Every Rig with a One-Variable Fixed Point Presentation is the Burnside Rig of a Prextensive Category
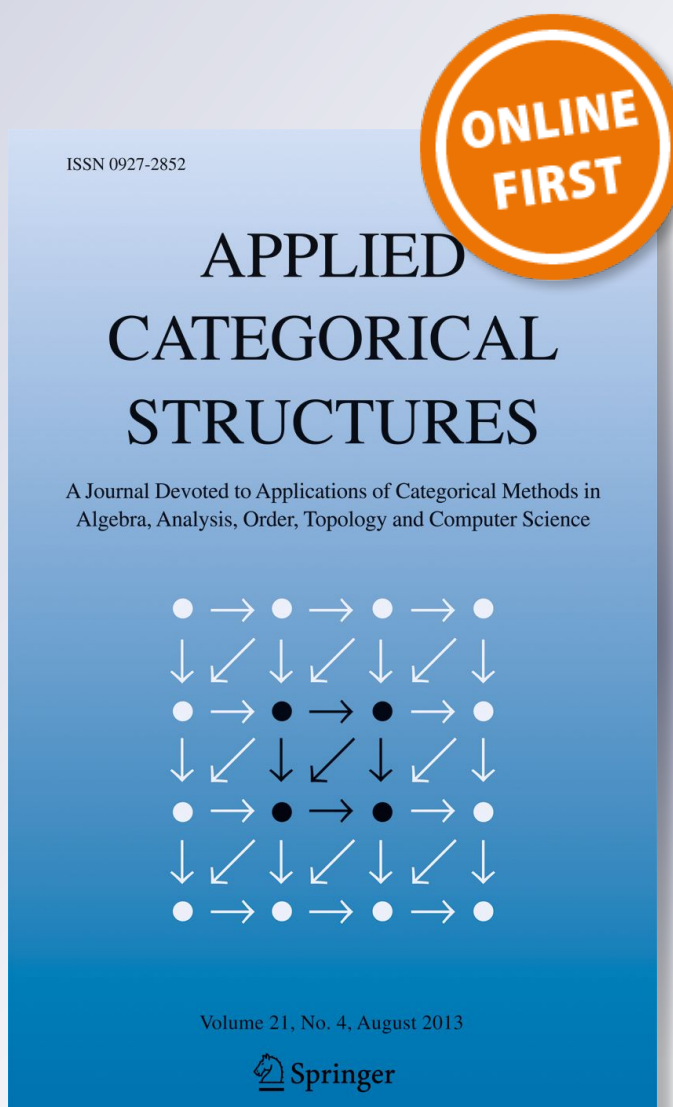
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Every Rig with a One-Variable Fixed Point Presentation is the Burnside Rig of a Prextensive Category

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Abstract We extend the work of Schanuel, Lawvere, Blass and Gates in Objective Number Theory by proving that, for any $L(X) \in \mathbb{N}[X]$, the rig $\mathbb{N}[X]/(X = L(X))$ is the Burnside rig of a prextensive category.

Keywords Objective number theory · Extensive category · Topos

1 Outline

The present paper solves a problem in *objective number theory*, a subject initiated by S. H. Schanuel through his 1990 work on negative sets [16], and whose collaboration with F. W. Lawvere for 20 more years advanced the subject [17, 18] and inspired others to do so. (We will use the terminolgy in [17], so that *prextensive* means extensive with finite products.) From the linear equation for Negative Sets, the next advance [11] concerned the quadratic equation descriptive of lists of words or trees. The key question became, that while it is trivial that these polynomial equations have solutions in the category of abstract countable sets, whether there exist examples of objects in suitable categories which satisfy no other equations than consequences of the given equation; for example, the tree equation $T = 1 + T^2$ implies that $T^7 = T$, but what else does it imply? It was partly to give a precise sense to this notion of 'consequence' that Lawvere and Schanuel had invented the theory of rigs and of prextensive categories. With the explicit use of classifying toposes, A. Blass was able to prove that indeed there are no polynomial isomorphisms in the concrete prexten- sive setting beyond those that are guaranteed by the abstract Rig-theoretic setting, a striking 'completeness theorem' which shed light on the classification of the difficulty of certain transformations of data types. Shortly thereafter a student of R. F. C. Walters in Australia,

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R. Gates, was able to show that a similar result holds, not only for the Tree equation, but for a vast number of other polynomial fixed point equations. (Namely, for polynomials that are not constant and have a non-zero constant term.) The objective of the present paper is to remove all restrictions on the polynomial involved in the fixed point equation and to begin to point the way toward multi-variable extensions.

We now describe the contents of the paper in some technical detail. The reader will be assumed to be familiar with (pr)extensive and distributive categories and their Burnside rigs as discussed in [16] and [14]. If $\mathcal{C}$ is a small distributive category, then its Burnside rig will be denoted by $\mathfrak{B}\mathcal{C}$. We also need to assume that the reader is familiar with algebraic theories [13] and some topos theory including the construction of classifying toposes as in Section D3.1 of [10].

Let $R = \mathbb{N}[X]/(X = L(X))$ for some $L(X) \in \mathbb{N}[X]$.

Is R the Burnside rig of a distributive category?

If $L(X)$ is constant the answer is 'yes' because in this case $R = \mathbb{N}$, which is the Burnside rig of the category of finite sets. As recalled above, the main result in [16] shows that the answer is positive for $L(X) = X + 1 + X = 2X + 1$; indeed, in this case R is the Burnside rig of the category of bounded polyhedra. Using a different technique, [3] proves that the answer is also positive for $L(X) = 1 + X^2$. A related but different approach is used in [8] to show that the answer is positive if $L(X)$ is not constant and $L0 \neq 0$.

We show that the answer is positive for all $L(X) \in \mathbb{N}[X]$. Our proof is a combination of the techniques used in [3] and [8], synthesising the use of toposes and of calculi of fractions. (We will only deal with one-variable presentations, but it must be mentioned that it is also proved in [16] that $\mathbb{N}[X, Y]/(X = 2X + 1, Y = X + 1 + Y, Y^2 = 2Y^2 + Y)$ is the Burnside rig of the category of unbounded polyhedra.)

Let $\mathcal{T}$ be a small category with finite products and let $\widehat{\mathcal{T}}$ be the induced topos of presheaves. Denote the (finite-)coproduct completion of $\mathcal{T}$ by $\mathrm{Fam}\mathcal{T}$. Since $\mathcal{T}$ has products, $\mathrm{Fam}\mathcal{T}$ is prextensive (see paragraph before Proposition 4.6 in [5]) and the essentially unique functor $\mathrm{Fam}\mathcal{T} \to \widehat{\mathcal{T}}$ making the following diagram commute

$$\mathcal{T} \longrightarrow \mathrm{Fam}\mathcal{T}$$
$$\searrow \quad \downarrow$$
$$\widehat{\mathcal{T}}$$

is fully faithful and preserves products and coproducts. (Of course, the diagonal map is the Yoneda embedding and the horizontal one is the universal inclusion of its domain into the coproduct completion.)

Now let J be a Grothendieck topology on $\mathcal{T}$ and $j : \mathrm{Sh}(\mathcal{T}, J) \to \widehat{\mathcal{T}}$ be the induced subtopos. The inverse image of j may be precomposed with the inclusion $\mathrm{Fam}\mathcal{T} \to \widehat{\mathcal{T}}$ to obtain a functor $\mathrm{Fam}\mathcal{T} \to \mathrm{Sh}(\mathcal{T}, J)$.

Definition 1.1 The full image of the functor $\mathrm{Fam}\mathcal{T} \to \mathrm{Sh}(\mathcal{T}, J)$ will be denoted by

$$\mathrm{Fam}(\mathcal{T}, J) \to \mathrm{Sh}(\mathcal{T}, J)$$

as in the following diagram

$$\begin{array}{ccc} \mathrm{Fam}\mathcal{T} & \longrightarrow & \widehat{\mathcal{T}} \\ \downarrow & & \downarrow j^* \\ \mathrm{Fam}(\mathcal{T}, J) & \longrightarrow & \mathrm{Sh}(\mathcal{T}, J) \end{array}$$

where the left-bottom pair is the bo-ff (bijective on objects, fully faithful) factorization of $\mathrm{Fam}\mathcal{T} \to \mathrm{Sh}(\mathcal{T}, J)$.

Since both the top and right functors in the square of Definition 1.1 preserve finite products and coproducts, the category $\mathrm{Fam}(\mathcal{T}, J)$ is prextensive and the inclusion $\mathrm{Fam}(\mathcal{T}, J) \to \mathrm{Sh}(\mathcal{T}, J)$ preserves products and coproducts. In particular, we can apply the above to the case where $\mathcal{T}$ is a free algebraic theory. Let us quickly recall the basic definitions and introduce some notation.

Fix a small version **fSet** of the category of finite sets. An *(algebraic) theory* is a small category $\mathcal{T}$ with finite products together with a bijective-on-objects functor $T : \mathbf{fSet}^{\mathrm{op}} \to \mathcal{T}$ that preserves finite products [13]. When there is no risk of confusion we will omit the functor T and simply say that $\mathcal{T}$ is a theory. It is convenient to use 'exponential notation' so that if $f : A \to B$ is a map in **fSet** then $T^f : T^B \to T^A$ is the corresponding map in $\mathcal{T}$. In particular, we will write T instead of T^1 and, for each element $i : 1 \to I$ in **fSet**, we may let $\pi_i = T^i : T^I \to T$. A *morphism of theories* is a functor $F : \mathcal{T} \to \mathcal{T}'$ that preserves finite products and makes the following diagram

$$
\begin{array}{ccc}
 & \mathbf{fSet}^{\mathrm{op}} & \\
 {}^{T}\swarrow & & \searrow^{T'} \\
\mathcal{T} & \xrightarrow[\quad F \quad]{} & \mathcal{T}'
\end{array}
$$

commute. Let **Th** be the category of theories.

Fix a natural numbers object $\mathbb{N}$ in **Set** and consider it as a discrete category. An object P in the topos $\mathbf{Set}^{\mathbb{N}}$ may be thought of as a 'signature' such that for each $n \in \mathbb{N}$, Pn is the set of 'operations of arity n'. Now fix an inclusion $\mathbb{N} \to \mathbf{fSet}$ sending each $n \in \mathbb{N}$ to a finite set $\mathbf{n}$ of cardinality n. This inclusion induces a functor $U : \mathbf{Th} \to \mathbf{Set}^{\mathbb{N}}$ such that for any $\mathcal{T}$ in **Th** and $n \in \mathbb{N}$, $(U\mathcal{T})n = \mathcal{T}(T^{\mathbf{n}}, T)$. It is well-known that the functor U has a left adjoint $F : \mathbf{Set}^{\mathbb{N}} \to \mathbf{Th}$. (See Section II.2 in [13].) For any P in $\mathbf{Set}^{\mathbb{N}}$, FP is the *free theory* determined by P.

Any polynomial $L(X) \in \mathbb{N}[X]$ determines a 'signature' $\ell \in \mathbf{Set}^{\mathbb{N}}$ such that if m is the coefficient of degree n then $\ell n = \mathbf{m}$. The free theory determined by ℓ will be denoted by $\mathcal{L}$. (So that $\mathcal{L}$-algebras are sets S equipped with a function $L(S) \to S$.)

Let $\mathfrak{B}\mathcal{L}$ denote the (multiplicative) commutative monoid of iso clases of objects of $\mathcal{L}$. The product preserving inclusion $\mathcal{L} \to \mathrm{Fam}\mathcal{L}$ induces a (multiplicative-)monoid morphism $\mathfrak{B}\mathcal{L} \to \mathfrak{B}(\mathrm{Fam}\mathcal{L})$ as below

$$
\begin{array}{ccc}
n \longmapsto & & X^n \\
\Big\downarrow & & \\
 & (\mathbb{N}, +, 0) \longrightarrow (\mathbb{N}[X], \cdot, 1) & \\
 & {}_{\cong}\Big\downarrow \qquad\qquad \Big\downarrow{}_{\cong} & \\
[T^{\mathbf{n}}] & \mathfrak{B}\mathcal{L} \longrightarrow \mathfrak{B}(\mathrm{Fam}\mathcal{L}) &
\end{array}
$$

where $[T^{\mathbf{n}}]$ is the object (determined by $T^{\mathbf{n}}$) in the Burnside monoid $\mathfrak{B}\mathcal{L}$. This induces a morphism $\mathfrak{B}\mathcal{L} \to (\mathbb{N}[X], \cdot, 1)$ of multiplicative monoids and so, a canonical composite

$$
\mathfrak{B}\mathcal{L} \longrightarrow (\mathbb{N}[X], \cdot, 1) \longrightarrow (\mathbb{N}[X]/(L(X) = X), \cdot, 1)
$$

that sends $[T]$ to X.

For every $n \geq 0$, each $f \in \ell n$ induces a map in $\mathcal{L}$ that we denote by $f : T^{\mathbf{n}} \to T$. Let J_L be the least Grothendieck topology on $\mathcal{L}$ such that T is covered by the sieve generated by the family $(f : T^{\mathbf{n}} \to T \mid n \in \mathbb{N}, f \in \ell n)$. Our main result (Theorem 15.2) shows that: if $L(X) \in \mathbb{N}[X]$ is not constant, the canonical $\mathfrak{B}\mathcal{L} \to \mathbb{N}[X]/(L(X) = X)$ extends to a unique iso $\mathfrak{B}(\mathrm{Fam}(\mathcal{L}, J_L)) \to \mathbb{N}[X]/(L(X) = X)$ of rigs. We will also present the geometric theory classified by $\mathrm{Sh}(\mathcal{L}, J_L)$.

The purpose of Sections 2 to 5 is, roughly, to show that the more complicated part of the proof of the main result may be confined, as in Blass' paper [3], to a site with finite products (Proposition 5.7). The notion of *semi-saturated* subcategory (admitting a calculus of (right) fractions) plays here the most important role.

In Blass' paper, the more difficult part of the proof takes place in a site whose underlying category is the free algebraic theory generated by a constant and a binary operation. A key Lemma (see p. 16 in [3]) shows that the relevant covering sieves may be characterized as those that contain all constants. Of course, this must change since the theories we consider may lack constants. Yet, there is a sense in which the same idea works. We try to capture the essence of the idea via the notion of *ample family* introduced in Section 6 in the context of *forts*.

Sections 7 to 11 culminate in the definition of *ranked (algebraic) theory* and the result that free theories are ranked in a canonical way. Intuitively, ranked theories form a class of algebraic theories where most of Blass' argument makes sense. We decided to formulate such a concept because all the attempts to mimic Blass' proof in an arbitrary free theory described using syntactic terms led to calculations that were impossible to read. Fortunately, all that is needed about free theories may be expressed as in Proposition 11.5. Unfortunately, this requires some new auxiliary concepts such as that of *rigged theories* but, altogether, we believe that the abstract formulation is better than the alternative using strings of symbols.

Sections 12 to 14 mimic Blass' proof. In particular, the notion of *development* is the natural analogue of Blass' notion in our more general context.

In Section 15 we combine our results on calculi of fractions (inspired by Gates' work) with our generalization of Blass' ideas. This combination proves the main result (Theorem 15.2). In Section 16 we give a presentation of the theory classified by the topos used to prove the main result.

2 Calculi of Fractions

Here we recall some well-known material on calculi of fractions (see, e.g., [4, 7] or [9]).

Definition 2.1 A bijective-on-objects subcategory $\Gamma \to \mathcal{X}$ is said to *admit a calculus of (right) fractions* if:

CF1   Every cospan as on the left below

can be completed to a commutative square as on the right above.

CF2   For every commutative diagram as on the left below

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{s \in \Gamma}{\longrightarrow} Z \qquad W \overset{s' \in \Gamma}{\longrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

there exists an $s' : W \to X$ in Γ such that the diagram on the right above commutes.

For the rest of the section fix a bijective-on-objects subcategory $\Gamma \to \mathcal{X}$ admitting a calculus of fractions. Notice that if every map in Γ is mono then condition (CF2) is trivially satisfied.

The *category of fractions* $\mathcal{X}[\Gamma^{-1}]$ has the same objects as $\mathcal{X}$ and, as arrows $X \to Y$, equivalence classes of spans

$$\begin{array}{ccc} X_s & \overset{f}{\longrightarrow} & Y \\ {\scriptstyle s}\Big\downarrow & & \\ X & & \end{array}$$

with $s \in \Gamma$. Two such spans (f, s) and (g, t) are equivalent if there is a commutative diagram

with $sa = tb \in \Gamma$. The equivalence class determined by (f, s) will be denoted by $\frac{f}{s}$. The obvious functor from $\mathcal{X}$ to $\mathcal{X}[\Gamma^{-1}]$ sending f to $\frac{f}{id}$ is many times denoted by $P_\Gamma : \mathcal{X} \to \mathcal{X}[\Gamma^{-1}]$ and it is universal among functors from $\mathcal{X}$ sending all maps in Γ to isos.

Lemma 2.2 *Let $f : X \to Y$ be a map in $\mathcal{X}$. Let $u : U \to X$ in Γ and $g : U \to Y$. Then $\frac{f}{id} = \frac{g}{u}$ if and only if there is a $w : W \to U$ such that $uw : W \to X$ is in Γ and the following diagram*

$$W \overset{w}{\longrightarrow} U \underset{g}{\overset{fu}{\rightrightarrows}} Y$$

commutes.

Proof The condition $\frac{f}{id} = \frac{g}{u}$ means that there is a commutative diagram

with $w' = uw \in \Gamma$. □

The following result is essentially that appearing in Section I.3.5 of [7].

Lemma 2.3 *Let $f : X \to Y$ be a map in $\mathcal{X}$. The map $\frac{f}{id}$ is an iso in $\mathcal{X}[\Gamma^{-1}]$ if and only if there is a diagram as below*

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
{\scriptstyle \alpha}\downarrow & {\scriptstyle g}\swarrow & \downarrow{\scriptstyle \beta} \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

with α and β in Γ and such that both triangles inside the square commute.

Let $F : \mathcal{X}_0 \to \mathcal{X}$ be a full subcategory and $\Gamma_0 \to \mathcal{X}_0$ be a bijective-on-objects subcategory admitting a calculus of fractions such that, for every f in Γ_0, Ff is in Γ. Then there exists a unique functor $G : \mathcal{X}_0[\Gamma_0^{-1}] \to \mathcal{X}[\Gamma^{-1}]$ such that the following diagram

$$
\begin{array}{ccc}
\mathcal{X}_0 & \xrightarrow{\ F\ } & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{X}_0[\Gamma_0^{-1}] & \xrightarrow[\ G\ ]{} & \mathcal{X}[\Gamma^{-1}]
\end{array}
$$

commutes.

