-1}(z)$. $\square$

4. Remarks on the zeros of certain Fourier transforms

Lemma 3. *Assume that* $\sum_{k=0}^{N} a_{k,N} z^k$ *is a family of polynomials with its roots on* $|z| = 1$*. Furthermore assume that there is a continuous function* $b(x) : [0,1] \to \mathbf{R}$ *such that* $|b(k/N) - a_{k,N}|$ *can be made arbitrarily small for sufficiently large* N*, uniformly in* k*.*

If $p(z)$ *is a real polynomial with only real roots then*

$$\int_{-1}^{1} b\left(\frac{u+1}{2}\right) p(z+iu)du$$

has only real zeros or it is identically equal to zero.

Proof. In fact one knows that $q_{N,\lambda}(z) := \sum_{k=0}^{N} a_{k,N} p(z + (2k - N)i\lambda)$ has real roots, Theorem 4 pg. 201 [4]. Note that as $N \to \infty$

$$\frac{1}{N} q_{N,1/N}(z) = \frac{1}{N} \sum_{k=0}^{N} a_{k,N} p(z + (2k/N - 1)i) \to \int_0^1 b(\tau) p(z + (2\tau - 1)i) d\tau.$$

The result follows changing variables $2\tau - 1 = u$ □

Lemma 4. *Assume that* $b(x)$ *is as in the last lemma. Then*

$$\int_{-1}^{1} b\left(\frac{u+1}{2}\right) e^{izu} du \tag{4.1}$$

has only real roots.

Proof. By the last lemma the integral $I(z) = \int_{-1}^{1} b\left(\frac{u+1}{2}\right)(z+iu)^n du$ has real roots as does $z^n I(1/z) =: J(z)$. Now $J(z/n)$ tends to the integral (4.1) as $n \to \infty$. □

Theorem 3. Assume that $b(x)$ satisfies one of the following two conditions:
i) $b(x) := f(x) + f(1-x)$ where $f(x) : [0,1] \to \mathbf{R}$ is non-negative, bounded and non-decreasing function.
ii) $b(x) : [0,1] \to \mathbf{R}$, $b(x)$ is non-negative and $b(x) = b(1-x)$. Furthermore there exists $0 < \alpha < 1/2$ such that $b(x)$ is non-increasing on $[0, \alpha]$, $b(x)$ is non-decreasing on $[\alpha, 1]$ with $b(1/2) \leq 2b(\alpha)$.

If the function (4.1) is not identically zero then it has only real roots.

Proof. *i*) Assume for the moment that $f(x)$ is continuous. As $0 \leq f(x)$ is non-decreasing by the Eneström-Kakeya theorem the polynomial $q(z) = \sum_{k=0}^{N} f(k/N)z^k$ has its roots in $|z| \leq 1$. Then by a theorem of Schur [13] we know that $q(z) + z^N q(1/z) = \sum_{k=0}^{N} \{f(k/N) + f(1-k/N)\} z^k$ has its roots on $|z| = 1$. We are in the conditions of the last lemma with $b(x) = f(x) + f(1-x)$ and then (4.1) has real roots.

The general theorem follows using a limiting argument and Hurwitz's theorem.

ii) Again assume that $b(x)$ is continuous and that $0 < \alpha = k_0/N_0 < 1/2$ for some positive integers k_0, N_0. By Theorem 3 of [5] the polynomial $q(z) = \sum_{k=0}^{N_0} b(k/N_0)z^k$ has its roots in $|z| = 1$ as does $q_M(z) = \sum_{k=0}^{2^M N_0} b(k/(2^M N_0))z^k$ with $M = 1, 2, 3, \ldots$ (diadic subdivision). Thus using the above lemmas then the function (4.1) has real roots.

Again a limit argument yields the general result. $\square$

5. Appendix

Here we show a connection between the polynomials given in Theorem 2 with some results proved in [11] about certain approximations of the Riemann's zeta function (see [14] or [7]). For the sake of completeness, we provide ad-hoc proofs, correct a minor error that appear in [11] and we add some new results.

