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Quantum Mechanics: Modal Interpretation and Galilean Transformations

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Abstract The aim of this paper is to consider in what sense the modal-Hamiltonian interpretation of quantum mechanics satisfies the physical constraints imposed by the Galilean group. In particular, we show that the only apparent conflict, which follows from boost-transformations, can be overcome when the definition of quantum systems and subsystems is taken into account. On this basis, we apply the interpretation to different well-known models, in order to obtain concrete examples of the previous conceptual conclusions. Finally, we consider the role played by the Casimir operators of the Galilean group in the interpretation.

Keywords Modal-Hamiltonian interpretation · Galilean group · Casimir operators

1 Introduction

In the physical literature on quantum mechanics, the Galilean covariance of the theory is not one the most discussed topics. This situation can also be noticed in the ample literature on interpretation: the relationships between interpretation and Galilean transformations is a question usually not addressed with enough detail. However, it seems quite clear that the Galilean group imposes definite constraints to interpretation, in the sense that interpretative conclusions should not violate the physical meaning of the requirement of covariance.

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In our modal-Hamiltonian interpretation (see [1, 2]), the Galilean group plays a relevant role in endowing the formal structure of quantum mechanics with a physical content. In this context, the study of the space-time symmetry transformations serves a dual purpose: the identification of the fundamental physical magnitudes of the theory and the explanation of the central role played by the Hamiltonian in the selection of the constants of motion of the quantum system. Nevertheless, the behavior of the interpretative postulates under the Galilean transformation was not explored. The aim of this paper is to address this issue by considering in what sense the modal-Hamiltonian rule of definite-value ascription satisfies the physical constraints imposed by the Galilean group. In particular, we will show that the only apparent conflict, which follows from boost-transformations, can be overcome when the definition of quantum systems and subsystems is taken into account. On this basis, we will apply the interpretation to different well-known models, in order to obtain concrete examples of the previous conceptual conclusions. Finally, we will consider the role played by the Casimir operators of the Galilean group in the interpretation.

2 The Modal-Hamiltonian Interpretation

During the last decades, the research on the formal properties of the mathematical structure of quantum mechanics has shown a great advance: many results, unknown by the founding fathers of the theory, have been obtained, and this work has greatly improved the understanding of the deep obstacles that any interpretation must face. However, this interest in the features of the formalism has sometimes led to forget the physical content of quantum mechanics. In fact, in the last times, realist interpretations usually rely on mathematical results and focus their attention mainly on the formal model of the measurement problem. But quantum mechanics is a physical theory by means of which an impressive amount of experimental evidence has been accounted for.

Our recently proposed modal-Hamiltonian interpretation belongs to the modal family (whose roots can be found in certain works of van Fraassen, [3–5]): like previous modal interpretations, it is a realist, non-collapse approach according to which the quantum state describes the possible properties of the system but not its actual properties (see [6]). However, our interpretation moves away from the present trend in the research on the subject in the sense that it places an element with a clear physical meaning, the Hamiltonian of the system, at the heart of the proposal. From our interpretational perspective, the Hamiltonian is decisive in the definition of the quantum system and in the selection of its definite-valued observables.

2.1 Systems and Subsystems

In order to study the physical world, we have to identify the systems that populate it. We can cut out the physical reality in many different ways, but only when a portion of reality does not interact with others we obtain a system that obeys the dynamical postulate of quantum mechanics. For this reason, we conceive as quantum systems only those pieces of reality non-interacting with other pieces. On this basis, and by

adopting an algebraic perspective, we define a *quantum system* S as a pair $(\mathcal{O}, H)$ such that (i) $\mathcal{O}$ is a space of self-adjoint operators on a Hilbert space $\mathcal{H}$, representing the *observables* of the system, (ii) $H \in \mathcal{O}$ is the time-independent *Hamiltonian* of the system, and (iii) if $\rho_0 \in \mathcal{O}'$ (where $\mathcal{O}'$ is the dual space of $\mathcal{O}$) is the *initial state* of S, it evolves according to the Schrödinger equation in its von Neumann version.

Of course, any quantum system can be decomposed in parts in many ways; however, not any decomposition will lead to parts which are, in turn, quantum systems. This will be the case only when there is no interaction among the components and, then, the components' behaviors are *dynamically independent* to each other (see [7, 8]). On this basis, we say that a quantum system $S : (\mathcal{O}, H)$ with initial state $\rho_0 \in \mathcal{O}'$ is *composite* when it can be partitioned into two quantum systems $S^1 : (\mathcal{O}^1, H^1)$ and $S^2 : (\mathcal{O}^2, H^2)$ such that (i) $\mathcal{O} = \mathcal{O}^1 \otimes \mathcal{O}^2$, and (ii) $H = H^1 \otimes I^2 + I^2 \otimes H^1$ (where I^2 and I^2 are the identity operators in the corresponding tensor product spaces). In this case, the initial states of S^1 and S^2 are obtained as the partial traces $\rho_0^1 = Tr_{(2)}\rho_0 \in \mathcal{O}^{1\prime}$ and $\rho_0^2 = Tr_{(1)}\rho_0 \in \mathcal{O}^{2\prime}$, and we say that S^1 and S^2 are *subsystems* of the composite system, symbolized as $S = S^1 \cup S^2$.

It has to be emphasized that, although this definition of composite quantum system is completely general, the decomposition of a quantum system into subsystems is not always possible: it may happen that there is no partition of the whole S such that the total Hamiltonian can be expressed as a sum of component Hamiltonians. In this case, the quantum system is not composite, and we call it *elemental*.

On the other hand, given $S^1 : (\mathcal{O}^1, H^1)$ and $S^2 : (\mathcal{O}^2, H^2)$, with initial states $\rho_0^1 \in \mathcal{O}^{1\prime}$ and $\rho_0^2 \in \mathcal{O}^{2\prime}$ respectively, we can always define a system $S : (\mathcal{O}, H)$ with initial state ρ_0 such that (i) $\mathcal{O} = \mathcal{O}^1 \otimes \mathcal{O}^2$, (ii) $H = H^1 \otimes I^2 + I^1 \otimes H^2 + H_{int}$, where H_{int} is the *interaction Hamiltonian*, and (iii) $\rho_0 = \rho_0^1 \otimes \rho_0^2 \in \mathcal{O}'$. In this case, the initial state ρ_0 of S and the initial states ρ_0^1 of S^1 and ρ_0^2 of S^2 are still related by a partial trace. However, when the two systems S^1 and S^2 interact with each other, $H_{int} \neq 0$ and, therefore, ρ_0^1 and ρ_0^2 do not evolve according to the Schrödinger equation. This means that, strictly speaking, S^1 and S^2 are not subsystems of S but have to be considered as mere "parts" of S; we symbolize this fact as $S = S^1 + S^2$. Only in the particular case that $H_{int} = 0$, S^1 and S^2 will evolve unitarily, and they will properly be subsystems of the composite system S.

Summing up, the modal-Hamiltonian interpretation supplies a precise criterion for distinguishing between elemental and composite systems, and such a criterion is based on the system's Hamiltonian.

2.2 The Selection of the Preferred Context

Given the constraints imposed by the Kochen-Specker theorem ([9]), the subtlest point in any realist interpretation of quantum mechanics is the selection of the preferred context, that is, the set of the definite-valued observables of the system. In the modal-Hamiltonian interpretation this selection is based on the *actualization rule*, which defines, among all the observables of the system, those that acquire actual, and not merely possible, values.

