Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 209, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

RANGE OF SEMILINEAR OPERATORS FOR SYSTEMS AT RESONANCE

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Abstract. For a vector function $u:\mathbb{R}\to\mathbb{R}^N$ we consider the system

$$u''(t)+\nabla G(u(t))=p(t)$$
$$u(t)=u(t+T),$$

where $G:\mathbb{R}^N\to\mathbb{R}$ is a C^1 function. We are interested in finding all possible T-periodic forcing terms $p(t)$ for which there is at least one solution. In other words, we examine the range of the semilinear operator $S:H^2_{\rm per}\to L^2([0,T],\mathbb{R}^N)$ given by $Su=u''+\nabla G(u)$, where

$$H^2_{\rm per}=\{u\in H^2([0,T],\mathbb{R}^N); u(0)-u(T)=u'(0)-u'(T)=0\}.$$

Writing $p(t)=\overline{p}+\widetilde{p}(t)$, where $\overline{p}:=\frac{1}{T}\int_0^T p(t)\,dt$, we present several results concerning the topological structure of the set

$$\mathcal{I}(\widetilde{p})=\{\overline{p}\in\mathbb{R}^N;\overline{p}+\widetilde{p}\in{\rm Im}(S)\}.$$

1. Introduction

Let $G\in C^1(\mathbb{R}^N,\mathbb{R})$. A well known result establishes that if ∇G is bounded, then the Dirichlet problem

$$u''+\nabla G(u)=p(t) \tag{1.1}$$
$$u(0)=u(T)=0 \tag{1.2}$$

has at least one solution for any $p\in L^2([0,T],\mathbb{R}^N)$; that is to say, the operator $S(u):=u''+\nabla G(u)$, regarded as a continuous function from $H^2\cap H^1_0([0,T],\mathbb{R}^N)$ to $L^2([0,T],\mathbb{R}^N)$, is surjective. This is due to the fact that the associated linear operator $Lu:=-u''$ is invertible; thus, a simple proof follows as a straightforward application of Schauder's fixed point theorem. The boundedness condition ensures that the nonlinearity does not interact with the spectrum of L.

The situation is different at resonance, namely when the associated linear operator is non-invertible. In particular, if we consider the periodic problem for (1.1), then integrating we have

$$\frac{1}{T}\int_0^T\nabla G(u(t))\,dt=\overline{p}.$$

2000 *Mathematics Subject Classification.* 34B15, 34L30.

Key words and phrases. Resonant systems; semilinear operators; critical point theory.

Submitted October 7, 2011. Published November 27, 2012.

Thus, the geometric version of the Hahn-Banach Theorem implies that a necessary condition for the existence of solutions is that $\overline{p} \in \mathrm{co}(\mathrm{Im}(\nabla G))$, where 'co' stands for the convex hull. In particular, if we decompose $L^2([0,T],\mathbb{R}^N)$ as the orthogonal sum of $\mathbb{R}^N$ and the set $\widetilde{L}^2$ of zero-average functions; i.e.,

$$L^2([0,T],\mathbb{R}^N) = \mathbb{R}^N \oplus \widetilde{L}^2$$
$$p = \overline{p} + \widetilde{p},$$

with

$$\widetilde{L}^2 := \{v \in L^2([0,T],\mathbb{R}^N);\ \overline{v} = 0\},$$

then the range of S, now defined on H^2_{per}, is contained in $\mathrm{co}(\mathrm{Im}(\nabla G)) \oplus \widetilde{L}^2$. Thus, it is useful to study, for a given $\widetilde{p} \in \widetilde{L}^2$, the set

$$\mathcal{I}(\widetilde{p}) := \{\overline{p} \in \mathbb{R}^N : \overline{p} + \widetilde{p} \in \mathrm{Im}(S)\} \subset \mathrm{co}(\mathrm{Im}(\nabla G)).$$

When ∇G is bounded it can be proven, generalizing the arguments given in [5] for a scalar equation, that $\mathcal{I}(\widetilde{p})$ is non-empty and connected; if ∇G is also periodic, then $\mathcal{I}(\widetilde{p})$ is compact (see e.g. [6]). For example, a quite precise description of this set can be given when the radial limits

$$\lim_{s\to+\infty} \nabla G(sv) := \Gamma(v)$$

exist uniformly for $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. In this case, a well-known result by Nirenberg [9] implies that all the interior points of the field $\Gamma : S^{N-1} \to \mathbb{R}^N$ (i. e. those points $\overline{p}$ such that the winding number of Γ with respect to $\overline{p}$ is nonzero) is contained in $\mathcal{I}(\widetilde{p})$. If also $\mathrm{co}(\mathrm{Im}(\nabla G)) \subset int(\Gamma)$, then the condition $deg(\Gamma,\overline{p}) \neq 0$ is both necessary and sufficient, indeed:

$$\mathrm{Im}(S) = \mathrm{Int}(\Gamma) \oplus \widetilde{L}^2.$$

A different situation occurs when ∇G is unbounded; in particular, $\mathcal{I}(\widetilde{p})$ might be empty. The following result, adapted from the main theorem in [1], is useful to verify that this is not the case if G tends to $+\infty$ or to $-\infty$ as $|u| \to \infty$. More generally:

Theorem 1.1. *Let $G \in C^1(\mathbb{R}^N,\mathbb{R})$, $\widetilde{p} \in \widetilde{L}^2$ and $\overline{p} \in \mathbb{R}^N$. If*

$$\lim_{|u|\to\infty} G(u) - \overline{p}\cdot u = +\infty \quad or \quad \lim_{|u|\to\infty} G(u) - \overline{p}\cdot u = -\infty,$$

then $\overline{p} \in \mathcal{I}(\widetilde{p})$.

In particular, if $G(u) \to +\infty$ or $G(u) \to -\infty$ as $|u| \to \infty$, then $0 \in \mathcal{I}(\widetilde{p})$. Furthermore, if G is strongly convex, in the sense that $G(u) - c|u|^2$ is convex for some constant $c > 0$, then $\mathcal{I}(\widetilde{p}) = \mathbb{R}^N$ and hence S is surjective. The same conclusion is obviously true when G is strongly concave.

Remark 1.2. When $N = 1$, Theorem 1.1 generalizes the well-known Landesman-Lazer conditions. However, although [9] can be regarded as an extension of these conditions, Theorem 1.1 does not necessarily generalize Nirenberg's result.

