RESEARCH PAPER

TWO EQUIVALENT STEFAN'S PROBLEMS
FOR THE TIME FRACTIONAL DIFFUSION EQUATION

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Abstract

Two Stefan's problems for the diffusion fractional equation are solved, where the fractional derivative of order $\alpha \in (0,1)$ is taken in the Caputo sense. The first one has a constant condition on $x = 0$ and the second presents a flux condition $T_x(0,t) = q/t^{\alpha/2}$. An equivalence between these problems is proved and the convergence to the classical solutions is analyzed when $\alpha \nearrow 1$ recovering the heat equation with its respective Stefan's condition.

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1. Introduction

In 1695 L'Hôpital asked Leibnitz, the father of the concept of the classical differentiation, what meaning could be assigned to a derivative of order $\frac{1}{2}$. Leibnitz replied prophetically: "[...] this is an apparent paradox from which, one day, useful consequences will be drawn." Since 1819, mathematicians like Lacroix, Abel, Liouville, Riemann and later Grünwald and Letnikov, have attempted to establish various consistent definitions of derivative of fractional order.

Here we use the definition introduced by Caputo [2] in 1967, referred to as *fractional derivative in Caputo's sense*, given by

$$
{}_a^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,
$$

where $\alpha > 0$ is the order of differentiation, $n = \lceil \alpha \rceil$ and f is a differentiable function up to order n in $[a, b]$.

The one-dimensional heat equation has become the paradigm for the all-embracing study of parabolic partial differential equations, linear and nonlinear. Cannon [1] did a methodical development of a variety of aspects of this paradigm. Of particular interest are the discussions on the one-phase Stefan problem, one of the simplest examples of a free-boundary-value problem for the heat equation (see Datzeff [3]). In mathematics and its applications, particularly related to phase transitions in matter, a Stefan problem is a particular kind of boundary value problem for a partial differential equation, adapted to the case in which a phase boundary can move with the time. The classical Stefan problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, for example ice passing to water: this is accomplished by solving the heat equation imposing the initial temperature distribution on the whole medium, and a particular boundary condition, the Stefan condition, on the evolving boundary between its two phases. Note that in the one-dimensional case this evolving boundary is an unknown curve: hence, the Stefan problems are examples of free boundary problems. A large bibliography on free and moving boundary problems for the heat-diffusion equation was given in Tarzia [12].

In this paper, we study a one-phase Stefan problem with time fractional diffusion equation, obtained from the standard diffusion equation by replacing the first order time-derivative by a fractional derivative of order $\alpha > 0$ in the Caputo sense:

$$
{}_a^C D^\alpha u(x, t) = \lambda^2 \frac{\partial^2 u}{\partial x^2}(x, t), \quad -\infty < x < \infty, \ t > 0, \ 0 < \alpha < 1,
$$

and the Stefan condition $\dfrac{ds(t)}{dt} = ku_x(s(t), t), \ t > 0$, by the fractional Stefan condition

$$
{}_a^C D^\alpha s(t) = ku_x(s(t), t), \quad t > 0.
$$

This equation has been recently treated by a number of authors (see e.g. Gorenflo and Mainardi [4], Liu and Xu [6], Kilbas [5], Podlubny [9]) and, among the several applications that have been studied, Mainardi [7] studied the application to the theory of linear viscoelasticity.

The solutions of this equation are expressed in terms of two special functions that play a very important role in Fractional Calculus as the theory of differentiation and integration of arbitrary order: the Mittag-Leffler function (see e.g. [5], [7], [9])

$$E(z, \alpha, \beta) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (1.1)$$

and the Wright function [13] (see e.g. also [5])

$$W(z, \alpha, \beta) = \sum_{n=0}^{\infty} \frac{z^n}{n! \, \Gamma(\alpha n + \beta)}, \qquad (1.2)$$

respectively. A particular case of the Wright function is the so-called Mainardi function (see Podlubny [9])

$$M_\nu(z) = W(-z, -\nu, 1 - \nu) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \, \Gamma(-\nu n + 1 - \nu)}. \qquad (1.3)$$

This function is a part of the fundamental solution for the time fractional diffusion equation studied in [8]

$$G_\alpha(x, t) = \frac{1}{2\lambda t^{\alpha/2}} M_{\alpha/2}\left(\frac{x}{\lambda t^{\alpha/2}}\right).$$

