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Limited Packing and Multiple Domination problems: Polynomial time reductions☆

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ABSTRACT

The Limited Packing and Multiple Domination problems in graphs have closely-related definitions and the same computational complexity on several graph classes. In this work we present two polynomial time reductions between them. Besides, we take into consideration generalized versions of these problems and obtain polynomial time reductions between each one and its generalized version.

1. Introduction

The well-known concept of domination in graphs, introduced by Berge in 1962, was generalized by Harary and Haynes in 2000 [7]. On the other side, the notion of 2-packing in graphs, introduced by Meir and Moon in 1975, was recently generalized by Gallant et al. [6].

These concepts are good models for many utility location problems in operations research. In the Limited Packing problem (LP), the utilities are necessary but probably obnoxious. That is why we are interested in placing the maximum number of utilities in such a way that no more than a given number of them is near to each agent in a given scenario. In the Multiple Domination problem (MD), we can think that every agent has a minimum requirement of expensive utilities and we want to satisfy the requirements by placing at least a given number of them in its neighborhood, with the minimum cost.

Both problems are NP-complete for general graphs and several graph classes are known to be polynomial time solvable instances of both of them. In first place, separate linear time algorithms for MD and LP for trees were provided independently in [8,2], respectively. Later, Liao and Chang extended their results by providing a linear time algorithm for MD for strongly chordal graphs, a superclass of trees [9]. Also in [5], we proved that both problems are polynomial time solvable for spiders, quasi-spiders and P_4-tidy graphs. As regards graph classes where both problems are NP-complete, the authors in [9] showed that MD is NP-complete for split graphs and bipartite graphs. In [5], the same results were obtained for LP.

These "symmetric" results lead us naturally to wonder if there is some kind of equivalence between MD and LP or, on the contrary, if it is possible to find a graph class where one of them is polynomial time solvable and the other, NP-complete. While working on this question, we started looking for polynomial time reductions between them.

In Section 3, we present two polynomial time reductions: one from LP to MD and the other, in the other direction. By considering graph classes which are closed under the involved transformations, computational complexity results for one of the problems give rise to results for the other.

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Both problems have quite natural generalizations in the following sense. First, different agents may have different facility requirements. Second, some places where to locate the facilities may be "not allowed" (for the packing case) and "required" (for the covering case). In this sense, generalized versions for MD and LP were first introduced in [8,3], respectively.

In Section 4, we prove that, within graph classes that are closed under certain operations, LP and MD are, respectively, "as hard" as their generalizations.

The results in Section 3 have been already published – without proofs – in an electronic version [4].

2. Preliminaries, basic definitions and notation

Graphs in this work are simple and connected and, for a graph G, $V(G)$ and $E(G)$ denote respectively its vertex and edge sets.

Given a graph G, a set $C \subseteq V(G)$ is *complete* if $|C| \geq 1$ and every two distinct vertices in C are adjacent in G.

Given a graph class $\mathbf{G}$ and a graph transformation P, we say that $\mathbf{G}$ is *closed under* P if $P(G) \in \mathbf{G}$ for all $G \in \mathbf{G}$.

Given a family $\mathcal{F}$ of graphs, a graph G is $\mathcal{F}$*-free* if for any induced subgraph G' of G, $G' \notin \mathcal{F}$.

A *stable set* in G is a set of pairwise nonadjacent vertices and a *clique*, a set of pairwise adjacent vertices.

For $v \in V(G)$, $N_G(v)$ and $N_G[v]$ denote respectively its *neighborhood* and *closed neighborhood* and $d_G(v)$ the *degree* of v in G. The *minimum* and *maximum degree* in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *pendant vertex* in a graph is a vertex of degree one.

For a given positive integer k, a k*-limited packing* in G is a subset of vertices B verifying $|N_G[v] \cap B| \leq k$, for every $v \in V(G)$. From its definition, it is clear that every subset of a k-limited packing in G is also a k-limited packing in G.

The *Limited Packing* problem (LP) may be formulated as follows:

INSTANCE: A graph G; positive integers k, α.
QUESTION: Does G contain a k-limited packing of size at least α?