This paper is organized as follows. In the next section, we prove a basic criterion which ensures that $\overline{p} \in \mathbb{R}^N$ belongs to $\mathcal{I}(\widetilde{p})$ for some given $\widetilde{p}$. In section 3, we give sufficient conditions for a point $\overline{p}_0 \in \mathcal{I}(\widetilde{p})$ to be interior. In section 4, we extend a well known result by Castro [2] for the pendulum equation; more precisely, we prove that if ∇G is periodic then $\mathcal{I}$ regarded as a function from $\widetilde{L}$ to the set of compacts subsets of $\mathbb{R}^N$ (equipped with the Hausdorff metric) is continuous. Finally, in section 5, we prove that if G is strictly convex and satisfies some accurate growth assumptions, then $\mathcal{I}(\widetilde{p}) = \mathrm{Im}(\nabla G)$ for all $\widetilde{p}$.

2. A basic criterion for general G

Proposition 2.1. *Let $\overline{p} \in \mathbb{R}^N$ and define $\psi_{\overline{p}} : \mathbb{R}^N \to \mathbb{R}$ by $\psi_{\overline{p}}(u) := \overline{p} \cdot u - G(u)$. Assume that:*

1. *$\psi_{\overline{p}}$ is bounded from below,*
2. *$\liminf_{|u|\to+\infty} \psi_{\overline{p}}(u) > \inf_{u\in\mathbb{R}^N} \psi_{\overline{p}}(u) + \frac{T}{8\pi^2}\|\widetilde{p}\|^2_{L^2(0,T)}$.*

Then $\overline{p} \in \mathcal{I}(\widetilde{p})$.

Proof. Consider the functional $J : H^1_{\rm per} := \{u \in H^1([0,T],\mathbb{R}^N) : u(0) = u(T)\} \to \mathbb{R}$ given by

$$J(u) := \int_0^T \frac{|u'(t)|^2}{2} + \psi_{\overline{p}}(u(t)) + \widetilde{p}(t)\cdot u(t)\,dt.$$

It is readily seen that J is continuously Fréchet differentiable, and

$$DJ(u)(v) = \int_0^T u'(t)\cdot v'(t) - \nabla G(u(t))\cdot v(t) + p(t)\cdot v(t)\,dt. \tag{2.1}$$

Thus, if u is a minimum of J, u is a weak solution of (1.1), and by standard arguments we deduce that it is classical. Also, it is known that J is weakly lower semicontinuous; thus, due to Theorem 1.1 of [8], it suffices to prove that J has a bounded minimizing sequence. Without loss of generality, we may suppose that $G(0) = 0$.

Claim 1: $-\infty < \inf J \le T \inf \psi_{\overline{p}} \le 0$. Indeed, let us recall the well known Wirtinger inequality:

$$\|u - \overline{u}\|_2^2 \le \big(\frac{T}{2\pi}\big)^2 \|u'\|_2^2. \tag{2.2}$$

From (2.2) and Cauchy-Schwarz inequality we deduce:

$$J(u) \ge \frac{1}{2}\|u'\|_2^2 - \|\widetilde{p}\|_2\|u-\overline{u}\|_2 + \int_0^T \psi_{\overline{p}}(u(t))\,dt.$$

Thus,

$$J(u) \ge \frac{1}{2}\Big(\|u'\|_2 - \frac{T}{2\pi}\|\widetilde{p}\|_2\Big)^2 - \frac{T^2}{8\pi^2}\|\widetilde{p}\|_2^2 + T\inf_{u\in\mathbb{R}^N}\psi_{\overline{p}} \tag{2.3}$$

an the first inequality is proven. For the second inequality, it is sufficient to observe that

$$\inf_{u\in H^1_{\rm per}} J(u) \le \inf_{u\in\mathbb{R}^N} J(u) \le T \inf_{u\in\mathbb{R}^N}\psi_{\overline{p}}(u).$$

The third inequality is obvious since $\psi_{\overline{p}}(0) = 0$.

Next, consider a sequence $(u_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} J(u_n) = \inf J$.

Claim 2: The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{\rm per}$. From the previous claim, for any given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$J(u_n) < T\inf\psi_{\overline{p}} + \epsilon, \quad \text{for all } n \ge n_0. \tag{2.4}$$

Then (2.3) yields

$$\Big(\|u_n'\|_2 - \frac{T}{2\pi}\|\widetilde{p}\|_2\Big)^2 < \frac{T^2}{4\pi^2}\|\widetilde{p}\|_2^2 + 2\epsilon$$

so

$$\|u_n'\|_2^2 < \frac{T}{\pi}\|\widetilde{p}\|_2\|u_n'\|_2 + 2\epsilon.$$

Hence, there exists $\tau > 0$, independent of n, such that $\|u_n'\|_2 \le \frac{T}{\pi}\|\widetilde{p}\|_2 + \tau$.

As before,

$$J(u_n) \ge \frac{1}{2}\Big(\|u_n'\|_2 - \frac{T}{2\pi}\|\widetilde{p}\|_2\Big)^2 - \frac{T^2}{8\pi^2}\|\widetilde{p}\|_2^2 + \int_0^T \psi_{\overline{p}}(u_n(t))\,dt,$$

and from (2.4) we deduce that

$$\int_0^T \psi_{\overline{p}}(u_n(t))\,dt \le \frac{T^2}{8\pi^2}\|\widetilde{p}\|_2^2 + T\inf\psi_{\overline{p}} + \epsilon. \tag{2.5}$$

Suppose that $\|u_n\|_{H^1} \to \infty$, then from the bound for $\|u_n'\|_2$ and the standard inequality

$$\|u_n - \overline{u}_n\|_\infty \le \frac{\sqrt{T}}{2}\|u_n'\|_2$$

we deduce that $|\overline{u}_n| \to \infty$ and $|u_n(t)| \to \infty$ uniformly in t. Thus, for a given $\delta > 0$ there exists $n_1 \ge n_0$ such that $\psi_{\overline{p}}(u_n(t)) \ge \liminf_{|u|\to\infty}\psi_{\overline{p}}(u) - \frac{\delta}{T}$ for all $n \ge n_1$. Hence

$$\int_0^T \psi_{\overline{p}}(u_n(t))\,dt \ge T\liminf_{|u|\to\infty}\psi_{\overline{p}}(u) - \delta \quad \text{for all } n \ge n_1.$$

Then, by (2.5)

$$T\liminf_{|u|\to\infty}\psi_{\overline{p}}(u) \le T\inf\psi_{\overline{p}} + \frac{T^2}{8\pi^2}\|\widetilde{p}\|_2^2 + \epsilon + \delta, \tag{2.6}$$

which contradicts hypothesis 2 when $\epsilon + \delta$ is small enough. So $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{\rm per}$. $\square$

Remark 2.2. In particular, if

$$\liminf_{|u|\to+\infty}\psi_{\overline{p}}(u) - \inf\psi_{\overline{p}} = r > 0,$$

then $\overline{p} \oplus \tilde{B}_r(0) \subset \operatorname{Im}(S)$, where $\tilde{B}_r(0) \subset \widetilde{L}^2$ denotes the open ball of radius r centered at 0.

