

Zeta functions of the 3-dimensional almost-Bieberbach groups

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Abstract. The subgroup zeta function and the normal zeta function of a finitely generated virtually nilpotent group can be expressed as finite sums of Dirichlet series admitting Euler product factorization. We compute these series except for a finite number of local factors when the group is virtually nilpotent of Hirsch length 3. We deduce that they can be meromorphically continued to the whole complex plane and that they satisfy local functional equations. The complete computation (with no exception of local factors) is presented for those groups that are also torsion-free, that is, for the 3-dimensional almost-Bieberbach groups.

1 Introduction

The *subgroup zeta function* and the *normal zeta function* of a finitely generated group G are, respectively, the Dirichlet series

$$\zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} \frac{a_n^{\leq}(G)}{n^s} \quad \text{and} \quad \zeta_G^{\triangleleft}(s) = \sum_{n=1}^{\infty} \frac{a_n^{\triangleleft}(G)}{n^s},$$

where $a_n^{\leq}(G)$ denotes the number of subgroups and $a_n^{\triangleleft}(G)$ the number of normal subgroups of index n . They were introduced by Grunewald, Segal and Smith in [8] as a means to study groups of polynomial subgroup growth (PSG). Indeed, these are precisely the groups for which these series are not only formal gadgets but also define analytic functions on some complex right-half plane. A natural problem is to understand how structural information of a group of PSG is encoded into arithmetical and analytic properties of its zeta functions and vice versa.

Lubotzky, Mann and Segal characterized the finitely generated residually finite groups of PSG: these are precisely the virtually solvable groups of finite rank [11]. On the other hand, since they were introduced, zeta functions of groups have been extensively studied only when the group is a finitely generated torsion-free nilpotent group: a \mathfrak{T} -group. The theory for these groups is rich and continues to grow.

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We summarize some general results and refer the reader to the survey [26] and the references therein for more information and variations of the topic. Let N be a \mathfrak{T} -group, let $*$ \in $\{\leq, \triangleleft\}$ and let α_N^* denote the abscissa of convergence of $\zeta_N^*(s)$. Then the following holds.

- (N1) $\alpha_N^* \leq h(N)$ (= the Hirsch length of N) [8, Proposition 1].
- (N2) $\alpha_N^* \in \mathbb{Q}$ and there is $\delta > 0$ such that $\zeta_N^*(s)$ admits a meromorphic continuation to the region $\{s \in \mathbb{C} : \operatorname{Re}(s) > \alpha_N^* - \delta\}$ (see [5, Theorem 1.1]).
- (N3) $\zeta_N^*(s)$ has an Euler product factorization

$$\zeta_N^*(s) = \prod_{p \text{ prime}} \zeta_{N,p}^*(s),$$

where $\zeta_{N,p}^*(s) = \sum_{k=0}^{\infty} a_{p^k}^*(N) p^{-ks}$. In addition, each local factor $\zeta_{N,p}^*(s)$ is a rational function in p^{-s} with coefficients in \mathbb{Q} (see [8, Proposition 1.3 and Theorem 1]).

- (N4) For almost all prime p , $\zeta_{N,p}^{\leq}(s)$ satisfies the functional equation

$$\zeta_{N,p}^{\leq}(s)|_{p \rightarrow p^{-1}} = (-1)^h p^{\binom{h}{2} - hs} \zeta_{N,p}^{\leq}(s),$$

where $h = h(N)$ (see [25, Corollary 1.1]). If in addition N is a \mathfrak{T}_2 -group (a \mathfrak{T} -group of nilpotency class 2), then for almost all prime p , $\zeta_{N,p}^{\triangleleft}(s)$ satisfies the functional equation

$$\zeta_{N,p}^{\triangleleft}(s)|_{p \rightarrow p^{-1}} = (-1)^h p^{\binom{h}{2} - (d+h)s} \zeta_{N,p}^{\triangleleft}(s),$$

where d is the rank of $N/Z(N)$ (see [25, Theorem C]).

Remark 1.1. Properties (N1)–(N4) are, in fact, corollaries of the analogous properties established for *the subring zeta function* and *the ideal zeta function* of nilpotent Lie rings additively isomorphic to some \mathbb{Z}^h . The translation is done via the Mal'cev correspondence. The question of whether the ideal zeta function of a nilpotent Lie ring of nilpotency class greater than 2 satisfies local functional equations led to the introduction and investigation of *the submodule zeta function*; see, e.g., [10, 17, 27]. On the other hand, it has long been known that, for instance, the ideal zeta function of the filiform nilpotent Lie ring Fil_4 fails to satisfy local functional equations; see [7, Theorem 2.39].

Regarding the computation of zeta functions, the following are classical examples.