A nontrivial instance of LP is one for which $k \leq \alpha \leq |V(G)| - 1$ and $\Delta(G) \geq k$.

On the other side, a k*-tuple dominating set* in G is a subset of vertices D verifying $|N_G[v] \cap D| \geq k$, for every $v \in V(G)$. From its definition, it is clear that every superset of a k-tuple dominating set in G is also a k-tuple dominating set in G.

The *Multiple Domination* problem (MD) may be formulated as follows:

INSTANCE: A graph G; positive integers k, α.
QUESTION: Does G contain a k-tuple dominating set of size at most α?

A nontrivial instance of MD is one for which $k + 1 \leq \alpha \leq |V(G)| - 1$ and $\delta(G) + 1 \geq k$.

The concept of a k-limited packing in a graph was generalized in [3].

Let us denote by $\mathbb{Z}_+$ the set of nonnegative integers. Given a vector $\mathbf{c} = (c_v) \in \mathbb{Z}_+^{V(G)}$ and $\mathcal{A} \subseteq V(G)$, a subset B of vertices is a $(\mathbf{c}, \mathcal{A})$*-limited packing* in G if $B \subseteq \mathcal{A}$ and $|N_G[v] \cap B| \leq c_v$, for every $v \in V(G)$. When $\mathcal{A} = V(G)$, we will simply refer to $(\mathbf{c}, \mathcal{A})$-limited packings as $\mathbf{c}$-limited packings. Clearly, when $c_v = k$ for every $v \in V(G)$, $\mathbf{c}$-limited packings are k-limited packings.

Given $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$, let $\mathbf{c}' \in \mathbb{Z}_+^{V(G)}$ be defined by $c'_v = \min\{c_v, d_G(v) + 1\}$ for each $v \in V(G)$. It holds that B is a $(\mathbf{c}, \mathcal{A})$-limited packing in G if and only if B is a $(\mathbf{c}', \mathcal{A})$-limited packing in G. This fact leads us to define the *Generalized Limited Packing* problem (GLP) in the following way:

INSTANCE: A graph G; $\mathcal{A} \subseteq V(G)$, $\alpha \in \mathbb{N}$,
$\mathbf{c} = (c_v) \in \mathbb{Z}_+^{V(G)}$ with $c_v \leq d_G(v) + 1$ for all $v \in V(G)$.
QUESTION: Does G contain a $(\mathbf{c}, \mathcal{A})$-limited packing of size at least α?

We will consider an instance of GLP nontrivial, when $\mathcal{A} \neq \emptyset$, $\alpha \leq |\mathcal{A}| - 1$ and $c_v \leq d_G(v)$ for some $v \in V(G)$.

Observe that solving a nontrivial instance of LP given by G, k and α is equivalent to solving the nontrivial instance of GLP given by G, $\mathcal{A} = V(G)$, α and $\mathbf{c}$, where $c_v = k$ for each $v \in V(G)$. Then, the NP-completeness of LP for a graph class implies the NP-completeness of GLP for the same graph class.

In a similar way, the concept of a k-tuple dominating set in a graph has been generalized in [8]. In this work we have modified the original notation slightly in order to make it consistent with the one for the packing case.

Given a vector $\mathbf{r} = (r_v) \in \mathbb{Z}_+^{V(G)}$ and $\mathcal{R} \subseteq V(G)$, a subset D of vertices is a $(\mathbf{r}, \mathcal{R})$*-dominating set* in G if $\mathcal{R} \subseteq D$ and $|N_G[v] \cap D| \geq r_v$, for every $v \in V(G)$. When $\mathcal{R} = \emptyset$, we will simply refer to $(\mathbf{r}, \mathcal{R})$-dominating sets as $\mathbf{r}$-dominating sets. Clearly, when $r_v = k$ for every $v \in V(G)$, $\mathbf{r}$-dominating sets are k-tuple dominating sets.