Lemma 2.4 *If for every B in $\mathcal{X}_0$ and $f : X \to FB$ in Γ there exists a map $g : FA \to X$ such that $fg = Fh$ for some $h : A \to B$ in Γ_0, then $G : \mathcal{X}_0[\Gamma_0^{-1}] \to \mathcal{X}[\Gamma^{-1}]$ is fully faithful.*

Proof To prove that G is full, let A and B be in $\mathcal{X}_0$ and $\frac{f}{s} : FA \to FB$ in $\mathcal{X}[\Gamma^{-1}]$ with $s : X_s \to FA$ in Γ. By hypothesis there exists $g : FA' \to X_s$ such that $sg = Ft$ for some $t : A' \to A$ in Γ_0 as in the following diagram

$$
\begin{array}{ccc}
FA' & \xrightarrow{\ g\ } & X_s & \xrightarrow{\ f\ } & FB \\
 & {\scriptstyle Ft}\searrow & \downarrow{\scriptstyle s} & & \\
 & & FA & &
\end{array}
$$

and, since F is full, there exists $b : A' \to B$ such that $Fb = fg$. Clearly $\frac{f}{s} = G\frac{b}{t}$ so G is full. To prove that G is faithful let $\frac{f}{s}, \frac{g}{t} : A \to B$ in $\mathcal{X}_0[\Gamma_0^{-1}]$ with $s : A_s \to A$ and $t : A_t \to A$ both in Γ_0. Assume that $G\frac{f}{s} = G\frac{g}{t}$ so there exists a commutative diagram as on the left below

$$
\begin{array}{ccccc}
 & & FA_s & & \\
 & {\scriptstyle Fs}\swarrow & {\scriptstyle a}\uparrow & \searrow{\scriptstyle Ff} & \\
FA & \xleftarrow{\ c\ } & X' & & FB \\
 & {\scriptstyle Ft}\nwarrow & {\scriptstyle b}\downarrow & \nearrow{\scriptstyle Fg} & \\
 & & FA_t & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & FA_s & & \\
 & {\scriptstyle Fs}\swarrow & \uparrow{\scriptstyle Fa'} & \searrow{\scriptstyle Ff} & \\
FA & & FA' & & FB \\
 & {\scriptstyle Ft}\nwarrow & \downarrow{\scriptstyle Fb'} & \nearrow{\scriptstyle Fg} & \\
 & & FA_t & &
\end{array}
$$

such that $(Fs)a = (Ft)b \in \Gamma$. Let us call this map $c : X' \to FA$. By hypothesis there exists a map $k : FA' \to X'$ such that $ck = Fh$ for some $h : A' \to A$ in Γ_0. Since F is full there are

$a' : A' \to A_s$ and $b' : A' \to A_t$ in $\mathcal{X}_0$ such that $Fa' = ak$ and $Fb' = bk$ and the diagram on the right above commutes. We also have that $(Fs)(Fa') = Fh = (Ft)(Fb')$ and, since F is faithful, a' and b' witness that $\frac{f}{s} = \frac{g}{t}$. □

An important ingredient in the proof of our main result is a weakening of the following concept.

Definition 2.5 The subcategory $\Gamma \to \mathcal{X}$ (admitting a calculus of fractions) is called *saturated* if for every map f in $\mathcal{X}$, $\frac{f}{id}$ an iso implies that $f \in \Gamma$.

We end this section with a relevant example of a saturated subcategory. Let j be a (Lawere-Tierney) topology in a topos $\mathcal{E}$ and let $L : \mathcal{E} \to \text{Sh}_j\mathcal{E}$ be the associated-sheaf functor. We assume that the reader is familiar with the relation between Lawvere-Tierney topologies and closure operators, and with the concept of *(j-)dense* monos.

Definition 2.6 Let $f : X \to Y$ be a morphism in $\mathcal{E}$. It is called *almost epi* if its image is dense (as a subobject of Y). Also, let $\langle a, b \rangle : R \to X \times X$ be the kernel pair of f and $\tau : X \to R$ be the factorization of the diagonal $X \to X \times X$ through $\langle a, b \rangle : R \to X \times X$. The map f is called *almost mono* if τ is dense. Finally, f is called *bidense* if f is both almost epi and almost mono.

The next result follows from the material in Section 3.4 in [9].

Proposition 2.7 *The subcategory $\Xi \to \mathcal{E}$ of bidense morphisms admits a calculus of fractions and the composite $\text{Sh}_j\mathcal{E} \to \mathcal{E} \to \mathcal{E}[\Xi^{-1}]$ is an equivalence. Moreover, $\Xi \to \mathcal{E}$ is saturated.*

In other words, $\text{Sh}_j\mathcal{E}$ is a category of fractions.

3 Proper Families

Let $\mathcal{C}$ be a small category. We assume that the reader is familiar with the notion of Grothendieck topology but we recall the related notion of basis.

Definition 3.1 A *basis* (for a Grothendieck topology) on $\mathcal{C}$ is a function assigning to each object U of $\mathcal{C}$ a collection KU of families $(f_i : U_i \to U \mid i \in I)$ of maps in $\mathcal{C}$ (called *K-covering families*) such that:

1. If $f : U' \to U$ is an iso then the singleton family $(f \mid 1)$ is in KU.
2. If $(f_i : U_i \to U \mid i \in I)$ is a K-covering family and $g : V \to U$ is any map in $\mathcal{C}$ then there exists a K-covering family $(h_j : V_j \to V \mid j \in J)$ such that each gh_j factors through some f_i.
3. If $(f_i : U_i \to U \mid i \in I)$ is in KU and, for each $i \in I$, $(h_{i,j} : U_{i,j} \to U_i \mid j \in J_i)$ $\in KU_i$ then the family of composites $(f_i h_{i,j} \mid i \in I, j \in J_i)$ is in KU.

Let K be a basis on $\mathcal{C}$ such that every K-covering family is finite.

For any U in $\mathcal{C}$, a map $[f_i : U_i \to U \mid i \in I] : \sum_{i \in I} U_i \to U$ in $\text{Fam}\mathcal{C}$ is called *basic* if the family $(f_i : U_i \to U \mid i \in I)$ is in KU. We now extend this definition to maps with arbitrary codomain in $\text{Fam}\mathcal{C}$. Each object in $\text{Fam}\mathcal{C}$ is of the form $\sum_{i \in I} U_i$ for a finite set

I and, for each $i \in I$, U_i an object in $\mathcal{C}$. Any map with codomain $\sum_{i \in I} U_i$ is of the form $\sum_{i \in I} g_i : \sum_{i \in I} X_i \to \sum_{i \in I} U_i$. (Notice that we are not requiring the X_i's to be in $\mathcal{C}$.) We say that such a map is *selected* if $g_i : X_i \to U_i$ is basic for every $i \in I$.

Lemma 3.2 *The selected maps form a bijective-on-objects subcategory that we denote by $\Gamma_K \to \mathcal{C}$. If every K-covering family is finite then condition (CF1) in Definition 2.1 is satisfied.*

Proof The first condition for bases implies that all identities are selected. The third condition for bases implies that selected maps are closed under composition. The second condition for bases, together with finiteness, implies that (CF1) holds. □

In order to establish a sufficient condition for $\Gamma_K \to \mathcal{C}$ to admit a calculus of fractions we introduce the following.

Definition 3.3 A family $(f_i : C_i \to C \mid i \in I)$ of maps in $\mathcal{C}$ is called *monic* if the existence of a commutative diagram

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\;h\;} & C_j \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f_j} \\
C_i & \xrightarrow[\;f_i\;]{} & C
\end{array}
$$

implies that $i = j$ and $g = h$.

The following result gives two alternative formulations.

Lemma 3.4 *For any family $F = (f_i : C_i \to C \mid i \in I)$ of maps in $\mathcal{C}$ the following are equivalent:*

1. *The family F is monic.*
2. *The following two conditions hold:*

   (a) *the map $f_i : C_i \to C$ is mono for each $i \in I$ and*
   (b) *for every $i, j \in I$ the existence of a commutative diagram*

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\;h\;} & C_j \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f_j} \\
C_i & \xrightarrow[\;f_i\;]{} & C
\end{array}
$$

   in $\mathcal{C}$ implies that $i = j$.

   If, moreover, I is finite then the above are also equivalent to:
3. *The induced map $[f_i \mid i \in I] : \sum_{i \in I} C_i \to C$ is mono in $\mathrm{Fam}\mathcal{C}$.*

Proof The equivalence between the first two items is left for the reader. We prove that the last two items are equivalent when I is finite. So let $f = [f_i \mid i \in I] : \sum_{i \in I} C_i \to C$ in $\mathrm{Fam}\mathcal{C}$. Since every object of $\mathrm{Fam}\mathcal{C}$ is a finite coproduct of objects in $\mathcal{C}$, f is mono if and only if for every object D in $\mathcal{C}$, and maps $g, h : D \to \sum_{i \in I} C_i$, $fg = fh$ implies $g = h$.

Since D is connected in Fam$\mathcal{C}$, $g = in_i g_i$ for some $i \in I$ and $g_i : D \to C_i$ and, similarly, $h = in_j h_j$ for some $j \in I$ and $h_j : D \to C_j$. So, the equality $fg = fh$ simply means that the square below

$$
\begin{array}{ccc}
D & \xrightarrow{\ h_j\ } & C_j \\
{\scriptstyle g_i}\big\downarrow & & \big\downarrow{\scriptstyle f_j} \\
C_i & \xrightarrow[\ f_i\ ]{} & C
\end{array}
$$

commutes. It follows that the map f is mono in Fam$\mathcal{C}$ if and only if the family F is monic in the sense of Definition 3.3. $\qquad\square$

The next somewhat ad-hoc terminology will prove efficient.

Definition 3.5 A family $(f_i : C_i \to C \mid i \in I)$ of maps in $\mathcal{C}$ will be called *proper* if it is finite and monic.

We can exhibit our source of calculi of fractions coming from bases.

Lemma 3.6 *If every K-covering family is proper then the subcategory $\Gamma_K \to$ Fam$\mathcal{C}$ admits a calculus of fractions such that every map in Γ_K is mono.*

Proof Follows from Lemma 3.2 and the fact that if every K-covering family is monic then every map in Γ_K is mono. $\qquad\square$

Let Sh$(\mathcal{C}, K) \to \widehat{\mathcal{C}}$ be the associated topos of sheaves and $\mathbf{a} : \widehat{\mathcal{C}} \to$ Sh$(\mathcal{C}, K)$ be the associated sheaf functor. Adapting the notation in Definition 1.1 from topologies to bases, the full image of Fam$\mathcal{C} \to \widehat{\mathcal{C}} \to$ Sh$(\mathcal{C}, K)$ will be denoted by Fam$(\mathcal{C}, K) \to$ Sh$(\mathcal{C}, K)$.

Proposition 3.7 *If every K-covering family is proper then there is full and faithful* $(\text{Fam}\mathcal{C})[\Gamma_K^{-1}] \to$ Sh$(\mathcal{C}, K)$ *making the following diagram*

$$
\begin{array}{ccccc}
\mathcal{C} & \longrightarrow & \text{Fam}\mathcal{C} & \longrightarrow & (\text{Fam}\mathcal{C})[\Gamma_K^{-1}] \\
& \searrow & \big\downarrow & & \big\downarrow \\
& & \widehat{\mathcal{C}} & \xrightarrow[\ \mathbf{a}\ ]{} & \text{Sh}(\mathcal{C}, K)
\end{array}
$$

commute. Therefore, Fam$(\mathcal{C}, K)$ *is canonically equivalent to* $(\text{Fam}\mathcal{C})[\Gamma_K^{-1}]$.

Proof By Proposition 2.7, Sh$(\mathcal{C}, J)$ coincides with $\widehat{\mathcal{C}}[\Xi^{-1}]$ where $\Xi \to \widehat{\mathcal{C}}$ is the subcategory of bidense maps. So it is enough to check that Lemma 2.4 is applicable to Fam$\mathcal{C} \to \widehat{\mathcal{C}}$. First we need to check that the full inclusion Fam$\mathcal{C} \to \widehat{\mathcal{C}}$ sends maps in Γ_K to dense monos. This follows from the fact that the inclusion Fam$\mathcal{C} \to \widehat{\mathcal{C}}$ preserves monos and coproducts, and the fact that Yoneda sends K-covers to families of maps generating a dense subobject. It remains to prove that the condition in the statement of Lemma 2.4 holds. It is enough to restrict to bidense maps $X \to C$ in $\widehat{\mathcal{C}}$ with codomain in $\mathcal{C}$. Its image $s : S \to C$ is a dense subobject. Since K is a basis, the sieve S must contain the maps in a K-cover $(f_i : C_i \to C \mid i \in I)$. In other words, the induced map $f : \sum_{i \in I} C_i \to C$ factors through

$S \to C$ and, since the domain of f is projective in $\widehat{C}$, f factors through the bidense $X \to C$. That is, we obtain a map $\sum_{i \in I} C_i \to X$ such that the composite $\sum_{i \in I} C_i \to X \to C$ is in Γ_K. $\qquad \square$

We will need the following closure property.

Lemma 3.8 *Let G be a proper family of maps with codomain Y and let $f : X \to Y$. If the pullback family f^*G exists then it is proper.*

Proof If f^*G exists then it is clearly a finite family of monos. To prove that it is monic let $G = (g_i : A_i \to Y \mid i \in I)$ and let $f^*G = (f^*g_i : f^*A_i \to X \mid i \in I)$. Assume that the diagram on the left below

$$
\begin{array}{ccc}
W & \xrightarrow{w_1} & f^*A_j \\
{\scriptstyle w_0}\downarrow & & \downarrow{\scriptstyle f^*g_j} \\
f^*A_i & \xrightarrow{f^*g_i} & X
\end{array}
\qquad\qquad
\begin{array}{ccccc}
W & \xrightarrow{w_1} & f^*A_j & \longrightarrow & A_j \\
{\scriptstyle w_0}\downarrow & & & & \downarrow{\scriptstyle g_j} \\
f^*A_i & \longrightarrow & A_i & \xrightarrow{g_i} & Y
\end{array}
$$

commutes for some $i, j \in I$. Then the diagram on the right above commutes and, since G is monic, $i = j$. So f^*G is monic by Lemma 3.4. $\qquad \square$

For example, in an algebraic theory, pullbacks along projections always exist.

4 Semi-Saturation

Let $\Gamma \to \mathcal{X}$ be a bijective-on-objects subcategory admitting a calculus of fractions. The second paragraph of page 9 in [3] says that *very explicit bijections* are determined by two families of patterns $(p_i)_{i \in I}$ and $(q_i)_{i \in I}$ such that, for each i, "the same labels occur in p_i as in q_i". This condition motivates the following.

Definition 4.1 The subcategory $\Gamma \to \mathcal{X}$ (admitting a calculus of fractions) is called *semi-saturated* if for every iso $\frac{f}{s} : X \to Y$ in $\mathcal{X}[\Gamma^{-1}]$ there are $m, v \in \Gamma$ such that $\frac{f}{s} = \frac{m}{v}$.

The next result provides a useful alternative formulation.

Lemma 4.2 *The subcategory $\Gamma \to \mathcal{X}$ is semi-saturated if and only if for every map $f : X \to Y$, $\frac{f}{id}$ an iso implies that there are $m, v \in \Gamma$ such that $\frac{f}{id} = \frac{m}{v}$.*

Proof One direction is trivial, for the other assume that $\frac{f}{s} : X \to Y$ is an iso in $\mathcal{X}[\Gamma^{-1}]$, with $f : X_s \to Y$. Clearly, $\frac{f}{s} = \frac{f}{id}\frac{id}{s}$ and $\frac{id}{s} : X \to X_s$ is an iso, so $\frac{f}{id}$ is an iso. By hypothesis, there are $m, v \in \Gamma$ such that $\frac{f}{id} = \frac{m}{v}$. But then

$$
\frac{f}{s} = \frac{f}{id}\frac{id}{s} = \frac{m}{v}\frac{id}{s} = \frac{m}{sv}
$$

so Γ is semi-saturated. $\qquad \square$

Clearly, if Γ is saturated in the sense of Definition 2.5 then it is semi-saturated.

Definition 4.3 A bijective-on-objects subcategory $\Gamma \to \mathcal{X}$ is said to *admit a calculus of dense monos* if it admits a calculus of fractions, every map in Γ is mono and for every diagram of monos as below

$$
\begin{array}{ccc}
U & \xrightarrow{\quad} & V \\
& u \searrow & \downarrow v \\
& & X
\end{array}
$$

$u \in \Gamma$ implies $v \in \Gamma$.

Our source of semi-saturated subcategories is based on the following.

Proposition 4.4 *If $\Gamma \to \mathcal{X}$ admits a calculus of dense monos then it is semi-saturated.*

Proof Assume that $\frac{f}{id} : X \to Y$ is an iso. Then there is a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
\alpha \downarrow & \swarrow g & \downarrow \beta \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

as in the statement of Lemma 2.3 with α and β in Γ. By hypothesis, β is mono, so g is also mono. But then α, h and g form a triangle as in Definition 4.3 so $g \in \Gamma$ and we have a diagram

$$
A \xrightarrow{\ h\ } B \underset{\beta}{\overset{fg}{\rightrightarrows}} Y
$$

such that $gh \in \Gamma$. Lemma 2.2 implies that $\frac{f}{id} = \frac{\beta}{g}$, so Lemma 4.2 implies that Γ is semi-saturated. $\qquad\square$

The semi-saturated subcategories we are interested in come from bases.

Definition 4.5 A basis K on a small category $\mathcal{C}$ is called a *basis of dense monos* if:

1. every K-cover is proper and
2. for every C in $\mathcal{C}$, for any $F \in K(C)$ and any proper family P of maps with codomain C, if every map in F factors through some map in P then $P \in K(C)$.

Recall (Lemma 3.6) that any basis K on $\mathcal{C}$ such that all K-covers are proper determines a subcategory $\Gamma_K \to \mathrm{Fam}\mathcal{C}$ admitting a calculus of fractions such that every map in Γ_K is mono.

Corollary 4.6 *If K is a basis of dense monos on $\mathcal{C}$ then $\Gamma_K \to \mathrm{Fam}\mathcal{C}$ is semi-saturated.*

Proof By Proposition 4.4 it is enough to show that $\Gamma_K \to \mathrm{Fam}\mathcal{C}$ admits a calculus of dense monos. So consider a diagram of monos in $\mathrm{Fam}\mathcal{C}$ as below

$$
\begin{array}{ccc}
\sum_{i \in I} X_i & \xrightarrow{\quad} & \sum_{j \in J} Y_j \\
& [u_i | i \in I] \searrow & \downarrow [v_j | j \in J] \\
& & Z
\end{array}
$$

and assume that $[u_i \mid i \in I]$ is in the subcategory $\Gamma_K \to \text{Fam}\mathcal{C}$. We need to show that $[v_j \mid j \in J]$ is in Γ_K. It is enough to concentrate on the case when Z is in $\mathcal{C}$. By Lemma 3.4 the families $(u_i \mid i \in I)$ and $(v_j \mid j \in J)$ are proper and, by hypothesis, the former is in KZ. Since K is assumed to be a basis of dense monos, the latter family is also in KZ. So $[v_j \mid j \in J]$ is also in $\Gamma_K \to \mathcal{C}$. $\qquad\square$

5 Compatibility and Weights

Let $\mathcal{X}$ be a distributive category and $\Gamma \to \mathcal{X}$ a bijective-on-objects subcategory admitting a calculus of fractions. The only assumption that we make on $\Gamma \to \mathcal{X}$ is that $\mathcal{X}[\Gamma^{-1}]$ is distributive and that the universal functor $\mathcal{X} \to \mathcal{X}[\Gamma^{-1}]$ preserves finite coproducts. (Recall that this functor preserves finite products automatically. In fact, all finite limits by the 'right-fractions' analogue of Proposition I.3.1 in [7].) For example, this is what happens in the case of Proposition 3.7. See also remark after Definition 1.1.

Fix also a rig R.