2. Solving the two fractional Stefan's problems

Hereinafter, $_{0}^{C}D^{\alpha} = D^{\alpha}$. We consider the following two problems:

$$\begin{cases} D^{\alpha} u(x, t) = \lambda^2 \dfrac{\partial^2 u}{\partial x^2}(x, t) & 0 < x < s(t),\, t > 0,\, 0 < \alpha < 1,\ \lambda > 0 \\ u(0, t) = B & t > 0, \quad B > 0 \text{ constant} \\ u(s(t), t) = C < B & t > 0 \\ D^{\alpha} s(t) = -k u_x(s(t), t) & t > 0, \quad k > 0 \\ s(0) = 0 \end{cases}$$

$$(2.1)$$

and

$$\begin{cases} D^{\alpha} u(x, t) = \lambda^2 \dfrac{\partial^2 u}{\partial x^2}(x, t) & 0 < x < s(t),\, t > 0,\, 0 < \alpha < 1,\ \lambda > 0 \\ u_x(0, t) = -\dfrac{q}{t^{\alpha/2}} & t > 0, \quad q > 0 \\ u(s(t), t) = C & t > 0 \\ D^{\alpha} s(t) = -k u_x(s(t), t) & t > 0 \\ s(0) = 0 \end{cases}.$$

$$(2.2)$$

A pair $\{u, s\}$ is a solution of the problem (2.1) (or (2.2)) if:

(1) u and s satisfy (2.1) (or (2.2)),

(2) u_{xx} and $D^\alpha u$ are continuous for $0 < x < s(t)$, $0 < t < T$,

(3) u and u_x are continuous for $0 \leq x \leq s(t)$, $0 < t < T$,

(4) $0 \leq \liminf\limits_{x,t\to 0^+} u(x,t) \leq \limsup\limits_{x,t\to 0^+} u(x,t) < +\infty$,

(5) s is continuously differentiable in $[0, T)$ and
$\frac{\dot{s}(\tau)}{(t-\tau)^\alpha} \in L^1(0,t) \ \forall t \in (0, T)$.

Let us solve the problem (2.1). We show in Appendix that

$$u_1(x,t) = a_1 + b_1 \left[1 - W \left(-\frac{x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right], \quad a_1, b_1 \text{ constants,} \quad (2.3)$$

is a solution for the time-fractional-diffusion equation.

$$u_1(0,t) = a_1 + b_1 \left[1 - W \left(0, -\frac{\alpha}{2}, 1 \right) \right] = B \Rightarrow a_1 = B, \quad (2.4)$$

$$u_1(s_1(t), t) = a_1 + b_1 \left[1 - W \left(-\frac{s(t)}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right] = C. \quad (2.5)$$

Note that (2.5) must be verified for all $t > 0$, so we will ask for $s(t)$ to be proportional to $t^{\alpha/2}$, that is to say

$$s_1(t) = \lambda \xi t^{\alpha/2} \quad \text{for some} \ \xi > 0, \quad (2.6)$$

and from (2.5), (2.6) and Corollary **4.1**,

$$C = B + b_1 \left[1 - W \left(-\xi, -\frac{\alpha}{2}, 1 \right) \right] \Rightarrow b_1 = \frac{C - B}{1 - W \left(-\xi, -\frac{\alpha}{2}, 1 \right)}. \quad (2.7)$$

Now we will obtain ξ from the "fractional Stefan condition". Taking into account that

$$D^\alpha(t^\beta) = \frac{\Gamma(\beta + 1)}{\Gamma(1 + \beta - \alpha)} t^{\beta - \alpha} \quad \text{if} \ \ \beta > -1,$$

we have

$$D^\alpha s_1(t) = D^\alpha(\lambda \xi t^{\alpha/2}) = \lambda \xi \frac{\Gamma(\frac{\alpha}{2} + 1)}{\Gamma(1 - \frac{\alpha}{2})} t^{-\alpha/2}. \quad (2.8)$$

On the other hand,

$$u_{1x}(s_1(t), t) = b_1 \frac{1}{\lambda t^{\alpha/2}} M_{\alpha/2}(\xi) = \frac{C - B}{1 - W \left(-\xi, -\frac{\alpha}{2}, 1 \right)} \frac{1}{\lambda t^{\alpha/2}} M_{\alpha/2}(\xi). \quad (2.9)$$