Notice that G has $(\mathbf{r}, \mathcal{R})$-dominating sets if and only if $r_v \leq d_G(v) + 1$, for every $v \in V(G)$. This fact leads us to introduce the *Generalized Multiple Domination* problem (GMD) in the following way:

INSTANCE: A graph G; $\mathcal{R} \subseteq V(G)$, $\alpha \in \mathbb{N}$,
$\mathbf{r} = (r_v) \in \mathbb{Z}_+^{V(G)}$ with $r_v \leq d_G(v) + 1$ for all $v \in V(G)$.
QUESTION: Does G contain a $(\mathbf{r}, \mathcal{R})$-dominating set of size at most α?

We will consider an instance of GMD nontrivial, when $\mathcal{R} \neq V(G)$, $\alpha \geq \max\{r_v : v \in V(G)\}$ and $r_v \leq d_G(v)$ for some $v \in V(G)$.

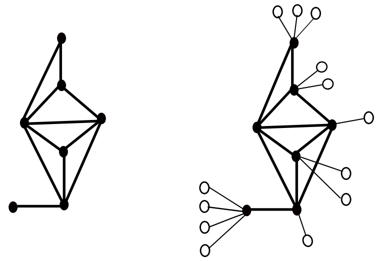

Fig. 1. A graph G and $P(G)$ in Definition 3.

Observe that solving a nontrivial instance of MD given by G, k and α is equivalent to solving the nontrivial instance of GMD given by G, $\mathcal{R} = \emptyset$, α and $\mathbf{r}$, where $r_v = k$ for each $v \in V(G)$. Thus, the NP-completeness of MD implies the NP-completeness of GMD.

GLP and GMD are equivalent in the following sense:

Proposition 1 ([3]). *Let G, $\mathcal{A}$ and $\mathbf{c}$ define an instance of GLP and $B \subseteq V(G)$. Then, B is a $(\mathbf{c}, \mathcal{A})$-limited packing in G if and only if $V(G) - B$ is a $(\mathbf{r}, V(G) - \mathcal{A})$-dominating set in G, where $r_v = d_G(v) + 1 - c_v$, for $v \in V(G)$.*

3. Polynomial reductions between MD and LP

Let us begin this section by stating a simple fact which shows the relevance of vertices of degree one in finding maximum k-limited packings.

For each $v \in V(G)$, we denote

$$N_G^1(v) := \{w \in N_G(v) : d_G(w) = 1\}.$$

Lemma 2. *Given a k-limited packing B in a graph G and $v \in V(G)$, there exists a k-limited packing B' in G such that $|B' \cap N_G^1(v)| = \min\{|N_G^1(v)|, k\}$ and $|B|' \geq |B|$.*

Proof. It is straightforward that $|B \cap N_G^1(v)| \leq \min\{|N_G^1(v)|, k\}$.

Therefore, it is enough to consider the case $|B \cap N_G^1(v)| < \min\{|N_G^1(v)|, k\}$ and prove that there exists a k-limited packing B' such that $|B|' \geq |B|$ and $|B' \cap N_G^1(v)| > |B \cap N_G^1(v)|$.

Observe that, since $B \cap N_G^1(v) \neq N_G^1(v)$, there exists $w \in N_G^1(v) \setminus B$.

If $B \cap N_G[v] \subseteq N_G^1(v)$, then $B' := B \cup \{w\}$ is a k-limited packing in G. If not, $B' := (B \setminus \{w'\}) \cup \{w\}$ is a k-limited packing in G, for each $w' \in (B \cap N_G[v]) \setminus N_G^1(v)$. Clearly, in both cases we have $|B|' \geq |B|$ and $|B' \cap N_G^1(v)| = |B \cap N_G^1(v)| + 1$.

Lemma 2 motivates us to define the following graph transformation P, which is the basis for the reduction of MD to LP in Theorem 4 below:

Definition 3. Given a graph G, $P(G)$ is the graph obtained from G by adding for each $v \in V(G)$, $\Delta(G) - d_G(v)$ vertices that are adjacent to v and define a stable set S_v (see Fig. 1).

Notice that

$$|V(P(G))| = |V(G)| + \sum_{v \in V(G)} [\Delta(G) - d_G(v)] = |V(G)|[\Delta(G) + 1] - 2|E(G)|.$$

If $|V(G)| = n$, $\Delta(G) \leq n - 1$ and, since G is connected, $|E(G)| \geq n - 1$. Then

$$|V(P(G))| \leq n^2 - 2(n - 1).$$

Notice that this bound is achieved when G is a *star* (a tree consisting of one vertex adjacent to all the others) on n vertices, since in this case $|E(G)| = \Delta(G) = n - 1$.