As it is well known, to say that the Hamiltonian is symmetric or invariant under a certain continuous transformation means that (see [10])

$$e^{iKs}He^{-iKs} = H \quad \text{then} \quad [H, K] = 0 \tag{1}$$

where s is the parameter of the transformation and K is the corresponding generator. This means that, when H is invariant under a certain transformation, the generator of that transformation is a constant of motion: each symmetry of H defines a conserved quantity. Moreover, each symmetry of the Hamiltonian leads to an energy degeneracy. In fact, if H is invariant under a continuous transformation with generator K, we can write

$$KH|n\rangle = K\omega_n|n\rangle \quad \Rightarrow \quad HK|n\rangle = \omega_n K|n\rangle. \tag{2}$$

This means that any vector $K|n\rangle$, obtained by applying the operator K on the eigenvector $|n\rangle$, is also an eigenvector of H with the same eigenvalue (see [11, 12]). As a consequence, H can be expressed as

$$H = \sum_n \omega_n P_n \tag{3}$$

where P_n is the projector operator onto the subspace spanned by the degenerate eigenvectors corresponding to ω_n.

Now we have the conceptual elements necessary to present our actualization rule. The basic idea is that the Hamiltonian of the system defines actualization; therefore, any observable that does not have the symmetries of the Hamiltonian cannot acquire an actual value, since this actualization would break the symmetry of the system in an arbitrary way. Precisely, given an elemental quantum system $S : (\mathcal{O}, H)$, (i) if the Hamiltonian H is zero, there is no actualization, that is, none observable acquires definite value, and (ii) if the Hamiltonian H is not zero, the definite-valued observables are H and the observables commuting with H and having, at least, the same symmetries as H. Let us see how the rule works in different cases:

- If the Hamiltonian H does not have symmetries (it is non-degenerate), then:

$$H|n\rangle = \omega_n|n\rangle \quad \text{with } \omega_n \neq \omega_{n'} \text{ if } n \neq n' \tag{4}$$

  where $\{|n\rangle\}$ is a basis of the Hilbert space $\mathcal{H}$. In this case, the definite-valued observables of the system are H and all the observables commuting with H.
- If the Hamiltonian H has certain symmetries that lead to energy degeneracy, it can be written as

$$\begin{aligned} &H|n, i_n\rangle = \omega_n|n, i_n\rangle \quad \Rightarrow \\ &H = \sum_{n,i_n} \omega_n|n, i_n\rangle\langle n, i_n| = \sum_n \omega_n \sum_{i_n} |n, i_n\rangle\langle n, i_n| = \sum_n \omega_n P_n \end{aligned} \tag{5}$$

  where $\omega_n \neq \omega_{n'}$ for $n \neq n'$, and the index i_n expresses the degeneracy of the eigenvalue ω_n. Any observable of the form

$$A = \sum_{n,i_n} a_n|n, i_n\rangle\langle n, i_n| = \sum_n a_n \sum_{i_n} |n, i_n\rangle\langle n, i_n| = \sum_n a_n P_n \tag{6}$$

is definite-valued, since $[H, A] = 0$ and A has, at least, the same degeneracy of H. On the contrary, any observable of the form

$$B = \sum_{n,i_n} b_{n,i_n} |n, i_n\rangle\langle n, i_n| \tag{7}$$

in spite of commuting with H, does not acquire a definite value, since the actualization of a particular eigenvalue b_{n,i_n} of B would discriminate among the degenerate eigenvectors corresponding to a single eigenvalue ω_n of H and, in this way, would introduce in the system an asymmetry not contained in the Hamiltonian.

The fact that the Hamiltonian always belongs to the preferred context agrees with the many physical cases where the energy has definite value. The modal-Hamiltonian actualization rule has been applied to several well-known physical situations (hydrogen atom, Zeeman effect, fine structure, etc.), leading to results consistent with experimental evidence (see [1], Sect. 5). Moreover, it has proved to be effective for solving the measurement problem, both in its ideal and its non-ideal versions (see [1], Sect. 6).

3 The Galilean Group

The space-time symmetry group of non-relativistic −classical or quantum− mechanics is the Galilean group, defined by ten symmetry generators K_α, with $\alpha = 1$ to 10: one time displacement K_τ, three space-displacements K_{r_i}, three space-rotations K_{θ_i}, and three boost-velocity components K_{u_i}. The Galilean group is a Lie group with its associated Galilean algebra of generators. The central extension of the Galilean algebra is obtained as a semi-direct product between the Galilean algebra and the algebra generated by a central charge, which in this case denotes the mass operator $M = mI$, where I is the identity operator (see [13, 14]). In this central extension, the symmetry generators represent the basic magnitudes of the theory: the energy $H = \hbar K_\tau$, the three momentum components $P_i = \hbar K_{r_i}$, the three angular momentum components $J_i = \hbar K_{\theta_i}$, and the three boost components $G_i = \hbar K_{u_i}$. The rest of the physical magnitudes can be defined in terms of these basic ones: for instance, the three position components are $Q_i = G_i/m$, the three orbital angular momentum components are $L_i = \varepsilon_{ijk} Q_j P_k$ (where ε_{ijk} is such that $i \neq k$, $j \neq k$, $\varepsilon_{xyz} = \varepsilon_{yzx} = \varepsilon_{zxy} = 1$, $\varepsilon_{xzy} = \varepsilon_{yxz} = \varepsilon_{zyx} = -1$, and $\varepsilon_{ijk} = 0$ if $i = j$), the three spin components are $S_i = J_i - L_i$. In order to simplify the presentation, from now on we will use the expression 'Galilean group' and 'Galilean algebra' to refer to the corresponding central extension, and we will take $\hbar = 1$.

The Galilean group is defined by the commutation relations between its generators:

$$\begin{aligned}
&[P_i, P_j] = 0, && (8_a) && [G_i, P_j] = i\delta_{ij} M, && (8_f)\\
&[G_i, G_j] = 0, && (8_b) && [P_i, H] = 0, && (8_g)\\
&[J_i, J_j] = i\varepsilon_{ijk} J_k, && (8_c) && [J_i, H] = 0, && (8_h)
\end{aligned} \tag{8}$$

$$[J_i, P_j] = i\varepsilon_{ijk} P_k, \quad (8_d) \qquad [G_i, H] = i P_i, \quad (8_i)$$
$$[J_i, G_j] = i\varepsilon_{ijk} G_k. \quad (8_e)$$

Moreover, each Galilean transformation $\mathcal{T}_\alpha$ acts on observables and states as

$$O \to O' = U_{s_\alpha} O U_{s_\alpha}^{-1}, \qquad |\varphi\rangle \to |\varphi'\rangle = U_{s_\alpha}|\varphi\rangle \tag{9}$$

where s_α is the parameter corresponding to the transformation $\mathcal{T}_\alpha$, and U_{s_α} is the family of unitary operators describing $\mathcal{T}_\alpha$. Since in any case s_α is a continuous parameter, each U_{s_α} can be expressed in terms of the corresponding symmetry generator K_α as

$$U_{s_\alpha} = e^{iK_\alpha s_\alpha}. \tag{10}$$

The combined action of all the transformations is given by

$$U_s = \prod_{\alpha=1}^{10} e^{iK_\alpha s_\alpha}. \tag{11}$$

When the state vector is represented as a function of space-time coordinates, there is an inverse relation between transformations on function space and transformations on coordinates (see [10]):

$$\varphi(\mathbf{x}, t) = U_s \varphi(\mathbf{x}', t'). \tag{12}$$

In the case of time-displacement, the transformation is $(\mathbf{x}, t_0) \to (\mathbf{x}, t_0 + \tau)$ and $U_s = U_\tau = e^{iH\tau}$, where H is the generator of the transformation and τ is the corresponding continuous parameter:

$$|\varphi(t_0)\rangle = e^{iH\tau}|\varphi(t_0 + \tau)\rangle. \tag{13}$$

Then, by making $t_0 = 0$ and $\tau = t$, we obtain

$$|\varphi(t)\rangle = e^{-iHt}|\varphi(0)\rangle. \tag{14}$$

This equation, which has the form of a solution of the Schrödinger equation, can be obtained only when H is independent of t and, as a consequence, it is the generator of time-displacements. This means that the Schrödinger equation has the physical meaning of describing time-displacements only for time-independent Hamiltonians, that is, for closed systems. On the other hand, H may have the remaining space-time symmetries or not. As we have seen, to say that the Hamiltonian is symmetric or invariant under a certain continuous transformation $\mathcal{T}_\alpha$ means that $[H, K_\alpha] = 0$ and, therefore, K_α is a constant of motion of the system (see (1)).