Example 2.3. Suppose that

$$\limsup_{|u|\to\infty}\frac{G(u)}{|u|} = -R < 0.$$

Then $B_R(0) \subseteq \mathcal{I}(\widetilde{p})$ for any $\widetilde{p}$.

Indeed, if $|\overline{p}| < R$ let $\epsilon = \frac{R-|\overline{p}|}{2}$ and fix r_0 such that $\frac{G(u)}{|u|} < -R + \epsilon$ for $|u| \ge r_0$. Hence

$$\psi_{\overline{p}}(u) = |u|\Big(\frac{u}{|u|}\cdot\overline{p} - \frac{G(u)}{|u|}\Big) > |u|(R - \epsilon - |\overline{p}|) = \epsilon|u| \to +\infty$$

as $|u| \to \infty$ and the result follows. This particular case is obviously covered by Theorem 1.1; however, Proposition 2.1 is still applicable for example if

$$\limsup_{|u|\to\infty}\frac{G(u)}{|u|} \le 0 \quad \text{and} \quad \limsup_{|u|\to\infty, u\in\mathcal{C}}\frac{G(u)}{|u|} = -R < 0$$

with

$$\mathcal{C} := \{u \in \mathbb{R}^N : u\cdot w > -c|u|\}$$

for some $w \in S^{n-1}$ and $c \in (0,1)$. In this case, $I(\widetilde{p})$ contains all the vectors $\overline{p} \in B_R(0)$ such that the angle between $\overline{p}$ and $-w$ is smaller than $\frac{\pi}{2} - \arccos(c)$.

3. Interior points

In this section we give sufficient conditions for a point $\overline{p}_0 \in \mathcal{I}(\widetilde{p})$ to be interior. Roughly speaking, we shall prove that if the Hessian matrix of G does not interact with the spectrum of the operator $Lu := -u''$ then $\mathcal{I}(\widetilde{p})$ is a neighborhood of $\overline{p}_0$. More precisely:

Theorem 3.1. *Let us assume that $G \in C^2(\mathbb{R}^N, \mathbb{R})$ and let $\overline{p}_0 \in \mathcal{I}(\widetilde{p})$ for some $\widetilde{p} \in \widetilde{L}^2$. Further, let u_0 be a solution of (1.1) for $\overline{p} = \overline{p}_0$ and assume there exist symmetric matrices A, $B \in \mathbb{R}^{N\times N}$ such that*

$$A \le d^2G(u_0(t)) \le B \quad t \in [0,T]$$

and

$$\Big(\frac{2\pi N_k}{T}\Big)^2 < \lambda_k \le \mu_k < \Big(\frac{2\pi(N_k+1)}{T}\Big)^2$$

for some integers $N_k \ge 0$, $k = 1, \dots, N$, where $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$ and $\mu_1 \le \mu_2 \le \dots \le \mu_N$ are the respective eigenvalues of A and B. Then there exists an open set $U \subset \mathbb{R}^N$ such that $\overline{p}_0 \in U \subseteq \mathcal{I}(\widetilde{p})$.

The proof relies in the following uniqueness result, which has been proven by Lazer in [7] using a lemma on bilinear forms.

Theorem 3.2. *Let Q be a real $N \times N$ symmetric matrix valued function with elements defined, continuous and 2π-periodic on the real line. Suppose there exist real constant symmetric $A, B \in \mathbb{R}^{N\times N}$ such that*

$$A \le Q(t) \le B, \quad t \in [0, 2\pi], \tag{3.1}$$

and such that if $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$ and $\mu_1 \le \mu_2 \le \dots \le \mu_N$ denote the eigenvalues of A and B respectively and there exist integers $N_k \ge 0$, $k = 1, \dots, N$, such that

$$N_k^2 < \lambda_k \le \mu_k < (N_k+1)^2. \tag{3.2}$$

Then, there exist no non-trivial $2\pi-$periodic solution of the vector differential equation

$$w'' + Q(t)w = 0. \tag{3.3}$$

Proof of Theorem 3.1. Let us consider the operator

$$F : H^2_{\rm per} \times \mathbb{R}^N \to L^2,$$
$$(u, \overline{p}) \mapsto u'' + \nabla G(u) - \widetilde{p} - \overline{p},$$

then clearly $F(u_0, \overline{p}_0) = 0$.

On the other hand, F is Fréchet differentiable, and its differential with respect to u at $(u_0, \overline{p}_0)$ is computed by

$$\begin{aligned} D_uF(u_0,\overline{p}_0)(\varphi) &= \lim_{t\to 0} \frac{F(u_0 + t\varphi, \overline{p}_0) - F(u_0, \overline{p}_0)}{t} \\ &= \lim_{t\to 0} \frac{t\varphi'' + \nabla G(u_0 + t\varphi) - \nabla G(u_0)}{t} \\ &= \varphi'' + \lim_{t\to 0} \frac{\nabla G(u_0 + t\varphi) - \nabla G(u_0)}{t} \\ &= \varphi'' + d^2G(u_0)\varphi \end{aligned}$$

Taking $Q(t) = d^2G(u_0(t))$ in Theorem 3.2 we deduce that $D_uF(u_0, \overline{p}_0) : H^2_{\rm per} \to L^2$ is a monomorphism; furthermore, from the Fredholm Alternative (see e. g. [3]) we conclude that $D_uF(u_0, \overline{p}_0)$ is an isomorphism.

By the Implicit Function Theorem (see [10, Theorem 1.6]), there exists an open neighborhood U of $\overline{p}_0$ and a unique function $u : U \to H^2_{\rm per}$ such that

$$F(u(\overline{p}), \overline{p}) = 0, \quad \text{for all } \overline{p} \in U.$$

Thus $U \subset \mathcal{I}(\widetilde{p})$ and the proof is complete. □

Remark 3.3. A simple computation shows that a similar result is obtained when $d^2G(u_0(t))$ lies at the left of the first eigenvalue. Indeed, it suffices to assume:

(1) $d^2G(u_0(t)) \le 0$ for all t.
(2) There exists $A \subset [0, T]$ with $meas(A) > 0$ such that $d^2G(u_0(t)) < 0$ for $t \in A$.

