ON CYCLES OF LENGTH THREE

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Abstract. We prove that if A is a string algebra then there are not three irreducible morphisms between indecomposable A-modules such that its composition belongs to $\Re^6\backslash\Re^7$, whenever the compositions of two of them are not in $\Re^3$. Moreover, for any positive integer $n \geq 3$, we show that there are n irreducible morphisms such that their composition is in $\Re^{n+4}\backslash\Re^{n+5}$.

Introduction

Introduced by Auslander and Reiten in the early 70's, the notion of irreducible morphisms plays an important role in the representation theory of artin algebras.

It is well-known that the composition of n irreducible morphisms between indecomposable modules over an artin algebra A belongs to $\Re^n$, the n-th power of the radical $\Re$ of the module category. Such a composition could be a non-zero morphism in $\Re^{n+1}$. This is still a problem of interest in the representation theory of artin algebras, and in the last years, there have been some advances in such a direction, see for example [7], [9], and [11].

In [9], Coelho, Trepode and the first named author characterized when the composition of two irreducible morphisms is non-zero and belongs to $\Re^3$. Moreover, they proved that if two irreducible morphisms between indecomposable A-modules such that their composition is non-zero and belongs to a greater power of the radical, greater than two, then such composition is at least in $\Re^4$.

Later in [1], Alvares and Coelho proved that if f and g are irreducible morphisms between indecomposable A-modules such that $0 \neq fg \in \Re^3$ then $fg \in \Re^5$. Furthermore, they showed an example of two irreducible morphisms whose composition is in $\Re^5\backslash\Re^6$. To prove such a result they used a result due to Hoshino, proved in [14], that if a module X in Γ_A is such that $DTrX = X$, then either the connected component of Γ_A which contains X is a homogeneous stable tube or A is a local Nakayama algebra.

Finally, in [8], the first named author generalized the result proven in [1]. Precisely, the author proved that given an artin algebra A where the configurations of almost split sequences have at most two indecomposable middle terms, then the non-zero composition of n irreducible morphisms on a left almost pre-sectional path is such that it belongs to $\Re^{n+3}$ for $n \geq 1$.

As a consequence of the above mentioned result, for any artin algebra, we know that if the non-zero composition of any three irreducible morphisms h_i between indecomposable A-modules, is such that $h_3h_2h_1 \in \Re^4$, $h_3h_2 \notin \Re^3$ and $h_2h_1 \notin \Re^3$ then $h_3h_2h_1 \in \Re^6$.

A natural question now is if the composition of three irreducible morphisms between indecomposable A-modules can be in $\Re^6\backslash\Re^7$, whenever the composition of any two of them are not in $\Re^3$, that is, behaves well.

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In this work, we prove that if A is a string algebra then there are not three irreducible morphisms such that their composition is in $\Re^6\backslash\Re^7$, if the composition of any two of them is not in $\Re^3$. Furthermore, for a string algebras we prove that the minimum for three irreducible morphisms in such a condition is seven.

We also find families of algebras where their module category have n irreducible morphisms between indecomposable modules such that its composition belongs to $\Re^{n+4}\backslash\Re^{n+5}$ for $n \geq 3$, whenever the compositions of $n-1$ of them belong to $\Re^{n-1}\backslash\Re^n$. It is still an open problem to see if the minimum n is equal to $n+3$, for $n \geq 3$.

The paper is organized as follows. The first section is dedicated to recall some preliminaries definitions and results. In section 2, we prove some general results concerning algebras which have cycles of length three. In section 3, we present some strings algebras that contains irreducible morphism from M to τM, for M an indecomposable A-module. In Section 4, we prove some technical lemmas and apply the results of the previous sections to prove that if we consider a string algebra there are not three irreducible morphisms such that their composition is in $\Re^6\backslash\Re^7$ whenever the composition of any two of them behaves well. Finally, in the last section we give families of algebras having n irreducible morphisms such that their composition belongs to $\Re^{n+4}\backslash\Re^{n+5}$, for $n \geq 3$ and such that the composition of $n-1$ of them behaves well.

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1. Preliminaries

1.1. A **quiver** Q is given by a set of vertices Q_0 and a set of arrows Q_1, together with two maps $s, e : Q_1 \to Q_0$. Given an arrow $\alpha \in Q_1$, we write $s(\alpha)$ the starting vertex of α and $e(\alpha)$ the ending vertex of α. For each arrow $\alpha \in Q_1$ we denote by α^{-1} its formal inverse, where $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$.

A **walk** in Q is a concatenation $c_1 \dots c_n$, with $n \geq 1$, such that c_i is either an arrow or the inverse of an arrow, and $e(c_i) = s(c_{i+1})$. We say that $c_1 \dots c_n$ is a **reduced walk** provided $c_i \neq c_{i+1}^{-1}$ for each i, $1 \leq i \leq n-1$.

If A is an algebra then there exists a quiver Q_A, called the **ordinary quiver of** A, such that A is the quotient of the path algebra kQ_A by an admissible ideal.

1.2. Let A be an artin algebra. We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ which consists of one representative of each isomorphism class of indecomposable A-modules.

Let X be a non-projective (non-injective) indecomposable A-module. By $\alpha(X)$ ($\alpha'(X)$, respectively) we denote the number of indecomposable summands in the middle term of an almost split sequence ending (starting, respectively) at X. We say that $\alpha(\Gamma) \leq 2$ if $\alpha(X)$ and $\alpha'(X)$ are less than or equal to two, whenever they are defined.

1.3. A morphism $f : X \to Y$, with $X, Y \in \operatorname{mod} A$, is called **irreducible** provided it does not split and whenever $f = gh$, then either h is a split monomorphism or g is a split epimorphism.

If $X, Y \in \operatorname{mod} A$, the ideal $\Re(X, Y)$ is the set of all the morphisms $f : X \to Y$ such that, for each $M \in \operatorname{ind} A$, each $h : M \to X$ and each $h' : Y \to M$ the composition $h'fh$ is not an isomorphism. For $n \geq 2$, the powers of $\Re(X, Y)$ are defined inductively. By $\Re^\infty(X, Y)$ we denote the intersection of all powers $\Re^i(X, Y)$ of $\Re(X, Y)$, with $i \geq 1$.

By [4], it is well-known that a morphism $f : X \to Y$, with $X, Y \in \operatorname{ind} A$, is irreducible if and only if $f \in \Re(X, Y) \setminus \Re^2(X, Y)$.

We recall the definition of degree of an irreducible morphism given by S. Liu in [15].

Let $f : X \to Y$ be an irreducible morphism in $\operatorname{mod} A$, with X or Y indecomposable. The **left degree** $d_l(f)$ of f is infinite, if for each integer $n \geq 1$, each module $Z \in \operatorname{ind} A$ and each morphism $g : Z \to X$ with $g \in \Re^n(Z, X)\backslash\Re^{n+1}(Z, X)$ we have that $fg \notin \Re^{n+2}(Z, Y)$. Otherwise, the left degree of f is the least natural number m such that there is an A-module Z and a morphism $g : Z \to X$ with $g \in \Re^m(Z, X)\backslash\Re^{m+1}(Z, X)$ such that $fg \in \Re^{m+2}(Z, Y)$.

The **right degree** $d_r(f)$ of an irreducible morphism f is dually defined.

We denote by Γ_A its Auslander-Reiten quiver, by τ the Auslander-Reiten translation, and τ^{-1} its inverse.

Let $X \to Y$ be an arrow in Γ_A. Assume that $f : X \to Y$ is an irreducible morphism in $\operatorname{mod} A$. Following [15], we define the left degree of the arrow $X \to Y$ to be $d_l(f)$, and the right degree of the arrow $X \to Y$ to be $d_r(f)$.

Lemma 1.4. *Let A be a finite dimensional k-algebra. Any cycle of irreducible morphisms between indecomposable A-modules has both a monomorphism and an epimorphism.*

Proof. By [15, Lemma 2.2], we know that every oriented cycle in Γ_A contains both an arrow of finite left degree and an arrow of finite right degree.

By [13, Corollary 3.2], the arrows of finite left degree and the ones of finite right degree correspond to irreducible epimorphisms and monomorphisms, respectively. Then we get the result. □

An indecomposable A-module M is **left (right) τ-stable** if for all positive integer n the module $\tau^n M$ ($\tau^{-n} M$) is defined. An indecomposable A-module M is **τ-stable** if it is both left and right τ-stable.

In particular, if a τ-stable module M satisfy that $\tau^m M \simeq M$ for some positive integer m, then we say that M is **τ-periodic**. Moreover, M is τ-periodic of rank m if $\tau^m M \simeq M$ and $\tau^k M \not\simeq M$ for ll $1 \leq k < m$.

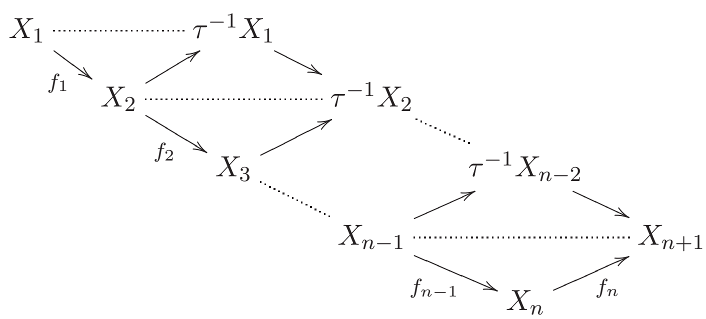
A path $M_1 \to M_2 \to \ldots \to M_n$ of irreducible morphisms with $M_j \in \operatorname{ind} A$ for $j = 1, ..., n$ and $n \geq 3$ is called **sectional** if for each $j = 3, ..., n$ we have that $M_{j-2} \not\simeq \tau M_j$.

A path $Y_0 \to Y_1 \to \cdots \to Y_n$ in Γ_A is **presectional** if for each i, with $1 \leq i \leq n-1$, such that $Y_{i-1} \simeq \tau Y_{i+1}$ then there is an irreducible morphism $Y_{i-1} \oplus \tau Y_{i+1} \to Y_i$. Equivalently, if $\tau^{-1} Y_{i-1} \simeq Y_{i+1}$, then there is an irreducible morphism $Y_i \to \tau^{-1} Y_{i-1} \oplus Y_{i+1}$. Note that a sectional path is also presectional.

A path $Y_0 \to Y_1 \to \cdots \to Y_n$ in Γ_A is **left almost presectional** if $Y_0 \to Y_1 \to \cdots \to Y_{n-1}$ is presectional in Γ_A and $Y_n \simeq \tau^{-1} Y_{n-2}$. Dually, we can define a right almost presectional path.

In [8], the first named author gave a generalization of the result proven in [1]. Moreover, as a consequence of such result the author got Corollary 1.6.

Theorem 1.5. *Let A be an artin algebra and assume that there is a configuration of almost split sequences as follows*



where $f_1 : X_1 \to X_2, \dots, f_n : X_n \to X_{n+1}$ are irreducible morphisms between indecomposable A-modules with $f_1 \dots f_{n-1}$ in a left almost pre-sectional path such that $f_{n-1} \dots f_1 \notin \Re^n$. Let $h_i : X_i \to X_{i+1}$ be irreducible morphisms for $i = 1, \dots, n$ such that $0 \neq h_n ... h_1 \in \Re^{n+1}$. Then, $h_n ... h_1 \in \Re^{n+3}$.

Corollary 1.6. *Let A be an artin algebra and $h_i : X_i \to X_{i+1}$ be irreducible morphisms with $X_i \in ind A$ for $i = 1, 2, 3$ such that $h_3h_2h_1 \in \Re^4(X_1, X_4)$. Then, $h_3h_2h_1 \in \Re^6(X_1, X_4)$.*

1.7. Let A be an algebra such that $A \cong kQ_A/I_A$. The algebra A is called a **string algebra** provided:

(1) Any vertex of Q_A is the starting point of at most two arrows.
(1') Any vertex of Q_A is the ending point of at most two arrows.
(2) Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\gamma\beta \notin I_A$.
(2') Given an arrow γ, there is at most one arrow β with $s(\beta) = e(\gamma)$ and $\gamma\beta \notin I_A$.
(3) The ideal I_A is generated by a set of paths of Q_A.