It is known (see Lemma 6) that if $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann's zeta function, then in the strip $0 < \Re(s) < 1$,

$$\zeta(s)\frac{\pi(1-2^{1-s})}{\sin(\pi s)} = \int_0^{\infty} f(\lambda)\lambda^{-s}d\lambda, \tag{5.1}$$

where

$$f(\lambda) := \int_0^1 \frac{1}{1+x}x^{\lambda}dx = \frac{1}{1+\lambda} - \frac{1}{2+\lambda} + \frac{1}{3+\lambda} - \cdots.$$

Set

$$g(s) := \zeta(s)\frac{\pi(1-2^{1-s})}{\sin(\pi s)}. \tag{5.2}$$

As it is shown below, the function $g(s)$ is approximated in the strip $0 < \Re(s) < 1$ by

$$-g_n(s) := -\int_0^{\infty} \frac{A_n(\lambda)}{B_n(\lambda)}\lambda^{-s}d\lambda, \tag{5.3}$$

where, for $\lambda \geq 0$ and $n \geq 0$, $A_n(\lambda)$ and $B_n(\lambda)$ are defined in terms of the (Jacobi) polynomials

$$F_n(x) = F_n(x,\lambda) := \sum_{j=0}^{n} (-1)^{n-j} x^j \frac{(j+\lambda+1)_n}{(n-j)!j!}, \tag{5.4}$$

as

$$A_n(\lambda) := \int_0^1 \frac{F_n(x) - F_n(-1)}{1+x} x^\lambda dx, \tag{5.5}$$

$$B_n(\lambda) := F_n(-1) = (-1)^n \sum_{j=0}^{n} \frac{(j+\lambda+1)_n}{(n-j)!j!}. \tag{5.6}$$

Here $(\alpha)_n$ stands for $\alpha(\alpha+1)\cdots(\alpha+n-1)$ and $(\alpha)_0 = 1$.

The key observation is that, if we set $z = 2\lambda$, then

$$2^n n!(-1)^n B_n(\lambda - n - 1/2) = \sum_{j=0}^{n} \binom{n}{j} (z+2j-2n+1)(z+2j-2n+3)\cdots(z+2j-1),$$

which is the polynomial appearing in Theorem 2.

It turns out that A_n and B_n satisfy a three-term recurrence relation (Theorem 5). Since polynomials $B_n(\lambda)$ appear in these formulas as denominators, we state some of their properties in Theorem 6. In Theorem 7 we give a closed form formula for $g_m(s)$ using $B_n(\lambda)$.

The two main ingredients of the proof below are the Euler's hypergeometric formula and the orthogonality of the $F_n(x)$ with respect to a power weight (see Lemma 5). Although orthogonality of Jacobi polynomials $F_n(x)$ is well known, we provide a simple proof that only uses Cauchy's residue theorem, following the lines of [3, Ch. 3].

6. Statement of the results

With the definitions and notation given we have the following results.

Theorem 4. Let $0 < \epsilon < 1/2$ and $s = \sigma + it$ such as $\epsilon \leq \sigma \leq 1 - \epsilon$ then

$$|g_n(s) + g(s)| \leq \frac{2}{\epsilon} \frac{(\sqrt{2}-1)^{2n}}{\binom{2n}{n}} \sim \frac{2\sqrt{\pi}}{\epsilon} \sqrt{n}\, 0.04289^n.$$

Note: a constant $\frac{1}{2\pi\epsilon}$ was stated in [11] instead of the correct $\frac{2}{\epsilon}$.