As before, it suffices to prove that $L\varphi := \varphi'' + d^2G(u_0)\varphi$ is a monomorphism. Suppose that $L\varphi = 0$, then

$$0 = -\int_0^T L\varphi(t) \cdot \varphi(t)\,dt = \int_0^T |\varphi'(t)|^2\,dt - \int_0^T d^2G(u_0(t))\varphi(t) \cdot \varphi(t)\,dt,$$

Then

$$\int_0^T |\varphi'(t)|^2\,dt = \int_0^T d^2G(u_0(t))\varphi(t) \cdot \varphi(t)\,dt \le \int_A d^2G(u_0(t))\varphi(t) \cdot \varphi(t)\,dt,$$

and we conclude that $\varphi \equiv 0$.

The following corollary is immediate.

Corollary 3.4. *Let $\widetilde{p} \in \widetilde{L}^2$ and assume that*

$$d^2G(u) < 0 \ \text{ for all } \ u \in \mathbb{R}^N$$

or that

$$A \le d^2G(u) \le B \ \text{ for all } \ u \in \mathbb{R}^N$$

with A and B as in Theorem 3.1. Then $\mathcal{I}(\widetilde{p})$ is open.

4. Continuity of the function $\mathcal{I}$

In this section we assume that ∇G is periodic and give a characterization of the set $\mathcal{I}(\widetilde{p})$ which, in particular, will allow us to prove the continuity of the function $\mathcal{I} : \widetilde{L} \to \mathcal{K}(\mathbb{R}^N)$, where $\mathcal{K}(\mathbb{R}^N)$ denotes the set of compacts subsets of $\mathbb{R}^N$ equipped with the Hausdorff metric. In fact, we prove a little more.

Theorem 4.1. *Assume that $G \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfies:*

(1) *∇G is periodic, that is: for every $j = 1, \dots, N$ there exists $T_j > 0$ such that $\nabla G(u + T_je_j) = \nabla G(u)$.*
(2) *There exists a discrete set $S \subset \mathbb{R}^N$ such that*

$$(\nabla G(u) - \nabla G(v)) \cdot (u - v) < \Big(\frac{T}{2\pi}\Big)^2 \|u - v\|_2^2 \quad \text{for } u, v \in \mathbb{R}^N \backslash S. \tag{4.1}$$

Then for every $\widetilde{p} \in \widetilde{L}^2$ there exists a periodic function $F_{\widetilde{p}} \in C(\mathbb{R}^N, \mathbb{R}^N)$ such that $\mathcal{I}(\widetilde{p}) = \operatorname{Im}(F_{\widetilde{p}})$. Furthermore, if $\widetilde{p}_n \to \widetilde{p}$ weakly in $\widetilde{L}^2$, then $\mathcal{I}(\widetilde{p}_n) \to \mathcal{I}(\widetilde{p})$ for the Hausdorff metric.

Remark 4.2. In particular, it follows that $\mathcal{I}(\widetilde{p})$ is compact and arcwise connected. This has been proven also by topological methods in [6]. As mentioned in the introduction, we also know that $\mathcal{I}(\widetilde{p}) \subset \operatorname{co}(\operatorname{Im}(\nabla G))$.

For convenience, let us consider the decomposition $H^1_{\rm per} = \mathbb{R}^N \oplus \tilde{H}^1_{\rm per}$, where $\tilde{H}^1_{\rm per} = H^1_{\rm per} \cap \widetilde{L}^2$, and denote the functional defined in section 2 by $J_p : H^1_{\rm per} \to \mathbb{R}$. The proof of Theorem 4.1 shall be based on a series of lemmas. From now on, we shall assume that all the preceding assumptions on G are satisfied.

Lemma 4.3. *For each $x \in \mathbb{R}^N$ and $p \in L^2([0,T],\mathbb{R}^N)$, there exists a unique $\phi(x,p) \in \tilde{H}^1_{\rm per}$ such that*

$$DJ_p(x+\phi(x,p))(v) = 0 \quad \text{for all } v \in \tilde{H}^1_{\rm per}. \tag{4.2}$$

Moreover, the function $\phi(\cdot,p) : \mathbb{R}^N \to \tilde{H}^1_{\rm per}$ is continuous.

Proof. Let us first prove the uniqueness of $\phi(x,p)$. Suppose u_1, $u_2 \in \tilde{H}^1_{\rm per}$ are such that

$$DJ_p(x+u_1)(v) = 0 = DJ_p(x+u_2)(v) \quad \text{for all } v \in \tilde{H}^1_{\rm per}.$$

Taking $v = u_1 - u_2$, using (2.1) it follows that

$$\int_0^T |(u_1-u_2)'|^2\,dt = \int_0^T (\nabla G(x+u_1) - \nabla G(x+u_2)) \cdot (u_1-u_2)\,dt. \tag{4.3}$$

This fact, (2.2) and (4.1) imply that $u_1 = u_2$.

Next we prove the existence of $\phi(x,p)$. Let $I_x : \tilde{H}^1_{\rm per} \to \mathbb{R}$ be the functional defined by $I_x(v) = J_p(x+v)$, then

$$\begin{aligned} I_x(v) &= \frac{1}{2}\|v'\|_2^2 + \int_0^T p(t)\cdot(x+v(t)) - G(x+v(t))\,dt \\ &\geq \frac{1}{2}\|v'\|_2^2 - \|\widetilde{p}\|_2\|v\|_2 + T(\overline{p}\cdot x - \|\nabla G\|_\infty) \end{aligned} \tag{4.4}$$

It follows that I_x is coercive and hence it achieves an absolute minimum, which satisfies (4.2).