Theorem 4. *Given G, k and α defining a nontrivial instance of MD, let $\alpha' := |V(P(G))| - \alpha$ and $k' := \Delta(G) - k + 1$. Then G has a k-tuple dominating set of size at most α if and only if $P(G)$ has a k'-limited packing of size at least α'.*

Proof. Let G, k and α define a nontrivial instance of MD, $\alpha' := |V(P(G))| - \alpha$ and $k' := \Delta(G) - k + 1$. Notice that $k' \geq 2$.

Let R be a k-tuple dominating set in G with $|R| \leq \alpha$. We define $B := V(P(G)) - R$. Clearly, $|B| \geq \alpha'$. To prove that B is a k'-limited packing in $P(G)$, take $w \in V(P(G))$. If $w \notin V(G)$, $|N_{P(G)}[w] \cap B| \leq 2 \leq k'$. If $w \in V(G)$,

$$|N_{P(G)}[w] \cap B| = |S_w \cap B| + |N_G[w] - R| = |S_w| + |N_G[w] - R|.$$

Since R is a k-tuple dominating set in G, $|N_G[w] \cap R| \geq k$, thus $|N_G[w] - R| \leq d_G(w) + 1 - k$. Therefore, $|N_{P(G)}[w] \cap B| \leq k'$.

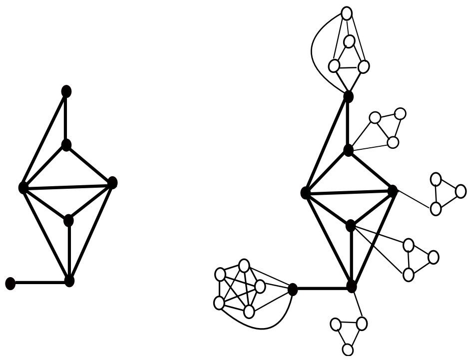

Fig. 2. A graph G and $D_3(G)$ in Definition 6.

To see the converse, let B be a k'-limited packing in $P(G)$ of size at least α'. Taking into account Lemma 2, since $S_v \subseteq N^1_{P(G)}[v]$ and $|S_v| \leq k'$ for each $v \in V(G)$, we can assume w.l.o.g that $S_v \subseteq B$, for all $v \in V(G)$. Let $R := V(G) - B$. We will prove that R is a k-tuple dominating set in G.

For $v \in V(G)$, $|N_G[v] \cap R| = d_G(v) + 1 - |N_G[v] \cap B|$ and $|N_G[v] \cap B| = |N_{P(G)}[v] \cap B| - |S_v|$.

Then,

$$|N_G[v] \cap R| = 1 - |N_{P(G)}[v] \cap B| + \Delta(G).$$

Since B is a k'-limited packing in $P(G)$, $|N_G[v] \cap R| \geq 1 - k' + \Delta(G) = k$ as desired.

We have the following immediate corollary:

Corollary 5. *Let* **G** *be a graph class that is closed under P. If LP is polynomial time solvable on* **G***, then MD is also polynomial time solvable on* **G***. Besides, if MD is NP-complete on* **G***, then LP is also NP-complete on* **G***.*

As regards NP-completeness results, recall that MD is NP-complete on split graphs [9], and therefore, on the superclass of chordal graphs. Since the class of chordal graphs is closed under P, from Corollary 5 we obtain that LP is NP-complete on this class as well, a result already obtained in [5] from the NP-completeness of LP on split graphs. However, Corollary 5 does not allow us to derive the NP-completeness of LP on split graphs since it is not a closed under P class.

The remainder of this section is devoted to give a reduction from LP to MD. For this purpose, we introduce the following transformation:

Definition 6. Given a graph G and a positive integer k with $1 \leq k \leq \Delta(G)$, $D_k(G)$ is the graph obtained from G by adding, for each $v \in V(G)$ with $d_G(v) < \Delta(G)$, $\Delta(G) + 1 - \min\{k, d_G(v)\}$ vertices defining a clique Q_v, and making v adjacent to exactly $\Delta(G) - d_G(v)$ vertices of Q_v (see Fig. 2).