- (1) For the free abelian group \mathbb{Z}^h , $\zeta_{\mathbb{Z}^h}^{\leq}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-h+1)$, where $\zeta(s)$ is the Riemann zeta function. In [12, Chapter 15], there are five different proofs of this elementary fact.
- (2) Let $\mathbf{H}(\mathbb{Z})$ denote the integral points of the Heisenberg unipotent group scheme \mathbf{H} . Then

$$\zeta_{\mathbf{H}(\mathbb{Z})}^{\leq}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)},$$

$$\zeta_{\mathbf{H}(\mathbb{Z})}^{\triangleleft}(s) = \zeta(s)\zeta(s-1)\zeta(3s-2)$$

[8, Section 8]. More generally, a calculation of $\zeta_{\mathbf{H}(\mathcal{O}_K)}^{\triangleleft}(s)$, for K a number field and \mathcal{O}_K its ring of integers, is presented in [20, 21].

Finding explicit formulae for the zeta functions of a \mathfrak{T} -group is in general a difficult task. A substantial list of examples is recorded in [7], and new ones have emerged more recently; see, e.g., [1, 20, 21, 28]. In [16, 19], Rossmann developed a method for computing certain zeta functions associated with \mathfrak{T} -groups and other algebraic structures. This algorithm was implemented in [18].

This work studies and presents formulae for the zeta functions of the virtually nilpotent groups of Hirsch length 3 (see the next paragraph for a detailed description of the content). This is not the first attempt in dealing with zeta functions of groups that are not nilpotent. In [4], du Sautoy investigated zeta functions of compact p -adic analytic groups (= virtually uniform pro- p -groups) and proved that they are rational functions in p^{-s} . Then, in [6], the authors studied zeta functions of virtually abelian groups, proving for example, that they can be meromorphically continued to the whole complex plane (this property is not shared by zeta functions of \mathfrak{T} -groups in general, as discussed in [7, Chapter 7]). The paper [24] contains general properties of zeta functions of virtually nilpotent groups that we now recall. Let G be a finitely generated virtually nilpotent group, and let $N \triangleleft G$ be a finite-index normal subgroup that is a \mathfrak{T} -group. It is easy to check that

$$\zeta_G^{\leq}(s) = \sum_{N \leq H \leq G} [G : H]^{-s} \zeta_G^{H, \leq}(s),$$

$$\zeta_G^{\triangleleft}(s) = \sum_{N \leq H \triangleleft G} [G : H]^{-s} \zeta_G^{H, \triangleleft}(s),$$

where

$$\zeta_G^{H, \leq}(s) := \sum_{A \leq G : AN = H} [H : A]^{-s},$$

$$\zeta_G^{H, \triangleleft}(s) := \sum_{A \triangleleft G : AN = H} [H : A]^{-s}.$$

The series $\zeta_G^{H, \leq}(s)$ and $\zeta_G^{H, \triangleleft}(s)$ will be referred to as *partial zeta functions* of $\zeta_G^{\leq}(s)$ and $\zeta_G^{\triangleleft}(s)$ (with respect to N). Each of them is an Euler product of *cone integrals* [24, Theorem 1], and hence it satisfies properties (N2) and (N3) above by the general theory of cone integrals developed in [5]. Property (N1) also holds with the bound $h(N) = h(G)$ replaced by $h(G) + 1$ (see [24, Proposition 2.12]). The partial zeta functions were computed explicitly, except for a finite number of local factors, when the group N is abelian; see [6, Section 2]. Computing the exceptional factors is, in general, more involved. The complete computation was done for the plane crystallographic groups. This is the main result in [13], and it is also summarized in [6, Section 4]. To the author's knowledge, these are the only existing examples of computations of zeta functions of infinite groups that are not nilpotent.

The article is organized as follows. In Section 2, we obtain formulae for the local factors of $\zeta_G^{G, \leq}(s)$ and $\zeta_G^{N, \triangleleft}(s)$ when $p \nmid [G : N]$ for every virtually nilpotent group of Hirsch length 3 (see Table 1 and Table 2). This suffices to conclude that the subgroup and normal zeta functions of each of these groups extend to meromorphic functions on the whole plane (Corollary 2.8 and Corollary 2.19). We also deduce local functional equations, similar to but different from those presented in (N4) (Corollary 2.7 and Corollary 2.18). The arguments in this section are of a group-theoretical nature, akin to [6, Section 2]. In Section 3, we review a method for expressing local zeta functions of groups in terms of p -adic integrals. This method is used later to compute local factors at primes $p \mid [G : N]$ (those not considered in Section 2). Section 4 presents the formulae for the zeta functions of the torsion-free virtually nilpotent groups of Hirsch length 3. A group like this is either a 3-dimensional Bieberbach group or a 3-dimensional almost-Bieberbach group (the fundamental group of a 3-dimensional infra-nilmanifold). Section 5 contains proofs of the formulae for the 3-dimensional Bieberbach groups. It also contains formulae for the zeta functions of a large family of Bieberbach groups with holonomy C_2 , to illustrate how involved the method for computing local factors at bad primes is, even in the case of virtually abelian groups. Finally, Section 6 contains proofs of the formulae for the 3-dimensional almost Bieberbach groups.