As it is well-known, the dynamical law of quantum mechanics is the Schrödinger equation, which only depends on the Hamiltonian. Therefore, the invariance or non-invariance of the Schrödinger equation under a certain symmetry transformation depends on the invariance or non-invariance of H. The invariance of the Schrödinger equation under time-displacements is guaranteed by the respective invariance of H ($[H, H] = 0$), and it expresses the homogeneity of time. The invariance of the Schrödinger equation under space-displacements and/or space-rotations amounts to

the respective invariance of H ($[P_i, H] = 0$ and/or $[J_i, H] = 0$), and this expresses the homogeneity and/or the isotropy of space, respectively. In turn, the homogeneity and the isotropy of space is guaranteed by the validity of the Galilean group under the form given by (8), since $[P_i, H] = 0$ and $[J_i, H] = 0$ belong to the set of commutation relations that define the group (see (8_g) and (8_h)).

Nevertheless, space is not always homogeneous and isotropic. In non-relativistic quantum mechanics, fields are not quantized: they are treated as classical fields that act on the quantum system by breaking the homogeneity and/or the isotropy of space. This breaking turns out to be expressed in the form of the Hamiltonian: the non-homogeneity of space leads to the fact that, at least, some P_i is not a constant of motion ($[P_i, H] \neq 0$); the non-isotropy of space leads to the fact that, at least, some J_i is not a constant of motion ($[J_i, H] \neq 0$). And this, in turn, amounts to the breaking of the full validity of the Galilean group under the form of (8); then, the commutation relations involving the Hamiltonian ((8_g), (8_h) and (8_i)) are, in general, no longer valid (see [10]).

4 Modal-Hamiltonian Interpretation and Boost-Transformations

In this section we will consider the modal-Hamiltonian interpretation in the light of the symmetry transformations of the Galilean group. For this reason, we will analyze the situation where the group is not broken by external fields. The purpose will be to see whether the definite-value ascription resulting from the actualization rule satisfies the physical meaning of the symmetries expressed by the Galilean group.

Let us begin with recalling the physical meaning of a symmetry transformation. A continuous transformation, as in the case of the Galilean group, admits two interpretations. Under the active interpretation, the transformation corresponds to a change from one system to another—transformed—system; under the passive interpretation, the transformation consists in a change of the viewpoint—reference frame—from which the system is described (see [15]). Nevertheless, in both cases the validity of a group of symmetry transformations expresses the fact that the identity and the behavior of the system are not altered by the application of the transformations: in the active interpretation language, the original and the transformed systems are equivalent; in the passive interpretation language, the original and the transformed reference frames are equivalent.

In the case of the Galilean group, and adopting the passive interpretation language, the validity of the group amounts to the equivalence between reference frames time-displaced, space-displaced or space-rotated with respect to each other, and between inertial reference frames: the application of a Galilean transformation does not introduce a modification in the physical situation, but only expresses a change of the perspective from which the system is described. It is quite clear that any adequate interpretation of quantum mechanics should not violate this physical meaning of the Galilean transformations. In particular, if a realist interpretation determines the preferred context that selects the definite-valued observables of the system, such a context should not change under the transformations of the group: from a realist viewpoint, it would be unacceptable that the set of definite-valued observables were

different as the mere result of a change in the reference frame from which the system is described. Therefore, one is entitled to ask whether the modal-Hamiltonian interpretation satisfies this constraint.

As we have seen, the preferred context selected by the modal-Hamiltonian actualization rule only depends on the Hamiltonian of the system. Then, at first sight, the requirement of invariance of the preferred context under the Galilean transformations would amount to the requirement of invariance for the Hamiltonian. It is easy to see that this requirement is fulfilled in the case of time-displacement, space-displacement and space-rotation:

- $H' = e^{iH\tau} H e^{-iH\tau} = H$ (since $[H, H] = 0$).
- $H' = e^{iP_i r_i} H e^{-iP_i r_i} = H$ (since $[P_i, H] = 0$, see (8_g))
- $H' = e^{iJ_i\theta_i} H e^{-iJ_i\theta_i} = H$ (since $[J_i, H] = 0$, see (8_h))

However, it is not clear that the requirement of invariance of the preferred context completely holds, since the Hamiltonian is not invariant under boost-transformations. In fact, under a boost-transformation corresponding to a velocity u_x, H changes as

$$H' = e^{iG_x u_x} H e^{-iG_x u_x} \neq H \quad (\text{since } [G_x, H] = iP_x \neq 0, \text{ see } (8_i)). \tag{15}$$

This seems to lead to the conclusion that the preferred context selected when the system is described in the reference frame RF is different that the preferred context selected in the boost-transformed reference frame RF': the set of definite-valued observables would change in different inertial frames, and this fact would make an objective feature of the system to depend on the particular descriptive viewpoint adopted. Of course, this conclusion would be unacceptable on physical grounds. Nevertheless, the conflict can be solved when the modal-Hamiltonian interpretation is considered as a whole.

By means of the only tools of group theory it can be proved that, under a boost-transformation of velocity u_x, the Hamiltonian changes in a value T_B and the momentum changes in a value P_B (see Appendix A):

$$H' = H + T_B, \quad \text{where } T_B = -u_x P_x + M u_x^2/2, \tag{16}$$

$$P' = P + P_B, \quad \text{where } P_B = (-M u_x, 0, 0). \tag{17}$$

In turn, since the original Hamiltonian can be expressed as

$$H = W + \frac{P^2}{2m} \tag{18}$$

where W is the internal energy, the transformed Hamiltonian results (see Appendix A)

$$H' = W + \frac{(P + P_B)^2}{2m}. \tag{19}$$

For simplicity, let us consider a system S described in the reference frame RF_0 at rest with respect of the center of mass of S, in such a way that its Hamiltonian is H_0.