Notice that

$$|V(D_k(G))| = |V(G)| + \sum_{v:\, d_G(v) < \Delta(G)} [\Delta(G) + 1 - \min\{k, d_G(v)\}].$$

Since $\min\{k, d_G(v)\} \geq 1$ and $|\{v \in V(G) : d_G(v) < \Delta(G)\}| \leq |V(G)| - 1$, if $|V(G)| = n$ we have:

$$|V(D_k(G))| \leq n + (n-1)(n-1).$$

Again, when G is a star on n vertices, the bound is achieved.

This transformation allows us to prove:

Theorem 7. *Given G, k and α defining a nontrivial instance of LP, let*

$$\alpha' := |V(D_k(G))| - \sum_{v:\, d_G(v) \leq k-1} (k - d_G(v)) - \alpha \quad \textit{and} \quad k' := \Delta(G) - k + 1.$$

Then G has a k-limited packing of size at least α if and only if $D_k(G)$ has a k'-tuple dominating set of size at most α'.

Proof. Let G, k and α define a nontrivial instance of LP.

Notice that $\delta(D_k(G)) = \Delta(G) - \min\{k, \tilde{\Delta}(G)\}$, where $\tilde{\Delta}(G) = 0$ if G is regular and $\tilde{\Delta}(G) = \max\{d_G(v) : d_G(v) \leq \Delta(G) - 1\}$ otherwise. Since $k' \leq \delta(D_k(G)) + 1$, there exists a k'-tuple dominating set in $D_k(G)$. Besides, observe that $|Q_v| = k'$ for each v such that $d_G(v) > k$ and $|Q_v| = k' + [k - d_G(v)]$ for each v such that $d_G(v) \leq k - 1$.

Let B be a k-limited packing in G with $|B| \geq \alpha$.

For each $v \in V(G)$ with $d_G(v) \le k-1$, let $A_v \subseteq N_{D_k(G)}[v] \cap Q_v$ with $|A_v| = k'$. By defining

$$R := (V(G) - B) \cup \bigcup_{v:\, k \le d_G(v) < \Delta(G)} Q_v \cup \bigcup_{v:\, d_G(v) \le k-1} A_v,$$

we will prove that R is a k'-tuple dominating set in $D_k(G)$ of size at most α'.

Notice that for every $v \in V(G)$, $|N_{D_k(G)}[v] \cap R \cap Q_v| \ge k'$ and therefore $|N_{D_k(G)}[v] \cap R| \ge k'$.

Otherwise, if $v \in Q_w$ for some w, from the construction of R clearly follows that $|R \cap N_{D_k(G)}[v]| \ge k'$.

Let us now prove that $|R| \le \alpha'$. Clearly,

$$\begin{aligned} |R| &= (|V(G)| - |B|) + \sum_{v:\, k \le d_G(v) < \Delta(G)} |Q_v| + \sum_{v:\, d_G(v) \le k-1} |A_v| \\ &= |V(G)| + \sum_{v:\, k \le d_G(v) < \Delta(G)} |Q_v| + \sum_{v:\, d_G(v) \le k-1} [|Q_v| - (k - d_G(v))] - |B| \\ &\le |V(D_k(G))| - \sum_{v:\, d_G(v) \le k-1} [k - d_G(v)] - \alpha = \alpha'. \end{aligned}$$

Conversely, let R be a k'-tuple dominating set in $D_k(G)$ with $|R| \le \alpha'$.

First, we will prove that we can assume $|R \cap Q_v| = k'$ for each v with $d_G(v) < \Delta(G)$, i.e. $|R \cap Q_v| = |Q_v|$ if $d_G(v) \ge k$ and $|R \cap Q_v| = |Q_v| + k - d_G(v)$ otherwise.

Let $v \in V(G)$ with $d_G(v) < \Delta(G)$. From Definition 6 follows that there exists $w \in Q_v$ not adjacent to v, i.e. $N_{D_k(G)}[w] = Q_v$. Then, since $|N_{D_k(G)}[w] \cap R| \ge k'$, we have $|Q_v \cap R| \ge k'$.