Finally, let $x_n \to x$ and suppose that $\phi(x_n,p) \not\to \phi(x,p)$. From (4.4), the sequence $(\phi(x_n,p))_n$ is bounded in $\tilde{H}^1_{\rm per}$. Taking a subsequence, if necessary, we may assume that it converges weakly to some $w \in H^1_{\rm per}$, uniformly and $\|\phi(x_n,p) - \phi(x,p)\|_{H^1} \geq \epsilon > 0$ for all n. Passing to the limit in the equalities

$$DJ_p(x_n+\phi(x_n,p))(v) = 0 \quad \text{for all } v \in \tilde{H}^1_{\rm per}$$

we deduce that $DJ_p(x+w)(v) = 0$ for all $v \in \tilde{H}^1_{\rm per}$ and hence $w = \phi(x,p)$. Moreover, as

$$J_p(x_n+\phi(x_n,p)) \leq J_p(x_n+\phi(x,p)) \quad \text{and} \quad J_p(x+\phi(x,p)) \leq J_p(x+\phi(x_n,p))$$

for all n, we deduce that

$$\limsup_{n\to\infty} \int_0^T |\phi(x_n,p)'|^2\,dt \leq \int_0^T |\phi(x,p)'|^2\,dt \leq \liminf_{n\to\infty} \int_0^T |\phi(x_n,p)'|^2\,dt$$

and hence $\|\phi(x_n,p)'\|_2 \to \|\phi(x,p)'\|_2$. Thus,

$$\|\phi(x_n,p)' - \phi(x,p)'\|_2^2 = \|\phi(x_n,p)'\|_2^2 + \|\phi(x,p)'\|_2^2 - 2\int_0^T \phi(x_n,p)'\cdot\phi(x,p)'\,dt \to 0$$

as $n \to \infty$, which contradicts the fact that $\phi(x_n,p) \not\to \phi(x,p)$. ∎

Lemma 4.4. *The function $\phi(\cdot, p)$ depends only on $\widetilde{p}$.*

Proof. Let $c \in \mathbb{R}^N$, then

$$\begin{aligned} DJ_{p+c}(x+\phi(x,p))(v) &= \int_0^T \phi(x,p)' \cdot v' - \nabla G(x+\phi(x,p)) \cdot v + (p+c) \cdot v \, dt \\ &= \int_0^T \phi(x,p)' \cdot v' - \nabla G(x+\phi(x,p)) \cdot v + p \cdot v \, dt = 0 \end{aligned}$$

for all $v \in \tilde{H}^1_{\rm per}$. From uniqueness, we deduce that $\phi(\cdot, p) = \phi(\cdot, p + c)$. □

Let us denote by $\tilde{J}_p : \mathbb{R}^N \to \mathbb{R}$ the function defined by

$$\tilde{J}_p(x) = J_p(x + \phi(x,p)).$$

It is readily seen that $\tilde{J}_p \in C^1(\mathbb{R}^N, \mathbb{R})$ and

$$\nabla \tilde{J}_p(x) \cdot y = DJ_p(x+\phi(x,p))(y+v) \quad \text{for all } y \in \mathbb{R}^N,\ v \in \tilde{H}^1_{\rm per}. \tag{4.5}$$

The following lemma will allow us to reduce the problem of finding a critical point in $H^1_{\rm per}$ to a finite-dimensional problem.

Lemma 4.5. *The element $x + v \in \mathbb{R}^N \oplus \tilde{H}^1_{\rm per}$ is a critical point of J_p if and only if $v = \phi(x,p)$ and x is a critical point of $\tilde{J}_p$.*

Proof. By Lemma 4.3, if $x + v$ is a critical point of J_p, then $v = \phi(x,p)$. From (4.5), $\nabla \tilde{J}_p(x) \cdot y = 0$ for every $y \in \mathbb{R}^N$ and hence x is a critical point of $\tilde{J}_p$.

Conversely, suppose $v = \phi(x,p)$ and $\nabla \tilde{J}_p(x) = 0$. For $u \in H^1_{\rm per}$, let us write $u = \overline{u} + \tilde{u}$ with $\overline{u} \in \mathbb{R}^N$ and $\tilde{u} \in \tilde{H}^1_{\rm per}$. Then

$$DJ_p(x+v)(u) = DJ_p(x+\phi(x,p))(\overline{u}+\tilde{u}) = \nabla \tilde{J}_p(x) \cdot \overline{u} = 0,$$

so $x + v$ is a critical point of J_p. □

Lemma 4.6. *The function $\phi(\cdot, p)$ is periodic.*

Proof. Let $x \in \mathbb{R}^N$. From the periodicity of ∇G we deduce that

$$DJ_p(x + T_j e_j + \phi(x,p))(v) = DJ_p(x+\phi(x,p))(v) = 0$$

for all $v \in \tilde{H}^1_{\rm per}$. By Lemma 4.3, $\phi(x + T_j e_j, p) = \phi(x,p)$. □

The following proposition will provide the proof of Theorem 4.1.

Proposition 4.7. *Let $\widetilde{p} \in \widetilde{L}^2$ and define the function $F_{\widetilde{p}} : \mathbb{R}^N \to \mathbb{R}^N$ by*

$$F_{\widetilde{p}}(x) = \int_0^{2\pi} \nabla G(x + \phi(x, \widetilde{p}(t)))\, dt.$$

Then $F_{\widetilde{p}}$ is continuous and $\mathcal{I}(\widetilde{p}) = \operatorname{Im}(F_{\widetilde{p}})$. Moreover, if $\widetilde{p}^n \in \widetilde{L}^2$ converges weakly to some $\widetilde{p} \in \widetilde{L}^2$, then $\mathcal{I}(\widetilde{p}^n)$ converges to $\mathcal{I}(\widetilde{p})$ for the Hausdorff topology.

Proof. The continuity of $F_{\widetilde{p}}$ is clear from the continuity of $\phi(\cdot, \widetilde{p})$ and the embedding $\tilde{H}^1_{\rm per} \hookrightarrow C([0,T], \mathbb{R}^N)$. Let us prove that $\mathcal{I}(\widetilde{p}) = \operatorname{Im}(F_{\widetilde{p}})$. According to (4.5), Lemma 4.4 and Lemma 4.5, problem (1.1) has a weak solution if and only if there exists $x \in \mathbb{R}^N$ such that $DJ_p(x+\phi(x,\widetilde{p}))(y)=0$ for every $y \in \mathbb{R}^N$. From (2.1), this is equivalent to

$$0=\int_0^T -\nabla G(x+\phi(x,\widetilde{p}))\cdot y + p\cdot y\,dt = y\cdot\int_0^T -\nabla G(x+\phi(x,\widetilde{p}))+\overline{p}\,dt$$

for all $y \in \mathbb{R}^N$. Thus, the problem has a solution if and only if

$$\overline{p}=\int_0^T \nabla G(x+\phi(x,\widetilde{p}))\,dt,$$

for some $x \in \mathbb{R}^N$; that is, $\overline{p} \in \operatorname{Im}(F_{\widetilde{p}})$.