Let $A = kQ_A/I_A$ be a string algebra. A **string** in Q_A is either a trivial path ε_v with $v \in Q_0$, or a reduced walk $C = c_1 \dots c_n$ of length $n \geq 1$ such that no sub-walk $c_i \dots c_{i+t}$ nor its inverse belongs to I_A. We say that a string $C = c_1 \dots c_n$ is **direct (inverse)** provided all c_i are arrows (inverse of arrows, respectively). We consider the trivial walk ε_v a direct as well as an inverse string.

We say that a string C has length n if the number of arrows and inverse of arrows in its composition is n.

For each string $C = c_1 \dots c_n$ in Q_A, an indecomposable string A-module $M(C)$ is defined. Conversely, given M an indecomposable string A-module there exists a "unique" string C such that $M = M(C) = M(C^{-1})$. The band modules are defined over strings C such that all powers C^n, with $n \in \mathbb{N}$ are defined, see [6]. Every module over a string algebra is defined either as a string module or as a band module, see [6]. Moreover, if A is a representation-finite string algebra then all the indecomposable A-modules are strings ones.

We say that a string C **starts in a deep (on a peak)** provided there is no arrow β such that $\beta^{-1}C$ (βC, respectively) is a string. Dually, a string C **ends in a deep (on a peak)** provided there is no arrow β such that $C\beta$ ($C\beta^{-1}$, respectively) is a string.

By [6] we know that given a string algebra A then $\alpha(\Gamma) \leq 2$. Moreover, the authors also described all the almost split sequences of $\operatorname{mod} A$ in terms of strings.

Consider $I(u)$ to be the injective module corresponding to the vertex $u \in (Q_A)_0$. Then, $I(u) = M(D_1 D_2)$ where D_1 is a direct string starting on a peak and D_2 is an inverse string ending on a peak.

Dually, if $P(u)$ is the projective corresponding to $u \in Q_0$ then $P(u) = M(C_1 C_2)$ where C_1 is an inverse string and C_2 is a direct string. Moreover, $C_1 C_2$ is a string that starts and ends in a deep.

For a detail account on these algebras see [6] and for general Auslander-Reiten theory we refer the reader to [2] and [3].

2. General results

Consider the following family of quivers Q_n

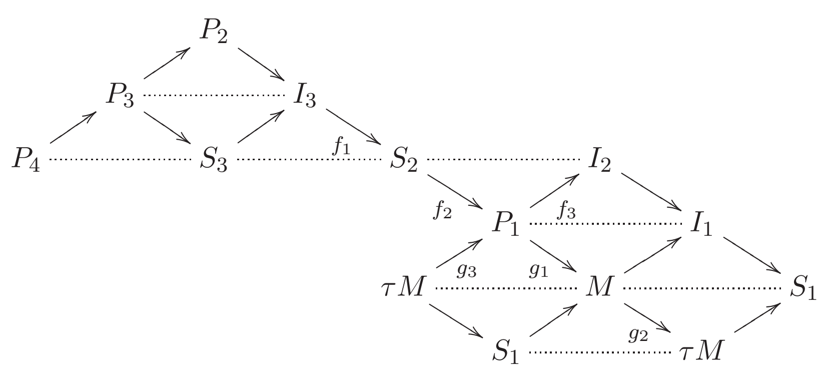
$$\alpha \circlearrowright 1 \xrightarrow{\beta_1} 2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_n} n+1$$

for $n \geq 2$ and the ideal $I =< \alpha^2, \beta_1\beta_2 >$. We denote the algebras kQ_n/I by $(W(n), I)$.

Fix an integer $n \geq 3$, and consider any algebra $A \simeq (W(n), I)$. In such algebras there is a composition of n irreducible morphisms $h_i : X_i \to X_{i+1}$ for $i = 1, \dots, n$ between indecomposable A-modules such that $h_n \dots h_1 \in \Re^{n+3}(X_1, X_{n+1}) \backslash \Re^{n+4}(X_1, X_{n+1})$, with $h_n \dots h_2 \in \Re^n(X_2, X_{n+1})$.

We illustrate the above situation in the next example.

Example 2.1. Consider the algebra $A \simeq (W(3), I)$. The Auslander-Reiten quiver Γ_A is the following:



where we identify the modules which are the same.

Consider the irreducible morphisms $f_1 : I_3 \to S_2$, $f_2 : S_2 \to P_1$ and $f_3 : P_1 \to I_2$.

We define $h_2 : S_2 \to P_1$ as follows $h_2 = f_2 + g_3 g_2 g_1 f_2$, where $g_1 : P_1 \to M$, $g_2 : M \to \tau M$ and $g_3 : \tau M \to P_1$ are irreducible morphisms. Then h_2 is irreducible. Indeed, otherwise, $h_2 \in \Re^2(S_2, P_1)$. Therefore, $f_2 \in \Re^2(S_2, P_1)$ a contradiction since f_2 is an irreducible morphism between indecomposable modules. Note that the composition $f_3 h_2 f_1 \in \Re^6(P_2, I_2) \backslash \Re^7(P_2, I_2)$, but the composition $f_3 h_2 \in \Re^3(I_3, I_2)$.

We are interested in finding three irreducible morphisms between indecomposable A-modules such that their composition belongs to $\Re^6\backslash\Re^7$, and moreover, with the property that the composition of two of such morphisms does not belong to $\Re^3$.

In Section 4, we shall prove that if A is a string algebra then there are not irreducible morphisms h_i for $i = 1, 2, 3$ between indecomposable A-modules in $\Re^6\backslash\Re^7$, with $h_2h_1 \notin \Re^3$ and $h_3h_2 \notin \Re^3$.

Throughout this paper, we shall prove all our results for the composition of three irreducible morphisms h_i for $i = 1, 2$ and 3, such that $d_l(h_3) = 2$. We observe that, with similar arguments one can prove the results for the case where $d_r(h_3) = 2$.

Now, we show that if for some artin algebra A, there are morphisms as described above, then there must be a cycle of irreducible morphisms between indecomposable A-modules of length three.

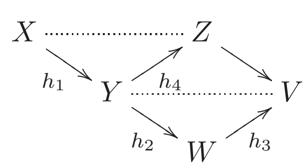
Proposition 2.2. *Let A be an artin algebra and let $f_1 : X \to Y$, $f_2 : Y \to W$, and $f_3 : W \to V$ be irreducible morphisms between indecomposable A-modules such that $f_3f_2f_1 \in \Re^6(X, V)\backslash\Re^7(X, V)$ with $f_2f_1 \notin \Re^3(X, W)$ and $f_3f_2 \notin \Re^3(Y, V)$. Then, there exists a cycle of length three.*

Proof. Since $f_3f_2f_1 \in \Re^6(X, V)\backslash\Re^7(X, V)$, then there is a path ψ of irreducible morphisms

$$\psi : X \xrightarrow{g_1} A_1 \xrightarrow{g_2} A_2 \xrightarrow{g_3} A_3 \xrightarrow{g_4} A_4 \xrightarrow{g_5} A_5 \xrightarrow{g_6} V$$

such that $\psi \notin \Re^7(X, V)$. Moreover, since $0 \neq f_3f_2f_1 \in \Re^4(X, V)$, $f_2f_1 \notin \Re^3(X, W)$ and $f_3f_2 \notin \Re^3(Y, V)$ then by [10, Theorem 2.2] there is a configuration of almost split sequences as follows:

(1)



such that $h_3h_2h_1 = 0$, $\alpha'(X) = 1$ and $\alpha'(Y) = 2$ or its dual.

By [11, Lemma 2.3] and the fact that $d_l(h_4) < \infty$, the dimension of the irreducible morphisms involved in (7) is one. Since $\alpha'(X) = 1$ and $g_1 : X \to A_1$ is irreducible then $A_1 \simeq Y$.

We claim that $A_2 \simeq W$. In fact, if $A_2 \simeq Z$ then $g_1 = \alpha_1 h_4 + \mu_1$ and $g_2 = \alpha_2 h_4 + \mu_2$ with $\alpha_1, \alpha_2 \in k^*$ and $\mu_1, \mu_2 \in \Re^2$. Since $h_4h_1 = 0$ we have that $g_2g_1 = \alpha_2 h_4\mu_1 + \alpha_1\mu_2 h_1 + \mu_2\mu_1 \in \Re^3(X, Z)$. Therefore, we get that $\psi \in \Re^7(X, V)$ a contradiction to our assumption. This establishes our claim.

With similar arguments as above we can prove that $A_3 \not\simeq V$.

On the other hand, since $\alpha(V) = 2$ and there are irreducible morphisms $A_5 \to V$, $Z \to V$ and $W \to V$, then $A_5 \simeq Z$ or $A_5 \simeq W$. If $A_5 \simeq W$ is easy to see that there is a cycle $W \to A_3 \to A_4 \to W$ of length three. Now, if $A_5 \simeq Z$, since $\alpha(Z) = 1$ then $A_4 \simeq Y$. Hence, the path ψ is as follows:

$$\psi : X \xrightarrow{g_1} Y \xrightarrow{g_2} W \xrightarrow{g_3} A_3 \xrightarrow{g_4} Y \xrightarrow{g_5} Z \xrightarrow{g_6} V.$$

Then, there is a cycle $Y \to W \to A_3 \to Y$ in $\operatorname{mod} A$ of length three. □

Next, we present a characterization for the existence of cycles of length three in $\operatorname{mod} A$.

Theorem 2.3. *Let A be an artin algebra. The following conditions are equivalent.*

(a) *There is a cycle in $\operatorname{mod} A$ which is a composition of irreducible morphisms between indecomposable A-modules of length three.*
(b) *There is an indecomposable not projective A-module M and an irreducible morphism from M to τM.*

Proof. $(a) \Rightarrow (b)$. By hypothesis there is a cycle of irreducible morphisms between indecomposable A-modules of length three. Let $M \to M_1 \to M_2 \to M$ be such a cycle. By [5, Theorem 7], any path of the form $M \to M_1 \to M_2 \to M \to M_1$ is not sectional. Therefore, one of the following conditions hold.

(1) $M \simeq \tau M_2$;
(2) $M_1 \simeq \tau M$; or
(3) $M_2 \simeq \tau M_1$.

In the former case, there is an irreducible morphism from M_2 to τM_2. In case (2) there is an irreducible morphism from M to τM. Finally, in the latter case, we have an irreducible morphism M_1 to τM_1.

In conclusion, in all the cases, there is an indecomposable A-module which is not projective and an irreducible morphism from that module to the Auslander-Reiten translate of such a module, proving (b).

$(b) \Rightarrow (a)$. Let M be a module as in Statement (b). First, suppose that M is not injective. Then $\tau^{-1}M$ is defined and there is an irreducible morphism from τM to $\tau^{-1}M$. Moreover, there is an irreducible morphism from $\tau^{-1}M$ to M. Hence there is a path of irreducible morphisms between indecomposable A-modules

$$\tau^{-1}M \to M \to \tau M \to \tau^{-1}M$$

which is a cycle in $\operatorname{mod} A$ of length three.

Secondly, if M is injective, then τM is not projective. In fact, otherwise, we get to the contradiction that the irreducible morphism from M to τM is both a monomorphism and an epimorphism. Hence, $\tau^2 M$ is defined.

With similar arguments as before, there is an irreducible morphism from $\tau^2 M$ to M and an irreducible morphism from τM to $\tau^2 M$. Therefore, there is a path of irreducible morphisms

$$M \to \tau M \to \tau^2 M \to M$$

which is clearly a cycle of length three, getting (a). □

Remark 2.4. We observe that for any positive integer n, condition (b) state below implies condition (a).

(a) There is a cycle in $\operatorname{mod} A$ which is a composition of irreducible morphisms between indecomposable A-modules of length $2n + 1$.
(b) There are indecomposable not projective A-modules $\tau^i M$ for $i = 1, \ldots, n-1$ and an irreducible morphism from M to $\tau^n M$.