In RF_0 the momentum P_0 of the system is zero and, as a consequence,

$$P_0 = 0 \quad \Rightarrow \quad H_0 = W + \frac{P_0^2}{2m} = W. \tag{20}$$

Let us now consider the system S described in a reference frame RF_1 in motion with a constant velocity u_x with respect to RF_0. The new Hamiltonian H_1 is (see (16))

$$H_1 = H_0 + T_B = W + Mu_x^2/2. \tag{21}$$

This means that, in RF_1, H_1 is the sum of two terms: a Hamiltonian $H_0 = W$ relative to the center of mass of the system, and a Hamiltonian $H_K = T_B$ representing the total kinetic energy of translation. In turn, since $H_0 = W$ does not depend on the position nor on the momentum of the center of mass, but only on the differences of positions and their respective conjugate momenta, and H_K only depends on the boost-velocity, we can guarantee that $[H_0, H_K] = 0$. If $\mathcal{H}^Q$ is the Hilbert space of the wavefunctions of the differences of coordinates and $\mathcal{H}^P$ is the Hilbert space of the wavefunctions of the coordinates of the center of mass, then H_1 can be expressed as

$$H_1 = H_0 + H_K = H_W^Q \otimes I^P + I^Q \otimes H_K^P \tag{22}$$

where H_W^Q is the internal energy Hamiltonian acting on the Hilbert space $\mathcal{H}^Q$, H_K^P is the kinetic Hamiltonian acting on the Hilbert space $\mathcal{H}^P$, and I^Q and I^P are the identity operators on the respective spaces. But, according to the modal-Hamiltonian definitions of elemental and composite system, (22) expresses the fact that S is a composite system, whose elemental subsystems are:

- a system S_0, defined by the Hamiltonian H_W^Q relative to the center of mass, which represents the internal energy W.
- a system S_K, defined by the Hamiltonian H_K^P representing the translational kinetic energy.

In turn, the modal-Hamiltonian actualization rule applies to elemental quantum systems. This means that, according to the interpretation, both elemental systems, S_0 and S_K "actualize" independently:

- In S_0, the Hamiltonian H_W^Q (or $H_0 = W$) determines the set of definite-valued observables, which is the same set selected by the actualization rule in the reference frame RF_0.
- In S_K, the Hamiltonian H_K^P (or H_K) acquires a definite value and, with it, the total kinetic energy of translation T_B also turns out to be definite-valued.

This shows that, by contrast to what was originally supposed, in the context of the modal-Hamiltonian interpretation a boost-transformation does not modify the preferred context in a way that violates the physical meaning of the transformation. The only change resulting from passing from an inertial frame to another consists in the appearance of an elemental system S_K, non-interacting with S_0, where the kinetic energy acquires a definite value.

Although the above argument was developed by starting from a description in a reference frame relative to the center of mass, the argument is completely general since it can be applied to any description of a quantum system S not affected by external fields. In fact, by means of an adequate change of variables, the Hamiltonian of S can always be referred to the center of mass and, in a generic reference frame RF_1, its Hamiltonian H_1 reads

$$H_1 = H_0 + H_{K1} = W + \frac{P_1^2}{2m} \tag{23}$$

where now the momentum P_1 of the center of mass relative to RF_1 does not need to be zero. When a boost-transformation with velocity u_x is applied to this system, in the new reference frame RF_2 the Hamiltonian results (see (16))

$$H_2 = H_1 + T_B = H_0 + H_{K1} + T_B = W + \frac{P_1^2}{2m} - u_x P_{1x} + M u_x^2/2. \tag{24}$$

Therefore, the new Hamiltonian can be expressed again as the sum of two terms

$$H_2 = H_0 + H_{K2} \tag{25}$$

where the kinetic energy H_{K2} relative to RF_2 is (see (17))

$$H_{K2} = H_{K1} + T_B = \frac{P_1^2}{2m} - u_x P_{1x} + M u_x^2/2 = \frac{(P_1 + P_B)^2}{2m}. \tag{26}$$

In this case, $H_0 = W$ only depends on the differences of positions and their respective conjugate momenta, and H_{K2} only depends on the total momentum of the center of mass; therefore, again $[H_0, H_{K2}] = 0$. This means that, also in the generic case, the transformed Hamiltonian turns out to be the sum of a Hamiltonian representing the internal energy, and a kinetic Hamiltonian that includes the original kinetic energy and the kinetic energy of the boost. As a consequence, S is a composite system and the actualization rule applies to its elemental components: again, the only difference in passing from RF_1 to RF_2 is confined to the change in the total kinetic energy.

Summing up, when space is homogeneous and isotropic, in general a quantum system S can be decomposed as $S = S_0 \cup S_K$. The subsystem S_0, defined by the internal energy $H_0 = W$, is invariant under boost-transformations and, therefore, its set of definite-valued observables is invariant under all the Galilean transformations. The subsystem S_K, defined by the kinetic energy H_K, is modified by a boost-transformation: H_K changes in the value of the kinetic energy of the boost. This change has a manifestation in the energy spectrum of the composite system. In fact, the Hamiltonian of S is (see (22))

$$H = H_0 + H_K = H_W^Q \otimes I^P + I^Q \otimes H_K^P \tag{27}$$

where $H_W^Q \in \mathcal{H}^Q \otimes \mathcal{H}^Q$ and $H_K^P \in \mathcal{H}^P \otimes \mathcal{H}^P$ can be written as[1]

$$H_W^Q = \int \omega_W |\omega_W\rangle\langle\omega_W| d\omega_W, \qquad H_K^P = \int \omega_K |\omega_K\rangle\langle\omega_K| d\omega_K. \tag{28}$$

Therefore, if $\mathcal{H} = \mathcal{H}^Q \otimes \mathcal{H}^P$, $H \in \mathcal{H} \otimes \mathcal{H}$ results

$$H = \int\int (\omega_W + \omega_K) |\omega_W, \omega_K\rangle\langle\omega_W, \omega_K| d\omega_W d\omega_K \tag{29}$$

where the eigenvalues of H are $\omega = \omega_W + \omega_K$. Since a boost-transformation only acts on the kinetic Hamiltonian, the transformed H_K^P results

$$H_K^P \longrightarrow H_K^{P\prime} = e^{iG_x u_x} H_K^P e^{-iG_x u_x} = \int \omega_K e^{iG_x u_x} |\omega_K\rangle\langle\omega_K| e^{-iG_x u_x} d\omega_K. \tag{30}$$

Then,

$$H_K^{P\prime} = \int \omega_K |\omega_K + \omega_B\rangle\langle\omega_K + \omega_B| d\omega_K = \int (\omega_K - \omega_B) |\omega_K\rangle\langle\omega_K| d\omega_K. \tag{31}$$

This means that the action of the boost produces a Doppler shift on the composite system, that is, a displacement of the energy spectrum in a constant value (see [16]). In other words, the information of the structure of the energy spectrum is carried by the internal energy W of the composite system, and the kinetic energy only introduces a shift of that spectrum. This is completely consistent with the result of the application of the actualization rule: according to it, in the subsystem S_0, $H_0 = W$ is a definite-valued observable that determines the invariant structure of the energy spectrum, and the definite-valued H_K of S_K is what produces the shift of the energy spectrum of the composite system S.

When the homogeneity and/or the isotropy of space is broken by external fields, the above arguments are no longer valid. In fact, although a generic Hamiltonian can always be expressed as the sum of a potential term and a kinetic term, the potential term may depend on positions and, therefore, the non-commutation of the two terms cannot be guaranteed: even if the Hamiltonian is referred to the center of mass, in general the absolute position Q_C of the center of mass and the relative positions $Q_{\alpha\beta}$ are coupled in such a way that the kinetic term and the potential term cannot be conceived as non-commuting Hamiltonians belonging to two quantum elemental—non-interacting—subsystems of a composite system. In this case, not only a boost-transformation, but also a space-displacement and a space-rotation may modify the result of the actualization rule based on the Hamiltonian. In fact,[2]

[1] For simplicity, non-degenerate Hamiltonians are considered, but the same argument can be developed for degenerate Hamiltonians.