Notation and conventions. $|S|$ denotes the cardinality of a set S . For a group G , $Z(G)$ denotes its center; $A \leq G$ means that A is a subgroup; $A \triangleleft G$ means that A is a normal subgroup; $[G : A]$ denotes the index of A in G ; $N_G(A)$ denotes the normalizer of A in G . In sums involving $[G : A]$, only subgroups of finite index are considered. For a subset $S \subset G$, $\langle S \rangle$ denotes the subgroup generated by S . When G is profinite, $\overline{\langle S \rangle}$ denotes the closed subgroup generated by S . For $x, y \in G$, we denote ${}^x y = xyx^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$ ($= {}^x y \cdot y^{-1}$). For $S, T \subset G$, we denote $[S, T] = \langle [s, t] \mid s \in S, t \in T \rangle$. We use without mentioning the fact that if

G is a finitely generated profinite group, then every finite-index subgroup is open, and if H and K are closed subgroups, then $[H, K]$ is also closed; see [15].

The letter p will be reserved for prime numbers. By $v_p: \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{\infty\}$, we denote the usual p -adic valuation, and by $|\cdot|_p = p^{-v_p(\cdot)}$ the p -adic norm. Moreover, $\zeta_p(s)$ denotes $\frac{1}{1-p^{-s}}$, the local factor of the Riemann zeta function $\zeta(s)$ at p .

2 Local factors at good primes and meromorphic continuation

Let G be a finitely generated virtually nilpotent group, and let $N \triangleleft G$ be a finite-index normal subgroup that is a \mathfrak{T} -group. For a prime p , let G_p be the completion of G with respect to the family $\{B \triangleleft G : B \subseteq N, [N : B] \text{ a power of } p\}$. For $X \leq G$, we denote by X_p the closure of X in G_p . For a subquotient X/Y of G , we denote $(X/Y)_p = X_p/Y_p$. Note that N_p is the pro- p completion of N , and that the inclusion $G \hookrightarrow G_p$ induces an isomorphism $G/N \cong G_p/N_p$. Note also that G_p is not the pro- p completion of G unless $[G : N]$ is a power of p .

As noted in the introduction, to study $\zeta_G^{\leq}(s)$, it is enough to focus on the partial zeta functions $\zeta_G^{H, \leq}(s) = \zeta_H^{H, \leq}(s)$ for $N \leq H \leq G$. There is no loss of generality if we only consider $\zeta_G^{G, \leq}(s)$. Similarly, to study $\zeta_G^{\triangleleft}(s)$, it is enough to consider $\zeta_G^{H, \triangleleft}(s)$ for $N \leq H \triangleleft G$. The advantage is that we have

$$\zeta_G^{G, \leq}(s) = \prod_{p \text{ prime}} \zeta_{G_p}^{G_p, \leq}(s) \quad \text{and} \quad \zeta_G^{H, \triangleleft}(s) = \prod_{p \text{ prime}} \zeta_{G_p}^{H_p, \triangleleft}(s),$$

where $\zeta_{G_p}^{G_p, \leq}(s)$ and $\zeta_{G_p}^{H_p, \triangleleft}(s)$ are computed with respect to N_p (see [24, Proposition 2.2]).

Throughout the section, unless otherwise specified, we assume that N is a \mathfrak{T}_2 -group, that is, a \mathfrak{T} -group of nilpotency class 2. We also fix the following notation:

$$Z = Z(N), \quad T = N/Z, \quad P = G/N, \quad \Lambda = \mathbb{Z}[P] \quad \text{and} \quad \Lambda_p = \mathbb{Z}_p[P].$$

We will first obtain general expressions for $\zeta_{G_p}^{G_p, \leq}(s)$ and $\zeta_{G_p}^{H_p, \triangleleft}(s)$ when $p \nmid |P|$. Then we shall specialize to the case of Hirsch length 3, giving explicit formulae in terms of well-known series. The case $N \cong \mathbb{Z}^3$ (and more generally $N \cong \mathbb{Z}^h$ for any h) was settled in [6].