Theorem 5. For $n \geq 2$, $A_n(\lambda)$ and $B_n(\lambda)$ satisfy the recurrence relation

$$X_n + (\beta_n - \alpha_n)X_{n-1} + \gamma_n X_{n-2} = 0, \tag{6.1}$$

where

$$\begin{aligned} \alpha_n(\lambda) &= -\tfrac{(2n+\lambda)(2n-1+\lambda)}{n(n+\lambda)}, \\ \beta_n(\lambda) &= (2n-1+\lambda) - \tfrac{(2n+\lambda)(2n-1+\lambda)(n-1+\lambda)(n-1)}{n(n+\lambda)(2n-2+\lambda)}, \\ \gamma_n(\lambda) &= \tfrac{(n-1)(n+\lambda-1)(n+\lambda/2)}{n(n+\lambda)(n+\lambda/2-1)}. \end{aligned} \tag{6.2}$$

Furthermore

$$A_0(\lambda) = 0,\ A_1(\lambda) = \frac{2+\lambda}{1+\lambda},$$

$$B_0(\lambda) = 1,\ B_1(\lambda) = -3 - 2\lambda.$$

The next result gives some information about the polynomials $B_n(\lambda)$.

Theorem 6. $B_n(\lambda)$ is a polynomial of n degree in λ such that,

1. $(-1)^n B_n(\lambda)$ is positive and increasing on $[0, +\infty)$.
2. $(-1)^n B_n(\lambda) \geq \binom{2n}{n}$, $\lambda \in [0, +\infty)$.
3. All the roots of $B_n(\lambda)$ are located on the line $-n - \frac{1}{2} + it$, $t \in \mathbf{R}$.
4. If n is even, $B_n(\lambda) = \frac{2^n}{n!} \prod_{j=1}^{n/2} ((\lambda + n + 1/2)^2 + \Delta_j)$, with $\Delta_j > 0$.
5. If n is odd, $B_n(\lambda) = -\frac{2^n}{n!}(\lambda + n + 1/2) \prod_{j=1}^{(n-1)/2} ((\lambda + n + 1/2)^2 + \Delta_j)$, with $\Delta_j > 0$.

We give an alternative formula for (5.3).

Theorem 7. Let s be in the strip $0 < \Re(s) < 1$, then if $m = 1, 2, \ldots$ there exists a real polynomial $Q_{m-1}(\lambda)$ of degree $\leq m - 1$ so that

$$\begin{aligned} g_m(s) &= \int_0^\infty \left(\sum_{n=1}^m \tfrac{(2n+\lambda)}{n(n+\lambda)} \tfrac{1}{B_n(\lambda) B_{n-1}(\lambda)} \right) \lambda^{-s} d\lambda \\ &= \int_0^\infty \left(\left(\sum_{n=1}^m \tfrac{(-1)^n}{n+\lambda} \right) + \tfrac{Q_{m-1}(\lambda)}{B_m(\lambda)} \right) \lambda^{-s} d\lambda. \end{aligned}$$

Remark 1. *The values of* $\frac{Q_{m-1}(\lambda)}{B_m(\lambda)}$ *if* $m = 1, 2, 3, 4, 5$ *are given respectively by*

$$\begin{array}{l} \frac{1}{3+2\lambda}, \\ -\frac{(5+2\lambda)}{2(13+10\lambda+2\lambda^2)}, \\ \frac{4}{5(7+2\lambda)} + \frac{(7+2\lambda)}{10(27+14\lambda+2\lambda^2)}, \\ -\frac{(423+256\lambda+54\lambda^2+4\lambda^3)}{4(963+792\lambda+250\lambda^2+36\lambda^3+2\lambda^4))}, \\ \frac{64}{89(11+2\lambda)} + \frac{(20273+9736\lambda+1650\lambda^2+100\lambda^3)}{356(2295+1496\lambda+378\lambda^2+44\lambda^3+2\lambda^4)}. \end{array}$$

If one sets $h_m(1/2 + it) := g_m(1/2 + it) + g_m(1/2 - it)$ then one knows by Theorem 4 that

$$-h_m(1/2+it) \to \frac{\pi}{\cosh(\pi t)} \left(\zeta(1/2 + it)(1 - 2^{1/2-it}) + \zeta(1/2 - it)(1 - 2^{1/2+it}) \right),$$

if $m \to \infty$ in the strip $|\Im t| < 1/2$.