[2] Let us remember that, in the presence of external fields, the commutation relations involving the Hamiltonian (see (8_g), (8_h) and (8_i)) are, in general, not valid. This means that the Hamiltonian is no longer the generator of time-displacements: it only retains its role as the generator of the dynamic evolution (see [17]).

- If space is non-homogeneous in the direction defined by r_i, then $[P_i, H] \neq 0$ (the Galilean commutation relation (8_g) is no longer valid) and, then, the space-displaced H' results

$$H' = e^{iP_i r_i} H e^{-iP_i r_i} \neq H. \tag{32}$$

- If space is non-isotropic in the direction defined by θ_i, then $[J_i, H] \neq 0$ (the Galilean commutation relation (8_h) is no longer valid) and, then, the space-rotated H' results

$$H' = e^{iJ_i \theta_i} H e^{-iJ_i \theta_i} \neq H. \tag{33}$$

- If space is non-homogeneous and/or non-isotropic in direction i, then $[G_i, H] \neq iP_i$ (the Galilean commutation relation (8_i) is no longer valid) and, then, the boost-transformed H' results

$$H' = e^{iG_i u_i} H e^{-iG_i u_i} \neq W + \frac{(P + P_B)^2}{2m}. \tag{34}$$

But this fact has not to be considered problematic, since the breaking of the homogeneity and/or the isotropy of space amounts to the breaking of the Galilean group. In this situation, one cannot longer expect that a system preserves its behavior under any Galilean transformation. For instance, a space-displaced system in a non-homogeneous space cannot be strictly viewed as the same original system to the extent that the behavior of the displaced system may be completely different from the behavior of the original one. Therefore, it should not be surprising that the set of definite-valued observables also changes with space-displacement. The modal-Hamiltonian interpretation accounts for this fact by making the identity of a quantum system and its preferred context to be defined by the Hamiltonian.

5 The Modal-Hamiltonian Interpretation at Work

In this section we will illustrate the conceptual arguments just developed by analyzing different well-known physical models. In particular, we will show how a system non affected by external fields can be described in such a way that its composite nature becomes manifest. In turn, in the case of the action of external fields, we will consider the conditions under which the system can nevertheless be conceived as a composite system, and we will explicitly describe the subsystems.

5.1 The Harmonic Oscillator

In general, an harmonic oscillator is an object affected by a quadratic potential energy, which produces a restoring force against displacement from equilibrium that is proportional to the displacement. Under a usual description, a quantum harmonic oscillator is a system of two bodies interacting through a potential quadratic in the relative displacement. The Hamiltonian of the system reads

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + k(Q_1 - Q_2)^2 \tag{35}$$

where k measures the strength of the interaction. Now we can define the coordinates of the center of mass and the relative coordinates,

$$Q_C = \frac{m_1 Q_1 + m_2 Q_2}{m_1 + m_2}, \qquad Q_R = Q_1 - Q_2, \tag{36}$$

$$P_C = M\dot{Q}_C = P_1 + P_2, \qquad P_R = \mu \dot{Q}_R = \frac{m_2 P_1 - m_1 P_2}{m_1 + m_2} \tag{37}$$

where $M = m_1 + m_2$ is the total mass, and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. In this new coordinate system, the Hamiltonian can be written as

$$H = \frac{P_C^2}{2M} + \frac{P_R^2}{2\mu} + k Q_R^2. \tag{38}$$

Then, H can be expressed as the sum of two terms, $H = H_0 + H_K$, such that

$$H_0 = W = H - \frac{P_C^2}{2M} = \frac{P_R^2}{2\mu} + k Q_R^2, \qquad H_K = \frac{P_C^2}{2M}. \tag{39}$$

It is quite clear that H_0 and H_K commute, since H_0 only depends on the relative coordinates and H_K only depends on the coordinates of the center of mass. Therefore, the total Hamiltonian can be written as[3]

$$[H_0, H_K] = 0 \quad \Longrightarrow \quad H = H_0 \otimes I_K + I_0 \otimes H_K. \tag{40}$$

In this case it is easy to see that H is the Hamiltonian of a composite system S, whose subsystems, S_0 with Hamiltonian H_0 and S_K with Hamiltonian H_K, do not interact with each other; thus, they "actualize" independently. This means that H_0 (internal energy) acquires a definite value in S_0, and H_K (kinetic energy) acquires a definite value in S_K.

5.2 The Hydrogen Atom

The hydrogen atom is conceived as a two-body system consisting of an electron and a proton interacting with each other through a Coulombian potential. In this case, the Hamiltonian reads

$$H = \frac{P_e^2}{2m_e} + \frac{P_p^2}{2m_p} - \frac{e^2}{|Q_e - Q_p|} \tag{41}$$

where e is the electric charge of the electron, and the indexes e and p refer to the electron and the proton respectively. As usual, if we take the center of mass coordinates and the relative coordinates as independent variables, we will obtain (36) and (37),

[3]For notation simplicity, from now on we will ignore the difference between H_K and H_K^P and between H_0 and H_W^Q.

with the indexes e and p instead of 1 and 2 respectively. Then, in the new coordinates, the total Hamiltonian results

$$H = \frac{P_C^2}{2M} + \frac{P_R^2}{2\mu} - \frac{e^2}{|Q_R|} \tag{42}$$

where we can identify H_0 and H_K as

$$H_0 = W = H - \frac{P_C^2}{2M} = \frac{P_R^2}{2\mu} - \frac{e^2}{|Q_R|}, \qquad H_K = \frac{P_C^2}{2M}. \tag{43}$$

Here it is also clear that

$$[H_0, H_K] = 0 \quad \Longrightarrow \quad H = H_0 \otimes I_K + I_0 \otimes H_K. \tag{44}$$

Again, the hydrogen atom is a composite system that can be analyzed into a subsystem S_0, defined by the internal energy $H_0 = W$, and a subsystem S_K, defined by the kinetic energy H_K. And, according to the modal-Hamiltonian actualization rule, both subsystems "actualize" independently.

5.3 Three-Body System

The two previous examples were two-body systems. Here we will show that the same conclusions can be drawn for three-body systems with internal potentials.

Let us consider a system of three particles that interact through a generic potential depending on the relative positions. The Lagrangian of the system reads

$$\begin{aligned} L = {} & \frac{1}{2}m_1\dot{Q}_1^2 + \frac{1}{2}m_2\dot{Q}_2^2 + \frac{1}{2}m_3\dot{Q}_3^2 - V(|Q_1 - Q_2|) - V(|Q_2 - Q_3|) \\ & - V(|Q_1 - Q_3|). \end{aligned} \tag{45}$$

Let us define the new position coordinates,

$$Q_C = \frac{m_1 Q_1 + m_2 Q_2 + m_3 Q_3}{m_1 + m_2 + m_3}, \tag{46}$$

$$Q_{12} = Q_1 - Q_2, \qquad Q_{13} = Q_1 - Q_3, \qquad Q_{23} = Q_2 - Q_3. \tag{47}$$

But since Q_{12}, Q_{13} and Q_{23} are not independent ($Q_{23} = Q_{12} + Q_{13}$), we will select Q_C, Q_{12} and Q_{13} to express the Hamiltonian. If we define the total mass $M = m_1 + m_2 + m_3$ and the reduced masses $\mu_{ij} = m_i m_j / M$, the Lagrangian in the new coordinate system results

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2}M\dot{Q}_C^2 + \frac{1}{2}\mu_{12}\dot{Q}_{12}^2 + \frac{1}{2}\mu_{13}\dot{Q}_{13}^2 + \frac{1}{2}\mu_{23}(\dot{Q}_{12} - \dot{Q}_{13})^2 \\ & - V(|Q_{12}|) - V(|Q_{13}|) - V(|Q_{12} - Q_{13}|). \end{aligned} \tag{48}$$