The action by conjugation of G on $N ((g, n) \mapsto {}^g n = gng^{-1})$ induces structures of left Λ -modules on the abelian groups T and Z (and hence T_p and Z_p become Λ_p -modules). For this reason, we use additive notation when working with them. The commutator operation $[x, y] = xyx^{-1}y^{-1}$ induces a bilinear map $[\cdot, \cdot]: T \times T \rightarrow Z$ (and hence also a \mathbb{Z}_p -bilinear map $[\cdot, \cdot]: T_p \times T_p \rightarrow Z_p$) that is compatible with the action of P , that is, $[\alpha \cdot x, \alpha \cdot y] = \alpha \cdot [x, y]$ for all $\alpha \in P$

and $x, y \in T$. If U_1 and U_2 are \mathbb{Z}_p -submodules of T_p , then $[U_1, U_2]$ denotes the \mathbb{Z}_p -submodule (=subgroup) of Z_p generated by $[x_1, x_2]$ with $x_i \in U_i$.

For a ring R and a left R -module M , we use the notation $X \leq_R M$ to indicate that X is a left R -submodule of M . For $r \in R$, $r_M: M \rightarrow M$ denotes the left multiplication by r . If $R = \Lambda$ or Λ_p , we denote by $\text{Der}(P, M)$ the set of derivations from P to M . For a subset $S \subseteq P$, we denote

$$C_M(S) = \{x \in M : \alpha \cdot x = x \text{ for all } \alpha \in S\}.$$

We will use the fact that if $p \nmid |P|$ and M is a finitely generated Λ_p -module that is torsion-free as a \mathbb{Z}_p -module, then M is a projective Λ_p -module, and all finite-index Λ_p -submodules of M are isomorphic. This follows from the fact that, when $p \nmid |P|$, Λ_p is a maximal order of $\mathbb{Q}_p[P]$.

2.1 Counting subgroups

Proposition 2.2 below presents a general expression for the local factors of $\zeta_G^{\mathcal{G}, \leq}(s)$ at “good” primes. We will need the following lemma.

Lemma 2.1. *Let R be a ring, let M be a left R -module, and fix $X \leq_R M$. If there is a complement of X in M (i.e., there is $Y \leq_R M$ such that $X + Y = M$ and $X \cap Y = 0$), then the set of complements of X in M is in a bijection with $\text{Hom}_R(M/X, X)$. If $R = \mathbb{Z}_p[P]$ with $p \nmid |P|$, and if M/X is finitely generated and torsion-free as a \mathbb{Z}_p -module, then the set of (P -invariant) complements of X is in a bijection with $\text{Hom}_{\mathbb{Z}_p[P]}(M/X, X)$.*

Proof. Assume that there is a complement of X , say Y . Given any other complement $K \leq_R M$, we define $\varphi_K: Y \rightarrow X$ as follows. If $y \in Y$, then we can write $y = k + x$ uniquely with $k \in K$ and $x \in X$. We set $\varphi_K(y) = x$. It is clear that $\varphi_K \in \text{Hom}_R(Y, X)$. Conversely, given $\varphi \in \text{Hom}_R(Y, X)$, we define

$$K_\varphi = \{y - \varphi(y) : y \in Y\}.$$

It is easy to check that K_φ is a complement of X , and that $\varphi_{K_\varphi} = \varphi$ and $K_{\varphi_K} = K$. Therefore, the set of complements of X is in a bijection with $\text{Hom}_R(Y, X)$, which in turn is in a bijection with $\text{Hom}_R(M/X, X)$ since M/X and Y are isomorphic. This proves the first part of the lemma

We now show the second part, so we assume now that $R = \mathbb{Z}_p[P]$ with $p \nmid |P|$ and that M/X is finitely generated and torsion-free as a \mathbb{Z}_p -module. Then M/X is a free \mathbb{Z}_p -module of finite rank, and hence it is a projective $\mathbb{Z}_p[P]$ -module (here we use the fact that $p \nmid |P|$). This implies that there is at least one complement of X ; therefore, the set of complements is in a bijection with $\text{Hom}_{\mathbb{Z}_p[P]}(M/X, X)$, as shown in the first part of the lemma. This completes the proof. \square

Proposition 2.2. *If $p \nmid |P|$, then*

$$\begin{aligned} \zeta_{G_p}^{G_p, \leq}(s) = & \sum_{\substack{U \leq_{\Lambda_p} T_p, V \leq_{\Lambda_p} Z_p \\ [U, U] \subseteq V}} [T_p : U]^{-s} [T_p : C_{T_p}(P) + U] \\ & \times [Z_p : V]^{-s} [Z_p : C_{Z_p}(P) + V] \\ & \times |\text{Hom}_{\Lambda_p}(T_p, Z_p/V)|. \end{aligned}$$