Proposition 2.5. *Let A be an artin algebra. Consider an indecomposable A-module M such that there is an irreducible morphism from M to τM. If M is τ-stable, then M is τ-periodic of rank three.*

Proof. Consider M an indecomposable τ-stable A-module such that there is an irreducible morphism from M to τM. Assume that M is not τ-periodic. Then for all integer n, the modules $\tau^n M$ are defined. Moreover, for all integers r and s such that $r \neq s$, then $\tau^s M \not\simeq \tau^r M$.

Since there is an irreducible morphism from M to τM, then there is an irreducible morphism from $\tau^k M$ to $\tau^{k+1} M$ for every integer k. Furthermore, there is an irreducible morphism from $\tau^2 M$ to M. Hence, for all integer k there is an irreducible morphism from $\tau^k M$ to $\tau^{k-2} M$.

Consider a full subquiver Γ of Γ_A consisting of modules of the form $\tau^k M$ for all integer k. Observe that all the modules in Γ are neither projective nor injective. Then for all module $\tau^k M$ in Γ, we have that the morphism $\tau^k M \to \tau^{k+1} M \oplus \tau^{k-2} M$ is irreducible. Since $\tau^{k+1} M \not\simeq \tau^{k-2} M$, then all the almost split sequences in Γ have at least two indecomposable middle terms. By Theorem [15, Teorema 2.3] there are not oriented cycles in Γ, a contradiction to Theorem 2.3. Then M is τ-periodic.

We claim that M has τ-period three. In fact, let n be the τ-period of M, that is, $M \simeq \tau^n M$ and $M \not\simeq \tau^k M$ for $1 \leq k < n$. Since there are irreducible morphisms from M to τM and from $\tau^2 M$ to M and there are not loops in Γ_A, then $n > 2$.

On the other hand, since there is an irreducible morphism M to τM there is a cycle in Γ_A of the form

$$\psi : M \to \tau M \to \tau^2 M \to \ldots \to \tau^{n-1} M \to \tau^n M \simeq M.$$

By [5, Theorem 7], we know that the path $M \overset{\psi}{\rightsquigarrow} M \to \tau M$ is not sectional. Then $\tau^k M \simeq \tau(\tau^{k+2} M) \simeq \tau^{k+3} M$ for some $k \leq n$. In conclusion, for any k satisfying the above condition, we have that $M \simeq \tau^3 M$, proving the result. □

Remark 2.6. In case that M is an indecomposable τ-stable A-module such that there is an irreducible morphism from M to $\tau^n M$ for $n \geq 3$, then M is τ-periodic of rank $n + 3$.

3. On some string Algebras

We shall present some string algebras such that their module category has an irreducible morphism from M to τM, with M an indecomposable module. This results shall be fundamental to prove that if we consider a string algebra A then there are not three irreducible morphisms between indecomposable A-modules in $\Re^6 \backslash \Re^7$, when the composition of two of then behaves well.

We start given a characterization of the string algebras which have an irreducible morphism from M to τM, where M is an indecomposable not τ-stable module.

Proposition 3.1. *Let $A = kQ_A/I_A$ be a string algebra. The following conditions are equivalent.*

1. *There is an indecomposable not τ-stable A-module M, and an irreducible morphism from M to τM.*
2. *The quiver Q_A has one of the following full subquivers.*

(a)

$$\widetilde{Q}_1: \quad \alpha \circlearrowleft a \xrightarrow{\beta} x$$

with $\alpha^n \in I_A$ *for* $n \geq 2$, *and* $\alpha\beta \in I_A$. *If there is an arrow* $\delta : x \to y$ *in* Q_A, *then* $\beta\delta \in I_A$. *Moreover, there are no arrows in* Q_A *going out or coming in from the vertices of* $\widetilde{Q}_1$*; or*

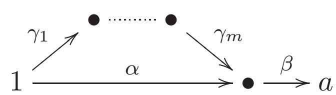
(b)

$$\widetilde{Q}_2: \quad x \xrightarrow{\beta} a \circlearrowright \alpha$$

with $\alpha^n \in I_A$ *for* $n \geq 2$, *and* $\beta\alpha \in I_A$. *If there is an arrow* $\delta : y \to x$ *in* Q_A, *then* $\delta\beta \in I_A$. *Moreover, there are no arrows in* Q_A *going out or coming in from the vertices of* $\widetilde{Q}_2$*; or*

(c)

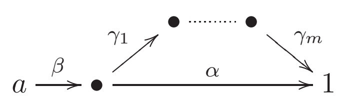
$\widetilde{Q}_3$:



with $m \geq 1$, $\alpha\beta \in I_A$ *and* $\gamma_1 \ldots \gamma_m\beta \notin I_A$. *If there is an arrow* $\delta : a \to y$ *in* Q_A, *then* $\gamma_1 \ldots \gamma_m\beta\delta \in I_A$. *If there is an arrow* $\lambda : \bullet \to 1$ *in* Q_A, *then* $\lambda\gamma_1 \ldots \gamma_m \in I_A$. *Moreover, the vertex* a *is not the end point of any other arrow; or*

(d)

$\widetilde{Q}_4$:



with $m \geq 1$, $\beta\alpha \in I_A$ *and* $\beta\gamma_1 \ldots \gamma_m \notin I_A$. *If there is an arrow* $\delta : y \to a$ *in* Q_A, *then* $\delta\beta\gamma_1 \ldots \gamma_m \in I_A$. *If there is an arrow* $\lambda : 1 \to \bullet$ *in* Q_A, *then* $\gamma_1 \ldots \gamma_m\lambda \in I_A$. *Moreover, the vertex* a *is not the start point of any other arrow.*

Proof. Let M be an indecomposable A-module, not τ-stable, and such that there is an irreducible morphism from M to τM. Since M is not τ-stable then there is an integer m such that $\tau^m M$ is either projective or injective.

Without loss of generality, we may assume that $\tau^m M = I_a$ where I_a is the injective corresponding to the vertex a in Q_A. Moreover, with our notations, we have that there is an irreducible morphism from I_a to τI_a.

Since A is a string algebra, by [6] we know that $I_a = M(\overline{D_1}\ \overline{D_2})$ with $\overline{D_1}$ a direct string starting on a peak and $\overline{D_2}$ an inverse string ending on a peak. Assume that $\tau I_a = M(D_1)$, where $D_1 = \alpha_r \ldots \alpha_2$ if $\overline{D_1} = \alpha_r \ldots \alpha_1$.

Now, depending on the string D_1, we shall analyze the possible almost split sequences starting in $M(D_1)$.

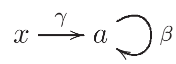
Firstly, assume that D_1 does not start on a peak and neither ends on a peak. Then the almost split sequences starting in $M(D_1)$ is as follows:

$$0 \to M(\alpha_r \ldots \alpha_2) \to M(C^{-1}\beta\alpha_r \ldots \alpha_2) \oplus M(\alpha_r \ldots \alpha_2\gamma^{-1}C') \to M(C^{-1}\beta\alpha_r \ldots \alpha_2\gamma^{-1}C') \to 0.$$

Therefore $I_a = M(C^{-1}\beta\alpha_r \ldots \alpha_2\gamma^{-1}C')$. Since I_a is injective then $l(C) = l(C') = 0$. Moreover, since $l(C) = 0$, then $s(\beta)$ is not the start point of any other arrow in Q_A. Similarly, since $l(C') = 0$ then $s(\gamma)$ is not the start point of any other arrow in Q_A. Then we conclude that

$$I_a = M(\alpha_r \dots \alpha_1 \overline{D_2}) = M(\beta\alpha_r \dots \alpha_2\gamma^{-1})$$

with $\alpha_i = \beta$ for all $i = 1, \dots, r$, $\beta^{r+1} \in I_A$, and $\gamma\beta \in I_A$. Moreover, if there is an arrow $\delta : \bullet \to s(\gamma)$ then $\delta\gamma \in I_A$. Hence in Q_A there is a full subquiver of the following form:



with $\beta^{r+1} \in I_A$ for $, r \geq 1$ and $\gamma\beta \in I_A$.

Secondly, suppose that D_1 starts and ends on a peak. In such a case $M(D_1)$ is injective, getting a contradiction to the fact that $\tau^{-1}M(D_1) = I_a$.

Now, suppose that D_1 does not start on a peak, but ends on a peak. Then the almost split sequence starting in $M(D_1)$ is as follows:

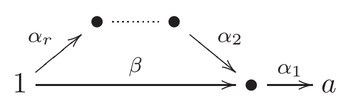
$$0 \to M(\alpha_r \dots \alpha_2) \to M(C^{-1}\beta\alpha_r \dots \alpha_2) \oplus M(\alpha_r \dots \alpha_3) \to M(C^{-1}\beta\alpha_r \dots \alpha_3) \to 0.$$

Then $I_a = M(C^{-1}\beta\alpha_r \dots \alpha_3)$ with $l(C) = 0$. Then there is a path of $r - 1$ arrows, while $\overline{D_1}$ has r, a contradiction.

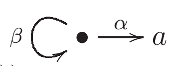
Finally, assume that D_1 starts on a peak and does not end on a peak. In this case, since D_1 is a direct string then the almost split sequence starting in $M(D_1)$ has only one indecomposable middle term and it is as follows:

$$0 \to M(\alpha_r \dots \alpha_2) \to M(\alpha_r \dots \alpha_2\beta^{-1}C) \to M(C) \to 0$$

with C a direct string ending in a deep. Then $I_a = M(C) = M(\alpha_r \dots \alpha_1)$ is uniserial. Hence Q_A has a subquiver of the form



with $\beta\alpha_1 \in I_A$. In case that there is an arrow $\lambda : \bullet \to 1$ then $\lambda\alpha_r \dots \alpha_2 \in I_A$, because $\alpha_r \dots \alpha_2$ starts on a peak. Note that in this case, α_i can be all trivial for $i = 2, \dots, r$. In such a case, we have a subquiver as follows:



where $\beta\alpha \in I_A$ (otherwise, $I_a \neq M(\alpha)$) and $\beta^n \in I_A$ (in order to be a finite dimensional algebra).

With a similar analysis as before and assuming that $\tau^m M$ is projective, we obtain the subquivers (a) and (d).

For the converse, it is enough to show that for each configuration there is an indecomposable A-module M and an irreducible morphism from M to τM. □

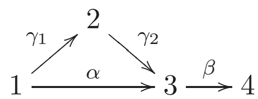
As an immediate consequence of Proposition 3.1, we get the following corollary.

Corollary 3.2. *With the notation introduced in Proposition 3.1, the following conditions hold.*

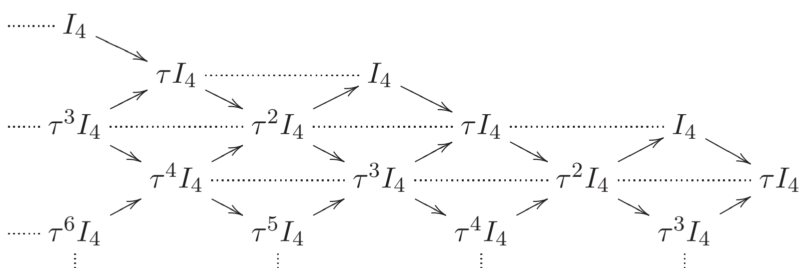
(1) *In $\widetilde{Q}_1$ there are irreducible morphisms from I_x to τI_x and from $\tau^{-1}P_a$ to P_a.*
(2) *In $\widetilde{Q}_2$ there are irreducible morphisms from I_a to τI_a and from $\tau^{-1}P_x$ to P_x.*
(3) *In $\widetilde{Q}_3$ there is an irreducible morphism from I_a to τI_a.*

(4) *In $\widetilde{Q}_4$ there is an irreducible morphism from $\tau^{-1}P_a$ to P_a.*

Example 3.3. Let A be the algebra given by the presentation



with $I =< \alpha\beta >$. Let Γ be a component of Γ_A having the injective I_4. Then Γ is as follows:



where we identify the modules in Γ which are the same. Observe all the modules M that belong to the sectional path starting in I_4 have the property that there is an irreducible morphism from M to τM.

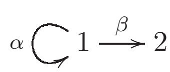
Now, we concentrate our attention in the algebras which have an irreducible morphism from M to τM, where M is an indecomposable τ-stable module.