The conjugate momenta are obtained from

$$P_C = \frac{\partial L}{\partial \dot{Q}_C} = M\dot{Q}_C, \tag{49}$$

$$P_{12} = \frac{\partial L}{\partial \dot{Q}_{12}} = -\mu_{23}(\dot{Q}_{12} - \dot{Q}_{13}) + \mu_{12}\dot{Q}_{12}, \tag{50}$$

$$P_{13} = \frac{\partial L}{\partial \dot{Q}_{13}} = \mu_{23}(\dot{Q}_{12} - \dot{Q}_{13}) + \mu_{13}\dot{Q}_{13}. \tag{51}$$

Then, the total Hamiltonian associated with the Lagrangian L reads

$$\begin{aligned} H &= P_C\dot{Q}_C + P_{12}\dot{Q}_{12} + P_{13}\dot{Q}_{13} - L \\ &= \frac{P_C^2}{2M} + \frac{P_{12}^2}{2m_2} + \frac{P_{13}^2}{2m_3} + \frac{(P_{12}+P_{13})^2}{2m_1} \\ &\quad - V(|Q_{12}|) - V(|Q_{13}|) - V(|Q_{12}-Q_{13}|) \end{aligned} \tag{52}$$

where, again, we can identify

$$\begin{aligned} H_0 = W &= H - \frac{P_C^2}{2M} = \frac{P_{12}^2}{2m_2} + \frac{P_{13}^2}{2m_3} + \frac{(P_{12}+P_{13})^2}{2m_1} \\ &\quad - V(|Q_{12}|) - V(|Q_{13}|) - V(|Q_{12}-Q_{13}|), \\ H_K &= \frac{P_C^2}{2M} \end{aligned} \tag{53}$$

such that $[H_0, H_K] = 0$.

This procedure can easily be generalized to an n-body system, where we will always obtain the total Hamiltonian as the sum of two non-commuting terms: the internal energy term, $H_0 = W$, only depending on the relative coordinates, and the kinetic term, $H_K = P_C^2/2M$, only depending on the kinetic energy of the center of mass. The possibility of decomposing the n-body system into those two subsystems is strongly associated with the validity of the Galilean group (for the constraints imposed by the Galilean group on the allowed potential, see Appendix B).

5.4 Two-Body System with External Field

In the previous subsections we have analyzed systems not affected by external fields, that is, situations where one can guarantee the full validity of the Galilean group due to the homogeneity and the isotropy of space. In this subsection we will begin the study of systems with external fields, in order to see how the conclusions drawn in the previous sections break down when space looses the Galilean properties.

Let us consider a system of two particles interacting through an internal potential and affected by an external potential depending on their positions. The general Lagrangian for this system reads

$$\mathcal{L} = \frac{1}{2}m_1\dot{Q}_1^2 + \frac{1}{2}m_2\dot{Q}_2^2 - V_{12}(|Q_1 - Q_2|) - V_1(Q_1) - V_2(Q_2). \tag{54}$$

If we introduce the same change of coordinates as in the case of the harmonic oscillator (see (36) and (37)), the corresponding Hamiltonian results

$$H = \frac{P_C^2}{2M} + \frac{P_{12}^2}{2\mu} - V_{12}(|Q_{12}|) - V_1\left(Q_C + \frac{m_2}{M}Q_{12}\right) - V_2\left(Q_C + \frac{m_1}{M}Q_{12}\right). \tag{55}$$

In this Hamiltonian we could define H_K as $P_C^2/2M$ and H_0 as the remaining terms, but in this case H_0 would not be the internal energy W due to its dependence on the coordinates Q_C of the center of mass. In other words, in general $[H_0, H_K] \neq 0$ and, as a consequence, the system cannot be decomposed into quantum subsystems as in the previous cases.

Nevertheless, the system is composite in particular cases, when the potentials V_1 and V_2 have at least one of the following features:

1. Both V_1 and V_2 are linear in the positions, i.e. in the case of scalar potentials of the form $V_i = \alpha Q_i + \beta$. In this case,

$$V_1(Q_1) + V_2(Q_2) = \alpha_1 Q_1 + \beta_1 + \alpha_2 Q_2 + \beta_2 \tag{56}$$

and, since the change of coordinates is linear, the two last terms of (55) result

$$\begin{aligned} &V_1(Q_C, Q_{12}) + V_2(Q_C, Q_{12}) \\ &\quad = (\alpha_1 + \alpha_2) Q_C + \frac{Q_{12}}{M}(\alpha_1 m_2 + \alpha_2 m_1) + (\beta_1 + \beta_2). \end{aligned} \tag{57}$$

Therefore, the position Q_C of the center of mass and the relative positions Q_{12} turn out to be uncoupled, and the system can be decomposed into two subsystems S_R and S_C with their respective Hamiltonians

$$H_R = \frac{P_{12}^2}{2\mu} - V_{12}(|Q_{12}|) - \frac{Q_{12}}{M}(\alpha_1 m_2 + \alpha_2 m_1) - (\beta_1 + \beta_2), \tag{58}$$

$$H_C = \frac{P_C^2}{2M} + (\alpha_1 + \alpha_2) Q_C. \tag{59}$$

In this case, a boost-transformation only affects the subsystem S_C by adding the kinetic energy of the boost to its Hamiltonian H_C.

2. Both V_1 and V_2 are non-linear in the positions, but their sum cancels the term that crosses the coordinates Q_C and Q_{12}. In fact, let us suppose that

$$V_1(Q_1) + V_2(Q_2) = \alpha_1 Q_1^2 + \alpha_2 Q_2^2. \tag{60}$$

In this case, the two last terms of (55) result

$$\begin{aligned} &V_1(Q_C, Q_{12}) + V_2(Q_C, Q_{12}) \\ &\quad = (\alpha_1 + \alpha_2) Q_C^2 + \frac{Q_{12}^2}{M^2}(\alpha_1 m_2^2 + \alpha_2 m_1^2) + 2 Q_C \frac{Q_{12}}{M}(\alpha_1 m_2 + \alpha_2 m_1). \end{aligned} \tag{61}$$

It is easy to see that, if $\alpha_1 m_2 = -\alpha_2 m_1$, the term including the product between Q_C and Q_{12} cancels and, again, the system can be decomposed into two subsystems S_R and S_C, whose respective Hamiltonians read

$$H_R = \frac{P_{12}^2}{2\mu} - V_{12}(|Q_{12}|) - \frac{Q_{12}}{M}(\alpha_1 m_2^2 + \alpha_2 m_1^2), \tag{62}$$

$$H_C = \frac{P_C^2}{2M} + (\alpha_1 + \alpha_2) Q_C^2 \tag{63}$$

and, again, a boost-transformation will only affect the subsystem S_C.

5.5 Zeeman Effect

As we have seen in the previous subsection, although the breaking of the Galilean group by external fields in general precludes the decomposition of the whole system into quantum subsystems, in certain particular cases such a decomposition is possible. This is the case of a two-body system affected by a uniform magnetic field, as in the case of the Zeeman effect.