Proof. For a subgroup $U \leq T_p$, we denote by \tilde{U} its pre-image in N_p . We set

$$\mathcal{A} := \{A \leq G_p : AN_p = G_p, [G_p : A] < \infty\},$$

$$\mathcal{U} := \{(U \leq_{\Lambda_p} T_p, V \leq_{\Lambda_p} Z_p) : [T_p : U] < \infty, [Z_p : V] < \infty, [U, U] \subseteq V\},$$

$$\mathcal{V} := \{(U, V, C) : (U, V) \in \mathcal{U}, C \in \mathcal{A}, C \cap N_p = \tilde{U}\}$$

and define maps $\Phi: \mathcal{A} \rightarrow \mathcal{V}, \Psi: \mathcal{V} \rightarrow \mathcal{U}$ by

$$\Phi(A) = (((A \cap N_p)Z_p)/Z_p, A \cap Z_p, Z_pA) \quad \text{and} \quad \Psi(U, V, C) = (U, V).$$

It is straightforward to check that Φ is well-defined. Note that

$$\begin{aligned} [G_p : A] &= [N_p : A \cap N_p] = [N_p : \tilde{U}][Z_p : V] \\ &= [T_p : U][Z_p : V] \quad \text{for } A \in (\Psi\Phi)^{-1}(U, V); \end{aligned}$$

therefore,

$$\zeta_{G_p}^{G_p, \leq}(s) = \sum_{A \in \mathcal{A}} [G_p : A]^{-s} = \sum_{(U, V) \in \mathcal{U}} [T_p : U]^{-s} [Z_p : V]^{-s} |(\Psi\Phi)^{-1}(U, V)|.$$

We now fix $(U, V) \in \mathcal{U}$ and show that $|\Psi^{-1}(U, V)| = |\text{Der}(P, T_p/U)|$. Note first that $\Psi^{-1}(U, V)$ is in bijection with the set of complements of N_p/\tilde{U} in G_p/\tilde{U} . Since N_p/\tilde{U} is a normal Hall subgroup of G_p/\tilde{U} , there is at least one complement by the Schur–Zassenhaus theorem, and therefore the number of complements is $|\text{Der}(P, N_p/\tilde{U})| = |\text{Der}(P, T_p/U)|$ (cf. [22, Proposition 1, Chapter 3]).

We now fix $(U, V, C) \in \mathcal{V}$ and show that

$$|\Phi^{-1}(U, V, C)| = |\text{Hom}_{\Lambda_p}(U, Z_p/V)| |\text{Der}(P, Z_p/V)|.$$

The subgroups \tilde{U} and V are normal closed subgroups of G_p , and the condition $[U, U] \subseteq V$ implies that the quotient \tilde{U}/V is abelian. Thus, \tilde{U}/V is a finitely generated \mathbb{Z}_p -module and Z_p/V is clearly the torsion submodule. We use C to give \tilde{U}/V a structure of $\mathbb{Z}_p[P]$ -module. Firstly, the action by conjugation of C on \tilde{U} induces a structure of C/\tilde{U} -module on \tilde{U}/V . Secondly, the inclusion $C \rightarrow G_p$ induces an isomorphism $C/\tilde{U} \cong G_p/N_p = P$. Therefore, \tilde{U}/V becomes a $\mathbb{Z}_p[P]$ -module. Note that Z_p/V has two structures of $\mathbb{Z}_p[P]$ -module: one as a subobject

of \tilde{U}/V and one as a quotient of Z_p . It is easy to check that these two structures coincide. Similarly, the structure of $\mathbb{Z}_p[P]$ -module on $\tilde{U}/Z_p = U$ as a quotient of \tilde{U}/V and the one as a subobject of T_p are the same.

Given $A \leq \Phi^{-1}(U, V, C)$, observe that $(A \cap N_p)/V$ is a complement of Z_p/V in \tilde{U}/V . We claim that this complement is P -invariant. Indeed, $A \cap N_p$ is normal in A and in $(A \cap N_p)Z_p$, so it is normal in $AZ_p = C$. Therefore, $(A \cap N_p)/V$ is C/\tilde{U} -invariant and hence P -invariant. Now, according to Lemma 2.1, the number of P -invariant complements of Z_p/V in \tilde{U}/V is $|\text{Hom}_{\Lambda_p}(U, Z_p/V)|$. Fix one such complement, say B/V . Note that B is normal in C (this follows from the condition of B/V being P -invariant) and the set of those $A \in \Phi^{-1}(U, V, C)$ such that $A \cap N_p = B$ is in a bijection with the set of complements of \tilde{U}/B in C/B . Since \tilde{U}/B is a normal Hall subgroup of C/B , there is at least one such complement by the Schur–Zassenhaus theorem, and then the number of complements is $|\text{Der}(C/\tilde{U}, \tilde{U}/B)| = |\text{Der}(P, \tilde{U}/B)| = |\text{Der}(P, Z_p/V)|$. We conclude that $|\Phi^{-1}(U, V, C)| = |\text{Hom}_{\Lambda_p}(U, Z_p/V)||\text{Der}(P, Z_p/V)|$.