Proposition 3.4. *Let $A = kQ_A/I_A$ be a string algebra. The following conditions are equivalent.*

(1) *There is a τ-stable indecomposable A-module M with $\alpha'(M) = 1$ and an irreducible morphism $M \to \tau M$.*

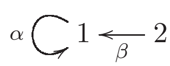
(2) *The quiver Q_A contains one of the following full subquivers:*

i)



with $\alpha^2 \in I_A$ and $\alpha\beta \notin I_A$. Moreover, there are no arrows coming in the vertex 1, if there is an arrow $\lambda : 2 \to \bullet$ then $\beta\lambda \in I_A$ and 2 is not the end point of any other arrow; or

ii)



with $\alpha^2 \in I_A$ and $\beta\alpha \notin I_A$. Moreover, there are no arrows going out from the vertex 1, if there is an arrow $\lambda : \bullet \to 2$ then $\lambda\beta \in I_A$ and 2 is not the starting point of any other arrow.

Proof. Let M be an indecomposable A-module as in (1). Since $\alpha'(M) = 1$ by [6], we get that $M = M(\gamma_1^{-1} \dots \gamma_r^{-1}) = M(B_2^{-1})$ and $\tau M = N(\beta_0) = M(\delta_s^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}) = M(B_1^{-1}\beta_0 B_2^{-1})$. Observe that τM can not be the starting of an almost split sequence with indecomposable middle term. Hence, the almost split sequence starting in τM has two indecomposable middle terms.

Now, we shall built such a sequence. We know that the string $B_1^{-1}\beta B_2^{-1}$ ends on a peak. Then we analyze two cases:

(a) if $B_1^{-1}\beta_0 B_2^{-1}$ starts on a peak; or
(b) if $B_1^{-1}\beta_0 B_2^{-1}$ does not start on a peak.

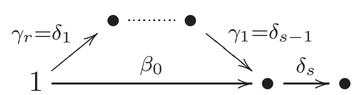
Assume that (a) holds, then there is no $\lambda \in Q_1$ such that $\lambda B_1^{-1}\beta_0 B_2^{-1}$ is a string. Since $\tau^{-1}M$ is not injective, then $s \geq 1$. The almost split sequence starting in τM is as follows:

$$0 \to M(B) \to M(B_1^{-1}) \oplus M(\delta_{s-1}^{-1} \dots \delta_1^{-1}\beta_0 B_2^{-1}) \to M(\delta_{s-1}^{-1} \dots \delta_1^{-1}) \to 0.$$

Therefore

$$M = M(B_2^{-1}) = M(\delta_{s-1}^{-1} \dots \delta_1^{-1}). \tag{2}$$

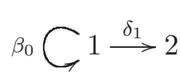
Hence $\gamma_1^{-1} \dots \gamma_r^{-1} = \delta_{s-1}^{-1} \dots \delta_1^{-1}$. Then $r = s-1$ and $\gamma_i = \delta_{s-i}$. If $r \geq 1$ then there is a subquiver of the form:



where $\beta_0\delta_s \in I_A$ since $\gamma_1\delta_s \notin I_A$. Observe that since the string B_2^{-1} ends on a peak and $B_2^{-1} = \delta_{s-1}^{-1} \dots \delta_1^{-1}$ then $B_1^{-1} = \delta_s^{-1} \dots \delta_1^{-1}$ also ends on a peak.

On the other hand, since we assume (a) then $B_1^{-1}\beta_0 B_2^{-1}$ starts on a peak and therefore B_1^{-1} starts on a peak. Then $M(B_1^{-1})$ is injective since B_1^{-1} starts and ends on a peak, a contradiction to the fact that M is τ-stable and $M(B_1^{-1}) = \tau M$.

Then $r = 0$ and $B_2^{-1} = e(\beta_0)$ and $B_1^{-1} = \delta_1$. From (2) we get that $B_2^{-1} = e(\beta_0) = s(\beta_0)$, and therefore β_0 is a loop. Since $e(\beta_0)$ is not the end point of any other arrow, we have that if $\beta_0\delta_1 \in I_A$ then $M(\delta_1)$ is injective getting a contradiction. Thus $\beta_0\delta_1 \notin I_A$ and then $\beta_0^2 \in I_A$. Then in Q_A we have a subquiver of the form:

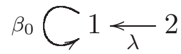


such that $\beta_0^2 \in I_A$, $\beta_0\delta_1 \notin I_A$, there are no arrows coming in or going out of the vertex 1, if there is an arrow $\lambda : e(\delta_1) \to \bullet$ then $\delta_1\lambda \in I_A$ and $e(\delta_1)$ is not the end point of any other arrow, since the string $B_1^{-1}\beta_0 B_2^{-1} = \delta_1^{-1}\beta_0$ starts on a peak.

Assume now, that $B_1^{-1}\beta_0 B_2^{-1}$ satisfies (b). That is, there is a an arrow $\lambda \in Q_1$ such that $\lambda B_1^{-1}\beta_0 B_2^{-1}$. In this case, the almost split sequence starting in $B_1^{-1}\beta_0 B_2^{-1}$ is as follows:

$$0 \to M(B_1^{-1}\beta_0 B_2^{-1}) \to M(B_1^{-1}) \oplus M(D^{-1}\lambda B_1^{-1}\beta_0 B_2^{-1}) \to M(D^{-1}\lambda B_1^{-1}) \to 0.$$

Then $M = M(B_2^{-1}) = M(D^{-1}\lambda B_1^{-1})$. In this case, we obtain that $B_2^{-1} = B_1\lambda^{-1}D$. Thus, we deduce that B_1 and D are trivial and $B_2^{-1} = \lambda^{-1}$. Since D is trivial, $s(\lambda)$ is not the starting point of any other arrow. Similarly, since B_1 is trivial then $s(\beta_0)$ is not the starting point of any other arrow. Furthermore, since the string $\lambda\beta_0\lambda^{-1}$ is defined, β_0 is a loop and $\beta_0^2 \in I_A$ because $\beta_0\lambda \notin I_A$. Then we have a subquiver as follows:



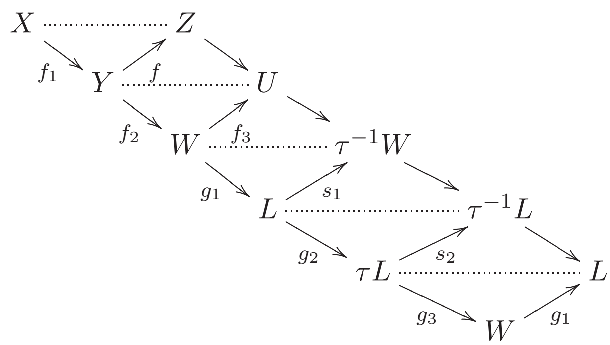
with $\beta_0^2 \in I_A$, $\lambda\beta_0 \notin I_A$, and where there are not arrows going out the vertex 1, and if there is $\rho : \bullet \to 2$ then s $\rho\lambda \in I_A$ and 2 is not the starting point of any other arrow. □

4. On the composition of three irreducible morphisms

We shall prove several lemmas in order to prove the main result of this work.

Lemma 4.1. *Let A be a string algebra. A configuration of almost split sequences as follows:*

(3)



is a forbidden configuration in Γ_A.

Proof. Since f is an epimorphism then $g_1 : W \to L$ so is. The A-modules X and Z are the end points of an almost split sequence with indecomposable middle term Y. Then $Y = N(\beta_0) = M(\delta_s^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}) = M(C)$ with C a string that starts in a deep and ends in a peak. Moreover, $X = M(\gamma_1^{-1} \dots \gamma_r^{-1})$ and $Z = M(\delta_s^{-1} \dots \delta_1^{-1})$.

By [6] and from (3), we know that f_2, g_1, g_2 and g_3 are the irreducible morphisms obtained by analyzing the beginning of the string corresponding to the domain of such morphisms.

We start considering the case that C starts on a peak. Then C starts and ends on a peak. Since Y is not injective, then $s \geq 1$. Hence, $W = M(\delta_{s-1}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$, $U = M(\delta_{s-1}^{-1} \dots \delta_1^{-1})$. Consider $D_1 = \delta_{s-1}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$. Then $W = M(D_1) = M(D_1^{-1})$.

Since g_1 is an epimorphism, then the string corresponding to W starts on a peak (otherwise, g_1 is a monomorphism). Therefore, $\delta_{s-1}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ is a string that starts and ends on a peak. Now, since W is not injective then $s \geq 2$. Thus, $L = M(\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$ and $\tau^{-1}W = M(\delta_{s-1}^{-1} \dots \delta_1^{-1})$.

Now, we analyze how is the string corresponding to τL. In order to do that, we may consider how is the beginning of the string corresponding to L. Assume that $\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ does not start in a peak. Then $\tau L = M(\nu_t^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$ with $t \geq 1$ or $\tau L = M(\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$.

Assume that $\tau L = M(\nu_t^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$ with $t \geq 1$. Suppose that $\nu_t^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ starts on a peak. Since $t \geq 1$, then there exists an irreducible morphism from τL to $M(\nu_{t-1}^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$ and, by construction, this module is W.

Since D_1 has only one direct arrow then $D_1 \neq \nu_{t-1}^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$. If $D_1^{-1} = \nu_{t-1}^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ and since $s \geq 2$ then the arrows γ_i are trivial. Thus $t \geq 2$. Then D_1^{-1} has length s, while $\nu_{t-1}^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ has at least length $s+1$, a contradiction. Then $\nu_t^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ does not start on a peak. Therefore there is an irreducible morphism as follows:

$$\tau L \to M(\lambda_k^{-1} \dots \lambda_1^{-1}\beta_2\nu_t^{-1} \dots \nu_1^{-1}\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$$

where the length of such a string is different from the length of D_1 (and D_1^{-1}), proving that this case is not possible.

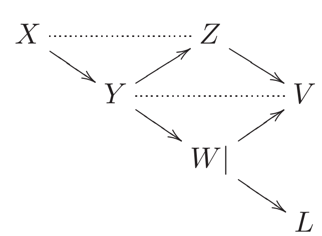
Now, consider that $\tau L = M(\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$. If $\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ does not start on a peak, then we can add an arrow and the string should have at least length $r+s+1$, while the string corresponding to W has length $r+s$, a contradiction. If $\beta_1\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{1} \dots \gamma_r^{-1}$ starts on a peak, since τL is not injective then $s \geq 3$. Therefore, there is an irreducible morphism from τL to $M(\delta_{s-3}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$ where the length of such a string and the string corresponding to W are different, proving that this case is not possible.

Assume that the string corresponding to L starts on a peak. Then g_2 is an epimorphism and the string $\delta_{s-2}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ starts and ends in a peak. Hence, $s \geq 3$, otherwise L is injective. Then $\tau L = M(\delta_{s-3}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$.

By Lemma 1.4, since g_1 and g_2 are epimorphisms then g_3 is a monomorphism. Since the existence of g_3 is due to the fact of how is the beginning of the string corresponding to τL, then $\delta_{s-3}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$ does not start in a peak. Thus, there is an irreducible morphism from τL to $M(\lambda_k^{-1} \dots \lambda_1^{-1}\beta_1\delta_{s-3}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$. It is clear that this string is different from D_1, since D_1 has only one arrow. Then we have that $D_1^{-1} = \lambda_k^{-1} \dots \lambda_1^{-1}\beta_1\delta_{s-3}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}$. Since $s \geq 3$, this implies that the arrows γ_i are trivial. If $s \geq 4$ then D_1^{-1} has at least three arrows, a contradiction, because the obtained string has two arrows. Therefore $s = 3$ and, in consequence, $k = 1$. Then $\lambda_1^{-1}\beta_1\beta_0 = \beta_0^{-1}\delta_1\delta_2$, and we get that $\beta_0 = \delta_2$ and $\beta_0^{-1} = \lambda_1^{-1}$ is a string that starts in a deep. Note that the string $\delta_3^{-1}\delta_2^{-1} = \delta_3^{-1}\beta_0^{-1}$ is defined, hence we get a contradiction. In conclusion, C can not start on a peak.