Definition 5.1 A rig morphism $\gamma : \mathfrak{B}\mathcal{X} \to R$ is called *compatible* (with the inclusion $\Gamma \to \mathcal{X}$) if for every map $X \to Y$ in Γ, $\gamma[X] = \gamma[Y]$.

Compatibility interacts well with semi-saturation.

Proposition 5.2 *If Γ is semi-saturated and $\gamma : \mathfrak{B}\mathcal{X} \to R$ is compatible then there exists a unique map $\gamma' : \mathfrak{B}(\mathcal{X}[\Gamma^{-1}]) \to R$ such that the following diagram*

$$\mathfrak{B}\mathcal{X} \longrightarrow \mathfrak{B}(\mathcal{X}[\Gamma^{-1}])$$
$$\gamma \searrow \qquad \downarrow \gamma'$$
$$R$$

commutes.

Proof Uniqueness follows because $\mathcal{X} \to \mathcal{X}[\Gamma^{-1}]$ is bijective on objects and so the induced $\mathfrak{B}\mathcal{X} \to \mathfrak{B}(\mathcal{X}[\Gamma^{-1}])$ is surjective. We need to define γ'. For any X in $\mathcal{X}$, the induced object in $\mathfrak{B}\mathcal{X}$ will be denoted by $[X]$ and that in $\mathfrak{B}(\mathcal{X}[\Gamma^{-1}])$ by $[[X]]$. The only possible definition is $\gamma'[[X]] = \gamma[X]$. To prove that it is well defined assume that $[[X]] = [[Y]]$. So there exists an iso $\frac{f}{s} : X \to Y$ in $\mathcal{X}[\Gamma^{-1}]$, say, with $s : X_s \to X$ and $f : X_s \to Y$. Because Γ is semi-saturated, we can assume that f (as well as s) is in Γ. Our hypothesis on γ implies that $\gamma[X] = \gamma[X_s] = \gamma[Y]$, so $\gamma'[[X]] = \gamma'[[Y]]$. The function γ' is a rig morphism because $\mathfrak{B}\mathcal{X} \to \mathfrak{B}(\mathcal{X}[\Gamma^{-1}])$ is surjective. $\qquad\square$

Let $\mathcal{C}$ be a small category with finite products and denote by $\mathfrak{B}\mathcal{C}$ the multiplicative monoid of iso-clases of objects.

Lemma 5.3 *For any morphism $\gamma : \mathfrak{B}\mathcal{C} \to R$ of multiplicative monoids there exists a unique map of rigs $\gamma' : \mathfrak{B}(\text{Fam}\mathcal{C}) \to R$ such that the following diagram*

$$\mathfrak{B}\mathcal{C} \longrightarrow \mathfrak{B}(\text{Fam}\mathcal{C})$$
$$\gamma \searrow \qquad \downarrow \gamma'$$
$$R$$

commutes.

Proof The morphism γ' sends coproducts in Fam$\mathcal{C}$ to sums in R. We leave the details for the reader. □

Let us fix a map $\gamma : \mathfrak{B}\mathcal{C} \to R$ of multiplicative monoids. We now explain how the compatibility of the extension $\mathfrak{B}(\text{Fam}\mathcal{C}) \to R$ with a subcategory (admitting a calculus of fractions) induced by a basis on $\mathcal{C}$ can be reduced to a condition in terms of the basis.

Definition 5.4 For any finite family $F = (f : C_i \to C \mid i \in I)$ of maps in $\mathcal{C}$ with common codomain, the *weight (relative to γ)* of F is the element $\mathbf{w}F = \sum_i \gamma[C_i]$ in R.

This notion of weight is analogous to the one introduced in p. 11 of Blass' paper. It plays essentially the same role in the proof.

Assume now that $\mathcal{C}$ is equipped with a basis K such that every K-cover is proper. (So that Fam$\mathcal{C}$ is equipped with the subcategory $\Gamma_K \to$ Fam$\mathcal{C}$, admitting a calculus of fractions, as explained in Lemma 3.6.)

Definition 5.5 The function $\gamma : \mathfrak{B}\mathcal{C} \to R$ is called *compatible* (with K) if for every $G \in K\mathcal{C}$, $\mathbf{w}G = \gamma[C]$.

This is justified by the following.

Lemma 5.6 *The monoid map $\gamma : \mathfrak{B}\mathcal{C} \to R$ is compatible with K if and only if the rig map $\gamma' : \mathfrak{B}(\text{Fam}\mathcal{C}) \to R$ is compatible with Γ_K.*

Proof Assume first that $\gamma : \mathfrak{B}\mathcal{C} \to R$ is compatible with K. A morphism m in the subcategory $\Gamma_K \to$ Fam$\mathcal{C}$ (admitting a calculus of fractions) is given by a coproduct $\sum_{i \in I} m_i : \sum_i Y_i \to \sum_i C_i$ where each C_i is in $\mathcal{C}$ and, for each $i \in I$, $m_i : Y_i \to C_i$ is a 'basic' morphism in the sense discussed after Definition 3.1. That is, $Y_i = \sum_{j \in I_i} U_{i,j}$ and $m_i = [n_{i,j} \mid j \in J] : \sum_{j \in J_i} U_{i,j} \to C_i$ for a family $(n_{i,j} : U_{i,j} \to C_i \mid j \in J_i) \in KC_i$. By hypothesis, $\sum_{j \in J_i} \gamma[U_{i,j}] = \gamma[C_i]$ for each $i \in I$. Therefore,

$$\gamma'\left(\left[\sum_{i \in I} Y_i\right]\right) = \sum_{i \in I} \gamma'\left[\sum_{j \in I_i} U_{i,j}\right] = \sum_{i \in I}\left(\sum_{j \in J_i} \gamma\left[U_{i,j}\right]\right) = \sum_{i \in I} \gamma[C_i] = \gamma'\left[\sum_{i \in i} C_i\right]$$

as we needed to prove. The converse holds because if γ' is compatible then, for every $(U_i \to C \mid i \in I)$ in KC,

$$\sum_{i \in I} \gamma[U_i] = \gamma'\left[\sum_{i \in I} U_i\right] = \gamma'[C] = \gamma[C]$$

so the proof is complete. □

Let us summarize what we have achieved so far.

Proposition 5.7 *Let $\mathcal{C}$ be a small category with finite products. Let R be a rig and let $\gamma : \mathfrak{B}\mathcal{C} \to R$ be a map of multiplicative monoids (inducing the unique extension to a rig map $\mathfrak{B}(\text{Fam}\mathcal{C}) \to R$). If K is a basis of dense monos on $\mathcal{C}$ and γ is compatible with it then*

there exists a unique map $\mathfrak{B}(\mathrm{Fam}(\mathcal{C}, K)) \to R$ *of rigs such that the right triangle below commutes.*

$$\mathfrak{B}\mathcal{C} \longrightarrow \mathfrak{B}(\mathrm{Fam}\mathcal{C}) \longrightarrow \mathfrak{B}(\mathrm{Fam}(\mathcal{C}, K))$$

$$\gamma \qquad \downarrow \qquad$$

$$R$$

Proof By Corollary 4.6, the basis of dense monos K induces the semi-saturated subcategory $\Gamma_K \to \mathrm{Fam}\mathcal{C}$ and, by Proposition 3.7, $\mathrm{Fam}(\mathcal{C}, K) = (\mathrm{Fam}\mathcal{C})[\Gamma_K^{-1}]$. By Lemma 5.6 the extension $\gamma' : \mathfrak{B}(\mathrm{Fam}\mathcal{C}) \to R$ is compatible with Γ_K and Proposition 5.2 implies the existence of a unique $\mathfrak{B}((\mathrm{Fam}\mathcal{C})[\Gamma_K^{-1}]) \to R$ as in the statement. $\qquad \square$

To prove the main result we will take $\mathcal{C}$ to be the free theory $\mathcal{L}$ determined by a non-constant polynomial $L(X) \in \mathbb{N}[X]$. The hard part is to find a basis of dense monos K such that the canonical $\mathcal{L} \to \mathbb{N}[X]/(X = L(X))$ is compatible with K in the sense of Definition 5.5. For this purpose the following will be useful eventually.

Lemma 5.8 (Weights of composite families) *Let* $G = (g_i : Y_i \to Z \mid i \in I)$ *be a finite family and, for each* $i \in I$, *let* $F_i = (f_{i,j} : X_{i,j} \to Y_i \mid j \in I_i)$ *be a finite family so that the composite family* $H = (g_i f_{i,j} \mid i \in I, j \in I_i)$ *is also finite. Then* $\mathbf{w}H = \sum_{i \in I} \mathbf{w}F_i$.

Proof Just calculate

$$\mathbf{w}H = \sum_{i \in I} \sum_{j \in I_i} \gamma[X_{i,j}] = \sum_{i \in I} \mathbf{w}F_i$$

using the definition of weight. $\qquad \square$

6 Amplitude

The Lemma in p. 16 of [3] deals with a Grothendieck topology on a free (algebraic) theory and, among other things, identifies the sieves in $J(C)$ as those that contain a finite family such that every constant $1 \to C$ in the theory factors through a map in that finite family. I claim that such a characterization is possible because the polynomial $L(X) = 1 + X^2$ is such that $L0 \neq 0$. Part of our proof was influenced by the wish to generalize this lemma to theories that may lack constants. This is the origin of the notion of 'ample family' that we introduce in this section.

We first recall the notion of *normed category* suggested in pages 139-140 of [12] and, after that, we define ample families in suitable normed categories.

Let $(\mathcal{V}, \otimes, \Bbbk)$ be a monoidal category.

Definition 6.1 A $\mathcal{V}$-*normed category* is a category $\mathcal{C}$ together with the assignment of an object $\partial_{X,Y} f$ of $\mathcal{V}$ to every map $f : X \to Y$ in $\mathcal{C}$, the assignment of a morphism $(\partial_{Y,Z} g) \otimes (\partial_{X,Y} f) \to \partial_{X,Z}(gf)$ in $\mathcal{V}$ for every pair of maps $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, and the assignment of a morphism $\Bbbk \to \partial_{X,X} id_X$ in $\mathcal{V}$ for every object X in $\mathcal{C}$, subject to the evident associativity and unit conditions (that we need not emphasize because they automatically hold in our main example of base monoidal category).

(Although I have not been able to obtain a copy of [2], it appears that normed categories have been also considered there. Indeed, it seems clear from the Zentralblatt *Autorreferat* and the AMS review by Linton that it is proved loc. cit. that normed categories may be seen as categories enriched in suitable monoidal categories; solving, in a general way, an exercise suggested in page 140 of [12].)

Let $(\mathbb{N}, +, 0)$ be the usual commutative monoid of natural numbers under addition and consider its extension $\mathbb{N}_\infty = (\mathbb{N} + \{\infty\}, +, 0)$ with an element ∞ such that, for every $n \in \mathbb{N} + \{\infty\}$, $\infty + n = \infty = n + \infty$. The monoid structure induces a total order $(\mathbb{N}_\infty, \leq)$ with ∞ as terminal object. Moreover, addition extends to a symmetric monoidal structure on the category $(\mathbb{N}_\infty, \leq)$. The resulting monoidal category $((\mathbb{N}_\infty, \leq), +, 0)$ will be denoted simply by $(\mathbb{N}_\infty, \leq)$.

In concrete terms, an $(\mathbb{N}_\infty, \leq)$-normed category is a category $\mathcal{C}$ equipped with a collection $(\partial_{X,Y} : \mathcal{C}(X, Y) \to \mathbb{N}_\infty \mid X, Y \in \mathcal{C})$ of functions such that:

$$(\partial_{Y,Z} g) + (\partial_{X,Y} f) \leq \partial_{X,Z}(gf)$$

holds for every $f : X \to Y$ and $g : Y \to Z$. (Notice that it automatically holds that, for every X in $\mathcal{C}$, $0 \leq \partial_{X,X} id_X$.) We will drop the subscripts and write ∂ instead of $\partial_{X,Y}$; so the key condition may be expressed as $\partial g + \partial f \leq \partial(gf)$.

Assume from now on that $\mathcal{C}$ is an $(\mathbb{N}_\infty, \leq)$-normed category with 'norm' ∂.

Lemma 6.2 *For any $f : X \to Y$ with ∂f finite, $\partial(id_X) = 0 = \partial(id_Y)$.*

Proof Simply observe that $\partial(id_Y) + \partial f \leq \partial f$ and similarly for the other equality. □

On the other hand, if we let $\partial f = \infty$ for all f then we obtain a somewhat extreme sort of $\mathbb{N}_\infty$-normed category.

Definition 6.3 Let F be a family of maps in $\mathcal{C}$ with codomain Y. For $k \in \mathbb{N}$, the family F is called *k-ample* if for every $f : X \to Y$ in $\mathcal{C}$, $\partial f \geq k$ implies that f factors through some map in F. The family F is called *ample* if it is k-ample for some $k \in \mathbb{N}$.

For brevity let us say that F is a family *on Y* if it is a family of maps with codomain Y. Also, if F is a family on Y then we will say that $f : X \to Y$ *factors through F* if it factors through some map in F.

Lemma 6.4 *Let $g : Y \to Z$ and H be a family on Z such that the pullback g^*H exists. If H is n-ample then so is g^*H.*

Proof Let $f : X \to Y$ be such that $\partial f \geq n$. Then $\partial(gf) \geq (\partial g) + (\partial f) \geq (\partial g) + n \geq n$ so $gf : X \to Z$ factors through H by hypothesis, but then f factors through g^*H. □

The family of all maps f with codomain Y and $\partial f \geq n$ is clearly n-ample. We are interested in categories that contain less trivial examples. In fact, we are going to be mainly interested in ample families that are also proper in the sense of Definition 3.5.

Definition 6.5 A family F on Y is *strictly n-ample* if it is n-ample and for every $f \in F$, $\partial f \geq n$. A family will be called *strictly ample* if it is strictly n-ample for some n.

Proper strictly ample families are unique up to iso in the following sense.

Lemma 6.6 *If* $F = (f_i : X_i \to Y \mid i \in I)$ *and* $H = (h_j : A_j \to Y \mid j \in J)$ *are both proper and strictly n-ample families then there exists a unique bijection* $\phi : I \to J$ *such that for every* $i \in I$, $h_{\phi i}$ *is iso to* f_i *over* Y.

Proof Let $i \in I$. Since $\partial f_i \geq n$ there exists a $j \in J$ and a $t : X_i \to A_j$ such that $h_j t = f_i$. Since H is monic, this j is unique so we may call it ϕi and in this way we obtain a function $\phi : I \to J$. Moreover, this t is unique (because $h_{\phi i}$ is mono) and mono (because f_i is mono). We claim that t is an iso. Indeed, since $\partial h_{\phi i} \geq n$, there exists a unique $k \in I$ and mono $u : A_{\phi i} \to X_k$ such that $f_k u = h_{\phi i}$. Then $f_k ut = h_{\phi i} t = f_i$ and, since F is monic, $i = k$ and $ut = id_{X_i}$. So u is a monic with section t and hence t is an iso.

It remains to show that $\phi : I \to J$ is bijective. To prove surjectivity let $j \in J$. As before, there exists an $i \in I$ such that h_j factors through f_i which, in turn, factors through $h_{\phi i}$. So h_j factors through $h_{\phi i}$ and, since H is monic, $j = \phi i$.

Finally, for $i_0, i_1 \in I$ assume that $\phi i_0 = j = \phi i_1$ for some $j \in J$. Once again, h_j factors through f_k for some $k \in I$ and then both f_{i_0} and f_{i_1} factor through f_k. Since F is monic, $i_0 = k = i_1$. So ϕ is injective. $\square$

Assume for the rest of the section that K is a function that assigns to each X in $\mathcal{C}$ a collection of ample and proper families on X.

Definition 6.7 We say *K has long covers* if for every Y in $\mathcal{C}$ and $k \in \mathbb{N}$ there exists an $F \in KY$ such that for all $f \in F$, $\partial f \geq k$.

We are interested in cases where K is the basis of a Grothendieck topology. For the moment we just show that having long covers is sufficient to ensure the key Coverage condition for bases.

Lemma 6.8 *The following hold:*

1. *If* $f : X \to Y$ *is an iso then the family* $(f \mid 1)$ *on* Y *is 0-ample and in* KY.
2. *If* K *has long covers then for every* $G \in KZ$ *and* $g : Y \to Z$ *there exists an* $F \in KY$ *such that for every* $f \in F$, gf *factors through* G.

Proof The first item is easy because every map with codomain Y factors through f. For the second item assume that G is k-ample. Since K has long covers there exists an $F \in KY$ such that for every f in F, $\partial f \geq k$. Then $\partial(gf) \geq (\partial g) + (\partial f) \geq (\partial g) + k \geq k$ and so, gf factors through G. $\square$

In a category with terminal object 1, a map $X \to Y$ is *constant* if it factors through the terminal object.

Definition 6.9 A *fort* is an $(\mathbb{N}_\infty, \leq)$-normed category $\mathcal{C}$ with terminal object such that for every constant $f : X \to Y$ in $\mathcal{C}$, $\partial f = \infty$. A fort will be called *strong* if, for every $f : X \to Y$, $\partial f = \infty$ implies that f is constant.

Sieves on an object containing all the points of that object play an important role in [3]. For this reason we highlight the following.

Lemma 6.10 *If* $\mathcal{C}$ *is a fort and* F *is an ample family on* Y *in* $\mathcal{C}$ *then every point* $1 \to Y$ *factors through* F.

Proof Assume that F is k-ample for $k \in \mathbb{N}$. If $p : 1 \to X$ in $\mathcal{C}$ then $\partial p = \infty > k$. $\qquad\square$

Part of the remaining work involves showing that free theories with some non-constant operation are strong forts in a canonical way. This will follow from the acquisition of more subtle information present in free theories.

7 Rigged Theories

Fix a rig $R = (R, \cdot, 1, +, 0)$ in **Set**. For any A in **fSet**, R^A will denote the exponential in **Set**.

Definition 7.1 A(n algebraic) theory $\mathcal{T}$ is *rigged (in R)* if it is equipped with a family of functions $((\_)_{A,B}^\flat = (\_)^\flat : \mathcal{T}(T^A, T^B) \to R^A \mid A, B \in \textbf{fSet})$ such that the following hold:

$$(T^a : T^I \to T^J)^\flat (i \in I) = (T^a)^\flat i = \sum_{aj=i} 1$$

for every $a : J \to I$ in **fSet** and,

$$(gf)^\flat (i \in A) = \sum_{j \in B} (g^\flat j) \cdot ((\pi_j f)^\flat i)$$

for every $f : T^A \to T^B$ and $g : T^B \to T^C$.

We are going to use the formulas as displayed above but it seems also useful to express them in more general terms. For any A in **fSet** we have a function $\sum_A : R^A \to R$ that sends $\phi \in R^A$ to $\sum_{i \in A} \phi i$. If B is also in **fSet** then the composite

$$R^{B \times A} \times A \xrightarrow{\cong} (R^B)^A \times A \xrightarrow{ev} R^B \xrightarrow{\sum_B} R$$

transposes to a map that, with little risk of confusion, we may call $\sum_B : R^{B \times A} \to R^A$. It sends $\psi \in R^{B \times A}$ to $\sum_{j \in B} \psi(j, \_)$ in R^A. So far this has nothing to do with any algebraic theory, but if we are given the family $((\_)_{A,B}^\flat = (\_)^\flat : \mathcal{T}(T^A, T^B) \to R^A \mid A, B \in \textbf{fSet})$ then there is a function

$$\mathcal{T}(T^B, T^C) \times \mathcal{T}(T^A, T^B) \xrightarrow{*} R^{B \times A}$$

such that the second equation of Definition 7.1 may be formulated as $(gf)^\flat = \sum_B (g * f)$ or by requiring that the diagram

$$
\begin{array}{ccc}
\mathcal{T}(T^B, T^C) \times \mathcal{T}(T^A, T^B) & \xrightarrow{\circ} & \mathcal{T}(T^A, T^C) \\
{\scriptstyle *} \downarrow & & \downarrow {\scriptstyle (\_)^\flat} \\
R^{B \times A} & \xrightarrow[\sum_B]{} & R^A
\end{array}
$$

commutes.