Therefore, if $d_G(v) \ge k$, since $|Q_v| = k'$ we have $|R \cap Q_v| = |Q_v| = k'$.

In the case $d_G(v) \le k-1$, since $|N_{D_k(G)}[v] \cap Q_v| = \Delta(G) - d_G(v) \ge k'$, we can take $A_v \subseteq N_{D_k(G)[v]} \cap Q_v$ with $|A_v| = k'$. Notice that $R^v := (R - Q_v) \cup A_v$ is a k'-tuple dominating set in $D_k(G)$, $|R^v| \le |R|$ and $|R^v \cap Q_v| = |Q_v| - [k - d_G(v)] = k'$.

Secondly, we consider $B := V(G) - R$. We will prove that B is a k-limited packing in G. Clearly, this is true if and only if for each $v \in V(G)$

$$d_G(v) + 1 - |N_G[v] \cap R| \le k. \tag{1}$$

Let $v \in V(G)$. Since R is a k'-tuple dominating set in $D_k(G)$, $|N_{D_k(G)}[v] \cap R| \ge k'$ for each $v \in V(D_k(G))$. On the one hand, if $d_G(v) = \Delta(G)$ it is clear from the definition of D_k that $N_G[v] = N_{D_k(G)}[v]$. Therefore, inequality (1) trivially holds. On the other hand, if $d_G(v) < \Delta(G)$ we have

$$\begin{aligned} d_G(v) + 1 - |N_G[v] \cap R| &= d_G(v) + 1 - |N_{D_k(G)}[v] \cap R| + |N_{D_k(G)}[v] \cap Q_v \cap R| \\ &\le d_G(v) + 1 - |N_{D_k(G)}[v] \cap R| + |N_{D_k(G)}[v] \cap Q_v|. \end{aligned}$$

Since by construction $|N_{D_k(G)}[v] \cap Q_v| = \Delta(G) - d_G(v)$, again inequality (1) holds.

Let us finally prove that $|B| \ge \alpha$. Clearly,

$$\begin{aligned} |B| &= |V(G)| - |R \cap V(G)| = |V(G)| - \left(|R| - \sum_{v: d_G(v) < \Delta(G)} |Q_v \cap R|\right) \\ &= |V(G)| + \sum_{v: k \le d_G(v) < \Delta(G)} |Q_v| + \sum_{v: d_G(v) \le k-1} \{|Q_v| - [k - d_G(v)]\} - |R| \\ &\ge |V(D_k(G))| - \sum_{v: d_G(v) \le k-1} [k - d_G(v)] - \alpha' = \alpha. \end{aligned}$$

Again, we have the following immediate corollary:

Corollary 8. *Let* **G** *be a graph class which is closed under D_k for every k. If MD is polynomial time solvable on* **G**, *then LP is also polynomial time solvable on* **G**. *Besides, if LP is NP-complete on* **G**, *then MD is also NP-complete on* **G**.

The class of strongly chordal graphs (a superclass of trees) is closed under D_k, for all k. As it was pointed out in the preliminaries of [5], an $O(n^3)$-algorithm for solving LP on a strongly chordal graph G can be derived from the total balancedness of the incidence matrix of the closed neighborhoods of the vertices of G. If now we take into account the linear time algorithm provided in [9] for solving MD in strongly chordal graphs, Theorem 7 allows us to state that there exists an $O(n^2)$-algorithm for solving LP in this graph class.

Finally, as a corollary of Theorems 4 and 7, we have:

Corollary 9. *LP and MD are polynomially equivalent on graph classes that are closed under transformations P and D_k, for every k.*

Looking for graph classes that are closed under both transformations, the class of regular graphs is a trivial example. Moreover, following the ideas of Theorem 9 in [1], we can state a more general result concerning graph classes defined by forbidden induced graphs:

Theorem 10. *Let $\mathcal{F}$ be a family of graphs satisfying the following property: for every graph G in $\mathcal{F}$, $|V(G)| \geq 2$ and, for every $v \in V(G)$, no connected component of $G - v$ is complete. Then LP and MD are polynomially equivalent in the class* **G** *of $\mathcal{F}$-free graphs.*

Proof. It is clear that no connected component of a graph in $\mathcal{F}$ is complete.