In case that C does not start on a peak, with similar arguments as above, we can conclude that there is not possible to have a configuration of almost split sequences as in (3), whenever A is a string algebra. □

Lemma 4.2. *A configuration of almost split sequences as follows:*

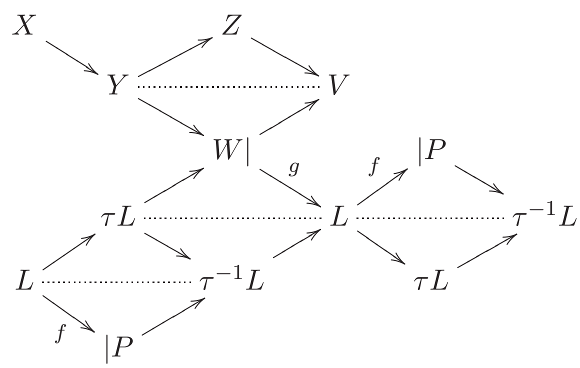


with W an indecomposable injective A-module, $L \not\simeq \tau W$ an indecomposable injective A-module such that there is an irreducible morphism from L to τL is not a possible configuration in Γ_A.

Proof. Since L is not injective then $\tau^{-1}L$ is defined. The existence of an irreducible morphism from L to τL, implies the existence of an irreducible morphism from τL to $\tau^{-1}L$. Moreover, there is an irreducible morphism from τL to W. Since $L \not\simeq \tau W$ then $\alpha'(\tau L) = 2$.

Assume that $\alpha'(L) = 2$. Then there is a configuration of almost split sequences as follows:

(4)



Since W is injective, then g is an epimorphism. By (4), we know that f is an epimorphism. On the other hand, since P is projective then f is a monomorphism, a contradiction. Hence, $\alpha'(L) = 1$.

Now, we analyze the string corresponding to such modules. Since X and Z are the start and end terms of an almost split sequence with indecomposable middle term Y, then $Y = N(\beta_0) = M(\delta_s^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}) = M(C)$ with C a string that starts on a deep and ends on a peak. Moreover, $X = M(\gamma_1^{-1} \dots \gamma_r^{-1})$ and $Z = M(\delta_s^{-1} \dots \delta_1^{-1})$.

Firstly, assume that C starts on a peak. Then C starts and ends on a peak, and therefore, since Y is not injective then $s \geq 1$. Hence, $W = M(\delta_{s-1}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$ and $U = M(\delta_{s-1}^{-1} \dots \delta_1^{-1})$. Since W is injective then $s - 1 = 0$, $W = M(\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}) = M(D_1)$ and D_1 is a string that starts and ends on a peak. If $r = 0$, then $W/\mathrm{soc}W$ is indecomposable, a contradiction to (4). Thus, $r \geq 1$ and $L = M(\gamma_2^{-1} \dots \gamma_r^{-1})$ (if $r = 1$, L is simple). Since $\alpha'(L) = 1$, then $\tau L = M(\lambda_k^{-1} \dots \lambda_1^{-1}\beta_1\gamma_2^{-1} \dots \gamma_r^{-1})$ and the string corresponding to τL starts in a deep and ends on a peak. Now, we analyze the beginning of the string corresponding to τL to determine the codomain of the irreducible morphisms whose domain is τL.

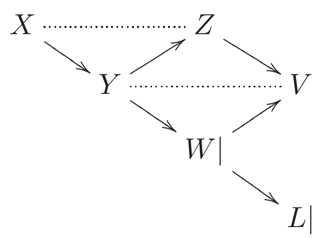
If $\lambda_k^{-1} \dots \lambda_1^{-1}\beta_1\gamma_2^{-1} \dots \gamma_r^{-1}$ starts on a peak, since τL is not injective then $k \geq 1$. Then there is an irreducible morphism from τL to $M(\lambda_{k-1}^{-1} \dots \lambda_1^{-1}\beta_1\gamma_2^{-1} \dots \gamma_r^{-1})$ and this module is W, therefore injective. Then we get that $W = M(\beta_1\gamma_2^{-1} \dots \gamma_r^{-1})$ and $\beta_1\gamma_2^{-1} \dots \gamma_r^{-1}$ has length r a contradiction.

If $\lambda_k^{-1} \dots \lambda_1^{-1}\beta_1\gamma_2^{-1} \dots \gamma_r^{-1}$ does not start on a peak, then there is an irreducible morphism from τL to $M(\epsilon_l^{-1} \dots \epsilon_1^{-1}\beta_2\lambda_k^{-1} \dots \lambda_1^{-1}\beta_1\gamma_2^{-1} \dots \gamma_r^{-1})$, and this module should be injective. Hence, there is an irreducible morphism from τL to $M(\beta_2\beta_1\gamma_2^{-1} \dots \gamma_r^{-1})$. Clearly, $\beta_2\beta_1\gamma_2^{-1} \dots \gamma_r^{-1}$ is not equal to D_1. Similarly, we can see that $\beta_2\beta_1\gamma_2^{-1} \dots \gamma_r^{-1}$ is not equal to D_1^{-1}.

Secondly, assume that C does not start on a peak. In this case, we have that $W = M(\nu_t^{-1} \dots \nu_1^{-1}\beta_1\delta_s^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$. Since W is injective, then $t = 0$, $s = 0$ and $W = M(\beta_1\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}) = M(D_2)$ with D_2 is a string that starts and ends on a peak. Then $r \geq 1$, otherwise, $W/\mathrm{soc}W$ is indecomposable. Then $L = M(\gamma_2^{-1} \dots \gamma_r^{-1})$. Since $\alpha'(L) = 1$, then $\tau L = M(\lambda_k^{-1} \dots \lambda_1^{-1}\beta_1\gamma_2^{-1} \dots \gamma_r^{-1})$ and the string corresponding to τL starts in a deep and ends on a peak.

Again, if we analyze the beginning of the string corresponding to τL to determine the codomain of the irreducible morphisms with domain τL, we can discard the case with similar arguments as before. □

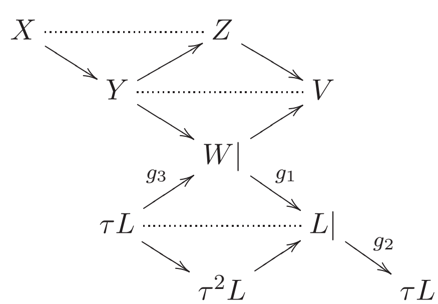
Lemma 4.3. *A configuration of almost split sequences as follows:*



with W and L indecomposable injective A-modules such that there is an irreducible morphism from L to τL is a forbidden configuration in Γ_A.

Proof. Since there are not morphisms from an injective to a projective, then $\alpha'(L) = 1$. Moreover, τL is not projective, then $\tau^2 L$ is defined and there is an irreducible morphism from τL to $\tau^2 L$. Since W is injective, then $\tau^2 L \not\simeq W$ and therefore, $\alpha'(\tau L) = 2$. Then there is a configuration of almost split sequences as follows:

(5)



Since $W \to W$ is a cycle and g_1, g_2 are epimorphisms then by Lemma 1.4, g_3 is a monomorphism.

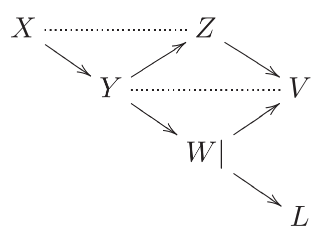
Since X and Z are the start and end terms of an almost split sequence with indecomposable middle term Y, then $Y = N(\beta_0) = M(\delta_s^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}) = M(C)$ with C a string that starts in a deep and ends on a peak. Moreover, $X = M(\gamma_1^{-1} \dots \gamma_r^{-1})$ and $Z = M(\delta_s^{-1} \dots \delta_1^{-1})$.

Assume that C starts on a peak. Then C starts and ends on a peak and therefore, since Y is not injective then $s \geq 1$. Hence, $W = M(\delta_{s-1}^{-1} \dots \delta_1^{-1}\beta_0\gamma_1^{-1} \dots \gamma_r^{-1})$, and $U = M(\delta_{s-1}^{-1} \dots \delta_1^{-1})$. Since W is injective, then $s - 1 = 0$, $W = M(\beta_0\gamma_1^{-1} \dots \gamma_r^{-1}) = M(D_1)$ and D_1 starts and ends on a peak. If $r = 0$, then $W/\mathrm{soc}W$ is indecomposable, a contradiction with (5). Then $r \geq 1$ and $L = M(\gamma_2^{-1} \dots \gamma_r^{-1})$. Since L is injective, but not simple then $r \geq 2$ and $\tau L = L/\mathrm{soc}\, L = M(\gamma_3^{-1} \dots \gamma_r^{-1})$.

Since the irreducible morphism from τL to W is a monomorphism, then $\gamma_3^{-1} \dots \gamma_r^{-1}$ either does not start on a peak or does not end on a peak. In the former case, there is an irreducible morphism from τL to $M(\lambda_k^{-1}\dots\lambda_1^{-1}\beta_1\gamma_3^{-1}\dots\gamma_r^{-1})$ and this module is W and therefore, injective. Then it is of the form $M(\beta_1\gamma_3^{-1}\dots\gamma_r^{-1})$ but the string corresponding to this module has length $r-1$, a contradiction. In the latter case, there is an irreducible morphism from τL to $M(\gamma_3^{-1}\dots\gamma_r^{-1}\beta_2^{-1}\epsilon_1\dots\epsilon_l)$ and this module must be injective. Then it is of the form $M(\gamma_3^{-1}\dots\gamma_r^{-1}\beta_2^{-1})$ but the string corresponding to this module is of length $r-1$, a contradiction.

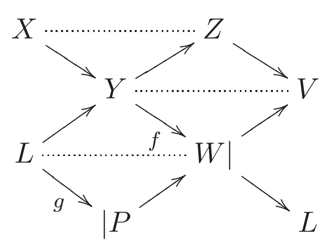
Assume that C does not start on a peak. Then $W = M(\nu_t^{-1}\dots\nu_1^{-1}\beta_1\delta_s^{-1}\dots\delta_1^{-1}\beta_0\gamma_1^{-1}\dots\gamma_r^{-1})$. Since W is injective, then $t=0$, $s=0$ and $W = M(\beta_1\beta_0\gamma_1^{-1}\dots\gamma_r^{-1}) = M(D_2)$ with D_2 a string that starts and ends on a peak. Then $r\geq 1$, otherwise, $W/\text{soc}W$ is indecomposable. Then $L = M(\gamma_2^{-1}\dots\gamma_r^{-1})$. Since L is injective but not simple, then $r\geq 2$ and $\tau L = L/\operatorname{soc} L = M(\gamma_3^{-1}\dots\gamma_r^{-1})$. With similar arguments as before, we get that this case is not possible, proving the lemma. □

Lemma 4.4. *A configuration of almost split sequences as follows:*



with W an indecomposable injective A-module, $L \simeq \tau W$ and $\alpha'(L) = 2$ is not a possible configuration in Γ_A.

Proof. Since $\alpha'(L) = 2$ and W is injective, then there is an irreducible morphism from L to a projective A-module P, where $P \not\simeq Y$. Then there is a configuration of almost split sequences as follows:



Since g is a monomorphism, then f is a monomorphism.

Since X and Z are the end points of an almost split with indecomposable middle term Y, then $Y = N(\beta_0) = M(\delta_s^{-1}\dots\delta_1^{-1}\beta_0\gamma_1^{-1}\dots\gamma_r^{-1}) = M(C)$ with C a string that starts in a deep and ends in a peak. Since f is a monomorphism, then the string corresponding to Y does not start on a peak. Then $W = M(\lambda_k^{-1}\dots\lambda_1^{-1}\beta_1\delta_s^{-1}\dots\delta_1^{-1}\beta_0\gamma_1^{-1}\dots\gamma_r^{-1})$. Since W is injective, then $k=s=0$ and $W = M(\beta_1\beta_0\gamma_1^{-1}\dots\gamma_r^{-1})$.

On the other hand, since $W/\operatorname{soc} W$ is not indecomposable, then $r\geq 1$ and $L = M(\gamma_2^{-1}\dots\gamma_r^{-1})$. Moreover, there is an irreducible morphism from L to Y.

If $r = 1$, then $L = M(s(\gamma_1))$ is simple. There are irreducibles morphisms from L to modules of the form $M(\epsilon_l^{-1}\dots\epsilon_1^{-1}\beta_2)$. The strings corresponding to such modules are equal to C or C^{-1}. In any case, it is a contradiction.