Fix a theory $\mathcal{T}$ rigged in R.

Lemma 7.2 *For any* $f : T^A \to T^B$ *in* $\mathcal{T}$

$$f^\flat i = \sum_{j \in B} (\pi_j f)^\flat i$$

for every $i \in A$.

Proof Calculate $(id_B f)^\natural$ using the second item in Definition 7.1. ☐

That is, the function $f^\natural : A \to R$ is determined by the projections $\pi_j f : T^A \to T$.

Definition 7.3 For any A, B in **fSet** define $\partial_{A,B} = \partial : \mathcal{T}(T^A, T^B) \to R$ by the formula

$$\partial f = \sum_{i \in A} f^\natural i$$

for each $f : T^A \to T^B$.

The operation ∂ can also be calculated in terms of the projections, in the following sense.

Lemma 7.4 *For any* $f : T^A \to T^B$ *the following holds:*

$$\partial f = \sum_{k \in B} \partial(\pi_k f)$$

Proof Just calculate:

$$\partial f = \sum_{i \in A} f^\natural i = \sum_{i \in A} \sum_{k \in B} (\pi_k f)^\natural i = \sum_{k \in B} \sum_{i \in A} (\pi_k f)^\natural i = \sum_{k \in B} \partial(\pi_k f)$$

using Definition 7.3 and Lemma 7.2. ☐

The next result explains how ∂ behaves with respect to composition.

Lemma 7.5 *For every* $f : T^A \to T^B$ *and* $g : T^B \to T^C$ *in* $\mathcal{T}$,

$$\partial(gf) = \sum_{j \in B} (g^\natural j) \cdot \partial(\pi_j f) \qquad \text{and} \qquad \partial(gf) + n = (\partial g) \cdot (\partial f)$$

for some $n \in R$.

Proof The calculation below

$$\partial(gf) = \sum_{i \in A} (gf)^\natural i = \sum_{i \in A} \sum_{j \in B} (g^\natural j) \cdot ((\pi_j f)^\natural i) = \sum_{j \in B} (g^\natural j) \cdot \sum_{i \in A} ((\pi_j f)^\natural i)$$

$$= \sum_{j \in B} (g^\natural j) \cdot \partial(\pi_j f)$$

shows that the left equality in the statement holds. Using Lemma 7.4 we may calculate as follows

$$(\partial g) \cdot (\partial f) = \left(\sum_{j \in B} g^\natural j \right) \cdot \left(\sum_{k \in B} \partial(\pi_k f) \right) = \sum_{j \in B} \sum_{k \in B} (g^\natural j) \cdot \partial(\pi_k f)$$

$$= \partial(gf) + \sum_{\substack{j,k \in B \\ j \neq k}} (g^\natural j) \cdot \partial(\pi_k f)$$

which proves that the right equality in the statement holds because we can take n to be the right summand in the last binary addition. ☐

The following will also be relevant.

Lemma 7.6 *If $\mathcal{T}$ is rigged in R and $F : \mathcal{T}' \to \mathcal{T}$ is a map of theories then $\mathcal{T}'$ is also rigged in R. (The 'rigging' of $\mathcal{T}'$ sends $\alpha : T^A \to T^B$ in $\mathcal{T}'$ to $(F\alpha)^\flat : A \to R$.)*

Proof For any $a : J \to I$ in **fSet**,

$$(F(T^a))^\flat i = (T^a)^\flat i = \sum_{aj=i} 1$$

and for $\alpha : T^A \to T^B$ and $\beta : T^B \to T^C$ in $\mathcal{T}'$,

$$(F(\beta\alpha))^\flat i = ((F\beta)(F\alpha))^\flat i = \sum_{j \in B}((F\beta)^\flat j) \cdot ((\pi_j(F\alpha))^\flat i) = \sum_{j \in B}((F\beta)^\flat j) \cdot ((F(\pi_j\alpha))^\flat i)$$

so $\mathcal{T}'$ is rigged in R. □

Let us now consider a concrete rig. The monoid $\mathbb{N}_\infty = (\mathbb{N} + \{\infty\}, +, 0)$ introduced in Section 6 may be extended to a rig $\mathbb{N}_\infty = (\mathbb{N}_\infty, +, 0, \wedge, \infty)$ where $\wedge$ denotes infima in the order $(\mathbb{N}_\infty, \leq)$. Notice that $+$ distributes over $\wedge$ so that $(\mathbb{N}_\infty, +, 0)$ is the 'multiplicative' part of the rig and $(\mathbb{N}_\infty, \wedge, \infty)$ is the 'additive' part.

An algebraic theory rigged in $\mathbb{N}_\infty$ is then an algebraic theory $\mathcal{T}$ equipped with a family $((\_)^\flat : \mathcal{T}(T^A, T^B) \to \mathbb{N}_\infty^A \mid A, B \in \mathbf{fSet})$ such that:

1. For every $a : J \to I$ in **fSet**,

$$(T^a : T^I \to T^J)^\flat(i \in I) = (T^a)^\flat i = \bigwedge_{aj=i} 0$$

2. For every $f : T^A \to T^B$ and $g : T^B \to T^C$,

$$(gf)^\flat(i \in A) = \bigwedge_{j \in B}(g^\flat j) + ((\pi_j f)^\flat i)$$

hold. Readers familiar with [3] may recognize the idea of 'depth of a node in a tree'. If we picture a map $f : T^A \to T$ as a tree with leaves in A then $f^\flat(i \in A)$ may be thought of as measure of how far is the leaf i from the root of f. We will give a precise meaning to this analogy later. For the moment, consider the following alternative formulation of the first item above.

Lemma 7.7 *For any $a : J \to I$ in **fSet** and the induced $T^a : T^I \to T^J$ in $\mathcal{T}$,*

$$(T^a)^\flat i = \begin{cases} 0 & \text{if } i \in aJ \subseteq I \\ \infty & \text{otherwise} \end{cases}$$

for every $i \in I$.

For example, for $k : 1 \to I$ in **fSet** and the induced projection $\pi_k = T^k : T^I \to T$,

$$(\pi_k)^\flat i = \begin{cases} 0 & \text{if } i = k \\ \infty & \text{otherwise} \end{cases}$$

for every $i \in I$. As another example of how the general facts manifest in the case of $\mathbb{N}_\infty$, and also for future reference, we state the following particular case of Lemma 7.2.

Lemma 7.8 *For every $f : T^A \to T^B$,*

$$f^\flat i = \bigwedge_{j \in B}(\pi_j f)^\flat i$$

for each $i \in A$.

Definition 7.3 says that for $f : T^A \to T^B$ in $\mathcal{T}$,

$$\partial f = \bigwedge_{i \in A} f^\flat i \in \mathbb{N}_\infty$$

and so the following holds.

Lemma 7.9 *For $a : J \to I$ in* **fSet**,

$$\partial(T^a) = \begin{cases} \infty & \text{if } J = 0 \\ 0 & \text{otherwise} \end{cases}$$

and, for every $f : T^A \to T^B$ and $g : T^B \to T^C$ in $\mathcal{T}$,

$$\partial(gf) = \bigwedge_{j \in B} (g^\flat j) + \partial(\pi_j f) \qquad \partial(gf) \geq (\partial g) + (\partial f)$$

hold in $\mathbb{N}_\infty$.

Proof We have

$$\partial(T^a) = \bigwedge_{i \in I} (T^a)^\flat i = \bigwedge_{i \in I} \bigwedge_{aj=i} 0 = \bigwedge_{j \in J} 0 = \begin{cases} \infty & \text{if } J = 0 \\ 0 & \text{otherwise} \end{cases}$$

and the other statements follow from Lemma 7.5 which, in particular, implies the existence of an $n \in \mathbb{N}_\infty$ such that $\partial(gf) \wedge n = (\partial g) + (\partial f)$. □

We can now relate theories rigged in $\mathbb{N}_\infty$ with the material of Section 6.

Proposition 7.10 *With the notation above, ∂ makes $\mathcal{T}$ into a fort.*

Proof Lemma 7.9 shows that ∂ makes $\mathcal{T}$ into an $(\mathbb{N}_\infty, \leq)$-normed category and

$$\partial p = \bigwedge_{i \in 0} p^\flat i = \infty$$

holds for any point $p : 1 = T^0 \to T^B$. □

In particular, ample families make sense inside theories rigged in $\mathbb{N}_\infty$. We now embark on the construction of rigged theories.

8 Extensivity of Fibered Categories

In this section we make explicit a particular case of the Grothendieck semi-direct product construction. Let $\mathcal{C}$ be a category with finite limits and equipped with a monoid $M = (M, \cdot, 1)$. The representable functor $\mathcal{C}(\_, M) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ has a natural lifting to a functor $\mathcal{C}(\_, M) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Mon}$ with the category of monoids as codomain and so, to a functor $\mathcal{C}(\_, M) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$. We may then consider the Grothendieck semi-direct product construction that we denote here by $\mathcal{C} \ltimes M$.

More explicitly, the objects of $\mathcal{C} \ltimes M$ are the objects of $\mathcal{C}$. A map $X \to Y$ in $\mathcal{C} \ltimes M$ is a pair (f, f') with $f : X \to Y$ and $f' : X \to M$ in $\mathcal{C}$. For any X, the identity on X is the pair

$(id_X, 1)$ where $1 : X \to M$ is the unique map that factors through the unit $1 : 1 \to M$. For $(f, f') : X \to Y$ and $(g, g') : Y \to Z$ as on the left below

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad\qquad X \xrightarrow{\langle g'f, f'\rangle} M \times M \xrightarrow{\cdot} M$$

with $f' : X \to M$, $g'f : X \to M$, $g' : Y \to M$

the composite $(g, g')(f, f')$ is defined by $(gf, (g'f) \cdot f')$ where the map $(g'f) \cdot f' : X \to M$ is the composite on the right above.

(If we force the notation a little bit then we could write $(gf)' = (g'f) \cdot f'$, which is reminiscent of the chain rule; but we stress: in the notation above, f' is not determined by f.)

There is an obvious functor $\mathcal{C} \ltimes M \to \mathcal{C}$ which is the identity on objects and sends (f, f') to f. This functor has a section $\mathcal{C} \to \mathcal{C} \ltimes M$ that sends $f : X \to Y$ in $\mathcal{C}$ to $(f, 1) : X \to Y$ in $\mathcal{C} \ltimes M$.

It is relevant to observe that maps in the image of $\mathcal{C} \to \mathcal{C} \ltimes M$ remain in $\mathcal{C}$ after pulling back along any map in $\mathcal{C} \ltimes M$. More precisely:

Lemma 8.1 *Let the square on the left below*

$$
\begin{array}{ccc}
P & \xrightarrow{\pi_1} & U \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle u} \\
Y & \xrightarrow{g} & Z
\end{array}
\qquad\qquad
\begin{array}{ccc}
P & \xrightarrow{(\pi_1, g'\pi_0)} & U \\
{\scriptstyle (\pi_0, 1)}\downarrow & & \downarrow{\scriptstyle (u, 1)} \\
Y & \xrightarrow{(g, g')} & Z
\end{array}
$$

be a pullback in $\mathcal{C}$ and $g' : Y \to M$ a map in $\mathcal{C}$. Then the square on the right above is a pullback in $\mathcal{C} \ltimes M$.

Proof The diagram on the right of the statement commutes because the left one does and because $(g'\pi_0) \cdot 1 = g'\pi_0 = 1 \cdot (g'\pi_0) = (1\pi_1) \cdot (g'\pi_0)$. Assume now that the following square commutes

$$
\begin{array}{ccc}
X & \xrightarrow{(h, h')} & U \\
{\scriptstyle (f, f')}\downarrow & & \downarrow{\scriptstyle (u, 1)} \\
Y & \xrightarrow{(g, g')} & Z
\end{array}
$$

in $\mathcal{C} \ltimes M$, which means that $gf = uh$ and $(g'f) \cdot f' = (1h) \cdot h' = h'$ in $\mathcal{C}$.

Consider now a map $(t, t') : X \to P$ in $\mathcal{C} \ltimes M$ such that the equations

$$(\pi_0, 1)(t, t') = (f, f') \qquad (\pi_1, g'\pi_0)(t, t') = (h, h')$$

hold. This means that the equations below

$$(\pi_0 t, (1t) \cdot t') = (\pi_0 t, t') = (f, f') \qquad (\pi_1 t, (g'\pi_0 t) \cdot t') = (h, h')$$

hold. Equivalently, the following hold

$$\pi_0 t = f \qquad t' = f' \qquad \pi_1 t = h \qquad (g'\pi_0 t) \cdot t' = h'$$

and it is clear that the first three equations uniquely determine t and t'. The last equation holds automatically, given the first one, because $(g'\pi_0 t) \cdot t' = (g'f) \cdot f' = h'$. Altogether, the map (t, t') exists and is unique. $\qquad \square$

If C has finite coproducts then so does $C \ltimes M$. More explicitly:

Lemma 8.2 *If 0 is initial in C then it is initial as an object in $C \ltimes M$. If the diagram on the left below is a coproduct in C*

$$X \xrightarrow{in_0} X + Y \xleftarrow{in_1} Y \qquad\qquad X \xrightarrow{(in_0, 1)} X + Y \xleftarrow{(in_1, 1)} Y$$

then the diagram on the right above is a coproduct in $C \ltimes M$.

Proof We concentrate on binary coproducts. Let $(f, f') : X \to Z$, $(g, g') : Y \to Z$ in $C \ltimes M$. Assume that $(h, h') : X + Y \to Z$ in $C \ltimes M$ is such that $(h, h')(in_0, 1) = (f, f')$ and $(h, h')(in_1, 1) = (g, g')$. That is, the equations below hold

$$(h \, in_0, (h'in_0){\cdot}1) = (h \, in_0, h'in_0) = (f, f') \quad (h \, in_1, (h'in_1){\cdot}1) = (h \, in_1, h'in_1) = (g, g')$$

or, equivalently, the equations below

$$h \, in_0 = h \quad h'in_0 = f' \quad h \, in_1 = g \quad h'in_1 = g'$$

hold; but these equations uniquely determine $h : X + Y \to Z$ and $h' : X + Y \to M$. $\square$

In the examples we are interested in C is extensive.

Proposition 8.3 *If C is extensive then so is $C \ltimes M$.*

Proof Recall that a category with finite coproducts is extensive if and only if coproducts are stable and disjoint [5]. Stability and disjointness are easy to prove using the description of coproducts in Lemma 8.2 and the description of the relevant pullbacks in Lemma 8.1. $\square$

We say that M is *conical* if the following square

$$
\begin{array}{ccc}
1 & \longrightarrow & 1 \\
{\scriptstyle \langle 1,1 \rangle} \downarrow & & \downarrow {\scriptstyle 1} \\
M \times M & \xrightarrow{\quad \cdot \quad} & M
\end{array}
$$

is a pullback.

Lemma 8.4 *If M is conical then, a map $(f, f') : X \to Y$ in $C \ltimes M$ is an iso if and only if f is an iso in C and $f' = 1$.*

Proof Let $(g, g') : Y \to Z$ in $C \ltimes M$ be such that $(g, g')(f, f') = (id, 1)$. That is $gf = id$ and $(g'f) \cdot f' = 1$. If C is conical then $f' = 1$. In other words, if (f, f') is a section then f is a section and $f' = 1$. It is easy to show that if $(f, 1)$ is a retraction then f is a retraction. $\square$

9 The Category of Restricted Spans

In this section we let C be an extensive category with finite limits and equipped with a monoid $M = (M, \cdot, 1)$. We can then consider the (non full) coproduct-preserving inclusion $C \to C \ltimes M$.
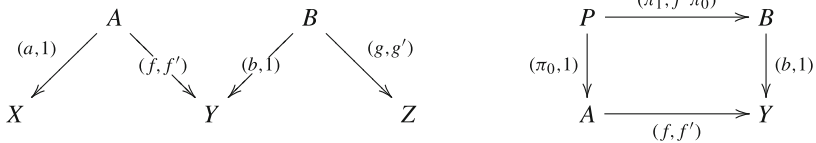
Definition 9.1 A *restricted span* (from X to Y in $\mathcal{C}$) is a span from X to Y in $\mathcal{C} \ltimes M$ whose left leg is in the subcategory $\mathcal{C} \to \mathcal{C} \ltimes M$.

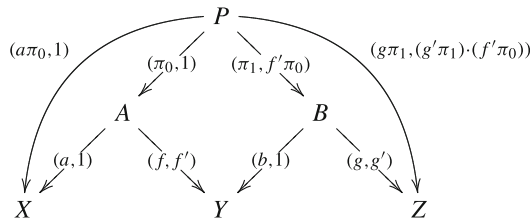More concretely, a restricted span is a span as on the left below

$$
\begin{array}{ccc}
 & A & \\
(a,1)\swarrow & & \searrow(f,f') \\
X & & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & X & \\
(id,1)\swarrow & & \searrow(id,1) \\
X & & X
\end{array}
$$

and, for example, for each object X in $\mathcal{C}$, we have the distinguished span on the right above from X to X.

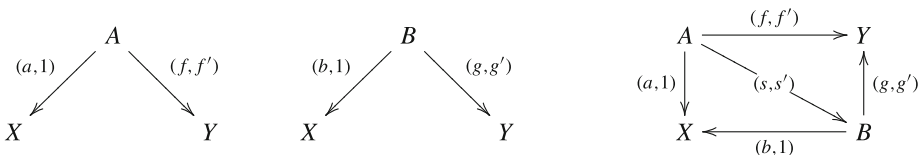Given consecutive spans as on the left below

$$
\begin{array}{ccccc}
 & A & & B & \\
(a,1)\swarrow & & \searrow(f,f') \quad (b,1)\swarrow & & \searrow(g,g') \\
X & & Y & & Z
\end{array}
\qquad
\begin{array}{ccc}
P & \xrightarrow{\ (\pi_1,f'\pi_0)\ } & B \\
{\scriptstyle(\pi_0,1)}\downarrow & & \downarrow{\scriptstyle(b,1)} \\
A & \xrightarrow[\ (f,f')\ ]{} & Y
\end{array}
$$

and a pullback square as on the right above (recall Lemma 8.1) then we define the *induced composite* as the span

$$
\begin{array}{ccc}
 & P & \\
(a\pi_0,1)\swarrow \quad {\scriptstyle(\pi_0,1)} & & {\scriptstyle(\pi_1,f'\pi_0)} \searrow (g\pi_1,(g'\pi_1)\cdot(f'\pi_0)) \\
 & A \qquad\qquad B & \\
 (a,1)\swarrow \ (f,f') \quad (b,1)\ \searrow (g,g') & & \\
X \qquad\qquad Y \qquad\qquad Z & &
\end{array}
$$

which is clearly a restricted span from X to Z.