Finally, suppose that $\widetilde{p}^n \to \widetilde{p}$ weakly in $\widetilde{L}^2$ and denote $J_n := J_{\widetilde{p}^n}$, $J := J_{\widetilde{p}}$, $\phi_n(\cdot) := \phi(\cdot,\widetilde{p}^n)$, $\phi(\cdot) := \phi(\cdot,\widetilde{p})$, $F_n := F_{\widetilde{p}^n}$ and $F := F_{\widetilde{p}}$.

We claim that $F_n \to F$ pointwise. Indeed, for fixed $x \in \mathbb{R}^N$ proceeding as in the proof of Lemma 4.3 it is easy to see that if $n \to \infty$, then $\phi_n(x) \to \phi(x)$. As ∇G is continuous, we deduce from the Lebesgue's dominated convergence theorem that $F_n(x) \to F(x)$.

To prove that $\mathcal{I}(\widetilde{p}^n) \to \mathcal{I}(\widetilde{p})$ as $n \to \infty$ for the Hausdorff topology, we need to see that:

(i) $\sup_{\overline{q}^n\in\mathcal{I}(\widetilde{p}^n)} \operatorname{dist}(\overline{q}^n,\mathcal{I}(\widetilde{p})) \to 0$,
(ii) $\sup_{\overline{q}\in\mathcal{I}(\widetilde{p})} \operatorname{dist}(\overline{q},\mathcal{I}(\widetilde{p}^n)) \to 0$.

For (i), denote $S_n = \sup_{\overline{q}^n\in\mathcal{I}(\widetilde{p}^n)} \operatorname{dist}(\overline{q}^n,\mathcal{I}(\widetilde{p}))$ and let $\overline{p}^n \in \mathcal{I}(\widetilde{p}^n)$ be chosen in such a way that $\operatorname{dist}(\overline{p}^n,\mathcal{I}(\widetilde{p})) \ge S_n - \frac{1}{n}$. We shall prove that $\operatorname{dist}(\overline{p}^n,\mathcal{I}(\widetilde{p})) \to 0$. By contradiction, suppose there exists a subsequence, still denoted $\{\overline{p}^n\}$, such that

$$\operatorname{dist}(\overline{p}^n,\mathcal{I}(\widetilde{p})) \ge \epsilon > 0. \tag{4.6}$$

Moreover, we know that $\mathcal{I}(\widetilde{p}) \subset \operatorname{co}(\operatorname{Im}(\nabla G))$; in particular, taking a convergent subsequence if necessary we may suppose that $\overline{p}^n \to \overline{p}$ for some $\overline{p} \in \mathbb{R}^N$. For each n, let $u_n \in H^1_{\rm per}$ be a solution of the problem for $\overline{p}^n$. From the periodicity of ∇G, we may assume that the sequence $\{\overline{u}_n\}$ is bounded in $\mathbb{R}^N$. Thus, $\{u_n\}$ is bounded in $H^1_{\rm per}$ and

$$\int_0^T u_n'\cdot v' - \nabla G(u_n)\cdot v + (\widetilde{p}^n+\overline{p}^n)\cdot v\,dt = 0 \tag{4.7}$$

for all $v \in H^1_{\rm per}$. Taking again a subsequence, we may assume that $u_n \to u_0$ weakly in $H^1_{\rm per}$ and hence

$$\int_0^T u_0'\cdot v' - \nabla G(u_0)\cdot v + (\widetilde{p}+\overline{p})\cdot v\,dt = 0$$

for all $v \in H^1_{\rm per}$. Then u_0 is a weak solution of (1.1) with $p=\widetilde{p}+\overline{p}$ and $\overline{p} \in \mathcal{I}(\widetilde{p})$, which contradicts (4.6). Thus $\operatorname{dist}(\overline{p}^n,\mathcal{I}(\widetilde{p})) \to 0$ and consequently $S_n \to 0$.

Next we prove (ii). Denote now $S_n = \sup_{\overline{q}\in\mathcal{I}(\widetilde{p})} \operatorname{dist}(\overline{q},\mathcal{I}(\widetilde{p}^n))$ and take $\overline{q}^n \in \mathcal{I}(\widetilde{p})$ such that $\operatorname{dist}(\overline{q}^n,\mathcal{I}(\widetilde{p}^n)) \ge S_n - \frac{1}{n}$. As before, suppose there exists a subsequence, still denoted $\{\overline{q}^n\}$, such that

$$\operatorname{dist}(\overline{q}^n,\mathcal{I}(\widetilde{p}^n)) \ge \epsilon > 0. \tag{4.8}$$

Passing to a subsequence if necessary, there exist $\overline{q} \in \mathcal{I}(\widetilde{p}) = \operatorname{Im}(F)$ and $n_1 \in \mathbb{N}$ such that $\operatorname{dist}(\overline{q}^n,\overline{q}) < \frac{\epsilon}{2}$ for all $n \ge n_1$. Fix $x_0 \in \mathbb{R}^N$ such that $F(x_0)=\overline{q}$ and let $\overline{p}^n = F_n(x_0) \in \mathcal{I}(\widetilde{p}^n)$. As $F_n(x_0) \to F(x_0)$, there exists $n_2 \in \mathbb{N}$ such that $\operatorname{dist}(\overline{p}^n, \overline{q}) < \frac{\epsilon}{2}$ for all $n \geq n_2$. Take $n_0 = \max\{n_1, n_2\}$ and hence

$$\operatorname{dist}(\overline{q}^n, \mathcal{I}(\widetilde{p}^n) \leq \operatorname{dist}(\overline{q}^n, \overline{p}^n) \leq \operatorname{dist}(\overline{q}^n, \overline{q}) + \operatorname{dist}(\overline{q}, \overline{p}^n) < \epsilon$$

for $n \geq n_0$. This contradicts (4.8), so we conclude that $S_n \to 0$. $\square$

5. Characterization of $\mathcal{I}$ for convex G

In this section, we shall assume that G is a strictly convex function, namely

$$G(sx + (1-s)y) < sG(x) + (1-s)G(y) \quad \text{for all } s \in (0,1),\ x, y \in \mathbb{R}^N.$$

Our main result reads as follows.

Theorem 5.1. *Assume that:*

1. *There exist $\alpha < \left(\frac{T}{2\pi}\right)^2$ and $\beta \in \mathbb{R}$ such that*
$$G(u) \leq \frac{\alpha}{2}|u|^2 + \beta \quad \text{for all } u \in \mathbb{R}^N \tag{5.1}$$
2. *For every $a \in \mathbb{R}^N$ there exists $r_0 > 0$ such that*
$$\frac{\partial G}{\partial w}(rw + x) \geq \frac{\partial G}{\partial w}(a) \tag{5.2}$$
for all $r \geq r_0$, $w \in S^{n-1}$ and $|x| \leq C$, where $C = C(a, \widetilde{p})$ is the constant defined below in (5.7).