Now, assume that $r \geq 2$. If the string corresponding to L starts or ends on a peak, then there is an irreducible morphism from L to a module whose string has length $r-3$ while C has length $r+2$, a contradiction.

If $\gamma_2^{-1}\dots\gamma_r^{-1}$ does not start on a peak, then there is an irreducible morphism from L to $M(\lambda_k^{-1}\dots\lambda_1^{-1}\beta_2\gamma_2^{-1}\dots\gamma_r^{-1})$ and this module should be Y. If

$$\lambda_k^{-1}\dots\lambda_1^{-1}\beta_2\gamma_2^{-1}\dots\gamma_r^{-1} = C = \beta_1\gamma_1^{-1}\dots\gamma_r^{-1}$$

then $k = 0$ and $r = 1$. Hence $\beta_2 = \beta_1\gamma_1^{-1}$, a contradiction. Now, if

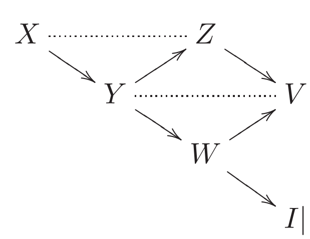
$$\lambda_k^{-1}\dots\lambda_1^{-1}\beta_2\gamma_2^{-1}\dots\gamma_r^{-1} = C^{-1} = \gamma_r\dots\gamma_1\beta_1^{-1}$$

then $k = 0$ and $r = 2$ getting a contradiction with the length of the strings.

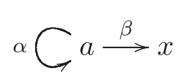
Finally, if $\gamma_2^{-1}\dots\gamma_r^{-1}$ does not end on a peak, then there is an irreducible morphism from L to $M(\gamma_2^{-1}\dots\gamma_r^{-1}\beta_2^{-1}\epsilon_1\dots\epsilon_l)$ and this module should be Y. With a similar analysis as before, we get that this case is not possible. □

Lemma 4.5. *Let A be a string algebra, and Γ be a component of Γ_A. Let I be an injective (non-projective) A-module such that there exists an irreducible morphisms from I to τI with $I \in \Gamma$. Then, there are not three irreducible morphisms between indecomposable modules $f_1 : X \to Y$, $f_2 : Y \to W$ and $f_3 : W \to V$ in Γ such that $f_3f_3f_1 \in \Re^6\backslash\Re^7$ and a configuration as follows:*

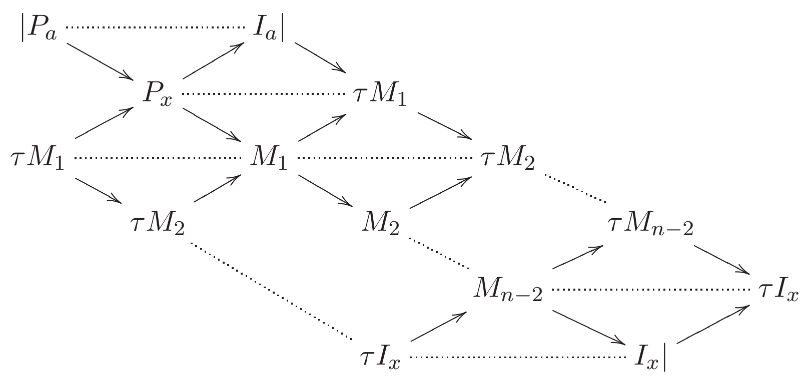
(6)



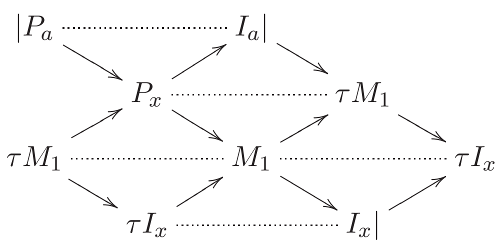
Proof. First, assume that A is representation-finite. Consider $\widetilde{Q}$ as described in Proposition 3.1 (a) or (b). We only analyze (a), since (b) follows similarly. If $\widetilde{Q}$ is the quiver



with $\alpha^n = 0$ for $n \geq 2$ and $\alpha\beta = 0$, then by Corollary 3.2 we know that there exists an irreducible morphism from I_x to τI_x. Consider the configuration of almost split sequences that involves such morphism:

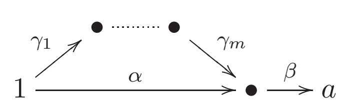


where we identify the modules which are the same. Observe that if there exist other arrows which start or end in some point of $\widetilde{Q}$, then the above configuration does not change. To obtain (6), we conclude that $n = 3$. Below, we illustrate the situation.

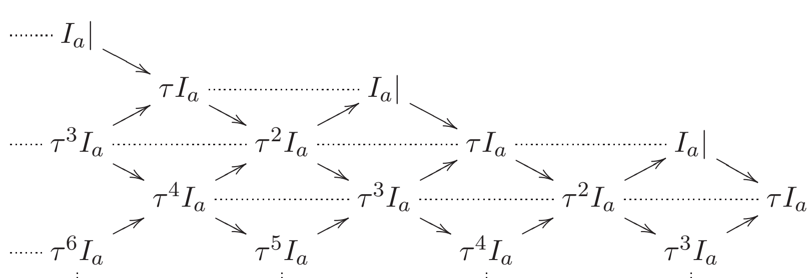


Even though there are cycles of length three, it is not hard to see that there are not three irreducible morphisms such that their composition is in $\Re^6 \backslash \Re^7$.

Now, if A is representation-infinite, by Proposition 3.1, we infer that $\widetilde{Q}$ is of the form (c). Then $\widetilde{Q}$ is the quiver



with $m \geq 1$, $\alpha\beta = 0$, and $\gamma_1 \cdots \gamma_m \beta \notin I_A$, and there exists an irreducible morphism from I_a to τI_a. Let Γ be the component of Γ_A such that $I_a \in \Gamma$. Then Γ is as follows:

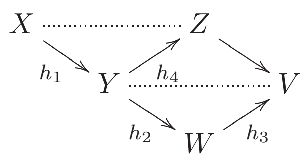


Observe that if there exist other arrows which start or end in some point of $\widetilde{Q}$, the quiver Γ does not change. In this case, we do not have a configuration as (6) since $\tau^4 I_a$ is not an injective module. Therefore, we dismiss this case. □

Now, we are in position to prove the main result of this paper.

Theorem 4.6. *Let A be a string algebra. There are not irreducible morphisms $f_1 : X \to Y$, $f_2 : Y \to W$, and $f_3 : W \to V$ between indecomposable A-modules such that $f_3 f_2 f_1 \in \Re^6(X,V)\backslash\Re^7(X,V)$ with $f_2 f_1 \notin \Re^3(X,W)$ and $f_3 f_2 \notin \Re^3(Y,V)$.*

Proof. Let $f_1 : X \to Y$, $f_2 : Y \to W$, and $f_3 : W \to V$ be irreducible morphisms as in the statement. By [10, Theorem 2.2] there is a configuration of almost split sequences as follows:

(7)


such that $h_3 h_2 h_1 = 0$, $\alpha'(X) = 1$ and $\alpha'(Y) = 2$ or its dual.

As we proved in Proposition 2.2, there exists a path of irreducible morphisms between indecomposable modules as follows:

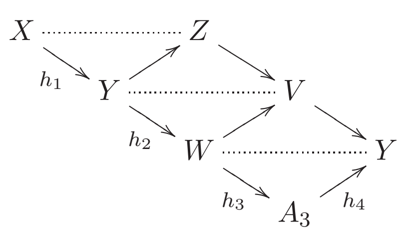
$$\psi : X \xrightarrow{g_1} Y \xrightarrow{g_2} W \xrightarrow{g_3} A_3 \xrightarrow{g_4} A_4 \xrightarrow{g_5} A_5 \xrightarrow{g_6} V$$

where $A_3 \not\simeq V$ and $A_3 \not\simeq X$, otherwise, $\psi \in \Re^7(X,V)$. Moreover, there is cycle of length three $Y \rightsquigarrow Y$ or $W \rightsquigarrow W$, if $A_5 \simeq Z$ or $A_5 \simeq W$, respectively. We claim that the first cycle is not possible in our situation. In fact, assume that $A_5 \simeq Z$, then $A_4 \simeq Y$. Following [5, Theorem 7], the path $Y \to W \to A_3 \to Y \to W$ is not sectional. Thus,

(1) $Y \simeq \tau A_3$, or
(2) $W \simeq \tau Y$, or
(3) $A_3 \simeq \tau W$.

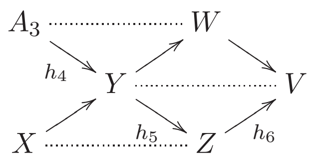
If $Y \simeq \tau A_3$ then $A_3 \simeq V$ contradicting that $\psi \notin \Re^7(X,V)$.

If $W \simeq \tau Y$, then there is a configuration as follows:



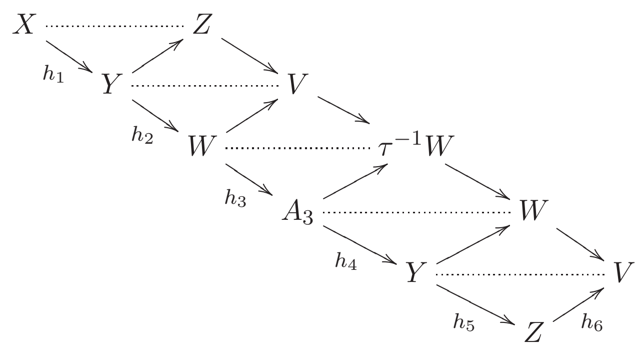
with $h_4 h_3 h_2 h_1 = 0$. Again, the dimension over k of the irreducible morphisms involved is one. Then $g_i = \alpha_i h_i + \mu_i$, with $\alpha_i \in k^*$ and $\mu_i \in \Re^2$ para $i = 1,2,3,4$. Hence $g_4 g_3 g_2 g_1 \in \Re^5(X,Y)$ a contradiction to the fact that $\psi \notin \Re^7(X,V)$. Therefore, $W \not\simeq \tau Y$.

Finally, suppose that $A_3 \simeq \tau W$. Then $\alpha'(A_3) = 2$, otherwise, $\alpha'(A_3) = 1$, and there is a configuration of almost split sequences as follows:



where $h_6h_5h_4 = 0$. Since any irreducible morphisms g_i between the involved modules is of the form $g_i = \alpha_i h_i + \mu_i$, with $\alpha_i \in k^*$ and $\mu_i \in \Re^2$ for $i = 4, 5, 6$ then $g_6g_5g_4 \in \Re^4(A_3, V)$, getting that $\psi \in \Re^7(X, V)$, a contradiction. Thus $\alpha'(A_3) = 2$.

By Lemma 4.4, W is not an injective module, then there is a configuration of almost split sequences as follows



Since $g_i = \alpha_i h_i + \mu_i$, with $\alpha_i \in k^*$ and $\mu_i \in \Re^2$ for $i = 1, \ldots, 6$, then $\psi \in \Re^7(X, V)$, a contradiction. Therefore $A_5 \not\simeq Z$.

In consequence, $A_5 \simeq W$ and the path ψ is as follows:

$$\psi : X \xrightarrow{g_1} Y \xrightarrow{g_2} W \xrightarrow{g_3} A_3 \xrightarrow{g_4} A_4 \xrightarrow{g_5} W \xrightarrow{g_6} V.$$

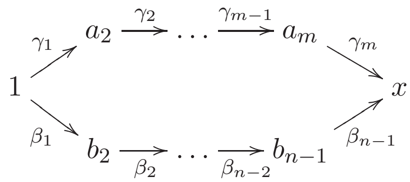
Observe, that there is a cycle $\varphi : W \to W$ of length three.