From (2.8) and (2.9)

$$\xi \left[1 - W \left(-\xi, -\frac{\alpha}{2}, 1 \right) \right] \frac{1}{M_{\alpha/2}(\xi)} = -\frac{k}{\lambda^2} \frac{\Gamma(1 - \frac{\alpha}{2})}{\Gamma(1 + \frac{\alpha}{2})} (C - B). \quad (2.10)$$

Let us define

$$H(\xi) = \xi \left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1 \right) \right] \frac{1}{M_{\alpha/2}(\xi)}. \qquad (2.11)$$

The function H has the following properties:

(1) $H(0^+) = 0$,
(2) $H(+\infty) = +\infty$,
(3) H is continuous and monotonically increasing.

Because of the asymptotic behavior of the Wright function (see [4]), it is easy to check the Properties (1) and (2).

For Property (3), we observe from Corollary **4.1** that, $1 - W\left(-\xi, -\frac{\alpha}{2}, 1 \right)$ is a positive and increasing function in $\mathbb{R}^+$. And from Lemma **4.2**, $\frac{1}{M_{\alpha/2}(\xi)}$ is a positive increasing function.

Observing that $-\frac{k}{\lambda^2} \frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})}(C - B) > 0$, we can assure that there exists a unique $\tilde{\xi}$ such that

$$H(\tilde{\xi}) = -\frac{k}{\lambda^2} \frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})}(C - B). \qquad (2.12)$$

So the solution of problem (2.1) is given by

$$\begin{cases} u_1(x,t) = B + \frac{C-B}{1 - W\left(-\tilde{\xi}, -\frac{\alpha}{2}, 1\right)} \left[1 - W\left(-\frac{x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right] \\ s_1(t) = \lambda \tilde{\xi} t^{\alpha/2}, \\ \qquad \text{where } \tilde{\xi} \text{ is the unique solution to the equation,} \\ H(\xi) = -\frac{k}{\lambda^2} \frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})}(C - B). \end{cases} \qquad (2.13)$$

Now let us solve (2.2). Here we consider

$$u_2(x,t) = a_2 + b_2 \left[1 - W\left(-\frac{x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right], \quad a_2, b_2 \text{ constants.} \qquad (2.14)$$

Then,

$$u_{2x}(0,t) = \frac{b_2}{\lambda t^{\alpha/2}} M_{\alpha/2}(0) = -\frac{q}{t^{\alpha/2}} \Rightarrow b_2 = -q\lambda\Gamma\left(1 - \frac{\alpha}{2} \right), \qquad (2.15)$$

$$u_2(s_2(t), t) = a_2 + b_2 \left[1 - W\left(-\frac{s_2(t)}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right) \right] = C. \qquad (2.16)$$

Note that (2.16) must be verified for all $t > 0$, so we will ask for $s_2(t)$ to be proportional to $t^{\alpha/2}$, that is to say

$$s_2(t) = \lambda\mu t^{\alpha/2}, \qquad \text{for some } \mu > 0. \qquad (2.17)$$

From (2.16) and (2.17) we have

$$a_2 = C + q\lambda\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\mu, -\frac{\alpha}{2}, 1\right)\right]. \tag{2.18}$$

Notice that

$$D^\alpha s_2(t) = \lambda\mu\frac{\Gamma(\frac{\alpha}{2} + 1)}{\Gamma(1 - \frac{\alpha}{2})}t^{-\alpha/2} \tag{2.19}$$

and

$$u_{2x}(s_2(t), t) = \frac{b_2}{\lambda t^{\alpha/2}}M_{\alpha/2}(\mu) = -\frac{q\Gamma\left(1 - \frac{\alpha}{2}\right)}{t^{\alpha/2}}M_{\alpha/2}(\mu). \tag{2.20}$$

So, from (2.19) and (2.20),

$$\lambda\mu\frac{\Gamma(\frac{\alpha}{2} + 1)}{\Gamma(1 - \frac{\alpha}{2})}t^{-\alpha/2} = -k\frac{(-q)\Gamma\left(1 - \frac{\alpha}{2}\right)}{t^{\alpha/2}}M_{\alpha/2}(\mu),$$

therefore

$$\mu\frac{1}{M_{\alpha/2}(\mu)} = \frac{kq}{\lambda}\frac{\Gamma\left(1 - \frac{\alpha}{2}\right)^2}{\Gamma(\frac{\alpha}{2} + 1)}. \tag{2.21}$$

Let us define

$$J(\mu) = \mu\,\frac{1}{M_{\alpha/2}(\mu)}. \tag{2.22}$$

The function J has the following properties:

(1) $J(0^+) = 0$,
(2) $J(+\infty) = +\infty$,
(3) J is continuous and monotonically increasing.

