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On the first Chvátal closure of the set covering polyhedron related to circulant matrices

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Abstract

We study the set covering polyhedron related to circulant matrices. In particular, our goal is to characterize the first Chvátal closure of the usual fractional relaxation. We present a family of valid inequalities that generalizes the family of minor inequalities previously reported in the literature. This family includes new facet-defining inequalities for the set covering polyhedron.

Keywords: set covering, circulant matrices, Chvátal closure

1 Introduction

The *weighted set covering problem* can be stated as

$$\text{(SCP)} \quad \min\{c^T x : Ax \geq \mathbf{1}, x \in \{0,1\}^n\}$$

where A is an $m \times n$ matrix with $0, 1$ entries, $c \in \mathbb{Z}^n$, and $\mathbf{1}$ is the m-vector having all entries equal to one. The SCP is a classic problem in combinatorial

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optimization with important applications (crew scheduling, facility location, vehicle routing, to cite a few prominent examples), but hard to solve in general. One established approach to tackle this problem is to study the polyhedral properties of the set of its feasible solutions.

The *set covering polyhedron* $Q^*(A)$ is defined as the convex hull of all feasible solutions of SCP. Its *fractional relaxation* $Q(A)$ is the feasible region of the linear programming relaxation of SCP, i.e., $Q(A) = \{x \in [0,1]^n : Ax \geq \mathbf{1}\}$.

It is known that SCP can be solved in polynomial time if A belongs to the particular class of *circulant matrices* defined in the next section. Hence, it is natural to ask whether $Q^*(A)$ has a compact description in terms of linear inequalities in this case, an issue that has been addressed in several recent studies by researchers in the field (see [2,4,5] among others).

Bianchi et al. obtained in [4] a family of facet-defining inequalities for $Q^*(A)$ which are associated with certain structures called circulant minors. Moreover, the authors presented two families of circulant matrices for which $Q^*(A)$ is completely described by this class of *minor inequalities*, together with the full-rank inequality and the inequalities defining $Q(A)$, usually denoted as boolean facets.

If an inequality $a^T x \leq b$ is valid for a polytope $P \subset \mathbb{R}^n$ and $a \in \mathbb{Z}^n$, then $a^T x \leq \lfloor b \rfloor$ is valid for the integer polytope $P_I := \text{conv}(P \cap \mathbb{Z}^n)$. This procedure is called Chvátal-Gomory rounding, and it is known that the system of all linear inequalities which can be obtained in this way defines a new polytope P', the *first Chvátal closure of* P. Moreover, iterating this procedure yields P_I in a finite number of steps. An inequality is said to have *Chvátal rank* of t if it is valid for the t-th Chvátal closure of a polytope. All inequalities mentioned above have Chvátal rank less than or equal to one.

With the aim of generalizing the result in [4], we have studied whether the system consisting of minor inequalities, boolean facets, and the full-rank inequality is sufficient for describing the first Chvátal closure of $Q(A)$ for any circulant matrix A. We have obtained a new class of valid inequalities for $Q^*(A)$ which contains minor inequalities as a proper subclass. All inequalities from this class have Chvátal rank equal to one, and besides, some of them define new facets of $Q^*(A)$, as we show by an example.

2 Notations, definitions and preliminary results

For $n \in \mathbb{N}$, let $[n]$ denote the additive group defined on the set $\{1, \ldots, n\}$, with integer addition modulo n. Throughout this article, if A is a $0,1$ matrix of order $m \times n$, then we consider the columns (resp. rows) of A to be indexed by $[n]$ (resp. by $[m]$). Two matrices A and A' are *isomorphic*, denoted by $A \approx A'$, if A' can be obtained from A by permutation of rows and columns. Moreover, we say that a row v of A is a *dominating row* if $v \geq u$ for some other row u of A, $u \neq v$.

Given $N \subset [n]$, the *minor of A obtained by contraction of N*, denoted by A/N, is the submatrix of A that results after removing all columns with indices in N and all dominating rows. In this work, when we refer to *a minor* of A we always consider a minor obtained by contraction.

In the following, e^i will denote the i-th canonical vector in $\mathbb{R}^n$.

Given $n, k \in \mathbb{N}$ with $2 \leq k \leq n-2$, let $C^i := \{i, i+1, \ldots, i+(k-1)\} \subset [n]$ for every $i \in [n]$. The *circulant matrix* C_n^k is the square matrix whose i-th row is the incidence vector of C^i. Observe that $C^i = \sum_{j=i}^{i+k-1} e^j$.

It is known that $Q^*(C_n^k)$ is a full dimensional polyhedron. Furthermore, for every $i \in [n]$, the constraints $x_i \geq 0$, $x_i \leq 1$ and $\sum_{j \in C^i} x_j \geq 1$ are facet defining inequalities of $Q^*(C_n^k)$ and we call them *boolean facets* ([6]). We will denote by $\mathcal{S}_0$ the system of linear inequalities corresponding to boolean facets.