With a similar analysis as before, it is not hard to see that $A_3 \not\simeq \tau W$ and $W \not\simeq \tau A_4$. Then $A_4 \simeq \tau A_3$. From Lemmas 4.2, 4.3 and 4.5 we have that W and A_3 are not injective. Moreover, if A_3 is not injective then we get a contradiction to Lemma 4.1. Analyzing all the cases we get that $W \not\simeq A_5$, proving the result. □

5. On the composition of n irreducible morphisms in $\Re^{n+1}$ which does not belong to the infinite radical

In this section, we show families of algebras, having n irreducible morphisms such that their compositions belong to $\Re^{n+t} \backslash \Re^{n+t+1}$, with $n \geq 3$ and $t \geq 4$, and moreover, with the condition that the composition of $n - 1$ of them is not in $\Re^n$.

We denote by $(U(m, n-1), I)$ the string algebras whose quiver is



with $I =< \gamma_{m-1}\gamma_m >$, for $m, n \geq 2$.

We shall prove that in the module category of such algebras there are n irreducible morphisms with composition in $\Re^{n+2m} \backslash \Re^{n+2m+1}$.

Remark 5.1. We define the following strings in $U(m, n-1)$:

(1) $G_j = \gamma_1 \dots \gamma_j$, and $\overline{G}_j = \gamma_j \dots \gamma_{m-1}$ for $1 \leq j \leq m-1$.
(2) $B_i = \beta_1 \dots \beta_i$, and $\overline{B}_i = \beta_i \dots \beta_{n-1}$ for $1 \leq i \leq n-1$.

Note that $G_{m-1} = \overline{G}_1$ and $B_{n-1} = \overline{B}_1$.

To prove the results of this section, we recall the following notation introduced in [12].

Let A be a string algebra and let $I = M(D_1D_2)$ be an indecomposable injective A-module, where $D_1 = \gamma_s \dots \gamma_1$ is a direct string that starts on a peak, $D_2 = \beta_1^{-1} \dots \beta_r^{-1}$ is an inverse string that ends on a peak. We consider the following set of strings:

$$\mathcal{C}_{D_2} = \{DD_2 \ \ \textit{where either} \ \ D \textit{ is trivial or } D = D'\gamma_1 \ \ \textit{with} \ \ D' \ \ \textit{a string}\}.$$

In a similar way, we can define $\mathcal{C}_{D_1}$ considering $I = M(D_2^{-1}D_1^{-1})$.

Next, we recall the quiver Q_u^e defined in [12], whose vertices are the strings involved in the sets $\mathcal{C}_{D_1}$ and $\mathcal{C}_{D_2}$.

Let $A \simeq kQ/I$ and consider the injective $I(u)$, with $u \in Q_0$. Then

(1) The vertices of $(Q_u^e)_0$ are the strings C in Q such that $e(C) = u$, where C is either the trivial walk ε_u or $C = C'\alpha$, with $\alpha \in Q_1$.
(2) If $a = C$ and $b = C'$ are two vertices of $(Q_u^e)_0$, then there is an arrow from $a \to b$ in Q_u^e if C' is the reduced walk of $\beta^{-1}C$, for some $\beta \in Q_1$.

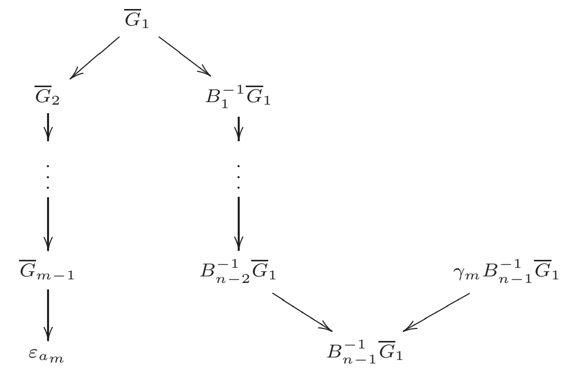
Dually, we can consider an indecomposable projective A-module, define the set of strings and the quiver Q_u^s, see [12].

The following results state below are essential to prove Theorem 5.7.

Lemma 5.2. *Let A be the algebra $(U(m, n-1), I)$, with $m, n \geq 2$. Consider the irreducible morphisms $\iota_{a_m} : \operatorname{rad} P_{a_m} \to P_{a_m}$ and $\theta_{a_m} : I_{a_m} \to I_{a_m}/\operatorname{soc} I_{a_m}$, where P_{a_m} and I_{a_m} are the projective and injective A-modules corresponding to the vertex a_m, respectively. Then $d_r(\iota_{a_m}) = m + n - 1$. Moreover, $d_r(\iota_{a_m}) = d_l(\theta_{a_m})$.*

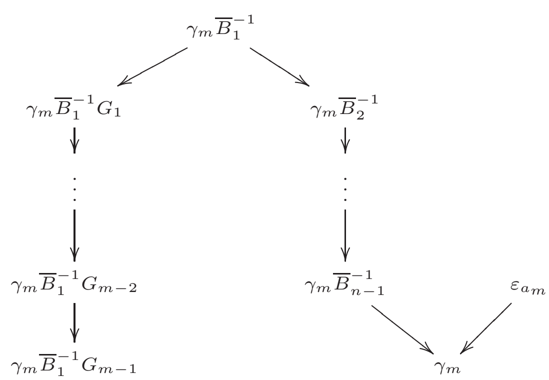
Proof. Consider $(U(m, n-1), I)$, with $m, n \geq 2$, and the irreducible morphisms ι_{a_m} and θ_{a_m}. By [12, Proposition 3.2], we know that $d_l(\theta_{a_m})$ and $d_r(\iota_{a_m})$ can be compute by the number of vertices of the quivers $Q_{a_m}^e$ and $Q_{a_m}^s$, respectively.

Recall that the vertices of the quiver $Q_{a_m}^e$ are the strings C such that $e(C) = a_m$, and $C = \varepsilon_{a_m}$ or C is of the form $C = C'\gamma_{m-1}$, with C' a string. With the notation of Remark 5.1, the quiver $Q_{a_m}^e$ is the following:



The cardinal of $(Q^e_{a_m})_0$ is $m+n$. By [12, Proposition 3.2], $d_l(\theta_{a_m}) = \mathrm{card}((Q^e_{a_m})_0) - 1$. Hence $d_l(\theta_{a_m}) = m+n-1$.

Dually, the quiver $Q^s_{a_m}$ is the following:



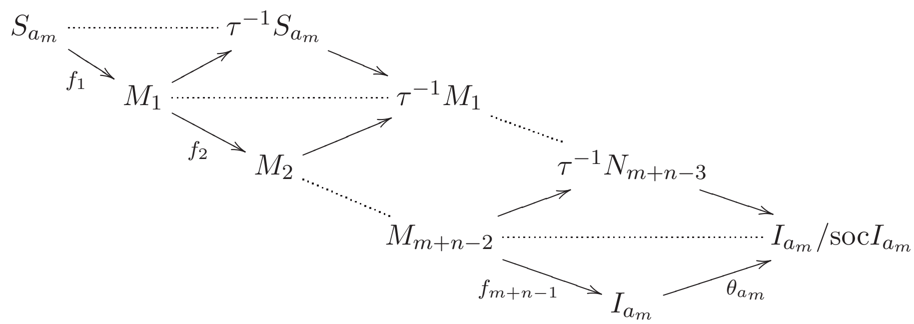
where $Q^s_{a_m}$ has $m+n$ vertices. Therefore, by [12, Proposition 3.2] we have that $d_r(\iota_{a_m}) = m+n-1$, proving the result. □

Remark 5.3. Observe that $\gamma_m \overline{B}_1^{-1} G_{m-1} = \gamma_m B_{n-1}^{-1} \overline{G}_1$ is a vertex in both quivers $Q^e_{a_m}$ and $Q^s_{a_m}$. We denote by L the A-module whose string is the mentioned one.

Given X, Y and Z indecomposable modules, we denote by $X \rightsquigarrow Y \rightsquigarrow Z$ a path of irreducible morphisms between indecomposable modules from X to Z, going through Y.

Proposition 5.4. *Let $A = (U(m, n-1), I)$, with $m, n \geq 2$, and P_{a_m}, S_{a_m} and I_{a_m} be the projective, simple, and injective module corresponding to the vertex a_m, respectively. Let L be the string module $M(\gamma_m \overline{B}_1^{-1} G_{m-1})$. Then, there is a sectional path $P_{a_m} \rightsquigarrow L \rightsquigarrow S_{a_m} \rightsquigarrow L \rightsquigarrow I_{a_m}$ in $\mathrm{mod}\, A$. Moreover, the cycle $L \rightsquigarrow S_{a_m} \rightsquigarrow L$ has length $2m$.*

Proof. Consider the irreducible morphism $\theta_{a_m} : I_{a_m} \to I_{a_m}/\mathrm{soc} I_{a_m}$. The module $I_{a_m}/\mathrm{soc} I_{a_m}$ is indecomposable. Moreover, $\mathrm{Ker}(\theta_{a_m}) = S_{a_m}$ and by Lemma 5.2, $d_l(\theta_{a_m}) = m+n-1$. By [12, Proposition 2.5], there is a configuration of almost split sequences as follows:



where the path $S_{a_m} \to M_1 \to \ldots \to M_{m+l-2} \to I_{a_m}$ is sectional.

On the other hand, the modules of such a path are in correspondence with the string modules $M(C)$, where C are vertices of $Q^e_{a_m}$. In particular, $L = M(\gamma_m \overline{B_1}^{-1} G_{m-1})$, is a module that appears in such a path. Moreover, $L \neq S_{a_m}$ and $L \neq I_{a_m}$. Hence, the path is of the form $S_{a_m} \rightsquigarrow L \rightsquigarrow I_{a_m}$.

We claim that the length of the path $S_{a_m} \rightsquigarrow L$ is m. To prove our claim, we order the strings of the set $\mathcal{C}_{\varepsilon_{a_m}}$ as follows; $C_i < C_{i+1}$ if there is an irreducible morphism $M(C_i) \to M(C_{i+1})$. To determine such order in the strings, we may analyze if the strings start on a peak.

Let $C_0 = \varepsilon_{a_m}^{-1}$. Since C_0 does not start on a peak, we define $C_1 = \gamma_{m-1}\varepsilon_{a_m}^{-1} = \overline{G}_{m-1}$. Then there is an irreducible morphism $S_{a_m} = M(C_0) \to M(C_1)$.

Observe that for $2 \leq j \leq m-1$, the strings $\overline{G}_j = \gamma_j \ldots \gamma_{m-1}$ do not start on a peak. Moreover, following [6], we observe that there exist irreducible morphisms $M(\overline{G_2}) \to M(B_{n-1}^{-1}\overline{G_1})$ and $M(\overline{G_j}) \to M(\overline{G_{j-1}})$, for $3 \leq j \leq m-1$. Continuing with the order in the set $\mathcal{C}_{\varepsilon_{a_m}}$, for $2 \leq i \leq m-2$, we define the strings $C_i = \overline{G}_{m-i}$ and $C_{m-1} = B_{n-1}^{-1}\overline{G}_1$.

Finally, C_{m-1} does not start on a peak. Then $C_m = \gamma_m B_{n-1}^{-1}\overline{G}_1$ and $M(C_m) = L$. Hence, we have a path of irreducible morphisms as follows

$$S_{a_m} = M(C_0) \to M(C_1) \to \ldots \to M(C_{m-1}) \to M(C_m) = L \rightsquigarrow I_{a_m} \tag{8}$$

where the path $S_{a_m} \rightsquigarrow L$ has length m.

Dually, if we consider the irreducible morphism $\iota_{a_m} : \operatorname{rad} P_{a_m} \to P_{a_m}$ then $\operatorname{rad} P_{a_m}$ is indecomposable and $\operatorname{Coker}(\iota_{a_m}) = S_{a_m}$. By [12, Proposition 2.5] and Lemma 5.2, there is a sectional path $P_{a_m} \rightsquigarrow S_{a_m}$ of length $m+n-1$. Again, the modules of such a path are in correspondence with the vertices of $Q^s_{a_m}$. In particular, $L = M(\gamma_m \overline{B_1}^{-1} G_{m-1})$ is a module of such a path. Moreover, $L \neq P_{a_m}$ and $L \neq S_{a_m}$. Hence, we have a path of the form $P_{a_m} \rightsquigarrow L \rightsquigarrow S_{a_m}$.