Observing that $\dfrac{kq}{\lambda}\dfrac{\Gamma\left(1 - \frac{\alpha}{2}\right)^2}{\Gamma(\frac{\alpha}{2} + 1)} > 0$, we can assure that there exists a unique $\tilde{\mu}$ such that

$$J(\tilde{\mu}) = \frac{kq}{\lambda}\frac{\Gamma\left(1 - \frac{\alpha}{2}\right)^2}{\Gamma(\frac{\alpha}{2} + 1)}.$$

So the solution of problem (2.2) is given by

$$\begin{cases} u_2(x, t) = & C + q\lambda\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\tilde{\mu}, -\frac{\alpha}{2}, 1\right)\right] \\ & \quad -q\lambda\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\frac{x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right], \\ s_2(t) = \lambda\tilde{\mu}t^{\alpha/2}, \\ \qquad \text{where } \tilde{\mu} \text{ is the unique solution to equation,} \\ J(\mu) = \frac{kq}{\lambda}\frac{\Gamma\left(1 - \frac{\alpha}{2}\right)^2}{\Gamma(\frac{\alpha}{2} + 1)}. \end{cases} \tag{2.23}$$

Finally, our goal is to show the relationship between the two fractional diffusion problems with temperature and flux conditions at $x = 0$, respectively, to obtain a similar result as the one given by [11].

THEOREM **2.1**. *Let us consider problems (2.1) and (2.2), where:*

(1) the constant C is the same in both problems,

(2) in problem (2.1): $B = C - q\lambda\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\tilde{\mu}, -\frac{\alpha}{2}, 1\right)\right]$, where $\tilde{\mu}$ is the unique solution to $J(\mu) = \frac{kq}{\lambda}\frac{\Gamma\left(1-\frac{\alpha}{2}\right)^2}{\Gamma\left(\frac{\alpha}{2}+1\right)}$ and $J(\mu)$ is defined by (2.22).

Then these problems are equivalent.

P r o o f. Let us define the following function

$$B(\xi) = C + q\lambda\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right].$$

Observe that $B(\xi) > C$, $\forall \xi$ and $B(\tilde{\mu}) = B$.

Now,

$$H(\xi) = -\frac{k}{\lambda^2}\frac{\Gamma(1 - \frac{\alpha}{2})}{\Gamma(1 + \frac{\alpha}{2})}(C - B(\xi)) \Longleftrightarrow$$

$$\xi\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right]\frac{1}{W\left(-\xi, -\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right)}$$

$$= -\frac{k}{\lambda^2}\frac{\Gamma(1 - \frac{\alpha}{2})}{1 + \frac{\alpha}{2}}(-q)a\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right]$$

$$\Longleftrightarrow \xi\left[1 - W\left(-\xi, -\frac{\alpha}{2}, 1\right)\right] = \frac{kq}{\lambda}\frac{\Gamma(1 - \frac{\alpha}{2})^2}{\Gamma\left(1 + \frac{\alpha}{2}\right)} \Longleftrightarrow$$

$$\Longleftrightarrow J(\xi) = \frac{kq}{\lambda}\frac{\Gamma(1 - \frac{\alpha}{2})^2}{\Gamma\left(1 + \frac{\alpha}{2}\right)}. \tag{2.24}$$

Then if $\tilde{\mu}$ is the unique solution of (2.24), we have

$$H(\tilde{\mu}) = \frac{k}{\lambda^2}\frac{\Gamma(1 - \frac{\alpha}{2})}{\Gamma(1 + \frac{\alpha}{2})}(C - B)$$

$$= \frac{k}{\lambda^2}\frac{\Gamma(1 - \frac{\alpha}{2})}{\Gamma(1 + \frac{\alpha}{2})}q\lambda\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\tilde{\mu}, -\frac{\alpha}{2}, 1\right)\right] > 0.$$

Due to the uniqueness of solution of (2.12), we can assure that $\tilde{\mu} = \tilde{\xi}$, and therefore $s_1 = s_2$.