Let k be a positive integer with $1 \leq k \leq \Delta(G)$. From Corollary 9, we only need to prove that **G** is closed under transformations D_k and P.

Let $G \in \mathbf{G}$. We will prove that $D_k(G) \in \mathbf{G}$, i.e. that every induced subgraph of $D_k(G)$ does not belong to $\mathcal{F}$.

Let G' be a subgraph of $D_k(G)$ induced by V' with $|V'| \geq 2$. Let us denote V^{new}, the set consisting of the vertices added to G in the construction of $D_k(G)$, i.e. $V^{new} := \bigcup_{v:\, d_G(v) < \Delta(G)} Q_v$.

If $V' \subseteq V(G)$, then G' is an induced subgraph of G and, as a consequence, $G' \notin \mathcal{F}$.

If G' has a connected component $C \subseteq V^{new}$, C is complete and then $G' \notin \mathcal{F}$.

In the remaining cases G' has a connected component C such that $C \cap V(G) \neq \emptyset$ and $C \cap V^{new} \neq \emptyset$. In this case, $C \cap Q_v \neq \emptyset$ for some $v \in V' \cap V(G)$. This implies that $C \cap Q_v$ is a complete connected component of $G' - v$ and therefore $G' \notin \mathcal{F}$.

To prove that **G** is closed under transformation P, it is enough to repeat the same proof by just replacing D_k by P and Q_v by S_v.

Following again the ideas in [1], Theorem 10 allows us to state that LP and MD are polynomially equivalent in the class of distance-hereditary graphs, for instance.

From the previous theorem we know that a candidate for being a graph class where one problem is NP-complete and the other polynomial time solvable, contains within its minimal forbidden induced subgraphs at least a graph G_0 with a vertex v such that $G_0 - v$ has a complete connected component.

4. Polynomial reductions between GLP and LP and between GMD and MD

In this section we will prove that, within graph classes that are closed under certain operations, LP and MD are, respectively, "as hard" as their generalizations.

Let us start by remarking some simple facts related to the role that vertices of degree one have in an instance of problems GLP and GMD.

Let G be a graph, $v \in V(G)$ with $d_G(v) = 1$ and w, the vertex adjacent to v in G. Consider the instance of GLP given by such G, a set $\mathcal{A}$ and a vector $\mathbf{c}$, where $c_v = 0$. It is easy to see that solving that instance is equivalent to solving the instance of GLP given by G, $\mathcal{A} - \{v, w\}$ and $\mathbf{c}$. As regards GMD, consider the instance given by such G, a set $\mathcal{R}$ and a vector $\mathbf{r}$ with $r_v = 2$. It is also not difficult to see that solving it is equivalent to solving the instance of GMD given by G, $\mathcal{R} \cup \{v, w\}$ and $\mathbf{r}$.

The above observations motivate us to introduce the following graph transformation:

Definition 11. Given a graph G and a set $F \subseteq V(G)$, $T_F(G)$ is the graph obtained from G by adding for each $v \in F$, a pendant vertex that is adjacent only to v.

The following straightforward lemmas, show respectively that we can restrict ourselves to work on instances of for GLP (GMD) where all vertices are allowed (nonrequired):

Lemma 12. *Given G, $\mathcal{A}$, α and $\mathbf{c}$ defining an instance of GLP, let $F := V(G) - \mathcal{A}$ and $\mathbf{c}' \in \mathbb{Z}_+^{V(T_F(G))}$ be defined by $c'_v = c_v$, for each $v \in V(G)$ and $c'_v = 0$, for $v \in V(T_F(G)) - V(G)$. Given $B \subseteq V(G)$, B is a $(\mathbf{c}, \mathcal{A})$-limited packing in G if and only if B is a $\mathbf{c}'$-limited packing of $T_F(G)$.*