To end the proof, we only need to show that

$$\begin{aligned} |\text{Der}(P, T_p/U)| &= [T_p : U + C_{T_p}(P)], \\ |\text{Der}(P, Z_p/V)| &= [Z_p : V + C_{Z_p}(P)], \\ |\text{Hom}_{\Lambda_p}(U, Z_p/V)| &= |\text{Hom}_{\Lambda_p}(T, Z_p/V)|. \end{aligned}$$

Here we use again the fact that $p \nmid |P|$. The first two equalities follow from [6, Lemma 2.4], and for the last one, we use the fact that U and T are isomorphic as Λ_p -modules. □

The following lemma will be used to calculate $|\text{Hom}_{\Lambda_p}(T_p, Z_p/V)|$ in some particular cases.

Lemma 2.3. *Let M be a Λ_p -module, and assume that $p \nmid |P|$. If there is $\alpha \in P$ such that $C_M(\alpha) = M$ and $C_{Z_p}(\alpha) = 0$, then*

$$\text{Hom}_{\Lambda_p}(M, Z_p/V) = \{0\} \quad \text{for all } V \leq_{\Lambda_p} Z_p.$$

Similarly, if $C_M(\alpha) = 0$ and $C_{Z_p}(\alpha) = Z_p$, then also $\text{Hom}_{\Lambda_p}(M, Z_p/V) = \{0\}$.

Proof. Since $p \nmid |P|$ and since Z_p is finitely generated and torsion-free as a \mathbb{Z}_p -module, Z_p is a projective Λ_p -module. Thus, the lemma in both cases follows from the equality $\text{Hom}_{\Lambda_p}(M, Z_p) = 0$, whose verification is straightforward. □

We now specialize to the case when N has Hirsch length 3. Recall that, given a finite subgroup $F \subset \text{GL}_2(\mathbb{Z})$, either F is included in $\text{SL}_2(\mathbb{Z})$, in which case

F is isomorphic to one of the cyclic groups C_1, C_2, C_3, C_4, C_6 , or else there is $\beta \in F$ of order 2 with determinant -1 , in which case F is isomorphic to one of the dihedral groups $D_1 (\cong C_2), D_2 (\cong C_2 \times C_2), D_3, D_4, D_6$; cf. [14, Chapter IX].

Recall that the action of G on N by conjugation induces an action of $P = G/N$ on $T = N/Z$, so there is a natural homomorphism $P \rightarrow \text{GL}(T)$.

Lemma 2.4. *Assume that N is a \mathfrak{T}_2 -group of Hirsch length 3 (so that $T \cong \mathbb{Z}^2$ and $Z \cong \mathbb{Z}$). Set $F := \text{Im}(P \rightarrow \text{GL}(T))$ and $\eta := \text{rank}_{\mathbb{Z}}(C_T(P))$. Then*

(1) $C_Z(P) = Z$ if and only if $F \subset \text{SL}(T)$.

(2) We have

$$\eta = \begin{cases} 2 & \text{if } F \text{ is trivial,} \\ 1 & \text{if } F \not\subset \text{SL}(T) \text{ and } F \cong D_1, \\ 0 & \text{otherwise.} \end{cases}$$

(3) If $p \nmid |P|$, then for any $V \leq_P Z_p$ of finite index, we have

$$|\text{Hom}_{\Lambda_p}(T_p, Z_p/V)| = [Z_p : V]^\eta.$$

Proof. Fix an ordered basis $\{x, y\}$ for T . Note that $[x, y] \in Z$ is non-zero. Given $\alpha \in P$, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of its image $\alpha_T \in \text{GL}(T)$ with respect to $\{x, y\}$. We have

$$\alpha \cdot [x, y] = [ax + cy, bx + dy] = (ad - bc)[x, y].$$

It follows that $C_Z(P) = Z$ if and only if $F \subset \text{SL}(T)$. It also follows that Z is naturally an F -module.

We now show (2) and (3). Fix a prime $p \nmid |P|$ and a subgroup $V \leq_{\Lambda_p} Z_p$ of finite index. Note that (2) and (3) hold clearly for $F = C_1$. Assume next that $F \cong D_1$ and $F \not\subset \text{SL}(T)$. Then the generator of F has eigenvalues 1 and -1 , whence the Λ -submodules $T_+ := C_T(P)$ and $T_- := \{x \in T : \beta \cdot x = -x\}$ have additive rank 1. It follows that $\eta = 1$. Next, since $p \neq 2$, the decomposition

$$x = \frac{x + \alpha \cdot x}{2} + \frac{x - \alpha \cdot x}{2}$$

holds in T_p and yields a decomposition $T_p = (T_+)_p \oplus (T_-)_p$. Since $C_Z(F) = 0$ by (1), Lemma 2.3 implies that $|\text{Hom}_{\Lambda_p}((T_+)_p, Z_p/V)| = 1$. Thus,

$$|\text{Hom}_{\Lambda_p}(T_p, Z_p/V)| = |\text{Hom}_{\Lambda_p}((T_-)_p, Z_p/V)| = [Z_p : V] = [Z_p : V]^\eta.$$