Notice that this is just the usual composition of spans, relying on Lemma 8.1, to conclude that restricted spans are closed under composition. We will not need general spans so, from now on, we use the word 'span' to mean 'restricted span'.

Given spans from X to Y as on the left below

$$
\begin{array}{ccccc}
 & A & & B & \\
(a,1)\swarrow & & \searrow(f,f') \quad (b,1)\swarrow & & \searrow(g,g') \\
X & & Y & & Y
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ (f,f')\ } & Y \\
{\scriptstyle(a,1)}\downarrow \quad {\scriptstyle(s,s')}\searrow & & \uparrow{\scriptstyle(g,g')} \\
X & \xleftarrow[\ (b,1)\ ]{} & B
\end{array}
$$

then a *map* from the first to the second is a morphism $(s,s') : A \to B$ in $\mathcal{C} \ltimes M$ such that the triangles in the right square above commute. This is the standard definition of morphism of spans, but notice that in this case, the lower triangle means that the composite $(b,1)(s,s') = (bs,(1s)\cdot s') = (bs,s')$ equals $(a,1)$. That is, $bs = a$ and $s' = 1$. Then, the upper triangle means that $(g,g')(s,1) = (gs,(g's)\cdot 1) = (gs,g's)$ equals (f,f'). That is, $gs = f : A \to Y$ and $g's = f' : A \to M$. So it is convenient to simplify the notation as follows.

From now on, spans from X to Y will be denoted as on the left below
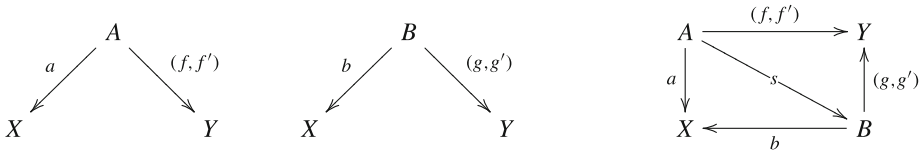
and a map from the first to the second will be just a map $s : A \to B$ in $\mathcal{C}$ such that $bs = a$, $gs = f$ and $g's = f'$.

Given a choice of pullbacks in $\mathcal{C}$ (which gives a choice of the relevant pullbacks in $\mathcal{C} \ltimes M$) we define 'the' bicategory $\mathfrak{Sp}(\mathcal{C}, \mathcal{C} \ltimes M)$ as in [1]. The objects of $\mathfrak{Sp}(\mathcal{C}, \mathcal{C} \ltimes M)$ are the objects of $\mathcal{C}$. For any pair of objects X, Y, the category $\mathfrak{Sp}(\mathcal{C}, \mathcal{C} \ltimes M)(X, Y)$ is the category of spans from X to Y and morphisms between them. Composition is defined using pullbacks as above and the rest of the structure is determined by the universal property of pullbacks as in the standard case.

(Surely, the above discussion generalizes to a setting starting with a category $\mathcal{D}$ with a suitable bijective on objects subcategory $\mathcal{C}$ that is closed under pullbacks in $\mathcal{D}$; but we have not found the details spelled out in the literature.)

The associated 'classifying category' (7.2 in [1]) will be denoted by $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$. Concretely, the objects of $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ are the objects of $\mathcal{C}$ and a map from X to Y is an equivalence class of spans from X to Y. Two spans being equivalent if they are iso in $\mathfrak{Sp}(\mathcal{C}, \mathcal{C} \ltimes M)(X, Y)$. More explicitly:

Lemma 9.2 *Two spans from X to Y as on the left below*
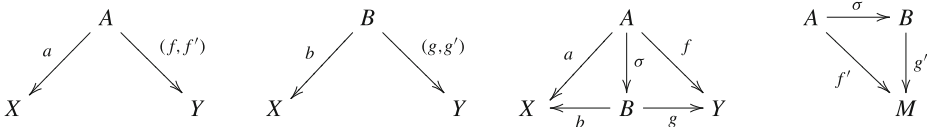
determine the same map from X to Y in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ if and only if there is an iso $\sigma : A \to B$ in $\mathcal{C}$ such that $b\sigma = a : A \to X$, $g\sigma = f : A \to Y$ and $g'\sigma = f' : A \to M$ in $\mathcal{C}$.

The equivalence class of the span $X \xleftarrow{\ a\ } A \xrightarrow{\ (f,f')\ } Y$ as in Lemma 9.2 will be denoted by $[a; f, f'] : X \to Y$.

Proposition 9.3 *The assignment that sends $a : Y \to X$ in $\mathcal{C}$ to $[a; id_Y, 1] : X \to Y$ in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ extends to a conservative functor $\mathcal{C}^{op} \to \mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$. On the other hand, the assignment $((f, f') : X \to Y) \mapsto ([id_X; f, f'] : X \to Y)$ extends to a faithful functor $\mathcal{C} \ltimes M \to \mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ which is conservative if M is conical. Moreover, every map in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ factors as a map in the subcategory $\mathcal{C}^{op} \to \mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ followed by a map in the subcategory $\mathcal{C} \ltimes M \to \mathbf{Span}(\mathcal{C}, M)$.*

Proof It is easy to check that the assignments in the statement are functorial. It is also easy to check that $\mathcal{C}^{op} \to \mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ is faithful. To prove that it reflects isos assume first that the map $[a; id, 1]$ has a retraction $[b; (g, g')]$ with $b : B \to Y$ and $g : B \to X$. That is, $[b; g, g'][a; id, 1] = [ba; g, g'] = [id_X, id_X, 1]$. So, without loss of generality, we may assume that $B = X$ and conclude that a is a section of b, $g = id_X$ and $g' = 1$. Now, assume

further that $[b; g, g'] = [b; id, 1]$ is a section of $[a; id, 1]$. Then it is easy to check that b is a section of a, so a is an iso.

To prove that $\mathcal{C} \ltimes M \to \mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ is faithful let $(f, f'), (g, g') : X \to Y$ in the category $\mathcal{C} \ltimes M$ and assume that $[id_X; f, f'] = [id_X; g, g']$. Then there exists an iso $\sigma : X \to X$ in $\mathcal{C}$ such that $id_X\sigma = id_X$, $g\sigma = f$ and $g'\sigma = f'$ in $\mathcal{C}$. So $\sigma = id_X$, $f = g$ and $f' = g'$. To prove that the functor $\mathcal{C} \ltimes M \to \mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ reflects isos assume first that $[id_X; f, f'] : X \to Y$ has a section. This means that there is a span
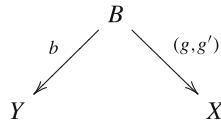
such that $[id; f, f'][b; g, g'] = [id; id, 1]$. This implies that $[b; fg, (f'g) \cdot g'] = [id; id, 1]$, so b is an iso, g is a section of f and $(f'g) \cdot g' = 1$. Since M is conical $g' = 1$. Without loss of generality, we may assume that b is the identity on Y. Now let us assume that $[b; g, g'] = [id_Y, g, 1]$ is also a retraction for $[id; f, f']$. That is, $[id; g, 1][id; f, f'] = [id; id, 1]$ and it implies that $gf = id$ and that $(1f) \cdot f' = 1$. So g is a retract of f and $f' = 1$. Altogether, if $[id; f, f']$ is an iso then f is an iso and $f' = 1$. Equivalently, by Lemma 8.4, (f, f') is an iso in $\mathcal{C} \ltimes M$.

Finally any map $[a; f, f'] : X \to Y$ with $a : A \to X$ in $\mathcal{C}$ factors as $[id_A; f, f'][a; id_A, 1]$ with $[a; id, 1] : X \to A$ and $[id; f, f'] : A \to Y$. $\qquad\square$

One may ask if the pair of subcategories $(\mathcal{C}^{\mathrm{op}}, \mathcal{C} \ltimes M)$ form a factorization system for $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$.

Lemma 9.4 *Let $b : B \to A$ in $\mathcal{C}$, $g : Y \to Z$ in $\mathcal{C} \ltimes M$ and assume that the square on the left below*
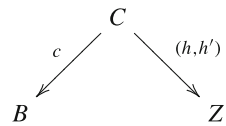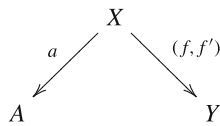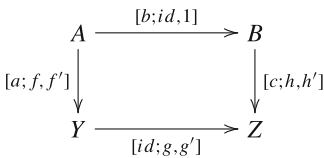
commutes in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$. *Then there exists a map* $B \to Y$ *making the two obvious triangles commute.*

Proof The top-right composite of the square equals $[bc; h, h'] : A \to Z$ and the left-bottom composite equals $[a; gf, (g'f) \cdot f'] : A \to Z$. Since both composites are assumed equal there exists an iso $\sigma : X \to C$ in $\mathcal{C}$ such that $bc\sigma = a : X \to A$, $gf = h\sigma : X \to Z$ and $h'\sigma = (g'f) \cdot f'$. Now consider the map $[c\sigma; f, f'] : B \to Y$. Easily,

$$[c\sigma; f, f'][b; id, 1] = [bc\sigma; f, f'] = [a; f, f']$$

and, on the other hand, $[id; g, g'][c\sigma; f, f'] = [c\sigma; gf, (g'f) \cdot f'] = [c\sigma; h\sigma, h'\sigma] = [c; h, h']$. $\qquad\square$

In general, though, the diagonal fill-in need not be unique. To see this it seems clearer to make explicit the phenomenon in the usual category of spans of sets. Take the commutative square

$$
\begin{array}{ccc}
1 & \xrightarrow{\ [!;id]\ } & 2 \\[4pt]
{\scriptstyle [!;id]}\Big\downarrow & & \Big\downarrow{\scriptstyle [id;!]} \\[4pt]
2 & \xrightarrow[\ [id;!]\ ]{} & 1
\end{array}
$$

so that the composite is just the map determined by the span $1 \leftarrow 2 \rightarrow 1$. Consider now the two maps $[id; id], [\tau; id] : 2 \rightarrow 2$ where $\tau : 2 \rightarrow 2$ is the only non-identity bijection. It is clear that the two maps are different and it is easy to check that they both make the relevant triangles commute.

Proposition 9.5 *If $\mathcal{C}$ is extensive then the category $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ has finite products and the functor $\mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ is product preserving. Also, every object in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ has a unique point.*

Proof We claim that the spans below

$$
\begin{array}{ccccc}
 & X & & & Y \\
{\scriptstyle (id,1)}\swarrow & & \searrow{\scriptstyle in_0} \qquad {\scriptstyle in_1}\swarrow & & \searrow{\scriptstyle (id,1)} \\
X & & X + Y & & Y
\end{array}
$$

induce a product in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$, i.e., that the maps $\pi_0 = [in_0; id_X, 1] : X + Y \rightarrow X$ and $\pi_1 = [in_1; id_Y, 1] : X + Y \rightarrow Y$ in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ are the projections of a product of X and Y. To prove this let the following spans

$$
\begin{array}{ccccc}
 & B & & C & \\
{\scriptstyle (f,f')}\swarrow & & \searrow{\scriptstyle b} \quad {\scriptstyle c}\swarrow & & \searrow{\scriptstyle (g,g')} \\
X & & A & & Y
\end{array}
$$

represent a span $X \xleftarrow{\ x\ } A \xrightarrow{\ y\ } Y$ in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$. We have the induced map $[b, c] : B + C \rightarrow A$ in $\mathcal{C}$ and $(f, f') + (g, g') = (f + g, [f', g']) : B + C \rightarrow X + Y$ in $\mathcal{C} \ltimes M$ by Proposition 8.3. It is easy to check that $\langle x, y \rangle = [[b, c]; f + g, [f', g']] : A \rightarrow X + Y$ is such that $\pi_0 \langle x, y \rangle = [b; f, f'] = x$ and $\pi_1 \langle x, y \rangle = [c; g, g'] = y$.

To prove uniqueness first observe that a map $A \rightarrow X + Y$ in $\mathbf{Span}(\mathcal{C}, \mathcal{C} \ltimes M)$ is determined by some span given by a map $d : D \rightarrow A$ and a map $(h, h') : D \rightarrow X + Y$ in $\mathcal{C} \ltimes M$. By Proposition 8.3 again, $D = D_0 + D_1$, $(h, h') = (h_0, h'_0) + (h_1, h'_1) : D_0 + D_1 \rightarrow X + Y, d = [d_0, d_1] : D_0 + D_1 \rightarrow A$ as in the diagram below

$$
\begin{array}{ccc}
 & D_0 + D_1 & \\
{\scriptstyle d=[d_0,d_1]}\swarrow & & \searrow{\scriptstyle h=h_0+h_1} \\
A & & X + Y
\end{array}
$$

and then, the conditions $\pi_0[d; h, h'] = x$ and $\pi_1[d; h, h'] = y$ imply that $[d; h, h'] = \langle x, y \rangle$.

The object 0 is terminal because every restricted span $X \leftarrow A \rightarrow 0$ is forced to be the unique $X \leftarrow 0 \rightarrow 0$. Finally, for every X, points $0 \rightarrow X$ are determined by spans $0 \leftarrow A \rightarrow X$; that is, by the unique span $0 \leftarrow 0 \rightarrow X$. □

10 Rigged Theories of Finite Spans

Fix a conical monoid $M = (M, \cdot, 1)$ in **Set** so that, by Proposition 9.5, we can consider the category **Span**(**Set**, **Set** $\ltimes M$) with finite products and the product preserving inclusion **Set**$^{\mathrm{op}} \rightarrow$ **Span**(**Set**, **Set** $\ltimes M$).

Let **fSet** $\rightarrow$ **Set** the fixed small version of the category of finite sets and functions that we used to introduce algebraic theories. We may consider the full subcategory of **Set** $\ltimes M$ determined by **fSet**. We will denote this subcategory by **fSet** $\ltimes M \rightarrow$ **Set** $\ltimes M$, even if M is not in **fSet**.

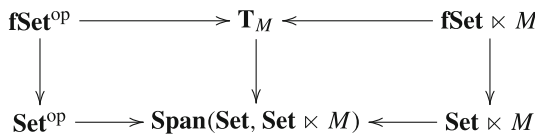**Definition 10.1** A *finite (restricted) span* in **Set** $\ltimes M$ is a span as on the left below

with A, X and Y in **fSet**. As before, we will denote the span as on the right above.

A map $X \rightarrow Y$ in **Span**(**Set**, **Set** $\ltimes M$) is called *finite* if it is represented by a finite span. Notice that if $[b; g, g'] : X \rightarrow Y$ is a finite map then, by Lemma 9.2, the representing span given by $b : B \rightarrow X$ and $(g, g') : B \rightarrow Y$ is finite in the sense of Definition 10.1.

If X is in **fSet** then the identity on X in **Span**(**Set**, **Set** $\ltimes M$) is finite. Also, since **fSet** has finite limits and the inclusion **fSet** $\rightarrow$ **Set** preserves them, it is easy to check that finite maps are closed under composition.

Let $\mathbf{T}_M \rightarrow$ **Span**(**Set**, **Set** $\ltimes M$) be the (non-full) subcategory determined by the finite maps. More concretely, the objects of $\mathbf{T}_M$ are the objects of **fSet** and, for X and Y in **fSet**, a map $X \rightarrow Y$ in **Span**(**Set**, **Set** $\ltimes M$) is in the subcategory $\mathbf{T}_M$ if and only if it is represented by a finite span.

Lemma 10.2 *The inclusions* **Set**$^{\mathrm{op}} \rightarrow$ **Span**(**Set**, **Set** $\ltimes M$) $\leftarrow$ **Set** $\ltimes M$ *restrict to conservative and bijective-on-objects functors as in the top span of the commutative diagram below*

$$
\begin{array}{ccccc}
\mathbf{fSet}^{\mathrm{op}} & \longrightarrow & \mathbf{T}_M & \longleftarrow & \mathbf{fSet} \ltimes M \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Set}^{\mathrm{op}} & \longrightarrow & \mathbf{Span}(\mathbf{Set}, \mathbf{Set} \ltimes M) & \longleftarrow & \mathbf{Set} \ltimes M
\end{array}
$$

Moreover, $\mathbf{T}_M$ *has finite products and* $\mathbf{fSet}^{\mathrm{op}} \rightarrow \mathbf{T}_M$ *preserves them, resulting in an algebraic theory with a unique constant.*

Proof The first part follows from the description of the functors in Proposition 9.3. Also, since **fSet** is extensive and the inclusion **fSet** → **Set** preserves finite coproducts then it is easy to check (from their construction in Proposition 9.5) that the non-full subcategory $\mathbf{T}_M \to \mathbf{Span}(\mathbf{Set}, \mathbf{Set} \ltimes M)$ is closed under the products. It is also easy to check that, given two maps in $\mathbf{T}_M$ with common domain, the induced pairing to the product in $\mathbf{Span}(\mathbf{Set}, \mathbf{Set} \ltimes M)$ is also a finite map. Therefore, $\mathbf{T}_M$ has finite products and the inclusion $\mathbf{T}_M \to \mathbf{Span}(\mathbf{Set}, \mathbf{Set} \ltimes M)$ preserves them. □

By Lemma 10.2 we have an algebraic theory that we denote by $\mathrm{E} : \mathbf{fSet}^{\mathrm{op}} \to \mathbf{T}_M$. So the objects of $\mathbf{T}_M$ will be denoted by E^X for X in **fSet**. The 'operations' $\mathrm{E}^X \to \mathrm{E}^1$ are in bijective correspondence with equivalence clases $[a; !, f'] : X \to 1$ of spans with $a : A \to X$ and $! : A \to 1$ in **fSet** and $f' : A \to M$ in **Set**. Among these we have the operations determined by equivalence classes of the form above but with a an iso. These are essentially the maps $(!, f') : X \to 1$ in the subcategory $\mathbf{fSet} \ltimes M \to \mathbf{T}_M$ (Lemma 10.2). In turn, these are essentially functions $f' : X \to M$. Roughly speaking, we have built an algebraic theory where the functions $X \to M$ are among the X-ary operations.

Now fix a rig $R = (R, +, 0, \cdot, 1)$ together with a morphism $M \to R$ of multiplicative monoids that, for simplicity, we assume that it is an inclusion. We will show that the theory $\mathrm{E} : \mathbf{fSet}^{\mathrm{op}} \to \mathbf{T}_M$ can be rigged in R.

Definition 10.3 For any finite span as in Definition 10.1 given by a function $a : A \to X$ and map $(f, f') : A \to Y$ we define $(a; f, f')^\natural : X \to R$ by the formula

$$(a; f, f')^\natural (x \in X) = \sum_{aj=x} f'j$$

where the $\sum$ is, of course, taken in R.

This process sending spans to R-valued functions is well behaved with respect to the equivalence of spans.

Lemma 10.4 *If $[a; f, f'] = [b; g, g'] : \mathrm{E}^X \to \mathrm{E}^Y$ in $\mathbf{T}_M$ then $(a; f, f')^\natural = (b; g, g')^\natural$.*

Proof The hypothesis implies the existence of an iso σ in **fSet** such that $b\sigma = a$, $g\sigma = f$ and $g'\sigma = f'$ in **Set**. The calculation

$$(a; f, f')^\natural i = \sum_{aj=i} f'j = \sum_{b(\sigma j)=i} g'(\sigma j) = \sum_{bk=i} g'k = (b; g, g')^\natural i$$

proves the result. □

So, if $\alpha : \mathrm{E}^X \to \mathrm{E}^Y$ is a morphism in $\mathbf{T}_M$ then it makes sense to write $\alpha^\natural : X \to R$. Moreover, we can equip the algebraic theory $\mathbf{T}_M$ with the indexed family of functions $((\_)^\natural : \mathbf{T}_M(\mathrm{E}^X, \mathrm{E}^Y) \to R^X \mid X, Y \in \mathbf{fSet})$.