Let us begin with the general Lagrangian of the system

$$\mathcal{L} = \frac{1}{2} m_1 \dot{Q}_1^2 + \frac{1}{2} m_2 \dot{Q}_2^2 + q_1 A \cdot Q_1 + q_2 A \cdot Q_2 - V(|Q_1 - Q_2|) \tag{64}$$

where A is the vector potential. Now we introduce the same change of coordinates as in the case of the harmonic oscillator (see (36) and (37)). In the case that $q_1 = -e$ and $q_2 = +e$ the Lagrangian reads

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} M \dot{Q}_C^2 + \frac{1}{2} \mu \dot{Q}_R^2 - eA \cdot \left(Q_C + \frac{m_2}{M} Q_R \right) + eA \cdot \left(Q_C - \frac{m_1}{M} Q_R \right) - V(Q_R) \\ &= \frac{1}{2} M \dot{Q}_C^2 + \frac{1}{2} \mu \dot{Q}_R^2 - e \left(\frac{m_1}{M} + \frac{m_2}{M} \right) A \cdot Q_R - V(Q_R). \end{aligned} \tag{65}$$

Then, the Hamiltonian reads

$$\begin{aligned} H &= P_C \dot{Q}_C + P_R \dot{Q}_R - L \\ &= \frac{P_C^2}{2M} + \frac{P_R^2}{2\mu} + \frac{e}{\mu} P_R \cdot A - \frac{e^2}{2\mu} A^2 + e \left(\frac{m_1}{M} + \frac{m_2}{M} \right) A \cdot Q_R + V(Q_R). \end{aligned} \tag{66}$$

If the magnetic field is uniform, the vector potential can be written as $A = \frac{1}{2} Q_R \times B$, and the term $A \cdot Q_R$ is zero. In turn, $P_R \cdot A$ can be computed as

$$P_R \cdot A = -\frac{1}{2} P_R \cdot (P_R \times B) = -\frac{1}{2} (Q_R \times P_R) \cdot B = -\frac{1}{2} L \cdot B. \tag{67}$$

Therefore, the Hamiltonian can be written as

$$H = \frac{P_C^2}{2M} + \frac{P_R^2}{2\mu} + V(Q_R) - \frac{\mu_B}{\hbar} L \cdot B - \frac{e^2}{8\mu} (Q_R \times B)^2 \tag{68}$$

where $\mu_B = \frac{e\hbar}{2\mu}$ is the Bohr magneton. In this Hamiltonian we can immediately identify two commuting Hamiltonians H_R and H_C,

$$H_R = \frac{P_R^2}{2\mu} + V(Q_R) - \frac{\mu_B}{\hbar} L \cdot B - \frac{e^2}{8\mu} (Q_R \times B)^2, \qquad H_C = \frac{P_C^2}{2M} \tag{69}$$

in such a way that the internal energy of the composite system is

$$W = H - H_C = H_R = \frac{P_R^2}{2\mu} + V(Q_R) - \frac{\mu_B}{\hbar} L \cdot B - \frac{e^2}{8\mu}(Q_R \times B)^2. \tag{70}$$

In this case, in spite of the action of the external field, the total system is a composite system $S = S_C \cup S_R$. The subsystem S_C, with Hamiltonian H_C, carries the total kinetic energy of translation. The Hamiltonian of the subsystem S_R is the internal energy $W = H_R$, and the set of definite-valued observables of S_R is independent of the kinetic energy and, therefore, invariant under boost-transformations. In general, the term $\frac{e^2}{2\mu} A^2$ is neglected, and the Hamiltonian is written as

$$H = \frac{P_C^2}{2M} + \frac{P_R^2}{2\mu} + V(Q_R) - \frac{\mu_B}{\hbar} L \cdot B. \tag{71}$$

This is the form in which it appears in the textbooks as the Hamiltonian of the Zeeman effect (see [10] p. 325, [16] p. 835, [18] p. 167).

The Zeeman effect is a particular case of the situation 1 of the previous subsection, since the external potential is linear in the position coordinates. From a conceptual viewpoint, a linear external potential can be conceived as introducing a completely uniform modification on space, that is, a modification that does not break space homogeneity.

6 Actualization Rule and Casimir Operators

As we have shown in the previous sections through conceptual arguments and well-known examples, when space is homogeneous and isotropic a boost-transformation only introduces a change in the subsystem that carries the kinetic energy of translation: the internal energy remains unaltered under the transformation. This should not sound surprising to the extent that the internal energy is a Casimir operator of the—central extension of the—Galilean group.

A Casimir operator of a Lie group is an operator that commutes with all the generators of the group and, therefore, is invariant under all the transformations of the group (see [19]). The Galilean group has three Casimir operators: the mass operator M, the operator S^2, and the internal energy $W = H - P^2/2m$. The eigenvalues of the Casimir operators label the irreducible representations of the group (see [20–22]). So, in each irreducible representation, the Casimir operators are multiples of the identity: $M = mI$, $S^2 = s(s+1)I$, where s is the eigenvalue of the spin S, and $W = wI$.

In Sect. 4 we have pointed out that, under the passive interpretation, the application of a Galilean transformation expresses a change in the perspective from which the system is described. Then, any realist interpretation should agree with this physical fact: the rule of definite-value ascription should select a set of definite-valued observables that remains unaltered under the transformations. Since the Casimir operators of the Galilean group are invariant under all the transformations of the group, one can reasonably expect that those Casimir operators belong to the set of the definite-valued observables.

When the problem of selecting the preferred context is viewed from this general perspective, based on group theory, it is easy to show that the modal-Hamiltonian interpretation satisfies that reasonable expectation:

- The actualization of M and S^2 follows from the actualization rule, since both observables commute with H, and they do not break its symmetries because, in non-relativistic quantum mechanics, both are multiples of the identity in any irreducible representation. The fact that M and S^2 always acquire definite values is completely natural from a physical viewpoint, since mass and spin are properties supposed to be always possessed by any quantum system and measurable in any physical situation.
- The actualization of W might seem to be in conflict with the rule because W is not the Hamiltonian: whereas W is invariant under all the Galilean transformations, H changes under the action of a boost. However, as we have seen in the previous sections, this is not a real obstacle when one considers the elemental subsystems to which the actualization rule applies.

Therefore, the modal-Hamiltonian interpretation agrees with the physical meaning of the Galilean transformations under the validity of the group, that is, when space is homogeneous and isotropic.

When the action of external fields breaks the homogeneity and/or the isotropy of space, the Galilean group is no longer valid and, as a consequence, the Casimir operators lose their physical relevance. Nevertheless, in this situation the modal-Hamiltonian interpretation still supplies the rule that selects the preferred context, and that rule includes the Hamiltonian as the main character.

7 Conclusions and Perspectives

The main obstacle to realist interpretations of quantum mechanics has been to find a preferred context adequate for facing the measurement problem. Those interpretations were so involved in this task that certain other relevant issues were scarcely considered. One of these issues is that referred to the group properties of the theory.

The aim of the present paper has been to address this traditionally forgotten topic. In particular, we have analyzed the relationship between the modal-Hamiltonian interpretation and the Galilean group. The purpose has been to argue that, in spite of the apparent conflict introduced by boost-transformations, the interpretation does not violate the physical constraints imposed by the Galilean group. We have shown that, when the conditions for the validity of the group hold—homogeneity and isotropy of space—, the preferred context does not change under a boost in a way that contradicts the physical meaning of the transformation; on the contrary, the modal-Hamiltonian rule of definite-value ascription selects a boost-invariant set of definite-valued observables in the elemental system that carries the information of the energy spectrum of the system.