The *rank constraint* $\sum_{i=1}^n x_i \geq \lceil \frac{n}{k} \rceil$ is always valid for $Q^*(C_n^k)$ and defines a facet if and only if n is not a multiple of k (see [6]). In [4] the authors obtained another family of facet-defining inequalities for $Q^*(C_n^k)$ associated with *circulant minors*, i.e., minors isomorphic to circulant matrices. Aguilera [1] completely characterized the subsets N of $[n]$ for which C_n^k/N is a circulant minor. In particular, it is known that if $i \in N$ then, $i+k \in N$ or $i+k+1 \in N$.

Lemma 2.1 *[4] Let $N \subset [n]$ such that $C_n^k/N \approx C_{n'}^{k'}$, and let $W = \{i \in N : i-k-1 \in N\}$. Then, the inequality*

$$\sum_{i \in W} 2x_i + \sum_{i \notin W} x_i \geq \left\lceil \frac{n'}{k'} \right\rceil \tag{1}$$

is a valid inequality for $Q'(C_n^k)$. Moreover, if $2 \leq k' \leq n'-2$, $\lceil \frac{n'}{k'} \rceil > \lceil \frac{n}{k} \rceil$ and $n' = 1 (\operatorname{mod} k')$, this inequality defines a facet of $Q^*(C_n^k)$.

The authors termed (1) as the *minor inequality corresponding to W*.

3 Computing the first Chvátal closure

In our attempt at finding a linear description of the first Chvátal closure of $Q(C_n^k)$, we use the following well-known result from integer programming:

Lemma 3.1 *Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ be a nonempty polyhedron with A integral and $Ax \geq b$ totally dual integral. Then, $P' = \{x \in \mathbb{R}^n : Ax \geq \lceil b \rceil\}$.*

As we shall see below, if all vertices of P are known then a totally dual integral system describing the polyhedron can be computed from this information. This is the case for $Q(C_n^k)$, whose vertices have been completely characterized by Argiroffo and Bianchi [2].

Lemma 3.2 ([2]) *Let x^* be a vertex of $Q(C_n^k)$. If x^* is not integral, then either $x^* = \frac{1}{k}\mathbf{1}$ or there exists $N \subset [n]$ with $C_n^k/N \approx C_{n'}^{k'}$ such that $x_i^* = \frac{1}{k'}$ if $i \notin N$ and $x_i^* = 0$ otherwise.*

In order to express $Q(C_n^k)$ via a totally dual integral system of linear inequalities, we use the method described below (see, e.g., [3, Ch. 8]). Given a polyhedral cone $K \subset \mathbb{R}^n$, consider the points in the lattice $L := K \cap \mathbb{Z}^n$. An *integral generating set* for L is a set $H \subseteq L$ having the property that every $x \in L$ can be written as a linear combination $\sum_{i=1}^{k} \alpha_i h_i$ of some elements $h_1, \ldots, h_k \in H$ with integral non negative coefficients $\alpha_1, \ldots, \alpha_k$.

The method consists in adding redundant inequalities to the original system $\mathcal{S}_0$ until the following property is verified: If $\left\{a_i^T x \geq b_i : i \in I(x^*)\right\}$ is the set of linear inequalities satisfied with equality by a vertex $x^* \in Q(C_n^k)$, then the set of vectors $\{a_i : i \in I(x^*)\}$ is an integral generating set of $K(x^*) := \text{cone}(\{a_i : i \in I(x^*)\}) \cap \mathbb{Z}^n$.

This idea leads to the following procedure for computing $Q'(C_n^k)$:

1. Let $\mathcal{S} := \mathcal{S}_0$.
2. For all x^* vertex of $Q(C_n^k)$) do
   - 2.1 Compute an integral generating set $H(x^*)$ of $K(x^*) \cap \mathbb{Z}^n$.
   - 2.2 For all $a_i \in H(x^*)$, let $b_i := a_i^T x^*$ and add the inequality $a_i^T x \geq \lceil b_i \rceil$ to $\mathcal{S}$.
3. Return $\mathcal{S}$ as a linear description of $Q'(C_n^k)$.

Observe that the inequality $a_i^T x \geq b_i$ at step 2.2 is valid for $Q(C_n^k)$, since x^* minimizes $a_i^T x$ over this polytope for any $a_i \in K(x^*)$. Moreover, if x^* is integral, then the new inequality added to the system $\mathcal{S}$ is redundant, as $b_i \in \mathbb{Z}$. Therefore, new inequalities for $Q'(C_n^k)$ may only arise from integer generating sets $H(x^*)$ related to fractional vertices belonging to one of the two clases described in Lemma 3.2.

4 Inequalities describing the first Chvátal closure

Firstly, we analyze all inequalities arising from the vertex $x^* = \frac{1}{k}\mathbf{1}$. The point $x^* = \frac{1}{k}\mathbf{1}$ is known to be a vertex of $Q(C_n^k)$ if and only if $\gcd(n,k) = 1$. In this case, $\{C^i x \geq 1 \, : \, i \in [n]\}$ are the inequalities of the original system $\mathcal{S}_0$ satisfied at equality by x^*. In order to find an integral generating set for $K(x^*) \cap \mathbb{Z}^n$ we need the following result.