Using that

$$\int_0^\infty \frac{1}{z+\lambda} \lambda^{-1/2} \left(\lambda^{it} + \lambda^{-it} \right) d\lambda = \frac{\pi}{\cosh(\pi t)} (1 + z^{2it}) z^{-1/2-it},$$

one gets, for example, that if $|\Im t| < 1/2$ then

$$h_1(1/2 + it) = 6^{-1/2-it} \left(2^{2it} - 2^{3/2+it} 3^{1/2+it} + 3^{2it} \right) \frac{\pi}{\cosh(\pi t)},$$

and $h_2(1/2 + it) = \frac{\pi 2^{-2-it} \operatorname{sech}(\pi t)}{\sqrt{13}} C$, where $C :=$

$$\begin{aligned} \left(\tfrac{5}{169} + \tfrac{i}{169} \right)^{it} (5 - i)^{\frac{1}{2}+2it} &+ \sqrt{5+i}(5 - i)^{it} + \left(\tfrac{5}{169} - \tfrac{i}{169} \right)^{it} (5 + i)^{\frac{1}{2}+2it} \\ &+ \sqrt{5-i}(5 + i)^{it} + \sqrt{13} 2^{3+it} - \sqrt{13} 2^{\frac{3}{2}+2it} - 2\sqrt{26}. \end{aligned}$$

7. Previous lemmas

The next two lemmas will be used in the proof of Theorem 4 and 5. The first one is about the polynomials $F_n(x)$ defined in (5.4).

Lemma 5. *Let* $n \geq 1$ *and* $\lambda \geq 0$. $F_n(x)$ *are orthogonal to* $1, x, x^2, \cdots, x^{n-1}$ *in* $[0, 1]$ *respect to the measure* x^λ, *that is*

$$\int_0^1 F_n(x) x^{k+\lambda} dx = 0,$$

for $k=0,\cdots,n-1$. *Also we have*

$$\int_0^1 F_n(x)x^{\lambda+n+j}dx=\frac{(j+1)_n}{(\lambda+n+j+1)_{n+1}},$$

for $j=0,1,2,\cdots$.

Proof. Using residues one easily gets

$$F_n(x)=\frac{1}{2\pi i}\int_\gamma \frac{(t+\lambda+1)(t+\lambda+2)\cdots(t+\lambda+n)}{t(t-1)\cdots(t-n)}x^t dt,$$

where γ is a curve enclosing $0,1,...,n$. By taking γ to be contained in $\Re(t)>-1/2$ we can applied Fubini's Theorem to obtain

$$\begin{aligned}\int_0^1 F_n(x)x^{\lambda+k}dx &= \frac{1}{2\pi i}\int_0^1\int_\gamma \frac{(t+\lambda+1)(t+\lambda+2)\cdots(t+\lambda+n)}{t(t-1)\cdots(t-n)}x^{t+\lambda+k}dtdx\\ &= \frac{1}{2\pi i}\int_\gamma \frac{(t+\lambda+1)\cdots(t+\lambda+n)}{t(t-1)\cdots(t-n)}\frac{dt}{(t+\lambda+k+1)}=0\end{aligned}$$

for $k=0,\cdots,n-1$. The last integral is zero since, after simplifying the fraction, γ can be changed by a circumference of arbitrary large radius.