Proposition 10.5 *The family $((\_)^\natural : \mathbf{T}_M(\mathrm{E}^X, \mathrm{E}^Y) \to R^X \mid X, Y \in \mathbf{fSet})$ makes the theory $\mathbf{T}_M$ rigged in R.*

Proof For every $a : J \to I$ in **fSet**, $\mathrm{E}^a = [a; id_J, 1] : \mathrm{E}^I \to \mathrm{E}^J$ so

$$(\mathrm{E}^a)^{\flat} i = \sum_{aj=i} 1j = \sum_{aj=i} 1$$

which is the first condition defining rigged theories (Definition 7.1). For the second condition let $\alpha : \mathrm{E}^X \to \mathrm{E}^Y$ and $\beta : \mathrm{E}^Y \to \mathrm{E}^Z$. We need to prove that

$$(\beta\alpha)^{\flat} i = \sum_{j \in Y} (\beta^{\flat} j) \cdot ((\pi_j \alpha)^{\flat} i)$$

holds in R. First let $\alpha = [a; f, f']$ with $a : A \to X$ and $j : 1 \to Y$ in **fSet**. Then the composite $\pi_j \alpha : \mathrm{E}^X \to \mathrm{E}$ can be calculated using a pullback

$$
\begin{array}{ccc}
A_j & \xrightarrow{(q_1, f'q_0)} & 1 \\
{\scriptstyle(q_0,1)}\downarrow & & \downarrow{\scriptstyle(j,1)} \\
A & \xrightarrow{(f,f')} & Y
\end{array}
$$

so that $\pi_j \alpha = [aq_0; q_1, f'q_0]$. Then

$$(\pi_j \alpha)^{\flat} i = \sum_{\substack{k \in A_j \\ a(q_0 k)=i}} f'(q_0 k) = \sum_{\substack{u \in A \\ fu=j \\ au=i}} f'u$$

for every $i \in X$. Now let $\beta = [b; g, g']$ with $b : B \to Y$. To calculate the composition $\beta\alpha$ construct the pullback

$$
\begin{array}{ccc}
P & \xrightarrow{(\pi_1, f'\pi_0)} & B \\
{\scriptstyle(\pi_0,1)}\downarrow & & \downarrow{\scriptstyle(b,1)} \\
A & \xrightarrow{(f,f')} & Y
\end{array}
$$

so that $\beta\alpha = [a\pi_0; g\pi_1, (g'\pi_1) \cdot (f'\pi_0)]$. Then

$$(\beta\alpha)^{\flat} i = \sum_{a(\pi_0 w)=i} (g'(\pi_1 w)) \cdot (f'(\pi_0 w)) = \sum_{\substack{(u,v) \in A \times B \\ fu=bv \\ au=i}} (g'v) \cdot (f'u) =$$

$$= \sum_{\substack{j \in Y \\ v \in B \\ bv=j}} \sum_{\substack{u \in A \\ fu=j \\ au=i}} (g'v) \cdot (f'u) = \sum_{\substack{j \in Y \\ v \in B \\ bv=j}} (g'v) \cdot \sum_{\substack{u \in A \\ fu=j \\ au=i}} f'u = \sum_{j \in Y} (\beta^{\flat} j) \cdot ((\pi_j \alpha)^{\flat} i)$$

for every $i \in X$. $\qquad\square$

A further relevant fact is that the morphism $M \to R$ interacts well with the inclusion **fSet** $\ltimes M \to \mathbf{T}_M$.

Lemma 10.6 *For any map $(f, f') : X \to Y$ in **fSet** $\ltimes M$ the function $[id; f, f']^{\flat} : X \to R$ factors through the inclusion $M \to R$.*

Proof Indeed, $[id_X; f, f']^{\flat}(i \in X) = \sum_{id_X j=i} f'j = f'i \in M$. $\qquad\square$

Bear in mind that, in the case $X = 0$, Lemma 10.6 only says that $0 \to R$ factors through $M \to R$; trivially, of course.

Consider now the 'multiplicative inclusion' $\mathbb{N} = (\mathbb{N}, +, 0) \to (\mathbb{N}_\infty, +, 0, \wedge, \infty) = \mathbb{N}_\infty$. In this case, for any $[a; f, f'] : E^X \to E^Y$ in $\mathbf{T}_\mathbb{N}$ with $a : A \to X$, we have that the function $[a; f, f']^\natural : X \to \mathbb{N}_\infty$ is defined by

$$[a; f, f']^\natural (x \in X) = \bigwedge_{aj=x} f'j$$

and the functions $\mathfrak{d}_{X,Y} : \mathbf{T}_\mathbb{N}(E^X, E^Y) \to \mathbb{N}_\infty$ considered in Section 7 are defined by

$$\mathfrak{d}[a; f, f'] = \bigwedge_{x \in X} [a; f, f']^\natural x = \bigwedge_{x \in X} \bigwedge_{aj=x} f'j = \bigwedge_{j \in A} f'j$$

so that we may conclude the following.

Corollary 10.7 *With the notation above, $\mathfrak{d}$ makes $\mathbf{T}_\mathbb{N}$ into a strong fort. Moreover, for every $(f, f') : X \to Y$ in the category* **fSet** $\ltimes \mathbb{N}$ *the function $[id; f, f']^\natural : X \to \mathbb{N}_\infty$ factors through the inclusion $\mathbb{N} \to \mathbb{N}_\infty$.*

Proof The theory $\mathbf{T}_\mathbb{N}$, rigged as in Proposition 10.5, induces a fort as in Proposition 7.10. Now, as explained above, for every $\alpha = [a; f, f'] : T^X \to T^Y$ in $\mathbf{T}_\mathbb{N}$ with $a : A \to X$ in **fSet**, $\mathfrak{d}\alpha = \bigwedge_{j \in A} f'j$ so it is clear that $\mathfrak{d}\alpha = \infty$ if and only if $A = 0$ if and only if α is constant. That is, the fort is strong (Definition 6.9). Lemma 10.6 completes the proof. $\square$

(Notice that for the unique point $[id_0; ! : 0 \to 1, ! : 0 \to \mathbb{N}] : 1 \to E^1$, $[id; !, !]^\natural : 0 \to \mathbb{N}_\infty$ factors trivially through $\mathbb{N} \to \mathbb{N}_\infty$ and that $\mathfrak{d}[id; !, !] = \infty \in \mathbb{N}_\infty$.)

It is fair to picture a map $f : X \to 1$ in **fSet** $\ltimes \mathbb{N}$ as the record of certain information related to a tree with leaves in X. In practice f will record the distances from the leaves to the roots. If we embed **fSet** $\ltimes \mathbb{N}$ into the algebraic theory $\mathbf{T}_\mathbb{N}$ then we need the rig $\mathbb{N}_\infty$ in order to extend the information recorded by maps in **fSet** $\ltimes \mathbb{N}$ to arbitrary maps in the theory $\mathbf{T}_\mathbb{N}$.

We are going to use $\mathbf{T}_\mathbb{N}$ to record relevant information of maps in free theories. On the other hand, it should be possible to generalize our construction of $\mathbf{T}_M$ in order to give an alternative construction of free theories; a construction that is intermediate between their conceptual construction as 'coproducts of operations' as in [13] and their construction in terms of strings of symbols.

11 Ranked Theories

If $\mathcal{T}$ is an algebraic theory such that $T : \mathbf{fSet}^{\mathrm{op}} \to \mathcal{T}$ is conservative then the maps in this subcategory will be called *bureaucratic*. The intuition is that bureaucratic maps just forget things, repeat things or permute things.

Definition 11.1 An algebraic theory $T : \mathbf{fSet}^{\mathrm{op}} \to \mathcal{T}$ is called *liberal* if the functor T is conservative and the pair $(\mathbf{fSet}^{\mathrm{op}}, (\mathbf{fSet}^{\mathrm{op}})^\downarrow)$ is a factorization system in $\mathcal{T}$.

If $\mathcal{T}$ is liberal then the maps in $(\mathbf{fSet}^{\mathrm{op}})^\downarrow \to \mathcal{T}$ will be called *efficient*.

It follows from the results in [19] that free theories are liberal. In relation to the work just cited some remarks on terminology are in order. Notice that if the subcategory $\mathbf{fSet}^{\mathrm{op}} \to \mathcal{T}$

is part of a factorization system then it must contain all the isos. So, in this case, the bureaucratic maps coincide with the class of *structural* maps as defined loc. cit. For this reason the maps in the subcategory $(\mathbf{fSet}^{\mathrm{op}})^{\downarrow} \to \mathcal{T}$ are exactly the *analytic* maps defined there. In other words, if structural maps are bureaucratic then efficient and analytic maps coincide. We keep the bureaucratic/efficient terminology in order to emphasize the condition that bureaucratic maps are part of a factorization system. (A theory is called *analytic* if structural and analytic maps form a factorization system. Marek Zawadowski observes that it follows from [19] that liberal theories are exactly the analytic theories with no non-trivial unary invertible operations. For example, any free theory. He also produced a characterization of the analytic theories which satisfy that analytic maps are mono. It follows easily from this characterization that efficient maps in free theories are mono.)

Lemma 11.2 *If $T : \mathbf{fSet}^{\mathrm{op}} \to \mathcal{T}$ is liberal then, $e : T^X \to T^Y$ is efficient if and only if there is a family $(e_i : T^{X_i} \to T \mid i \in Y)$ of efficient maps and an iso $b : \sum_{i \in Y} X_i \to X$ in* **fSet** *such that the following diagram*

$$T^X \xrightarrow{T^b} T^{\sum_{i \in Y} X_i} \xrightarrow{\cong} \prod_{i \in Y} T^{X_i}$$

with e down to T^Y and $\prod_{i \in Y} e_i$ into T^Y

commutes.

Proof First notice that, for general reasons about factorization systems, if the family $(e_i \mid i \in Y)$ of efficient maps exists then the product $\prod e_i$ is also efficient. Let $f_i : T^X \to T$ be the composite of $e : T^X \to T^Y$ followed by the projection $T^i : T^Y \to T^1 = T$, so that $e = \langle f_i \mid i \in Y \rangle$. Let $f_i = e_i(T^{b_i})$ with $b_i : X_i \to X$ in **fSet** and e_i efficient. Define b as the unique map $[b_i \mid i \in Y] : \sum_{i \in Y} X_i \to X$ determined by the universal property of the coproduct. Then the diagram in the statement commutes and since the product $\prod e_i$ is efficient then the top map T^b is an iso and hence b is an iso. $\square$

In other words, efficient maps are products of efficient maps with codomain T.

Before the next definition recall the algebraic theory $\mathbf{T}_{\mathbb{N}}$ rigged in $\mathbb{N}_{\infty}$ and exalted in Corollary 10.7.

Definition 11.3 An algebraic theory $\mathcal{T}$ is called *ranked* if it is liberal and it is equipped with a morphism of theories $\rho : \mathcal{T} \to \mathbf{T}_{\mathbb{N}}$ that sends efficient maps in $\mathcal{T}$ to maps in the subcategory $\mathbf{fSet} \ltimes \mathbb{N} \to \mathbf{T}_{\mathbb{N}}$.

We stress that $\mathbf{T}_{\mathbb{N}}$ is not liberal. (Recall Lemma 9.4 and discussion beneath it.) For our main result we only need the following source of examples.

Lemma 11.4 *Free theories are ranked in a canonical way.*

Proof We have already observed that free theories are liberal. The monoid $(\mathbb{N}, +, 0)$ has a distinguished element $1 \in \mathbb{N}$ and, for any X in **fSet**, we have the associated constant function $1 : X \to \mathbb{N}$. (To avoid a possible confusion we stress that $1 \in \mathbb{N}$ is not the multiplicative unit of the rig $\mathbb{N}_{\infty}$.) The map $[id_X; !, 1] : \mathrm{E}^X \to \mathrm{E}^1$ in the subcategory $\mathbf{fSet} \ltimes \mathbb{N} \to \mathbf{T}_{\mathbb{N}}$ will be denoted by $\mathrm{f}_X : \mathrm{E}^X \to \mathrm{E}$. Notice that $\mathrm{f}_0 : 1 \to \mathrm{E}^1$ is the unique point of $\mathbf{T}_{\mathbb{N}}$.

Recall the adjunction $F \dashv U : \mathbf{Th} \to \mathbf{Set}^{\mathbb{N}}$. For any 'signature' $P \in \mathbf{Set}^{\mathbb{N}}$ we have the constant function

$$Pn \to (U\mathbf{T}_{\mathbb{N}})n = \mathbf{T}_{\mathbb{N}}(\mathrm{E}^{\mathbf{n}}, \mathrm{E})$$

that sends everything in Pn to $f_{\mathbf{n}}$. These functions underlie a natural transformation $P \to U\mathbf{T}_{\mathbb{N}}$ and hence a morphism of theories $FP \to \mathbf{T}_{\mathbb{N}}$. It remains to show that this morphism sends efficient maps to maps in the subcategory $\mathbf{fSet} \ltimes \mathbb{N} \to \mathbf{T}_{\mathbb{N}}$. For this, notice that every efficient map in FP is a composite of products of maps coming from P. Since the elements in Pn are sent to $f_{\mathbf{n}} : \mathrm{E}^{\mathbf{n}} \to \mathrm{E}$ in the subcategory $\mathbf{fSet} \ltimes \mathbb{N} \to \mathbf{T}_{\mathbb{N}}$, the morphism of theories $FP \to \mathbf{T}_{\mathbb{N}}$ sends efficient maps to maps in the same subcategory. $\qquad\square$

The other basic fact we need about ranked theories is the following.

Proposition 11.5 *If $\rho : \mathcal{T} \to \mathbf{T}_{\mathbb{N}}$ is a ranked theory then the assignment that sends $f : T^A \to T^B$ in $\mathcal{T}$ to $(\rho f)^\natural : A \to \mathbb{N}_\infty$ makes $\mathcal{T}$ rigged in $\mathbb{N}_\infty$. Moreover, if f is efficient then $(\rho f)^\natural$ factors through $\mathbb{N} \to \mathbb{N}_\infty$.*

Proof The theory $\mathcal{T}$ is rigged in $\mathbb{N}_\infty$ by Lemma 7.6 and Corollary 10.7. Now let us assume that $\beta : T^A \to T^B$ is efficient in $\mathcal{T}$. By hypothesis $\rho\beta = [id; g, g']$ for some $(g, g') : A \to B$ in $\mathbf{fSet} \ltimes \mathbb{N}$ and so, by Corollary 10.7 again, $(\rho\beta)^\natural = [id; g, g']^\natural : A \to \mathbb{N}_\infty$ factors through $\mathbb{N} \to \mathbb{N}_\infty$. $\qquad\square$

We invite the reader to think of an efficient map $f : T^X \to T^1$ in $\mathcal{T}$ as a tree with leaves in X; and the function $(\rho f)^\natural$ as assigning to each $x \in X$ its distance from the root. The fact that $(\rho f)^\natural$ lands in $\mathbb{N}$ means that all these distances are finite. If X is non-empty then $\partial f \in \mathbb{N}$. If $X = 0$ then $(\rho f)^\natural : 0 \to \mathbb{N} \to \mathbb{N}_\infty$ and $\partial f = \infty$.

Fix a ranked category $\rho : \mathcal{T} \to \mathbf{T}_{\mathbb{N}}$. For any $f : T^X \to T^Y$ in $\mathcal{T}$ we write $f^\natural$ instead of $(\rho f)^\natural$. Similarly, we write $\partial f \in \mathbb{N}_\infty$ instead of $\partial(\rho f)$. As we have already mentioned, we can consider ample families in $\mathcal{T}$.

Lemma 11.6 *If G is a strictly ample and proper family on T^Z then every map in G is efficient.*

Proof Assume that G is strictly n-ample. Let $g : T^Y \to T^Z$ be a map in G and let $g = h(T^b)$ with $b : A \to Y$ in $\mathbf{fSet}$ and $h : T^A \to T^Z$ efficient. Before the next calculation notice that, for any $j : 1 \to A$, $\partial(T^{bj}) = 0$ by Lemma 7.9. Since G is strictly n-ample we have that:

$$n \le \partial(h(T^b)) = \bigwedge_{j \in A} h^\natural j + \partial(\pi_j(T^b)) = \bigwedge_{j \in A} h^\natural j + \partial(T^{bj}) = \bigwedge_{j \in A} h^\natural j + 0 = \bigwedge_{j \in A} h^\natural j = \partial h$$

so h factors through some map g' in G. In this case g also factors through g' but, since G is monic, $g = g'$. In other words, h factors through g, say, as $h = gr$ with $r : T^A \to T^Y$.

Since g is mono, $gr(T^b) = h(T^b) = g$ implies that r is a retraction of T^b. We also have $h(T^b)r = gr = h$ so the following diagram

$$
\begin{array}{ccc}
T^Y & \xrightarrow{\;T^b\;} & T^A \\
{\scriptstyle T^b}\big\downarrow & {\scriptstyle (T^b)r}\swarrow & \big\downarrow{\scriptstyle h} \\
T^A & \xrightarrow[\;h\;]{} & T^Z
\end{array}
$$

commutes and since T^b is bureaucratic and h is efficient $(T^b)r = id$ so T^b is an iso. (The last part is, of course, an instance of a more general fact about factorization systems.) □

Another feature of ranked theories is that they are naturally equipped with a notion of covering family that is closed under composition. In the cases we are interested in, these families will form the basis of a Grothendieck topology.

Definition 11.7 (The natural potential basis of a ranked theory) Let $\mathfrak{K}$ be the function that assigns to each T^X in $\mathcal{T}$, the collection of ample and proper families of efficient maps.

Notice that $\mathfrak{K}(T^0) = \mathfrak{K}1 = \{(id_1 \mid 1)\}$ because, for any X, the unique map $T^X \to 1$ is bureaucratic.

Before the next result recall that we could not prove that ample families compose in the general context of a fort.

Lemma 11.8 ($\mathfrak{K}$-families compose) *Let the family $G = (g_i : T^{Y_i} \to T^Z \mid i \in I)$ be in $\mathfrak{K}(T^Z)$ and, for each $i \in I$, let $F_i = (f_{i,j} : T^{X_{i,j}} \to T^{Y_i} \mid j \in I_i)$ be in $\mathfrak{K}(T^{Y_i})$. Then the composite family $H = (g_i f_{i,j} \mid i \in I, j \in I_i)$ is in $\mathfrak{K}(T^Z)$.*

Proof It is clear that H is a proper family of efficient maps. Assume that G is n-ample. Since all the maps in G are efficient then $\{(g_i)^{\flat} y \mid i \in I, y \in Y_i\}$ is a finite set of natural numbers by Proposition 11.5. So we can choose an $N \in \mathbb{N}$ that is above all these and also above n. Similarly, for each $i \in I$, if F_i is m_i-ample then we can choose an $M \in \mathbb{N}$ that is above every m_i. We claim that H is $(N + M)$-ample. Let $h : T^X \to T^Z$ be such that $\partial h \geq N + M$. Since $M + N \geq N \geq n$ there exists an $i \in I$ and an $f : T^X \to T^{Y_i}$ such that $g_i f = h$. Now calculate

$$
N + m_i \leq \partial h = \bigwedge_{y \in Y_i} ((g_i)^{\flat} y) + \partial(\pi_y f) \leq \bigwedge_{y \in Y_i} N + \partial(\pi_y f) = N + \bigwedge_{y \in Y_i} \partial(\pi_y f) = N + \partial f
$$

and, since $N \in \mathbb{N}$, conclude that $m_i \leq \partial f$. Then f factors through F_i and this implies that h factors through H. □

By Lemma 6.8, $\mathfrak{K}$ would be the basis of a Grothendieck topology if it had long covers in the sense of Definition 6.7.