It is easy now to check that $u_1 = u_2$. From (2.13) and (2.23)

$$u_2(x, t) = C + q\lambda\Gamma\left(1 - \frac{\alpha}{2}\right)\left[1 - W\left(-\tilde{\mu}, -\frac{\alpha}{2}, 1\right)\right]$$

$$-q\lambda\Gamma\left(1-\frac{\alpha}{2}\right)\left[1-W\left(-\frac{x}{\lambda t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right]$$

$$=B-q\lambda\Gamma\left(1-\frac{\alpha}{2}\right)\frac{1-W\left(-\tilde{\xi},-\frac{\alpha}{2},1\right)}{1-W\left(-\tilde{\xi},-\frac{\alpha}{2},1\right)}\left[1-W\left(-\frac{x}{\lambda t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right]$$

$$=B+\frac{C-B}{1-W\left(-\tilde{\xi},-\frac{\alpha}{2},1\right)}\left[1-W\left(-\frac{x}{\lambda t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right]=u_1(x,t).$$

$\square$

REMARK **2.1**.   Applying Theorem **4.1** from Appendix, to the given solutions (2.13) and (2.23) we recover the classical solutions:

$$\lim_{\alpha\nearrow 1}u_1(x,t)=\lim_{\alpha\nearrow 1}\left\{B+\frac{C-B}{1-W\left(-\tilde{\xi},-\frac{\alpha}{2},1\right)}\left[1-W\left(-\frac{x}{\lambda t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right]\right\}$$

$$=B+\frac{C-B}{\mathrm{erf}\left(\tilde{\xi}/2\right)}\,\mathrm{erf}\left(-\frac{x}{2\lambda t^{1/2}}\right),$$

$$\lim_{\alpha\nearrow 1}s_1(t)=\lim_{\alpha\nearrow 1}\lambda\tilde{\xi}t^{\alpha/2}=\lambda\tilde{\xi}\sqrt{t},$$

where $\tilde{\xi}$ is the unique solution to the equation

$$\frac{\xi}{2}\mathrm{erf}\left(\frac{\xi}{2}\right)e^{\xi^2/4}=\frac{k}{\lambda^2}\frac{(C-B)}{\sqrt{\pi}}.$$

3. Conclusions

We have studied the behavior of the two Wright functions in $\mathbb{R}_0^+$: $1-W(-x,-\frac{\alpha}{2},1)$ and $M_{\alpha/2}(x)$, and then we solved two fractional Stefan's problems for the time fractional diffusion equation with its respective fractional Stefan's conditions: the first one with a constant condition at $x=0$, and the second one with a flux condition $u_x(0,t)=-\dfrac{q}{t^{\alpha/2}}$. Finally, we proved the equivalence between these two problems (for a suitable constant condition) and we have analyzed the convergence when $\alpha\nearrow 1$, thus recovering the classical solution to the heat equation and its respective Stefan's condition.

4. Appendix: Working with the Wright function

Note that the Wright function (1.2) $W(z, \alpha, \beta) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!\, \Gamma(\alpha n + \beta)}$ is an entire function if $\Re(\alpha) > -1$.

Taking $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$, we get

$$W\left(-z, -\frac{1}{2}, \frac{1}{2}\right) = M_{1/2}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4}.$$

Due to the uniform convergence of the series,

$$\frac{\partial}{\partial z} W(z, \alpha, \beta) = W(z, \alpha, \alpha + \beta). \tag{4.1}$$

Then, for $x \in \mathbb{R}_0^+$, and taking account that

$$W\left(-\infty, -\frac{\alpha}{2}, 1\right) = 0, \qquad \text{if } \alpha \in (0, 2), \tag{4.2}$$

we have

$$W\left(-x, -\frac{1}{2}, 1\right) = W\left(-x, -\frac{1}{2}, 1\right) - W\left(-\infty, -\frac{1}{2}, 1\right)$$

$$= \int_{\infty}^{x} \left(\frac{\partial}{\partial x} W\left(-\xi, -\frac{1}{2}, 1\right) \right) d\xi = \int_{\infty}^{x} -W\left(-\xi, -\frac{1}{2}, \frac{1}{2}\right) d\xi$$

$$= \int_{x}^{\infty} W\left(-\xi, -\frac{1}{2}, \frac{1}{2}\right) d\xi = \int_{x}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\xi^2/4} d\xi$$

$$= \frac{2}{\sqrt{\pi}} \int_{x/2}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\xi^2} d\xi = \operatorname{erfc}\left(\frac{x}{2}\right).$$