Also, taking γ as above, we have, for $j\geq 0$,

$$\int_0^1 F_n(x)x^{\lambda+n+j}dx=\frac{1}{2\pi i}\int_\gamma \frac{(t+\lambda+1)\cdots(t+\lambda+n)}{t(t-1)\cdots(t-n)}\frac{dt}{(t+\lambda+n+j+1)},$$

which turns out to be $\frac{(j+1)_n}{(\lambda+n+j+1)_{n+1}}$ since the integrand has residue 0 at infinity. □

For $\lambda\geq 0$ we define

$$f(\lambda):=\int_0^1 \frac{1}{1+x}x^\lambda dx=\frac{1}{1+\lambda}-\frac{1}{2+\lambda}+\frac{1}{3+\lambda}-\cdots.$$

Then we have the following lemma.

Lemma 6. *Let* s *be in the strip* $0<\Re(s)<1$, *then, with the above definition,*

$$g(s):=\zeta(s)\frac{\pi(1-2^{1-s})}{\sin(\pi s)}=\int_0^\infty f(\lambda)\lambda^{-s}d\lambda.$$

Proof. One may prove this formula by making the change of variable $t=\lambda/n$ in

$$\int_0^\infty \frac{t^{-s}}{1+t}dt=\frac{\pi}{\sin(\pi s)},$$

which is valid for $0<\Re(s)<1$, and adding in n. □

8. Proof of Theorem 4

Let $0 < \epsilon < 1/2$ and $s = \sigma + it$ with $\epsilon \le \sigma \le 1-\epsilon$. First we observe that $g_n(s)$ are well defined since the rational functions $\frac{A_n(\lambda)}{B_n(\lambda)}$ are continuous on $[0, +\infty)$ and

$$\frac{A_n(\lambda)}{B_n(\lambda)} = O(\lambda^{-1}),$$

as $\lambda \to +\infty$.

By Lemma 6 we have to estimate

$$\begin{aligned} |g_n(s) + g(s)| &= \left| \int_0^\infty \left(\tfrac{A_n(\lambda)}{B_n(\lambda)} + f(\lambda) \right) \lambda^{-s} d\lambda \right| \\ &\le \tfrac{1}{|B_n(0)|} \int_0^\infty |A_n(\lambda) + B_n(\lambda) f(\lambda)| \lambda^{-\sigma} d\lambda, \end{aligned} \tag{8.1}$$

where we have used that $|B_n(\lambda)|$ is increasing on $[0, +\infty)$, which follows from (5.6).

Now recalling formulas (5.4), (5.5), (5.6) the key observation is that

$$\begin{aligned} \int_0^1 \tfrac{F_n(x)}{1+x} x^\lambda dx &= \int_0^1 \tfrac{F_n(x) - F_n(-1)}{1+x} x^\lambda dx + \int_0^1 \tfrac{F_n(-1)}{1+x} x^\lambda dx \\ &= A_n(\lambda) + B_n(\lambda) f(\lambda), \end{aligned}$$

which will allow us to prove

$$|A_n(\lambda) + B_n(\lambda) f(\lambda)| \le \frac{(\sqrt{2}-1)^{2n}}{\lambda + 1}. \tag{8.2}$$

The proof of this inequality is as follows. From Lemma 5, and recalling the Euler's formula for the hypergeometric function ${}_2F_1(a, b, c, z)$, with $\Re(c) > \Re(b) > 0$,

$$\sum_{m=0}^\infty \frac{(a)_m (b)_m}{(c)_m m!} z^m = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \tau^{b-1} (1-\tau)^{c-b-1} (1 - \tau z)^{-a} d\tau,$$

we have

$$\begin{aligned} \pm \int_0^1 \tfrac{F_n(x)}{1+x} x^\lambda dx &= \int_0^1 F_n(x) \left\{ x^{\lambda+n} - x^{\lambda+n+1} + x^{\lambda+n+2} - \cdots \right\} dx \\ &= \tfrac{(1)_n}{(\lambda+n+1)_{n+1}} - \tfrac{(2)_n}{(\lambda+n+2)_{n+1}} + \tfrac{(3)_n}{(\lambda+n+3)_{n+1}} - \cdots \\ &= \tfrac{n!}{(\lambda+n+1)_{n+1}} \sum_{m=0}^\infty (-1)^m \tfrac{(1+n)_m (n+\lambda+1)_m}{(2n+2+\lambda)_m m!} \\ &= \int_0^1 \tau^{n+\lambda} (1-\tau)^n (1+\tau)^{-n-1} d\tau. \end{aligned}$$