Then $\mathcal{I}(\widetilde{p}) = \operatorname{Im}(\nabla G)$.

Proof. Firstly, we shall prove the inclusion $\operatorname{Im}(\nabla G) \subseteq \mathcal{I}(\widetilde{p})$. For simplicity, from the rescaling $v(t) = u(\frac{T}{2\pi}t)$ we may assume that $T = 2\pi$. Let $K : \widetilde{L}^2 \to H^2 \cap \widetilde{L}^2$ be the inverse of the operator $Lu := u''$, namely $Kh = u$, where u is the unique solution of the problem

$$\begin{gathered} u'' = h \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \\ \overline{u} = 0. \end{gathered}$$

Claim 1: $\int_0^{2\pi} Kh(t) \cdot h(t)\,dt + \int_0^{2\pi} |h(t)|^2\,dt \geq 0$. Indeed, from (2.2) it is seen that

$$\int_0^{2\pi} |(Kh)'(t)|^2\,dt = -\int_0^{2\pi} Kh(t) \cdot h(t)\,dt \leq \|(Kh)'\|_2 \|h\|_2,$$

which implies that $\|(Kh)'\|_2 \leq \|h\|_2$, and hence

$$-\int_0^{2\pi} Kh(t) \cdot h(t)\,dt = \int_0^{2\pi} |(Kh)'(t)|^2\,dt \leq \|h\|_2^2.$$

For $\overline{p} \in \operatorname{Im}(\nabla G)$, fix $a \in \mathbb{R}^N$ such that $\nabla G(a) = \overline{p}$, and define the functions

$$F(t, u) := G(u) - p(t) \cdot u;$$

and, for given $\epsilon > 0$,

$$F_\epsilon(t, u) := G(u) - p(t) \cdot u + \frac{\epsilon}{2}|u|^2$$

where $p(t) = \widetilde{p}(t) + \overline{p}$. Next, consider the Fenchel transform F_ϵ^* of the function F_ϵ defined as

$$F_\epsilon^*(t, v) = \max_{w \in \mathbb{R}^N} \left(v \cdot w - F_\epsilon(t, w)\right). \tag{5.3}$$

Observe that F_ϵ^* is well defined, since F_ϵ is strongly concave; hence a unique global maximum w is achieved and satisfies the following properties:

(1) $v = \nabla F_\epsilon(t, w)$,
(2) $w = \nabla F_\epsilon^*(t, v)$,
(3) $v \cdot w - F_\epsilon(t, w) = F_\epsilon^*$.

Properties 1 and 2 are known as *Fenchel duality* (see [8]).

Define the functional $I_\epsilon : \widetilde{L}^2 \to \mathbb{R}$ given by

$$I_\epsilon(v) = \int_0^{2\pi} \frac{1}{2} K v(t) \cdot v(t) + F_\epsilon^*(t, v(t))\, dt.$$

From (5.3) and (5.1),

$$\begin{aligned} F_\epsilon^*(t, v) \geq |v|^2 - F_\epsilon(t, v) &= |v|^2 + p \cdot v - \frac{\epsilon}{2}|v|^2 - G(v) \\ &\geq |v|^2 + p \cdot v - \frac{\epsilon + \alpha}{2}|v|^2 - \beta \end{aligned}$$

and using Claim 1, Cauchy-Schwarz Inequality and the fact that $v \in \widetilde{L}^2$ we deduce:

$$I_\epsilon(v) \geq -\frac{1}{2}\int_0^{2\pi} |v(t)|^2\, dt + \int_0^{2\pi} |v(t)|^2 + \widetilde{p}(t) \cdot v(t) - \frac{\epsilon + \alpha}{2}|v(t)|^2 - \beta\, dt;$$

that is,

$$I_\epsilon(v) \geq \frac{1 - \alpha - \epsilon}{2}\|v\|_2^2 - \|\widetilde{p}\|_2\|v\|_2 - 2\pi\beta. \tag{5.4}$$

Thus I_ϵ is coercive for $\epsilon < 1 - \alpha$ and hence it achieves a minimum u_ϵ. As K is self-adjoint, it is easy to verify that

$$\int_0^{2\pi} [K u_\epsilon(t) + \nabla F_\epsilon^*(t, u_\epsilon(t))] \cdot \varphi(t)\, dt = 0, \quad \text{for all } \varphi \in \widetilde{L}^2.$$

Then $K(u_\epsilon) + \nabla F_\epsilon^*(s, u_\epsilon) = A \in \mathbb{R}^N$. Let $v_\epsilon = \nabla F_\epsilon^*(s, u_\epsilon) = A - K(u_\epsilon)$, then by the Fenchel duality $u_\epsilon = \nabla F_\epsilon(s, v_\epsilon)$. In other words, $u_\epsilon = \nabla G(v_\epsilon) - p(t) + \epsilon v_\epsilon$.

On the other hand, $v_\epsilon'' = (-K(u_\epsilon))'' = -u_\epsilon$; hence, v_ϵ satisfies

$$\begin{gathered} v_\epsilon'' + \nabla G(v_\epsilon) + \epsilon v_\epsilon = p(t) \\ v_\epsilon(0) = v_\epsilon(2\pi), \quad v_\epsilon'(0) = v_\epsilon'(2\pi). \end{gathered} \tag{5.5}$$

Moreover, if F^* denotes the Legendre transform of F defined by

$$F^*(t, v) = \sup_{w \in \mathbb{R}^N} (v \cdot w - F_\epsilon(t, w))$$

then it is obvious that $F_\epsilon^* \leq F^*$. As u_ϵ is the minimum, it follows that

$$I_\epsilon(u_\epsilon) \leq I_\epsilon(-\widetilde{p}) = \int_0^{2\pi} \frac{1}{2} K\widetilde{p}(t) \cdot \widetilde{p}(t) + F^*(t, -\widetilde{p}(t))\, dt. \tag{5.6}$$

For fixed t, let $\Psi(y) := -\widetilde{p} \cdot y - F(t, y) = \overline{p} \cdot y - G(y)$, then

$$\nabla\Psi(y) = -\widetilde{p} - \nabla F(t, y) = \overline{p} - \nabla G(y).$$