12 Developments

Let $\mathcal{T}$ be a ranked theory with associated natural potential basis $\mathfrak{K}$ (Definition 11.7). In this section we assume that $\mathcal{T}$ is equipped with a distinguished family $\mathrm{B} \in \mathfrak{K}(T)$ of 'basic operations'. Notice that for any $j : 1 \to Y$, the pullback $\pi_j^* \mathrm{B}$ exists.

Lemma 12.1 *For any* $j : 1 \to Y$ *in* **fSet***,* $\pi_j^* \mathrm{B} \in \mathfrak{K}(T^Y)$.

Proof The family $\pi_j^* \mathrm{B}$ is proper by Lemma 3.8 and ample by Lemma 6.4. Finally, the maps in $\pi_j^* \mathrm{B}$ are efficient by the basic properties of factorization systems. $\square$

The next concept is analogous to the notion of development introduced in p. 10 of [3].

Definition 12.2 (Development of a cover) For $G = (g_k : T^{Y_k} \to T^Z \mid k \in K) \in \mathfrak{K}(T^Z)$, $l \in K$ and $j \in Y_l$, the *development of G at g_l and j* is the family obtained by composing G with the families F_k on T^{Y_k}, where F_k is the trivial family $(id_{Y_k} \mid 1)$ if $k \neq l$ and $F_l = \pi_j^* \mathrm{B}$.

Roughly speaking the development of G at g_l and j is the result of replacing g_l with the family $g_l(\pi_j^* \mathrm{B})$. Lemmas 11.8 and 12.1 imply that any such development is in $\mathfrak{K}(T^Z)$.

Lemma 12.3 *Let* $G = (g_k : T^{Y_k} \to T^Z \mid k \in K) \in \mathfrak{K}(T^Z)$ *be n-ample. Let* $l \in K$ *and* $j \in Y_l$. *If* B *is 1-ample and* $(g_l)^\natural j < n$ *then the development H of G at g_l and j is also n-ample.*

Proof Let $h : T^X \to T^Z$ be such that $\eth h \geq n$. By hypothesis, h factors through G. That is, there exists $k \in K$ and $f : X \to Y_k$ such that $h = g_k f$. If $k \neq l$ then f factors through the trivial family and so h factors through H. If $k = l$ then it is enough to show that f factors through $\pi_j^* \mathrm{B}$. In turn, it is enough to prove that $\pi_j f : T^X \to T$ factors through B. Since B is 1-ample we are left to show that $\eth(\pi_j f) \geq 1$. Calculate:

$$n \leq \eth h = \eth(g_l f) = \bigwedge_{y \in Y_l} (g_l)^\natural y + \eth(\pi_y f) \leq (g_l)^\natural j + \eth(\pi_j f) < n + \eth(\pi_j f)$$

so $\eth(\pi_j f) \geq 1$. $\square$

Let us say that the family *G can be developed to H* if there exists a sequence of families $G = G_0, G_1, G_2, \ldots, G_n = H$ such that for each $m < n$, G_{m+1} is the development of G_m at some $g : T^Y \to T^Z$ in G_m and $j \in Y$.

Proposition 12.4 *If* B *is strictly 1-ample then any n-ample family in $\mathfrak{K}(T^Z)$ can be developed to a strictly n-ample family.*

Proof Let $G \in \mathfrak{K}(T^Z)$ be n-ample. If it is not strictly n-ample there exists a $g : T^Y \to T^Z$ in G such that $\eth g = \bigwedge_{k \in Y} g^\natural k < n$. Then there exists $j \in Y$ such that $g^\natural j < n$. Let G' be the result of developing G at g and j. If every map h in G' is such that $\eth h \geq n$ then we are done. If not, repeat the process. If the process terminates then the resulting family H is such that for every h in H, $\eth h \geq n$. Moreover, Lemma 12.3 implies that, at each stage of the development, n-amplitude is preserved; so H is strictly n-ample.

Now, why does the process terminate? Loosely speaking the reason is that since B is strictly 1-ample then, at each stage, the chosen 'tree' g is replaced by a family of trees whose 'leaves' are further from the root. Since this 'distance' can only grow up to n, and all the families involved are finite, then the process must terminate.

To be more precise let $g : T^Y \to T^Z$ in G and $j \in Y$ as above. Each map in $g(\pi_j^* \mathrm{B})$ is of the form $g(\pi_j^* f)$ for some $f : T^X \to T$ in B. Since B is strictly 1-ample, $\partial f \geq 1$. That is, for every $x \in X$, $f^\flat x \geq 1$. If we let $Y_j \to Y$ be the complement of $j : 1 \to Y$ then we can picture $g(\pi_j^* f)$ as follows

$$
\begin{array}{ccc}
 & T^{Y_j + X} \xrightarrow{\ T^{in_1}\ } & T^X \\
 T^{in_0}\nearrow \quad \downarrow \pi_j^* f & & \downarrow f \\
 T^{Y_j} \xleftarrow{\ T^{in_0}\ } T^{Y_j + 1} \xrightarrow{\ T^{in_1}\ } & T \\
 & \downarrow g & \\
 & T^Z &
\end{array}
$$

where the square is a pullback. For each $z \in Y_j + X$,

$$
(g(\pi_j^* f))^\flat z = \bigwedge_{w \in Y_j + 1} g^\flat w + (\pi_w(\pi_j^* f))^\flat z = \left(\bigwedge_{y \in Y_j} g^\flat y + (\pi_y(\pi_j^* f))^\flat z \right)
$$
$$
\wedge \ (g^\flat j + (\pi_j(\pi_j^* f))^\flat z) =
$$

$$
= \left(\bigwedge_{y \in Y_j} g^\flat y + (\pi_y)^\flat z \right) \wedge (g^\flat j + (f(T^{in_1}))^\flat z)
$$

and

$$
(f(T^{in_1}))^\flat z = \bigwedge_{x \in X} (f^\flat x) + (\pi_x(T^{in_1}))^\flat z = \bigwedge_{x \in X} (f^\flat x) + (\pi_x)^\flat z
$$

so we can distinguish two cases. Either $z \in Y_j$ or $z \in X$. If $z \in Y_j$ then

$$
(f(T^{in_1}))^\flat z = \bigwedge_{x \in X} (f^\flat x) + \infty = \infty
$$

and so

$$
(g(\pi_j^* f))^\flat z = \bigwedge_{y \in Y_j} g^\flat y + (\pi_y)^\flat z = g^\flat z
$$

which is reasonable because we have developed at $j \notin Y_j$.

On the other hand, if $z \in X$ then $\pi_y^\flat z = \infty$ for every $y \in Y_j$ so

$$
(g(\pi_j^* f))^\flat z = (g^\flat j) + \bigwedge_{x \in X} (f^\flat x) + (\pi_x)^\flat z = (g^\flat j) + (f^\flat z) > g^\flat j
$$

because $f^\natural z \geq 1$. Roughly speaking, the new labels that have appeared in the tree $g(\pi_j^* f)$, i.e. those in X, are further from the root than the label j in the tree g. (Of course, if f is a constant the X is empty.) □

The proof of Proposition 12.4 should be compared with the discussion in p. 10 of [3]. Also on the relation with [3], notice that one can develop the trivial family in $\mathfrak{K}(T^X)$ to a strictly n-ample one. These are analogous to the families denoted by S_n in [3].

Corollary 12.5 (Potential is actual) *If* B *is strictly 1-ample then* $\mathfrak{K}$ *is the basis for a Grothendieck topology on* $\mathcal{T}$.

Proof Lemma 11.8 implies that $\mathfrak{K}$-coverings compose. So, by Lemma 6.8, it is enough to check that $\mathfrak{K}$ has long covers in the sense of Definition 6.7. That is, that for every T^Z in $\mathcal{T}$ and $n \in \mathbb{N}$ there is a $H \in \mathfrak{K}(T^Z)$ such that for all $h \in H$, $\partial h \geq n$. The trivial family $(id \mid 1)$ on T^Z is 0-ample so it is also n-ample and, by Proposition 12.4, can be developed to a strictly n-ample family. □

The next result is analogous to the lemma in p. 16 of [3]. Notice that the structure of the proof is essentially the same.

Corollary 12.6 *If* B *is strictly 1-ample then the following Grothendieck topologies on* $\mathcal{T}$ *coincide:*

1. *The smallest topology for which* T *is covered by* B.
2. *The smallest topology in which, for each* $A \in$ **fSet**, T^A *is covered by the strictly ample proper families on* T^A.
3. *The topology where the covering sieves of* T^A *are those that include some strictly ample proper family.*
4. *The topology generated by* $\mathfrak{K}$.

Proof By our current hypotheses, B is in $\mathfrak{K}$ so the topology in the first item is included in that of the fourth.

To prove that the topology of the second item is included in that of the first we must show that, for every A in **fSet** and $n \in \mathbb{N}$, the essentially unique proper and strictly n-ample family on T^A is a cover with respect to the topology of the first item. This follows from Proposition 12.4 because it shows that using only B one can develop the trivial family $(id_{T^A} \mid 1)$ to 'the' strictly n-ample family (Lemma 6.6).

It is trivial that all the sieves described in the third item are in the topology of the second. (The fact that the sieves in the third item form a topology will follow once we show that it includes $\mathfrak{K}$, for then all four items are equal.)

So, to complete the proof, we consider an arbitrary sieve R on T^A containing a $\mathfrak{K}$-cover and show that this sieve is among the sieves described in the third item. Assume that R contains an n-ample family F in $\mathfrak{K}(T^A)$. It is clear that the maps in any development of F must be in R but then, by Proposition 12.4, the strictly n-ample family on T^Y must be in R. □

It must be stressed that Corollary 12.6 above is, in a sense, weaker than the lemma in p. 16 of [3] because the latter does not resort to efficient maps; something we used in the proof that $\mathfrak{K}$ is a basis (see Lemma 11.8). So it is natural to search for a result showing that, under

reasonable hypotheses, the efficiency requirement in the definition of $\mathfrak{K}$ is superfluous. We do this in the next section.

13 Inconstant Maps

Let $\mathcal{T}$ be a ranked theory with associated natural potential basis $\mathfrak{K}$ (Definition 11.7). In this section we assume that $\mathcal{T}$ is equipped with a distinguished strictly 1-ample family $B \in \mathfrak{K}(T)$ so that $\mathfrak{K}$ is the basis for a Grothendieck topology by Corollary 12.5. We will show that in certain cases the efficiency requirement in the definition of $\mathfrak{K}$ is unnecessary. First, a piece of very basic category theory.

Lemma 13.1 *Let the following diagram*

$$
\begin{array}{ccc}
P & \xrightarrow{\pi_1} & A \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle a} \\
C & \xrightarrow{r} & B
\end{array}
$$

be a pullback. If π_1 is an iso, r is split epi and π_0 is epi then r is an iso.

Proof Without loss of generality we can assume that the following diagram

$$
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle a} \\
C & \xrightarrow{r} & B
\end{array}
$$

is a pullback. Let $s : B \to C$ be a section of r. Since $rsa = a$ the pullback property implies that $sa = \pi_0$. Since, π_0 is epi, so is s. $\qquad\square$

By Lemma 11.2 every efficient map $f : T^A \to T^B$ is a B-indexed product of efficient maps. More explicitly, $f : T^A \to T^B$ is efficient if and only if there is a function $a : A \to B$ such that for every $i : 1 \to B$ and pullback as on the left below

$$
\begin{array}{ccc}
A_i & \xrightarrow{in_i} & A \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle a} \\
1 & \xrightarrow{i} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
T^A & \xrightarrow{f} & T^B \\
{\scriptstyle T^{in_i}}\downarrow & & \downarrow{\scriptstyle T^i} \\
T^{A_i} & \xrightarrow{f_i} & T^1
\end{array}
$$

there is an efficient $f_i : T^{A_i} \to T^1$ such the diagram on the right above commutes.

Definition 13.2 Such an efficient map $f : T^A \to T^B$ will be called *inconstant* if the map $a : A \to B$ is surjective.

Loosely speaking, the family of maps that determines f does not have constants. In particular, an efficient map $f : T^A \to T$ is inconstant if and only if $A \neq \emptyset$. In other words, $f : T^A \to T$ is inconstant if and only if it is not a constant.

Lemma 13.3 *Let* $f : T^A \to T^B$ *be inconstant and* $g : T^B \to T^C$ *be mono and bureaucratic. If the composite* $gf : T^A \to T^C$ *is efficient then* g *is an iso.*

Proof Let $f : T^A \to T^B$ be efficient with $a : A \to B$ as above. Now let $r : C \to B$ be any map in **fSet** and let the diagram on the left below

$$
\begin{array}{ccc}
P & \xrightarrow{p_1} & A \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle a} \\
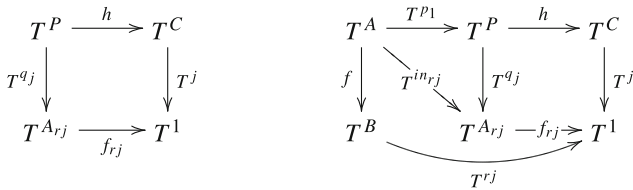C & \xrightarrow{r} & B
\end{array}
\qquad
\begin{array}{ccccc}
A_{rj} & \cdots\cdots\overset{q_j}{>} & P & \xrightarrow{p_1} & A \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle a} \\
1 & \xrightarrow{j} & C & \xrightarrow{r} & B
\end{array}
$$

with arc in_{rj} over the top of the right diagram.
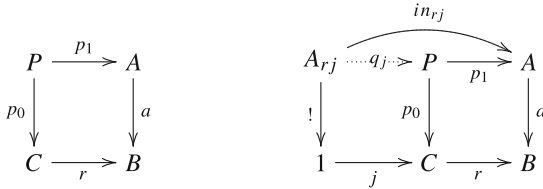
be a pullback in **fSet**. It follows that the rectangle on right above is also a pullback, and it induces a canonical $q_j : A_{rj} \to P$ as displayed there, which is mono because in_{rj} is. Let $h : T^P \to T^C$ be the unique map such that the square on the left below commutes

$$
\begin{array}{ccc}
T^P & \xrightarrow{h} & T^C \\
{\scriptstyle T^{q_j}}\downarrow & & \downarrow{\scriptstyle T^j} \\
T^{A_{rj}} & \xrightarrow{f_{rj}} & T^1
\end{array}
\qquad
\begin{array}{ccccc}
T^A & \xrightarrow{T^{p_1}} & T^P & \xrightarrow{h} & T^C \\
{\scriptstyle f}\downarrow & {\scriptstyle T^{in_{rj}}}\searrow & \downarrow{\scriptstyle T^{q_j}} & & \downarrow{\scriptstyle T^j} \\
T^B & & T^{A_{rj}} & \xrightarrow{f_{rj}} & T^1
\end{array}
$$

with arc T^{rj} under the bottom of the right diagram.

for every $j \in C$. Notice that h is a product of f_i's so it is efficient. We claim that $(T^r)f = h(T^{p_1}) : T^A \to T^C$. In other words, that h, T^{p_1} form the efficient/bureaucratic factorization of $(T^r)f$. To prove the claim calculate as on the right above and, together with $(T^j)(T^r) = T^{rj} : T^B \to T^1$, we may conclude that for every point $j : 1 \to C$, $(T^j)(T^r)f = (T^j)h(T^{p_1})$. That is, $(T^r)f = h(T^{p_1})$, so the claim is proved.

Consider now g as in the statement. Since g is bureaucratic it is of the form T^r for some $r : C \to B$ in **fSet** and, as g is mono, r must be epi. Since f is inconstant, there is an $a : A \to B$ as above which is also surjective. By the argument above, $gf = (T^r)f = h(T^{p_1})$ so, if gf is efficient, then T^{p_1} is an iso; so p_1 is an iso and, by Lemma 13.1, r is an iso. $\qquad\square$

Efficiency is 'reflected by developments' in the following sense.

Lemma 13.4 *Let* G *be a proper family on* T^Z *and let* H *be a development of* G *such that every map in* H *is efficient. If* $\mathbb{B}$ *contains an inconstant map then every map in* G *is efficient.*

Proof Let H be the development of G at $g : T^Y \to T^Z$ and $i \in Y$. Then, to prove the result, we need only check that g is efficient. So let $g = h(T^b)$ with h efficient and $b : B \to Y$. Since g is mono, so is (T^b). We need to prove that T^b is actually an iso.

By hypothesis there is an inconstant map in $\mathbb{B}$ and, since inconstant maps are closed under pullback, there is an inconstant map in $\pi_i^*\mathbb{B}$. Let us call this inconstant map $f : T^X \to T^Y$ and consider the composite $gf = h(T^b)f$. Since h is efficient, and gf is efficient by hypothesis, $(T^b)f$ is also efficient (by general considerations about factorizations systems). Then T^b is an iso by Lemma 13.3. $\qquad\square$

We believe that Corollary 12.6, together with the next result, form a fair generalization of the lemma in p. 16 of [3] to our context.

Proposition 13.5 *If* B *has an inconstant map then the basis* $\mathfrak{K}$ *may be described as assigning, to each* T^X *in* $\mathcal{T}$, *the collection of ample and proper families on* T^X.

Proof Definition 11.7 introduces $\mathfrak{K}$ as the function that assigns, to each T^X in $\mathcal{T}$, the collection of ample and proper families of efficient maps on T^X. So it is enough to show that the requirement that the maps are efficient is superfluous. Let G be a proper and n-ample family on T^X. Let H be the result of developing G to a strictly n-ample family. By Lemma 11.6, every map in H is efficient. By Lemma 13.4, every map in G is efficient. $\qquad\square$

We will use the following.

Corollary 13.6 *If* B *has an inconstant map then* $\mathfrak{K}$ *is a basis of dense monos in the sense of Definition 4.5.*

Proof Every $\mathfrak{K}$-cover is proper by definition of $\mathfrak{K}$ so to prove that $\mathfrak{K}$ is a basis of dense monos consider an $F \in \mathfrak{K}(T^Y)$ and a proper family P on T^Y such that every map in F factors through P. Since F is ample then so is P. By Proposition 13.5, P is in $\mathfrak{K}$. $\qquad\square$

14 Weights for a Ranked Theory

Let $\mathcal{T}$ be a ranked theory with natural potential basis $\mathfrak{K}$. Assume that $\mathcal{T}$ is equipped with a distinguished strictly 1-ample family B $\in \mathfrak{K}(T)$ so that $\mathfrak{K}$ is the basis for a Grothendieck topology. Moreover, let R be a rig and let $\gamma : \mathfrak{B}(\mathcal{T}) \to R$ be a multiplicative-monoid morphism so that we can consider weights as in Section 5.

Lemma 14.1 *If* $\mathbf{w}$B $= \gamma[T]$ *then, for any* Y *in* **fSet** *and* $j : 1 \to Y$, $\mathbf{w}(\pi_j^*$B$) = \gamma[T]^Y$.