Again we can prove that $L \rightsquigarrow S_{a_m}$ has length m, by considering an order on the strings of the set $\mathcal{D}_{\varepsilon_{a_m}}$ as follows, $D_i < D_{i+1}$ if there is an irreducible morphism $M(D_{i+1}) \to M(D_i)$. In this case, to order the strings, we have to analyze if the strings ends in a deep.

Similarly, we can prove that $M(D_m) = L$ and that there is a path of irreducible morphisms of the form:

$$P_{a_m} \rightsquigarrow L = M(D_m) \to M(D_{m-1}) \to \ldots \to M(D_1) \to M(D_0) = S_{a_m} \tag{9}$$

where the path $L \rightsquigarrow S_{a_m}$ has length m.

Now, from the paths (8) and (9) we obtain the path

$$P_{a_m} \rightsquigarrow L \rightsquigarrow S_{a_m} \rightsquigarrow L \rightsquigarrow I_{a_m}, \tag{10}$$

where the cycle $L \rightsquigarrow S_m \rightsquigarrow L$ clearly has length $2m$.

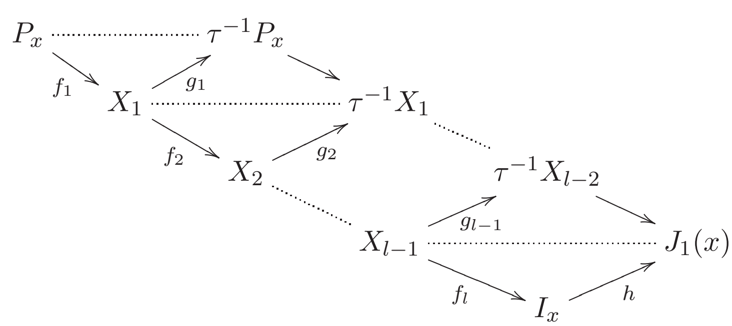
It is left to prove that the path (10) is sectional. By construction the paths $P_{a_m} \rightsquigarrow L \rightsquigarrow S_{a_m}$ and $S_{a_m} \rightsquigarrow L \rightsquigarrow I_{a_m}$ are sectional. We must analyze the path $M(D_1) \to S_{a_m} \to M(C_1)$. Note that $M(D_1)$ is injective, since $D_1 = \gamma_m B_{n-1}^{-1}$, where B_{n-1} and γ_m are string ending in a peak. More precisely, since $e(B_{n-1}) = x = e(\gamma_m)$, then $M(D_1) = I_x$. Therefore, $M(C_1) \not\simeq \tau^{-1} I_x$ and the path (10) is sectional, proving the result. □

Proposition 5.5. *Let $A = (U(m, n-1), I)$, with $m, n \geq 2$. Consider $L = M(\gamma_m B_{n-1}^{-1} \overline{G}_1)$ and $N = M(B_{n-2}^{-1} \overline{G}_1)$. Then $f : L \to N$ is an irreducible epimorphism with $d_l(f) = n-1$.*

Proof. Consider $L = M(C)$ and $N = M(D)$, where $C = \gamma_m \beta_{n-1}^{-1} \dots \beta_1^{-1} \gamma_1 \dots \gamma_{m-1}$ and $D = \beta_{n-2}^{-1} \dots \beta_1^{-1} \gamma_1 \dots \gamma_{m-1}$ is a string not starting in a deep. By [6] we have that $f : M(C) \to M(D)$ is an irreducible epimorphism, where $\mathrm{Ker}(f) \simeq M(C)/M(D) \simeq M(\gamma_m) = P_{a_m}$. Since A is representation finite, then $d_l(f) < \infty$.

Now, we compute the left degree of f. Since $e(\gamma_m) = x$, we consider the module $I_x = M(B_{n-1} \gamma_m^{-1})$. The indecomposable direct summands of $I_x / \mathrm{soc} I_x$ are $J_1(x) = M(B_{n-2})$ and $J_2(x) = S_{a_m}$.

Consider the irreducible morphism $h : I_x \to J_1(x)$, where $\mathrm{Ker}(f) \simeq M(\gamma_m) \simeq \mathrm{Ker}(h)$. Assume that $d_l(h) = l$. Then $f : L \to N$ is one of the morphisms $g_i : X_i \to \tau^{-1} X_{i-1}$ of the following configuration of almost split sequences:



where $\alpha'(P_x) = 1$ and $\phi : P_x \to X_1 \to \cdots \to X_{l-1} \to I_x$ is a sectional path. The modules that appear in ϕ are the string modules of the set $\mathcal{C}_{\gamma_m}$. In particular, L is one of such modules. Since $L \not\simeq P_x$ and $L \not\simeq I_x$, then $L \simeq X_j$, for some j, $1 \leq j \leq l-1$.

On the other hand, by the proof of Proposition 5.4, there is a sectional path

$$\rho : P_{a_m} \to M_1 \to \ldots \to L \to \ldots \to M_{m+n-3} \to I_x \to S_{a_m}$$

of length $m+n-1$ and where $L \rightsquigarrow S_{a_m}$ has length m.

Since $\dim_k(\mathrm{Hom}_A(P_{a_m}, S_{a_m})) = 1$, then $l = m+n-2$. We claim that for each i, $1 \leq i \leq l-1$ we have that $M_i \simeq X_i$. In fact, since $\alpha'(P_m) = 1$, then $X_1 \simeq M_1$. Now, since ρ is a sectional path, $M_2 \not\simeq \tau^{-1} P_x$. Then $X_2 \simeq M_2$. Following this argument, we get that $X_i \simeq M_i$, for $1 \leq i \leq l-1$. Since the path $L \rightsquigarrow S_{a_m}$ has length m, then $L \simeq X_{n-1}$ and therefore we obtain that $d_l(f) = n-1$. □

Remark 5.6. By the proofs of Proposition 5.4 and 5.5, we have the existence of a sectional path

$$P_{a_m} \overset{\phi}{\rightsquigarrow} L \rightsquigarrow S_{a_m} \rightsquigarrow L \rightsquigarrow I_{a_m}$$

where the path ϕ is of length $n-1$ and the cycle $L \rightsquigarrow L$ is of length $2m$. Moreover, we know that there exists an irreducible morphism $f : L \to N$. We claim that the module N belongs to such sectional path. In fact, in the proof of Proposition 5.4, we give an order for the strings $C_1, \ldots, C_m$ of the set $\mathcal{C}_{\varepsilon_{a_m}}$, where $C_m = \gamma_m B^{-1}_{n-1}\overline{G}_1$ and $M(C_m) = L$.

Observe that C_m is a string that starts on a peak. Hence $C_m = \gamma_m \beta^{-1}_{n-1} B^{-1}_{n-2}\overline{G}_1$. Then $C_{m+1} = B^{-1}_{n-2}\overline{G}_1$ and $N = M(C_{m+1})$. Moreover, N does not belong to the path $P_{a_m} \rightsquigarrow S_{a_m}$, because the string $B^{-1}_{n-2}\overline{G}_1$ is not a vertex of the quiver $Q^s_{a_m}$. Hence, we conclude that the above sectional path is of the form

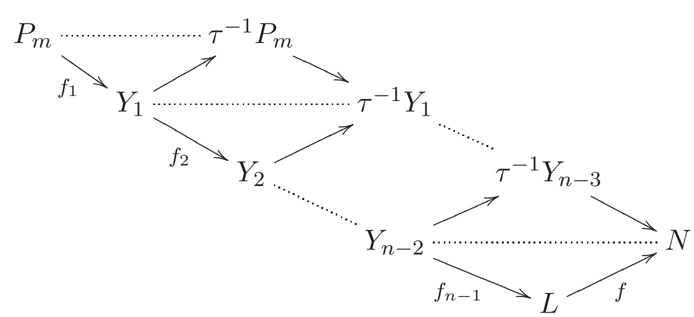
$$P_{a_m} \rightsquigarrow L \rightsquigarrow S_{a_m} \rightsquigarrow L \to N \rightsquigarrow I_{a_m}$$

where the arrow denotes an irreducible morphism.

Now, we are in position to prove the theorem.

Theorem 5.7. *Let $A = (U(m, n-1), I)$, with $m, m \geq 2$. Then there are irreducible morphisms $h_i : X_i \to X_{i+1}$ for $1 \leq i \leq n$, between indecomposable $A-$modules, such that $h_n \ldots h_1 \in \Re^{n+2m}(X_1, X_{n+1}) \backslash \Re^{n+2m+1}(X_1, X_{n+1})$, $h_{n-1} \ldots h_1 \notin \Re^n(X_1, X_n)$ and $h_n \ldots h_2 \notin \Re^n(X_2, X_{n+1})$.*

Proof. Consider the irreducible epimorphism $f : L \to N$ from Proposition 5.5. Then $d_l(f) = n-1$ and $\mathrm{Ker}(f) = P_m$. Then there is a configuration of almost split sequences as follows:



where $\delta : P_m \to Y_1 \to \ldots \to Y_{n-2} \to L$ is a sectional path of length $n-1$ and $f f_{n-1} \ldots f_1 = 0$. By Remark 5.6 there is a sectional path

$$P_m \overset{\phi}{\rightsquigarrow} L \overset{\rho_1}{\rightsquigarrow} S_m \overset{\rho_2}{\rightsquigarrow} L \to N \rightsquigarrow I_m$$

where $\ell(\phi) = n-1$ and $\ell(\rho_2\rho_1) = 2m$. Moreover, the modules in the path $\phi : P_m \rightsquigarrow L$ are the same that the ones in the path δ.

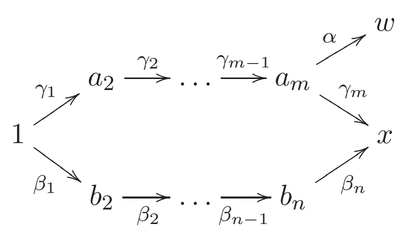
Consider $X_1 = P_m$, $X_n = L$, $X_{n+1} = N$ and $X_i = Y_{i+1}$ for $1 \leq i \leq n-1$. We define the irreducible morphisms $h_i = f_i$ for $1 \leq i \leq n-2$, $h_{n-1} = f_{n-1} + f_{n-1}\rho$, where $\rho : L \rightsquigarrow L$ is a composition of $2m$ irreducible morphisms which form part of the sectional path $\rho_2\rho_1$, and $h_n = f$. Then the composition

$$\begin{aligned} h_n \dots h_1 &= f(f_{n-1} + f_{n-1}\rho) f_{n-2} \dots f_1 \\ &= f f_{n-1} f_{n-2} \dots f_1 + f f_{n-1} \rho f_{n-2} \dots f_1 \\ &= f f_{n-1} \rho f_{n-2} \dots f_1. \end{aligned}$$

belongs to $\Re^{n+2m}(X_1, X_{n+1}) \backslash \Re^{n+2m+1}(X_1, X_{n+1})$, because the morphisms belong to a sectional path of length $n + 2m$. Furthermore, $h_{n-1} \dots h_1 \notin \Re^n(X_1, X_n)$ and by [8, Proposition 2.3] we have that $h_n \dots h_2 \notin \Re^n(X_2, X_{n+1})$, proving the result. □

In the families of algebras presented in Theorem 5.7, there are n irreducible morphisms such that their composition belong to $\Re^{n+t} \backslash \Re^{n+t+1}$, for $t \geq 4$ and moreover where t is an even number.

Below, we present a family of algebras for t an odd number. Consider $(V(m,n), J)$ for $n \geq 3$ and $m \geq 2$ as follows:



with $J = <\gamma_{m-1}\gamma_m>$.

We only state the result, since it can be proved with similar techniques as in Theorem 5.7.

Theorem 5.8. *Let $A = (V(m, n-2), J)$, with $m \geq 2$ and $n \geq 3$. Then there are irreducible morphisms $h_i : X_i \to X_{i+1}$ for $1 \leq i \leq n$, between indecomposable A-modules, such that $h_n \dots h_1 \in \Re^{n+2m+1}(X_1, X_{n+1}) \backslash \Re^{n+2m+2}(X_1, X_{n+1})$, $h_{n-1} \dots h_1 \notin \Re^n(X_1, X_n)$ and $h_n \dots h_2 \notin \Re^n(X_2, X_{n+1})$.*

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