Thus, a is a critical point of Ψ and, as Ψ is strictly concave, we conclude that a is the absolute maximum. Then

$$-\widetilde{p} \cdot a - F(t, a) = \max_{w \in \mathbb{R}^N} (-\widetilde{p}(t) \cdot w - F(t, w)) = F^*(t, -\widetilde{p}(t)).$$

Hence, from (5.6) and the fact that $\widetilde{p} \in \widetilde{L}^2$ we obtain:

$$I_\epsilon(u_\epsilon) \le \int_0^{2\pi} \frac{1}{2} K\widetilde{p}(t) \cdot \widetilde{p}(t) - F(t,a)\, dt = 2\pi\,(a \cdot \nabla G(a) - G(a)) - \frac{1}{2}\|(K\widetilde{p})'\|_2^2.$$

Fixing $c < (1-\alpha)/2$, we conclude from (5.4) that if ϵ is small enough then

$$c\|u_\epsilon\|_2^2 - \|\widetilde{p}\|_2\|u_\epsilon\|_2 \le 2\pi\,(a \cdot \nabla G(a) - G(a) + \beta) - \frac{1}{2}\|(K\widetilde{p})'\|_2^2.$$

As $v_\epsilon'' = -u_\epsilon$, it follows that $\tilde{v}_\epsilon$ is bounded for the H^2 norm; in particular,

$$\|v_\epsilon\|_\infty \le C \tag{5.7}$$

for some constant C, depending only on $\widetilde{p}$ and a.

Let us prove now that $\overline{v}_\epsilon$ is bounded. By direct integration of (5.5) we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi} \nabla G(v_\epsilon(t))dt + \epsilon\overline{v}_\epsilon = \overline{p}. \tag{5.8}$$

Writing $\overline{v}_\epsilon = rw$, where $r = |\overline{v}_\epsilon|$ and $|w| = 1$, and multiplying (5.8) by w, we obtain

$$\epsilon r + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial G}{\partial w}(rw + \tilde{v}_\epsilon(t))\, dt = \overline{p} \cdot w = \nabla G(a) \cdot w = \frac{\partial G}{\partial w}(a).$$

As $|\tilde{v}_\epsilon(t)| \le C$, for $r \ge r_0$ inequality (5.2) yields:

$$0 = \epsilon r + \frac{1}{2}\int_0^{2\pi} \Big(\frac{\partial G}{\partial w}(rw + \tilde{v}_\epsilon(t)) - \frac{\partial G}{\partial w}(a)\Big)dt \ge \epsilon r,$$

a contradiction. So, $|\overline{v}_\epsilon| \le r_0$ and v_ϵ is bounded for the H^2 norm.

From the compact embedding $H^2([0,2\pi],\mathbb{R}^N) \hookrightarrow C^1([0,2\pi],\mathbb{R}^N)$, there exists a sequence $\{v_{\epsilon_n}\}_{n\in\mathbb{N}}$ that converges in $C^1([0,2\pi],\mathbb{R}^N)$ to some function v. From (5.5),

$$\int_0^{2\pi} \Big(v_{\epsilon_n}''(t) + \nabla G(v_{\epsilon_n}(t)) + \epsilon_n v_{\epsilon_n}(t)\Big) \cdot \varphi(t)\, dt = \int_0^{2\pi} p(t) \cdot \varphi(t)\, dt \quad \forall\, \varphi \in \widetilde{L}^2.$$

Integrating by parts and passing to the limit, we obtain:

$$-\int_0^{2\pi} v'(t) \cdot \varphi'(t)\, dt + \int_0^{2\pi} \nabla G(v(t)) \cdot \varphi(t)\, dt = \int_0^{2\pi} p(t) \cdot \varphi(t)\, dt.$$

Then v is a solution of (1.1).

Finally, let us prove that $\mathcal{I}(\widetilde{p}) \subseteq \mathrm{Im}(\nabla G)$. As previously mentioned, we know that $\mathcal{I}(\widetilde{p}) \subseteq \mathrm{co}(\mathrm{Im}(\nabla G))$, so it remains to see that $\mathrm{Im}(\nabla G)$ is convex.

Claim 2: If $F \in C^1(\mathbb{R}^N,\mathbb{R})$ is strictly convex, then

$$0 \in \mathrm{Im}(F) \quad \iff \lim_{|x|\to+\infty} F(x) = +\infty.$$

The sufficiency is obvious. In order to prove the necessity, assume that $\nabla F(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$ and for each $w \in S^{n-1}$ define $\Phi_w(t) := \frac{\partial F}{\partial w}(x_0 + tw)$. From the convexity of F we deduce that Φ_w is strictly increasing. Furthermore, the function $\Phi : S^{n-1} \times [0,+\infty) \to \mathbb{R}$ given by $\Phi(w,t) := \Phi_w(t)$ is continuous and $\Phi(w,1) > 0$ for all $w \in S^{n-1}$. Hence, there exists a constant $c > 0$, such that $\Phi_w(1) \ge c > 0$ for all $w \in S^{n-1}$. As Φ_w is strictly increasing, we conclude that $\Phi_w(t) > c$ for all $t > 1$. Thus,

$$F(x_0 + Rw) - F(x_0 + w) = R\nabla F(x_0 + \xi w) \cdot w = R\frac{\partial F}{\partial w}(x_0 + \xi w) \ge cR.$$

We conclude that $F(x_0 + Rw) \geq F(x_0 + w) + cR$ and the claim is proved.

Next, let us consider $y_1,\, y_2 \in Im(\nabla G)$ and $y = a_1 y_1 + a_2 y_2$, with $a_1 + a_2 = 1$ and $a_1,\, a_2 \geq 0$. Define

$$F(x) = G(x) - y \cdot x = a_1\,(G(x) - y_1 \cdot x) + a_2(G(x) - y_2 \cdot x).$$

As $G(x) - y_1 \cdot x$ and $G(x) - y_2 \cdot x$ are strictly convex, it follows from Claim 2 that both functions tend to $+\infty$ as $|x| \to \infty$, and hence

$$\lim_{|x| \to +\infty} F(x) = +\infty. \tag{5.9}$$

Using Claim 2 again, (5.9) implies that $0 \in \mathrm{Im}(\nabla F) = \mathrm{Im}(\nabla G - y)$, then $y \in \mathrm{Im}(\nabla G)$ and so completes the proof. □

Acknowledgments. This research was partially supported by the projects PIP 11220090100637 from CONICET, and UBACyT 20020090100067. The authors want to thank J. Haddad for his fruitful comments.

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