Far from being an arriving point, this work opens a wide range of new questions. On the one hand, the philosophical problem of the relationship between objectivity and invariance deserves to be addressed once the relevance of the Galilean transformations to the interpretation of quantum mechanics is acknowledged. On the other

hand, the introduction of the Casimir operators of the group in the discussion leads us to wonder if the actualization rule may be reformulated in terms of the Casimir observables. Furthermore, the approach to quantum interpretative questions from the perspective of group theory allows us to speculate about the possibility of extrapolating the interpretation to quantum field theory, where group considerations play a central role from the very beginning. In short, this paper is only the departing point of a long path which, we think, is worth being followed.

Appendix A

In this appendix we will show how the Hamiltonian and the momentum are transformed by a boost when the commutation relations defining the Galilean group hold.

Under a boost-transformation with velocity u_x, the Hamiltonian changes as

$$H' = e^{iG_x u_x} H e^{-iG_x u_x}. \tag{72}$$

So, H' can be computed by means of the Hadamard's lemma applied to the Baker-Campbell-Hausdorff formula (see [23, 24]),

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \frac{1}{3!}[B, [B, [B, A]]] + \cdots. \tag{73}$$

Then, H' can be expressed as

$$H' = H + [iG_x u_x, H] + \frac{1}{2!}[iG_x u_x, [iG_x u_x, H]] + \frac{1}{3!}[iG_x u_x, [iG_x u_x, [iG_x u_x, H]]] + \cdots. \tag{74}$$

By the commutation relation (8_i), we know that

$$[iG_x u_x, H] = iu_x[G_x, H] = iu_x iP_x = -u_x P_x. \tag{75}$$

Then,

$$[iG_x u_x, [iG_x u_x, H]] = [iG_x u_x, -u_x P_x] = -iu_x^2[G_x, P_x]. \tag{76}$$

By the commutation relation (8_f), we obtain

$$-iu_x^2[G_x, P_x] = -iu_x^2 iM = Mu_x^2. \tag{77}$$

Then, by means of (76) and (77), the fourth term of (74) results

$$[iG_x u_x, [iG_x u_x, [iG_x u_x, H]]] = [iG_x u_x, Mu_x^2] = 0. \tag{78}$$

Therefore, all the terms following the fourth term of (74) are also zero. So, by introducing (75) to (78) into (74), we exactly obtain

$$H' = H - u_x P_x + \frac{1}{2}Mu_x^2 = H + T_B \tag{79}$$

where T_B is the boost contribution to the energy.

Under a boost-transformation with velocity u_x, the momentum in direction x changes as

$$P'_x = e^{iG_x u_x} P_x e^{-iG_x u_x}. \tag{80}$$

Again, on the basis of the Hadamard's lemma (see (73)), P'_x can be expressed as

$$\begin{aligned} P'_x = P_x + [iG_x u_x, P_x] + \frac{1}{2!}[iG_x u_x, [iG_x u_x, P_x]] \\ + \frac{1}{3!}[iG_x u_x, [iG_x u_x, [iG_x u_x, P_x]]] + \cdots. \end{aligned} \tag{81}$$

By the commutation relation (8_f), we know that

$$[iG_x u_x, P_x] = iu_x[G_x, P_x] = iu_x iM = -Mu_x. \tag{82}$$

Then,

$$[iG_x u_x, [iG_x u_x, P_x]] = [iG_x u_x, -Mu_x] = 0. \tag{83}$$

Therefore, all the terms following the third term of (81) are also zero. By introducing (82) and (83) into (81), we exactly obtain

$$P'_x = P_x - u_x M. \tag{84}$$

On the basis of (79), we can write

$$\begin{aligned} H' = H + T_B &= \frac{P^2}{2m} + W - u_x P_x + \frac{1}{2} M u_x^2 \\ &= W + \frac{1}{2m}(P_x^2 - 2Mu_x P_x + M^2 u_x^2 + P_y^2 + P_z^2). \end{aligned} \tag{85}$$

Therefore,

$$H' = W + \frac{(P_x - Mu_x)^2 + P_y^2 + P_z^2}{2m}. \tag{86}$$

If we call the boost momentum $P_B = (-Mu_x, 0, 0)$, the transformed Hamiltonian results

$$H' = W + \frac{(P + P_B)^2}{2m}. \tag{87}$$

It is interesting to note that, when we have to give up the commutation relations (8_g), (8_h) and (8_i) involving H due to the action of external fields, (79) and, as a consequence, (87) are no longer valid since they rely on the relation (8_i), but (84) still holds because its derivation does not need the abandoned relations.

Appendix B

In order to illustrate how the Galilean group constraints the form of the internal potential when there are no external fields acting on the system, in this appendix we will obtain the particular constraint imposed by the commutation relations given by (8) on the allowed potential in a two-particle system.

Let us consider two originally free particles S^1 and S^2, which interact through a potential $V(Q_1, Q_2)$ depending on the respective positions. The generators of the Galilean group of the resulting system $S = S^1 + S^2$ are (where the subindexes refer to the respective particles)

$$\begin{aligned} H &= H^1 \otimes I^2 + I^1 \otimes H^2 + H_{int}, \\ P_i &= P_i^1 \otimes I^2 + I^1 \otimes P_i^2, \\ J_i &= J_i^1 \otimes I^2 + I^1 \otimes J_i^2, \\ G_i &= G_i^1 \otimes I^2 + I^1 \otimes G_i^2, \\ M &= M^1 \otimes I^2 + I^1 \otimes M^2 \end{aligned} \tag{88}$$

where $H^1 = (P^1)^2/2m_1$, $H^2 = (P^2)^2/2m_2$ and $H_{int} = V(Q_j^1, Q_j^2)$. The commutation relation (8_g) for the system S states that

$$[P_i, H] = [P_i^1 \otimes I^2 + I^1 \otimes P_i^2, H^1 \otimes I^2 + I^1 \otimes H^2 + H_{int}] = 0. \tag{89}$$

Since the generators of S^1 and S^2 also satisfy the commutation relation (8_g), (89) results

$$\begin{aligned} [P_i, H] &= [P_i^1 \otimes I^2 + I^1 \otimes P_i^2, H_{int}] \\ &= [P_i^1 \otimes I^2, H_{int}] + [I^1 \otimes P_i^2, H_{int}] = 0. \end{aligned} \tag{90}$$

By knowing that $[A^1 \otimes B^2, C] = A^1 C \otimes B^2 C - C A^1 \otimes C B^2$, (90) can be rewritten as

$$[P_i^1, H_{int}] \otimes H_{int} + [P_i^2, H_{int}] \otimes H_{int} = 0. \tag{91}$$

Since we know that $[P_i, F(Q_i)] = -i \partial F / \partial Q_i$ (see [16], (48)), then

$$[P_i^1, H_{int}] = [P_i^1, V(Q_j^1, Q_j^2)] = -i \frac{\partial V}{\partial Q_i^1}, \tag{92}$$

$$[P_i^2, H_{int}] = [P_i^2, V(Q_j^1, Q_j^2)] = -i \frac{\partial V}{\partial Q_i^2}. \tag{93}$$

By introducing (92) and (93) into (91), we obtain

$$\frac{\partial V}{\partial Q_i^1} \otimes V + \frac{\partial V}{\partial Q_i^2} \otimes V = 0. \tag{94}$$

This equation means that

$$\frac{\partial V}{\partial Q_i^1} = -\frac{\partial V}{\partial Q_i^2}. \tag{95}$$

This equality expresses the constraint imposed by the Galilean group on the potential between two particles in the case of no external fields. The condition is obviously satisfied when $V = V(|Q_i^1 - Q_i^2|)$ and, in general, when V is any function of the relative position $(Q_i^1 - Q_i^2)$.

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