Consequently,

$$W\left(-x, -\frac{1}{2}, 1\right) = \operatorname{erfc}\left(\frac{x}{2}\right)$$

and

$$1 - W\left(-x, -\frac{1}{2}, 1\right) = \operatorname{erf}\left(\frac{x}{2}\right).$$

REMARK **4.1.** It is a fact that the Mainardi function $M_{\alpha/2}(z)$ (1.3) is an entire function of z (see [4]). So, any limit on the variable z can be calculated by interchanging limit and sum. However that is not always true if the limit is taken in the parameter α.

For example, the function

$$f_\alpha(z) = e^{-z/\alpha}$$

is an entire function on the variable z, whose series representation is $\sum_{n=0}^{\infty} \frac{(-z/\alpha)^n}{n!}$, and, for every z fixed,

$$\lim_{\alpha \searrow 0} e^{-z/\alpha} = 0$$

while

$$\lim_{\alpha \searrow 0} \frac{(-z/\alpha)^n}{n!} = \pm\infty,$$

and therefore the series diverges.

LEMMA **4.1**. *If* $x \in \mathbb{R}_0^+$ *and* $\alpha \in (0,1)$,

$$\lim_{\alpha \nearrow 1} M_{\alpha/2}(x) = M_{1/2}(x) = \frac{e^{-\frac{x^2}{4}}}{\sqrt{\pi}}.$$

P r o o f. Let α be such that $\frac{1}{2} < \alpha < 1$. Writing the series as a sum of even and odd terms subseries, it will be seen that each one of them is bounded by a convergent series which does not depend on α. For the even terms,

$$\left| \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!\Gamma(-\frac{\alpha}{2}2k + 1 - \frac{\alpha}{2})} \right| \leq \sum_{k=0}^{\infty} \frac{|x|^{2k}}{(2k)! \left| \Gamma(-\frac{\alpha}{2}2k + 1 - \frac{\alpha}{2}) \right|}$$

$$= \sum_{k=0}^{\infty} \frac{|x|^{2k}}{(2k)! \left| \Gamma(1 - \alpha(k + \frac{1}{2})) \right|}.$$

Recall that for all $x \in \mathbb{R}$,

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = \frac{\sin(\pi x)}{\pi},$$

and the Gamma function is increasing in $(\frac{3}{2}, +\infty)$. Then if $k \geq 3$,

$$0 < \Gamma(\alpha(k + \frac{1}{2})) \leq \Gamma(k + 1) , \quad \text{and therefore} \quad \frac{1}{\Gamma(k+1)} \leq \frac{1}{\Gamma(\alpha(k + \frac{1}{2}))}.$$

On the other hand, $(2k)! = (2k)...(k+1)\Gamma(k+1)$. So,

$$\frac{|x|^{2k}}{(2k)...(k+1)\Gamma(k+1) \left| \Gamma(1 - \alpha(k + \frac{1}{2})) \right|}$$

$$\leq \frac{|x|^{2k}}{(2k)...(k+1) \left| \Gamma(\alpha(k + \frac{1}{2}))\Gamma(1 - \alpha(k + \frac{1}{2})) \right|}$$

$$= \left| \frac{|x|^{2k} \sin(\pi\alpha(k+1))}{(2k)...(k+1)\pi} \right| \leq \frac{|x|^{2k}k!}{\pi(2k)!}, \quad \forall\, k \geq 3.$$

Then

$$\left| \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!\Gamma(-\frac{\alpha}{2}2k+1-\frac{\alpha}{2})} \right| \leq \left| \sum_{k=0}^{2} \frac{x^{2k}}{(2k)!\Gamma(-\frac{\alpha}{2}2k+1-\frac{\alpha}{2})} \right| + \sum_{k=3}^{\infty} \frac{|x|^{2k}k!}{\pi(2k)!}.$$
(4.3)

It is easy to see that this is an absolutely convergent series in $\mathbb{C}$.