Now (8.2) follows immediately since the above expression can be estimated by

$$\left(\max_{\tau\in[0,1]}\frac{\tau(1-\tau)}{1+\tau}\right)^n\int_0^1\tau^\lambda(1+\tau)^{-1}d\tau\leq\frac{(\sqrt{2}-1)^{2n}}{\lambda+1}.$$

Finally, from (8.1) and (8.2) we have

$$|g_n(s)+g(s)|\leq\frac{(\sqrt{2}-1)^{2n}}{|B_n(0)|}\int_0^\infty\frac{\lambda^{-\sigma}}{\lambda+1}d\lambda,$$

and the theorem follows since $\binom{2n}{n}\leq|B_n(0)|$ and

$$\int_0^\infty\frac{\lambda^{-\sigma}}{\lambda+1}d\lambda\leq\int_0^1\lambda^{-\sigma}d\lambda+\int_1^\infty\lambda^{-\sigma-1}d\lambda\leq 2/\epsilon.$$

9. Proof of Theorem 5

Observe that the coefficient x^n of $F_n(x)$ is non-zero. Therefore one can find constants α_n,β_n so that $F_n(x)+(\alpha_n x+\beta_n)F_{n-1}(x)$ is a polynomial of degree at most $n-2$, which in turn can be written as a linear combination of $F_{n-2}(x),\ldots,F_0(x)$. If this equality is multiplied by $x^{k+\lambda}$, $k=0,\cdots,n-3$, and orthogonality (Lemma 5) is used, then only the coefficient of $F_{n-2}(x)$ survives. This yields that, for $n\geq 2$, there exist $\alpha_n,\beta_n,\gamma_n$ such that

$$H_n(x):=F_n(x)+(\alpha_n x+\beta_n)F_{n-1}(x)+\gamma_n F_{n-2}(x)\equiv 0. \tag{9.1}$$

Coefficients $\alpha_n,\beta_n,\gamma_n$ can be calculated equating the coefficients of x^n,x^{n-1},x^{n-2} and using (5.4). This yields (6.2).

By letting $x=-1$ in (9.1) we obtain that B_n verify (6.1).

To prove that A_n also verify (6.1) we observe that

$$\begin{aligned}0 &= \int_0^1\frac{H_n(x)-H_n(-1)}{1+x}x^\lambda dx\\ &= A_n+(\beta_n-\alpha_n)A_{n-1}+\gamma_{n-1}A_{n-2}+\alpha_n\int_0^1F_{n-1}(x)x^\lambda dx.\end{aligned}$$

The last integral above is zero for $n\geq 2$ by Lemma 5. This completes the proof.

10. Proof of Theorem 6

We shall only prove Item *3* as the others are easy to check.

Observe that, if we set $z = 2\lambda$, then $2^n n!(-1)^n B_n(\lambda - n - 1/2)$ equals to

$$\sum_{j=0}^{n} \binom{n}{j} (z + 2j - 2n + 1)(z + 2j - 2n + 3) \cdots (z + 2j - 1).$$

Therefore, to prove Item *3*, it is enough to show that this polynomial has all its roots on the line $z = it$, $t \in \mathbf{R}$. But this is the content of Theorem 2.