Assume now that $C_1 \neq F = \langle \alpha_T \rangle \subset \text{SL}(T)$. We have $C_T(\alpha) = 0$ since otherwise 1 would be an eigenvalue of α_T , which in turn implies that α_T is the

identity since $\alpha_T \in \text{SL}(T)$. This is a contradiction as $F \neq C_1$. It follows that $\eta = 0$. On the other hand, $C_Z(\alpha) = Z$ by (1), and then Lemma 2.3 implies that $|\text{Hom}_{\Lambda_p}(T_p, Z_p/V)| = 1 = |\text{Hom}_{\Lambda_p}(T_p, Z_p/V)|^\eta$.

Assume finally that $F \cong D_d$ with $d \in \{2, 3, 4\}$. The intersection $F \cap \text{SL}(T)$ is not the trivial group. Indeed, the product of any two elements of F is in $\text{SL}(T)$ and F has more than two elements. It follows from the previous case that $\eta = 0$ and $|\text{Hom}_{\Lambda_p}(T_p, Z_p/V)| = 1 = |\text{Hom}_P(T_p, Z_p/V)|^\eta$. We have covered all the cases, so the proof is complete. \square

Lemma 2.5. *Assume that N is a \mathfrak{T}_2 -group of Hirsch length 3. If $U \leq T_p$ has finite index, then $[Z_p : [U, U]] = |[Z : [N, N]]|_p^{-1} [T_p : U]$*

Proof. Let $\{x, y\}$ be a basis for the \mathbb{Z}_p -module T_p , and let $U \leq T_p$ be a \mathbb{Z}_p -submodule of finite index, say generated by $ax + by$ and $cx + dy$. Note that $[T_p, T_p] = \mathbb{Z}_p[x, y]$ and $[U, U] = \mathbb{Z}_p[ax + by, cx + dy] = \mathbb{Z}_p(ad - bc)[x, y]$. Thus,

$$[Z_p : [U, U]] = [Z_p : [T_p, T_p]] |ad - bc|_p^{-1}.$$

On the other hand, clearly $[T_p : U] = |ad - bc|_p^{-1}$. Thus,

$$[Z_p : [U, U]] = [Z_p : [T_p, T_p]] [T_p : U].$$

Finally, it is clear that $[Z_p : [T_p, T_p]] = |[Z : [T, T]]|_p^{-1} = |[Z, [N, N]]|_p^{-1}$. \square

Theorem 2.6. *Assume that N is a \mathfrak{T}_2 -group of Hirsch length 3. Let*

$$F := \text{Im}(P \rightarrow \text{GL}(T))$$

and $E = T \rtimes F$, which is a plane crystallographic group. If $p \nmid |P|$, then

$$\begin{aligned} \zeta_{G_p^{p,\leq}}(s) &= \zeta_p(s - \eta - \epsilon) (\zeta_{E_p^{p,\leq}}(s) - p^{-s+\eta+\epsilon} |[Z : [N, N]]|_p^{s-\eta-\epsilon} \\ &\quad \times \zeta_{E_p^{p,\leq}}(2s - \eta - \epsilon)), \end{aligned}$$

where $\eta := \text{rank}_{\mathbb{Z}}(C_T(F))$, ϵ is 0 or 1 according to whether F is included in $\text{SL}(T)$ or not, and $\zeta_{E_p^{p,\leq}}(s)$ is computed with respect to $T_p \triangleleft E_p$.

Proof. Fix $p \nmid |P|$. Any finite-index subgroup $V \leq Z_p$ is P -invariant, and

$$[Z_p : C_{Z_p}(P) + V] \text{ is } 1 \text{ or } [Z_p : V]$$

according to whether $C_Z(P)$ is Z or 0, which in turn, by Lemma 2.4(1), is translated into whether $F \subset \text{SL}(T)$ or not. Thus, $[Z_p : C_{Z_p}(P) + V] = [Z_p : V]^\epsilon$, where ϵ is as in the theorem. Next, by Lemma 2.4(3),