Concerning the odd terms, reasoning in the same way, now with $k \geq 2$ we get

$$\left| \sum_{k=0}^{\infty} \frac{-x^{2k+1}}{(2k+1)!\Gamma(-\frac{\alpha}{2}2(k+1)+1-\frac{\alpha}{2})} \right|$$

$$\leq \left| \sum_{k=0}^{1} \frac{-x^{2k+1}}{(2k+1)!\Gamma(1-\alpha(k+1))} \right| + \sum_{k=2}^{\infty} \frac{|x|^{2k+1}}{(2k+1)! \, |\Gamma(1-\alpha(k+1))|}$$

$$\leq \left| \sum_{k=0}^{1} \frac{-x^{2k+1}}{(2k+1)!\Gamma(1-\alpha(k+1))} \right| + \sum_{k=2}^{\infty} \frac{|x|^{2k+1}k!}{(2k+1)!\pi}.$$
(4.4)

Again, this is an absolutely convergent series in $\mathbb{C}$.

From (4.3) and (4.4),

$$\lim_{\alpha \nearrow 1} M_{\alpha/2}(x)$$

$$= \sum_{k=0}^{\infty} \lim_{\alpha \nearrow 1} \frac{x^{2k}}{(2k)!\Gamma(-\frac{\alpha}{2}2k+1-\frac{\alpha}{2})} + \sum_{k=0}^{\infty} \lim_{\alpha \nearrow 1} \frac{-x^{2k+1}}{(2k+1)!\Gamma(1-\alpha(k+1))}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!\Gamma(-k+\frac{1}{2})} = \frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{4}}.$$

Moreover, convergence is uniform over compact sets in the variable x. $\quad\square$

THEOREM **4.1**. *If $x \in \mathbb{R}_0^+$ and $\alpha \in (0,1)$,*

$$\lim_{\alpha \nearrow 1} \left[1 - W\left(-x, -\frac{\alpha}{2}, 1\right) \right] = \mathrm{erf}\left(\frac{x}{2}\right).$$

P r o o f. Observe that

$$\lim_{\alpha \nearrow 1} \left[1 - W\left(-x, -\frac{\alpha}{2}, 1\right) \right] = \lim_{\alpha \nearrow 1} \int_0^x M_{\alpha/2}(t)dt$$

and apply Lemma **4.1**. $\quad\square$

LEMMA **4.2**. *The Mainardi function $M_{\alpha/2}(x)$ is a decreasing positive function if $0 < \alpha < 1$.*

P r o o f. Note that $M_{\alpha/2}(x) = W\left(-x, -\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right)$, see (1.3). From [10] we know that

$$x^{\beta-1} W\left(-x^{-\sigma}, -\sigma, \beta\right) > 0, \quad \text{if} \quad x > 0, \; \beta > 0, \; 0 < \sigma < 1,$$

then

$$W\left(-x^{-\sigma}, -\sigma, \beta\right) > 0, \quad \text{if} \quad x > 0, \; \beta > 0, \; 0 < \sigma < 1. \tag{4.5}$$

In our case, $\sigma = \frac{\alpha}{2} \in (0, 1)$, $\beta = 1 - \frac{\alpha}{2} > 0$, and $g(x) = x^{-\sigma}$ is a one to one function in $\mathbb{R}^+$, so

$$M_{\alpha/2}(x) > 0 \quad \text{if } x > 0.$$

On the other hand, $M_{\alpha/2}(0) = \frac{1}{\Gamma\left(1 - \frac{\alpha}{2}\right)} > 0$, $\lim\limits_{x \to \infty} M_{\alpha/2}(x) = 0$ and

$$\left(M_{\alpha/2}(x)\right)' = -W\left(-x, -\frac{\alpha}{2}, 1 - \alpha\right) < 0, \text{ because we can apply (4.5) again.}$$

Then the lemma is proved. $\qquad\qquad\square$

COROLLARY **4.1**. *If* $0 < \alpha < 1$, $1 - W\left(-x, -\frac{\alpha}{2}, 1\right)$ *is a positive and increasing function in* $\mathbb{R}^+$.