Lemma 13. *Given G, $\mathcal{R}$, α and $\mathbf{r}$ defining an instance of GMD, let $\mathbf{r}' \in \mathbb{Z}_+^{V(T_{\mathcal{R}}(G))}$ be defined by $r'_v = r_v$, for each $v \in V(G)$ and $r'_v = 2$, for $v \in V(T_{\mathcal{R}}(G)) - V(G)$. Given $D \subseteq V(G)$, D is a $(\mathbf{r}, \mathcal{R})$-dominating set in G if and only if $D \cup (V(T_{\mathcal{R}}(G)) - V(G))$ is a $\mathbf{r}'$-dominating set in $T_{\mathcal{R}}(G)$.*

In order to present the reduction from GLP to LP, we introduce the following graph transformation:

Definition 14. Given a graph G, $\mathbf{c} = (c_v) \in \mathbb{Z}_+^{V(G)}$ and a positive integer k with $k \geq c_v$ for every v, $P_{\mathbf{c},k}(G)$ is the graph obtained from G by adding for each $v \in V(G)$, $k - c_v$ vertices that are adjacent to v and define a stable set S_v.

Therefore, we have:

Proposition 15. *Let G, $\mathcal{A}$, α and $\mathbf{c}$ define an instance of GLP. Let also $k := \max\{c_v : v \in V(G)\}$, $\alpha' := \alpha + \sum_{v \in V(G)} (k - c_v)$ and $G' = P_{\mathbf{c},k}(G)$. Then G has a $\mathbf{c}$-limited packing of cardinality at least α if and only if G' has a k-limited packing of cardinality at least α'.*

Proof. From Lemma 12, we can assume $\mathcal{A} = V(G)$.

Given a $\mathbf{c}$-limited packing B in G of cardinality at least α, it is not difficult to see that

$$B' := B \cup \bigcup_{v \in V(G)} S_v$$

is a k-limited packing in G' of cardinality at least α'.

To see the converse, take a k-limited packing B' in G' of cardinality at least α'. From Lemma 2, we can assume that B' contains S_v for all v. Therefore for each $v \in V(G)$, $|B' \cap N_G[v]| = |B' \cap N_{G'}[v]| - |S_v| \le k - k + c_v = c_v$, i.e. $B' \cap V(G)$ is a $\mathbf{c}$-limited packing in G and besides, its size is at least α.

In order to present the reduction from GMD to MD, we introduce another graph transformation:

Definition 16. Given a graph G, $\mathbf{r} = (r_v) \in \mathbb{N}_{=}^{V(G)}$ and a positive integer k with $k \ge r_v$ for every v, $H_{\mathbf{r},k}(G)$ is the graph obtained from G by adding, for each $v \in V(G)$ with $k > r_v$, k vertices defining a clique Q_v, and making v adjacent to exactly $k - r_v$ vertices of Q_v.

In this case, we are able to prove:

Proposition 17. *Let G, $\mathcal{R}$, α and $\mathbf{r}$ define an instance of GMD. Let also $k := \max\{r_v : v \in V(G)\}$, $\alpha' := \alpha + k|V(G)|$ and $G'' = H_{\mathbf{r},k}(G)$. Then G has an $\mathbf{r}$-dominating set of cardinality at most α if and only if G'' has a k-tuple dominating set of cardinality at most α'.*

Proof. From Lemma 13, we can assume $\mathcal{R} = \emptyset$.

Given an $\mathbf{r}$-dominating set D in G of cardinality at most α, it is not difficult to see that $D' := B \cup \bigcup_{i \in V(G)} Q_v$ is a k-tuple dominating set of G'' of cardinality at most α'.

Conversely, take a k-tuple dominating set D' in G'' of cardinality at most α'. From Definition 16, for each $v \in V(G)$ there exists $w \in Q_v$ such that $N_{G''}[w] = Q_v$, thus not adjacent to v. Since $|N_{G''}[w] \cap D'| \ge k'$, we have $\bigcup_{v \in V(G)} Q_v \subseteq D'$ and thus $D' - \bigcup_{v \in V(G)} Q_v$ is an $\mathbf{r}$-dominating set in G of size at most α.

We have proved that the generalized problems considered in this work are, within graph classes that are closed under the transformations involved, not harder than those with uniform capacities. But, the question concerning the existence of a graph class where one of the latest is polynomial time solvable and the other NP-complete is still open.

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