11. Proof of Theorem 7

Observe that if $\Delta_n := B_{n-1}A_n - A_{n-1}B_n$ then, from Theorem 5, one gets $\Delta_n = \gamma_n \Delta_{n-1}$ (Hint: multiply the recurrence relation for A_n by B_{n-1} and the one for B_n by A_{n-1} and subtract). Then,

$$\Delta_n = \gamma_n \gamma_{n-1} \cdots \gamma_2 \Delta_1 = 2\frac{(n + \lambda/2)}{n(n + \lambda)}$$

where the last equality follows from the fact that $\gamma_n = \frac{(n-1)(n+\lambda-1)(n+\lambda/2)}{n(n+\lambda)(n+\lambda/2-1)}$ and $\Delta_1 = \frac{2+\lambda}{1+\lambda}$. Therefore

$$\begin{aligned} g_n(s) - g_{n-1}(s) &= \int_0^\infty \tfrac{A_n(\lambda)}{B_n(\lambda)} \lambda^{-s} ds - \int_0^\infty \tfrac{A_{n-1}(\lambda)}{B_{n-1}(\lambda)} \lambda^{-s} d\lambda \\ &= \int_0^\infty \tfrac{\Delta_n}{B_n(\lambda) B_{n-1}(\lambda)} \lambda^{-s} d\lambda, \end{aligned}$$

and the first formula of Theorem 7 follows.

To prove the second formula set

$$\sum_{n=1}^{m} \frac{(2n + \lambda)}{n(n + \lambda)} \frac{1}{B_n(\lambda) B_{n-1}(\lambda)} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n + \lambda} = \frac{P_1(\lambda)}{P_2(\lambda)}$$

where $P_2(\lambda) := (1+\lambda) \cdots (m+\lambda) B_1(\lambda) \cdots B_m(\lambda)$ and $P_1(\lambda)$ is a real polynomial whose degree is less than the degree of $P_2(\lambda)$. Recall that the roots of $B_n(\lambda)$ are on the line $-n - 1/2 + it$ with t real, so that it has no root in common with $B_j(\lambda)$ if $j \neq n$. We look at the singularities of $\frac{P_1(\lambda)}{P_2(\lambda)}$.

i) Due to the definition (5.6) then one has that $B_{n-1}(-n) = 1$, $B_n(-n) = (-1)^n$ if $n = 1, 2, \ldots$ (all terms are zero except one) and therefore

$$B_{n-1}(-n) B_n(-n) = (-1)^n,$$

which yields that $\frac{2n+\lambda}{n(n+\lambda)}+(-1)^{n-1}\frac{B_{n-1}(\lambda)B_n(\lambda)}{n+\lambda}$ is a real polynomial or, in other words, that

$$\frac{(2n+\lambda)}{n(n+\lambda)}\frac{1}{B_{n-1}(\lambda)B_n(\lambda)}+(-1)^{n-1}\frac{1}{n+\lambda},$$

is analytic around $\lambda=-n$. Therefore $(n+\lambda)|P_1(\lambda)$.
ii) The recurrence relation for B_n given by (6.1) can be written (divide by $B_nB_{n-1}B_{n-2}$) as

$$\frac{(2n-2+\lambda)}{(n-1)(n-1+\lambda)}\frac{1}{B_{n-1}(\lambda)B_{n-2}(\lambda)}+\frac{(2n+\lambda)}{n(n+\lambda)}\frac{1}{B_n(\lambda)B_{n-1}(\lambda)}$$
$$=-\frac{(\beta_n(\lambda)-\alpha_n(\lambda))\ (2n-2+\lambda)}{(n-1)(n-1+\lambda)B_n(\lambda)B_{n-2}(\lambda)}.$$

The right hand side of this expression is an analytic function in a neighbourhood of any root of $B_{n-1}(\lambda)$. This yields that $B_{n-1}(\lambda)|P_1(\lambda)$ if $n=1,\dots,m$.

Steps (i-ii) yield that we can write

$$\frac{P_1(\lambda)}{P_2(\lambda)}=\frac{Q_{m-1}(\lambda)}{B_m(\lambda)},$$

with $Q_{m-1}(\lambda)$ a polynomial of degree less than m.

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