P r o o f. It is obvious from

$$\left(1 - W\left(-x, -\frac{\alpha}{2}, 1\right)\right)' = M_{\alpha/2}(x) > 0 \quad \text{and} \quad 1 - W\left(0, -\frac{\alpha}{2}, 1\right) = 0.$$

$$\square$$

It is known that (see [8])

$$u(x, t) = \int_{-\infty}^{\infty} \frac{t^{-\frac{\alpha}{2}}}{2\lambda} M_{\frac{\alpha}{2}}\left(|x - \xi| \lambda^{-1} t^{-\frac{\alpha}{2}}\right) f(\xi) d\xi \tag{4.6}$$

is a solution for the problem

$$\begin{cases} {}_{0}^{C} D^{\alpha} u(x, t) = \lambda^2 \dfrac{\partial^2 u}{\partial x^2}(x, t) & -\infty < x < \infty, \; t > 0, \; 0 < \alpha < 1, \\ u(x, 0) = f(x) & -\infty < x < \infty. \end{cases} \tag{4.7}$$

Using this fact, is easy to see that

$$u_1(x, t) = \frac{1}{2\lambda t^{\frac{\alpha}{2}}} \int_0^{\infty} \left[M_{\frac{\alpha}{2}}\left(\frac{|x - \xi|}{\lambda t^{\frac{\alpha}{2}}}\right) - M_{\frac{\alpha}{2}}\left(\frac{x + \xi}{\lambda t^{\frac{\alpha}{2}}}\right) \right] f_0 \, d\xi \tag{4.8}$$

is a solution for the problem

$$\begin{cases} {}_{0}^{C} D^{\alpha} u_1(x, t) = \lambda^2 \dfrac{\partial^2 u}{\partial x^2}(x, t) & 0 < x < \infty, \; t > 0, \; 0 < \alpha < 1 \\ u_1(x, 0) = f_0 & 0 < x < \infty \\ u_1(0, t) = 0 & t > 0. \end{cases} \tag{4.9}$$

Working with (4.8),

$$u_1(x,t) = \frac{1}{2\lambda t^{\frac{\alpha}{2}}} \int_0^\infty \left[M_{\frac{\alpha}{2}} \left(\frac{|x-\xi|}{\lambda t^{\frac{\alpha}{2}}} - M_{\frac{\alpha}{2}} \left(\frac{x+\xi}{\lambda t^{\frac{\alpha}{2}}} \right) \right) \right] f_0 d\xi$$

$$= \frac{f_0}{2} \left[\int_0^x \frac{1}{\lambda t^{\frac{\alpha}{2}}} M_{\frac{\alpha}{2}} \left(\frac{x-\xi}{\lambda t^{\frac{\alpha}{2}}} \right) d\xi + \int_x^\infty \frac{1}{\lambda t^{\frac{\alpha}{2}}} M_{\frac{\alpha}{2}} \left(\frac{\xi-x}{\lambda t^{\frac{\alpha}{2}}} \right) d\xi \right.$$

$$\left. - \int_0^\infty \frac{1}{\lambda t^{\frac{\alpha}{2}}} M_{\frac{\alpha}{2}} \left(\frac{x+\xi}{\lambda t^{\frac{\alpha}{2}}} \right) d\xi \right]$$

$$= \frac{f_0}{2} \left[-W \left(-\frac{x}{\lambda t^{\frac{\alpha}{2}}}, -\frac{\alpha}{2}, 1 \right) + 2 - W \left(-\frac{x}{\lambda t^{\frac{\alpha}{2}}}, -\frac{\alpha}{2}, 1 \right) \right]$$

$$= f_0 \left[1 - W \left(-\frac{x}{\lambda t^{\frac{\alpha}{2}}}, -\frac{\alpha}{2}, 1 \right) \right],$$

and it is easy to check that

$$u_2(x,t) = g_0 \, W(-\frac{x}{\lambda t^{\alpha/2}}, -\frac{\alpha}{2}, 1) \tag{4.10}$$

is a solution for the problem

$$\begin{cases} {}_0^C D^\alpha u_2(x,t) = \lambda^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & 0 < x < \infty, \, t > 0, \, 0 < \alpha < 1 \\ u_2(x,0) = 0 & 0 < x < \infty \\ u_2(0,t) = g_0 & t > 0. \end{cases} \tag{4.11}$$

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