Lemma 4.1 *Let $x \in \mathbb{R}^n, b \in \mathbb{Z}^n$ be two vectors such that $\mathbf{0} \leq x < \mathbf{1}$ and $C_n^k x = b$, with* $\gcd(n,k) = 1$*. Then there exists $r \in \{0, 1, \ldots, k-1\}$ such that $x = \frac{r}{k}\mathbf{1}$ and $b = r\mathbf{1}$.*

With this result we can compute an integral generating set for $K(x^*)$.

Theorem 4.2 *Let C_n^k be a circulant matrix such that* $\gcd(n,k) = 1$ *and consider the vertex $x^* = \frac{1}{k}\mathbf{1}$ of $Q(C_n^k)$. Then an integral generating set for $K(x^*) \cap \mathbb{Z}^n$ is given by $H(x^*) = \{C^1, C^2, \ldots, C^n, \mathbf{1}\}$.*

When applying the procedure described in the previous section, the vectors C^i yield the inequalities $(C^i)^T x \geq 1$ from $\mathcal{S}_0$, while for the last vector we obtain the rank constraint of $Q^*(C_n^k)$:

Corollary 4.3 *If* $\gcd(n,k) = 1$*, then the inequality $\mathbf{1}^T x \geq \lceil \mathbf{1}^T x^* \rceil = \lceil \frac{n}{k} \rceil$ is valid for $Q'(C_n^k)$.*

On the other hand, for a vertex x^* corresponding to a circulant minor $C_{n'}^{k'}$, the task of finding an integral generating set for $K(x^*) \cap \mathbb{Z}^n$ turns out to be more complicated. We present here preliminary results concerning some special cases. Consider a circulant minor $C_{n'}^{k'} \approx C_n^k / N$ of C_n^k, and let x^* be the corresponding vertex of $Q(C_n^k)$, defined as in Lemma 3.2. One can show that x^* satisfies the following inequalities with equality:

$$(C^i)^T x \geq 1, \text{ for all } i \text{ such that } i+1 \notin N; \tag{2}$$

$$(e^j)^T x \geq 0, \text{ for all } j \in N. \tag{3}$$

There might be, however, other inequalities from $\mathcal{S}_0$ satisfied tightly by x^*. In the following we denote by K the subcone of $K(x^*)$ spanned by the normal vectors of the left-hand sides of (2) and (3).

Theorem 4.4 *An integral generating set for $K \cap \mathbb{Z}^n$ is given by*

$$\left\{C^i \, : \, i+1 \notin N\right\} \cup \left\{e^j \, : \, j \in N\right\} \cup \left\{r\mathbf{1} + \sum_{j \in W} e^j \, : \, 1 \leq r \leq k'-1\right\}.$$

Vectors in the first two sets give rise to boolean inequalities when the procedure of the previous section is applied. For the third set, we obtain:

Corollary 4.5 *Let $N \subset [n]$ be such that $C_n^k/N \approx C_{n'}^{k'}$. The inequalities*

$$\sum_{i \in W}(r+1)x_i + \sum_{i \notin W} rx_i \geq \left\lceil r\mathbf{1}^T x^* + \sum_{j \in W}(e^j)^T x^* \right\rceil = \left\lceil \frac{rn'}{k'} \right\rceil, \tag{4}$$

with $r \in \{1, \ldots, k'-1\}$, are valid for $Q'(C_n^k)$.

For $r = 1$, these inequalities are the minor inequalities described in [4]. Accordingly, we have called (4) as r-minor inequalities. In some cases, r-minor inequalities with $r > 1$ can be obtained from the addition of (classical) minor inequalities and the rank constraint, and are thus redundant for $Q'(C_n^k)$. However, this is not true in general. For instance, consider $W := \{6 + 5k : 0 \leq k \leq 10\}$ and $N := W \cup \{1\} \subset [59]$. One can verify that $C_{59}^4/N \approx C_{47}^3$ and the corresponding inequality (4) for $r = 2$ has the form $\sum_{i \in W} 3x_i + \sum_{i \notin W} 2x_i \geq 32$. We have shown that this inequality defines a facet of $Q^*(C_{59}^4)$. As a consequence, we disprove a former conjecture stating that 1-minor inequalities, together with boolean facets and the rank constraint, provide a complete linear description of $Q^*(C_n^k)$.

Theorem 4.6 *There are circulant matrices C_n^k for which minor inequalities, boolean facets, and the rank constraint, are not enough to describe $Q'(C_n^k)$.*

The study of necessary and sufficient conditions for an r-minor inequality to define a facet, as well as the question whether the system consisting of r-minor inequalities, boolean facets, and the rank constraint yields a complete linear description of $Q'(C_n^k)$ for any circulant matrix C_n^k, is a line of research for future work. All computational experiments we have conducted so far support this conjecture.

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