Proof Let B $= (\mathbf{b}_i : T^{X_i} \to T \mid i \in I)$ and $Y_j \subseteq Y$ be the complement of $j : 1 \to Y$. For each $i \in I$ let the following square

$$
\begin{array}{ccc}
T^{X_i+Y_j} & \longrightarrow & T^{X_i} \\
\pi_j^* \mathbf{b}_i \downarrow & & \downarrow \mathbf{b}_i \\
T^Y & \xrightarrow{\ \pi_j\ } & T
\end{array}
$$

be a pullback in $\mathcal{T}$. Of course, $[T^{X_i+Y_j}] = [T^{X_i} \times T^{Y_j}] = [T^{X_i}] \cdot [T^{Y_j}] = [T]^{X_i} \cdot [T]^{Y_j}$ so $\gamma([T^{X_i+Y_j}]) = \gamma[T]^{X_i} \cdot \gamma[T]^{Y_j}$ and we can calculate

$$
\mathbf{w}(\pi_j^* \mathrm{B}) = \sum_{i \in I} \gamma([T^{X_i+Y_j}]) = \sum_{i \in I} \gamma[T]^{X_i} \cdot \gamma[T]^{Y_j} =
$$

$$
= \gamma[T]^{Y_j} \cdot \sum_{i \in I} \gamma[T]^{X_i} = \gamma[T]^{Y_j} \cdot \mathbf{w}\mathrm{B} = \gamma[T]^{Y_j} \cdot \gamma[T] = \gamma[T]^Y
$$

to complete the proof. $\qquad\square$

The following is also easy.

Lemma 14.2 (Development does not change weight) *Assume that* $\mathbf{w}B = \gamma[T] \in R$. *If* $G = (g_k : T^{Y_k} \to T^Z \mid k \in K)$ *in* $\mathfrak{K}(T^Z)$ *and* H *is the development of* G *at* $g_l : T^{Y_l} \to T^Z$ *and* $j \in Y_l$ *then* $\mathbf{w}H = \mathbf{w}G$.

Proof By definition H is the family obtained by composing G with the families F_k on T^{Y_k}, where F_k is the trivial family $(id_{Y_k} \mid 1)$ if $k \neq l$ and $F_l = \pi_j^* B$. Clearly, for any $k \neq l$, $\mathbf{w}F_k = \gamma[T]^{Y_k}$ and, by Lemma 14.1, $\mathbf{w}F_l = \gamma[T]^{Y_l}$. Then, by Lemma 5.8,

$$\mathbf{w}H = \sum_{k \in K} \mathbf{w}F_k = \left(\sum_{l \neq k \in K} \mathbf{w}F_k \right) + \mathbf{w}F_l = \left(\sum_{l \neq k \in K} \gamma[T]^{Y_k} \right) + \gamma[T]^{Y_l} = \mathbf{w}G$$

and the proof is complete. $\qquad\square$

Notice how Proposition 12.4 appears again in the next result.

Lemma 14.3 *If* $\mathbf{w}B = \gamma[T]$ *then* γ *is compatible with* $\mathfrak{K}$.

Proof Let B in **fSet**. Clearly, $\mathbf{w}(id_{T^B} \mid 1) = \gamma[T]^B$. By Lemma 14.2 'the' strictly n-ample family H on T^B has weight $\gamma[T]^B$. On the other hand, by Proposition 12.4, any $G \in \mathfrak{K}(T^Y)$ can be developed to H so, again by Lemma 14.2, $\mathbf{w}G = \mathbf{w}H$. $\qquad\square$

It is time to summarize. We are assuming that $\mathcal{T}$ is a ranked theory equipped a strictly 1-ample family B in $\mathfrak{K}(T)$ so that $\mathfrak{K}$ is the basis for a Grothendieck topology by Corollary 12.5. We are also assuming a rig R and a morphism $\gamma : \mathfrak{B}\mathcal{T} \to R$ of multiplicative monoids.

Proposition 14.4 *If* B *contains an inconstant map and* $\mathbf{w}B = \gamma[T]$ *then there exists a unique* $\mathfrak{B}(\mathrm{Fam}(\mathcal{T}, \mathfrak{K})) \to R$ *such that the right triangle below commutes.*

$$\mathfrak{B}\mathcal{T} \longrightarrow \mathfrak{B}(\mathrm{Fam}\mathcal{T}) \longrightarrow \mathfrak{B}(\mathrm{Fam}(\mathcal{T}, \mathfrak{K}))$$
$$\gamma \searrow \quad \downarrow \quad \swarrow$$
$$R$$

Proof Corollary 13.6 implies that $\mathfrak{K}$ is a basis of dense monos and Lemma 14.3 implies that γ is compatible with $\mathfrak{K}$ so Proposition 5.7 is applicable. $\qquad\square$

15 The Main Result

It may be convenient to start by recalling a basic fact. A morphism in a category is iso if and only if it is epi and has a retraction. In particular, if R is a rig and $\mathcal{C}$ is a distributive category then, a surjective $R \to \mathfrak{B}\mathcal{C}$ is an iso if and only if it has a retraction. Most of the work in the paper was done to construct such a retraction. Lemma 15.1 below completes the construction. Theorem 15.2 makes explicit the simple surjection $R \to \mathfrak{B}\mathcal{C}$ and proves that it is a section of the previous morphism.

Fix a polynomial $L(X) \in \mathbb{N}[X]$. As discussed in Section 1, $L(X)$ determines a 'signature' $\ell \in \mathbf{Set}^{\mathbb{N}}$ and its associated free theory that we denote by $\mathcal{L}$.

We let $\mathcal{L}$ be ranked as in Lemma 11.4 and we consider its natural potential basis $\mathfrak{K}$ (Definition 11.7). For every $n \geq 0$, each $f \in \ell n$ induces an efficient monic map in $\mathcal{L}$ that we denote by $f : T^{\mathbf{n}} \to T$. (In free theories, efficient maps are mono. See discussion after Definition 11.1.) Let B be the (proper) family $(f : T^{\mathbf{n}} \to T \mid n \in \mathbb{N},\, f \in \ell n)$. The proof of Lemma 11.4 implies that $\partial f \geq 1$ for each f. The family B is also 1-ample because any map $h : T^X \to T$ with $\partial h \geq 1$ is of the form fg for some f in B. Altogether, B in $\mathfrak{K}(T)$ is strictly 1-ample so $\mathfrak{K}$ is the basis for a Grothendieck topology by Corollary 12.5 and we can consider the distributive category $\mathrm{Fam}(\mathcal{L}, \mathfrak{K})$.

Lemma 15.1 *If $L(X)$ is not constant then there exists a unique morphism of rigs $\mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K})) \to \mathbb{N}[X]/(X = L(X))$ such that the following diagram*

$$
\begin{array}{ccc}
\mathfrak{B}(\mathrm{Fam}\mathcal{L}) & \longrightarrow & \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K})) \\
{\scriptstyle \cong} \downarrow & & \vdots \\
(\mathbb{N}[X], \cdot, 1) & \longrightarrow & \mathbb{N}[X]/(X = L(X))
\end{array}
$$

commutes.

Proof As discussed in Section 1, the product preserving inclusion $\mathcal{L} \to \mathrm{Fam}\mathcal{L}$ induces a (multiplicative-)monoid morphism $\mathfrak{B}\mathcal{L} \to \mathfrak{B}(\mathrm{Fam}\mathcal{L})$ as below

$$
\begin{array}{ccccc}
[T^{\mathbf{n}}] & \mathfrak{B}\mathcal{L} & \longrightarrow & \mathfrak{B}(\mathrm{Fam}\mathcal{L}) & \\
\big| & {\scriptstyle \cong}\big\downarrow & & {\scriptstyle \cong}\big\downarrow & \\
\big| & (\mathbb{N}, +, 0) & \longrightarrow & (\mathbb{N}[X], \cdot, 1) & \longrightarrow & \mathbb{N}[X]/(X = L(X)) \\
\big\downarrow & & & & \\
n & \longmapsto & & X^n &
\end{array}
$$

and we may denote the composite

$$
\mathfrak{B}\mathcal{L} \longrightarrow (\mathbb{N}[X], \cdot, 1) \longrightarrow \mathbb{N}[X]/(X = L(X))
$$

by $\gamma : \mathfrak{B}\mathcal{L} \to \mathbb{N}[X]/(X = L(X))$. It is clearly a morphism of multiplicative monoids. If $L(X)$ is not constant then B contains an inconstant map and

$$
\mathbf{w}\mathrm{B} = \sum_{n \in \mathbb{N}} \sum_{f \in \ell n} \gamma[T]^n = L(\gamma[T]) = \gamma[T] \in \mathbb{N}[X]/(X = L(X))
$$

so Proposition 14.4 completes the proof. $\qquad\square$

We can now prove our main result.

Theorem 15.2 *If $L(X)$ is not constant then the rig morphism*

$$
\mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K})) \to \mathbb{N}[X]/(X = L(X))
$$

is an iso.

Proof The family $\mathbb{B}$ in $\mathfrak{K}(T)$ induces a map $L(T) \to T$ in $\Gamma_{\mathfrak{K}}$ and so, an iso $L(T) \to T$ in $(\mathrm{Fam}\mathcal{L})[\Gamma_{\mathfrak{K}}^{-1}] = \mathrm{Fam}(\mathcal{L}, \mathfrak{K})$. Then $L([T]) = [T]$ holds in $\mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K}))$ and the universal property of the rig quotient $\mathbb{N}[X] \to \mathbb{N}[X]/(X = L(X))$ implies the existence of a unique rig map $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K}))$ making the following diagram

$$
\begin{array}{ccccc}
\mathbb{N}[X] & \longrightarrow & \mathbb{N}[X]/(L(X) = X) & & X \\
\cong \downarrow & & \vdots & & \top \downarrow \\
\mathfrak{B}(\mathrm{Fam}\mathcal{L}) & \longrightarrow & \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K})) & & [T]
\end{array}
$$

commute. As $\mathrm{Fam}\mathcal{L} \to \mathrm{Fam}(\mathcal{L}, \mathfrak{K})$ is bijective on objects, the induced rig morphism $\mathfrak{B}(\mathrm{Fam}\mathcal{L}) \to \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K}))$ is surjective, and it follows that the induced rig morphism $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K}))$ is also surjective.

If we stack the rectangle in the previous paragraph on top of that in Lemma 15.1 then the universal property of the quotient $\mathbb{N}[X] \to \mathbb{N}[X]/(X = L(X))$ implies that the composite

$$\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K})) \to \mathbb{N}[X]/(X = L(X))$$

is the identity. It follows that $\mathbb{N}[X]/(X = L(X)) \to \mathfrak{B}(\mathrm{Fam}(\mathcal{L}, \mathfrak{K}))$ is an iso. $\qquad\square$

16 A Presentation of the Theory Classified by $\mathrm{Sh}(\mathcal{L}, \mathfrak{K})$

In this section we give a presentation of the theory classified by the topos $\mathrm{Sh}(\mathcal{L}, \mathfrak{K})$. The argument is a straightforward generalization of the argument used in the proof of Theorem 4 in [3], but it seems useful to split it in a way that gives a solution to a particular case of a more general problem

Let $\mathcal{T}$ be an algebraic theory and let $\mathcal{T}^{\ddagger}$ be the opposite of the category of finitely presented $\mathcal{T}$-models. It is well-known that the classifying topos for $\mathcal{T}$ may be identified with the presheaf topos $\widehat{\mathcal{T}^{\ddagger}}$. See, e.g., Corollary D3.1.1 in [10]. The full subcategory $\mathcal{T} \to \mathcal{T}^{\ddagger}$ induces a subtopos $\widehat{\mathcal{T}} \to \widehat{\mathcal{T}^{\ddagger}}$. What does $\widehat{\mathcal{T}}$ classify? Diaconescu's Theorem gives one answer, but it is natural to strive for a more specific one when starting with an algebraic theory instead of an arbitrary (internal) category. We don't know of a general satisfactory answer to this question but we observe that a simple variation of Theorem 4 in [3] provides an answer for the case of free algebraic theories.

Theorem 16.1 *Let $\mathcal{T}$ be the free algebraic theory generated by a 'signature' $\ell \in \mathbf{Set}^{\mathbb{N}}$. Then $\widehat{\mathcal{T}}$ classifies the (geometric) theory presented by the same operations and the following coherent sequents:*

1. *(Basic operations are injective)*

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \vdash_{x_1, \ldots, x_n, y_1, \ldots, y_n} \bigwedge_{i=1}^{n} x_i = y_i$$

   for every $n \in \mathbb{N}$ and $f \in \ell n$.
2. *(Basic operations are disjoint)*

$$f(x_1, \ldots, x_m) = f'(y_1, \ldots, y_n) \vdash_{x_1, \ldots, x_m, y_1, \ldots, y_n} \bot$$

   for all $m, n \in \mathbb{N}$, $f \in \ell m$, $f' \in \ell n$ and $f \neq f'$.
3. *(No proper 'subselves')*

> *For any term t that contains the variable x but is not just x,*
>
> $$t = x \vdash_{FV(t)} \perp$$
>
> *where FV(t) denotes a list of the distinct free variables that appear in t.*

Proof We show that the method of D3.1.10 in [10] to construct the classifying topos for the presentation in the statement leads to the site given by $\mathcal{T}$ with the trivial topology. Since the sequents are very simple it is easy to check that the method is indeed applicable. According to this method the classifying topos may be identified with $\mathrm{Sh}(\mathcal{C}, J)$ where $\mathcal{C}$ is the syntactic category determined by the signature ℓ and the axioms in the first item (equivalently, the opposite of the category of finitely presented models in **Set**) and J is the Grothendieck topology generated by the remaining axioms. (Notice that $\mathcal{C}$-models are $\mathcal{T}$-algebras whose basic operations are injective, and that free $\mathcal{T}$-algebras satisfy this property.) So we need to analyse the finitely presented models. These may be described as $\langle A \mid E \rangle$ where A is a finite set 'of generators' and E is a finite set of equations between terms built using the basic operations and the generators (see, e.g., D2.4 in [10]).

The axioms in the second and third items simply say that some objects of $\mathcal{C}$ are covered with the empty family and so may be removed from $\mathcal{C}$. We claim that all non free models are cocovered by the empty family. Consider a model $\langle A \mid E \rangle$.

1. Assume that a constant occurs as one side of an equation in E. Without loss of generality we can assume that the equation is of the form $f = t$ with $f \in \ell 0$ and t a term.

   (a) If $t = f$ then we can delete the equation from E leaving us with a 'smaller' presentation of the same model.

   (b) If $t = f'(t_1, \ldots, t_n)$ for some $f' \in \ell n$ with $f \neq f'$ then there is a morphism $\langle \{y_1, \ldots, y_n\} \mid \{f = f'(y_1, \ldots, y_n)\}\rangle \to \langle A \mid E \rangle$. Since basic operations are disjoint, the domain of this map is cocovered by the empty family and so, $\langle A \mid E \rangle$ is cocovered by the empty family too.

   (c) If $t \in A$ then $\langle A \mid E \rangle$ can also be presented as $\langle A - \{t\} \mid E' \rangle$ where E' is the result of removing the equation $f = t$ and replacing all the occurrences of t in the rest of the equations by f. So, again we obtain a 'smaller' presentation.

2. So we may assume that no constant occurs as the side of an equation. Suppose next that and element $a \in A$ occurs as a side of an equation in E. We can assume that the equation is of the form $a = t$.

   (a) If $t = a$ then we can delete the equation to obtain a smaller presentation.

   (b) If t is not a but involves a then there is a morphism $\langle FV(t) \mid \{a = t\}\rangle \to \langle A \mid E \rangle$. Since there are no proper subselves, the domain is cocovered by the empty family and then so is the codomain.

   (c) So we can assume that t does not involve a. We can then remove a from A, remove the equation $a = t$ and replace all other occurrences of a in E by t. Again, a smaller presentation.

3. We may now assume that every equation in E is $f(t_1, \ldots, t_m) = f'(u_1, \ldots, u_n)$ for some $f \in \ell m$ and $f' \in \ell n$.

   (a) If $f \neq f'$ then there is a morphism

   $$\langle \{x_1, \ldots, x_m, y_1, \ldots, y_n\} \mid \{f(x_1, \ldots, x_m) = f'(y_1, \ldots, y_n)\}\rangle \to \langle A \mid E \rangle$$

and, since basic operations are disjoint, $\langle A \mid E \rangle$ is cocovered by the empty family.

(b) If $f = f'$ then $m = n$ and, as basic operations are injective, we can replace this equation $f = f'$ with the 'smaller' equations $t_1 = u_1, \ldots, t_m = u_m$. So again we obtain a smaller presentation of $\langle A \mid E \rangle$.

At each set we have either proved that $\langle A \mid E \rangle$ is cocovered by the empty family or obtained a smaller presentation of the same model. In the second case we can repeat the steps. We can only do this a finite number of times because the size of A cannot decrease infinitely often and, after it stops decreasing, the 'total length' of E cannot decrease infinitely often. So the only way that the process can stop is if $\langle A \mid E \rangle$ is cocovered by the empty family or E has become empty.

Notice also that, at each step, the conclusion that $\langle A \mid E \rangle$ is cocovered by the empty family is the result of exhibiting a map from the model presented by the antecedent of one of the sequents in the second or third items in the statement. But free models cannot receive maps from these, so the topos $\mathrm{Sh}(\mathcal{C}, J)$ is equivalent to the topos $\mathrm{Sh}(\mathcal{T}, J')$ where $\mathcal{T} \to \mathcal{C}$ is the full subcategory determined by the free models and J' is the trivial topology. In other words, the presheaf topos $\widehat{\mathcal{T}}$ classifies the theory presented in the statement. □

Notice that the models of the theory classified by $\widehat{\mathcal{T}}$ are very similar to the Peano algebras discussed in [6]. See also Section 8 in [15].

We now give the analogue of Theorem 4 in [3] in the more general context of the present paper. Let $L(X) \in \mathbb{N}[X]$, let $\ell \in \mathbf{Set}^{\mathbb{N}}$ be the induced signature, and $\mathcal{L}$ be the resulting free algebraic theory.

Corollary 16.2 *If $L(X)$ is not constant then the topos $\mathrm{Sh}(\mathcal{L}, \mathfrak{K})$ classifies the theory presented by the sequents in Theorem 16.1 together with:*

4. *(Basic operations are jointly surjective)*

$$\top \vdash_x \bigvee_{n \in \mathbb{N}} \bigvee_{f \in \ell n} (\exists y_1, \ldots, y_n)(x = f(y_1, \ldots, y_n))$$

(we stress that this is a finite disjunction because $\ell n = \emptyset$ if $n > \deg L(X)$).

Proof The method of D3.1.10 in [10], together with the calculations performed in Theorem 16.1, show that the classifying topos for the 'extended' presentation of the present result may be described as the topos of sheaves on the site $(\mathcal{L}, J)$ where J is the Grothendieck topology generated by the new axiom; but this axiom simply says that the family $(f : T^{\mathbf{n}} \to T \mid n \in \mathbb{N}, f \in \ell n)$ covers T. So J is the smallest topology generated by this family. In other words, J is the topology generated by $\mathfrak{K}$ (Corollary 12.6). □

Much of the above seems to work without the one-sorted restriction. In particular, the notion of ample family. So it might be possible to extend the results in this paper to fixed-point quotients of $\mathbb{N}[X_1, \ldots, X_n]$.

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