$$|\text{Hom}_{\Lambda_p}(T_p, Z_p/V)| = [Z_p : V]^\eta.$$

Therefore, by Proposition 2.2,

$$\begin{aligned}
 \zeta_{G_p}^{G_p, \leq}(s) &= \sum_{\substack{U \leq_{\Lambda_p} T_p, V \leq Z_p \\ [U, U] \subseteq V}} [T_p : U]^{-s} [T_p : C_{T_p}(P) + U] [Z_p : V]^{-s+\epsilon+\eta} \\
 &= \sum_{U \leq_{\Lambda_p} T_p} [T_p : U]^{-s} [T_p : C_{T_p}(P) + U] \\
 &\quad \times \left(\sum_{V \leq Z_p} [Z_p : V]^{-s+\epsilon+\eta} - \sum_{V \leq_p [U, U]} [Z_p : V]^{-s+\epsilon+\eta} \right) \\
 &= \sum_{U \leq_{\Lambda_p} T_p} [T_p : U]^{-s} [T_p : C_{T_p}(P) + U] \\
 &\quad \times (\zeta_p(s - \epsilon - \eta) - [Z_p : p[U, U]]^{-s+\epsilon+\eta} \zeta_p(s - \epsilon - \eta)) \\
 &= \zeta_p(s - \epsilon - \eta) \left(\sum_{U \leq_{\Lambda_p} T_p} [T_p : U]^{-s} [T_p : C_{T_p}(P) + U] \right. \\
 &\quad \left. \times (1 - p^{-s+\epsilon+\eta} |[Z : [N, N]]|_p^{s-\eta-\epsilon} \times [T_p : U]^{-s+\epsilon+\eta}) \right),
 \end{aligned}$$

where, in the last equality, we used Lemma 2.5. To complete the proof, we have to show that $\sum_{U \leq_{\Lambda_p} T_p} [T_p : U]^{-s} [T_p : C_{T_p}(F) + U] = \zeta_{E_p}^{E_p, \leq}(s)$. However, this follows from [6, Proposition 2.3 and Lemma 2.4]. \square

Corollary 2.7. *If $p \nmid |P|$, then $\zeta_{G_p}^{G_p}(s)$ is given in Table 1, where $k = [Z : [N, N]]$, and for $d \in \{3, 4, 6\}$, $\chi_d : \mathbb{N} \rightarrow \mathbb{C}$ is the extended residue class character,*

$$\chi_d(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{d} \\ -1 & \text{if } n \equiv -1 \pmod{d} \\ 0 & \text{otherwise,} \end{cases}$$

and $L_p(s, \chi_d) = (1 - \chi_d(p)p^{-s})^{-1}$ is the local factor at p of the Dirichlet L -function of χ_d , $L(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n)n^{-s}$.

Therefore, if $p \nmid |P|$, then $\zeta_{G_p}^{G_p, \leq}(s)$ satisfies the functional equation

$$\zeta_{G_p}^{G_p, \leq}(s)|_{p \rightarrow p^{-1}} = \begin{cases} (-1)^3 p^{-3s+3} \zeta_{G_p}^{G_p, \leq}(s) & \text{if } F \cong C_1, C_2, D_2, \\ (-1)^3 p^{-3s+2} \chi_d(p) \zeta_{G_p}^{G_p, \leq}(s) & \text{if } F \cong C_d, d \in \{3, 4, 6\}, \\ p^{-3s+3} \zeta_{G_p}^{G_p, \leq}(s) & \text{if } F \cong D_d, d \in \{3, 4, 6\}. \end{cases}$$

Proof. The formula for $\zeta_{G_p}^{G_p, \leq}(s)$ in each case follows from Theorem 2.6 and uses the formula for $\zeta_E^{E, \leq}(s)$ obtained in [13, Chapter 5] as a step in the computation

F	$\zeta_{G_p}^{p, \leq}(s)$	
$F \subset \mathrm{SL}_2(T)$	C_1	$\zeta_p(s-2)(\zeta_p(s)\zeta_p(s-1) - p^{-s+2} k _p^{s-2}\zeta_p(2s-2)\zeta_p(2s-3))$ $\frac{\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)}{\zeta_p(3s-3)}$ $\zeta_p(s)(\zeta_p(s-1)\zeta_p(s-2) - p^{-s} k _p^s\zeta_p(2s-1)\zeta_p(2s-2))$ $\frac{\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-1)\zeta_p(2s-2)}{\zeta_p(3s-3)}$
	C_2	$\zeta_p(s)(\zeta_p(s-1)\zeta_p(s-2) - p^{-s} k _p^s\zeta_p(2s-1)\zeta_p(2s-2))$ $\frac{\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-1)\zeta_p(2s-2)}{\zeta_p(3s-3)}$
	$C_d, d = 3, 4, 6$	$\zeta_p(s)(L_p(s-1, \chi_d)\zeta_p(s-1) - p^{-s} k _p^s L_p(2s-1, \chi_d)\zeta_p(2s-1))$ $\frac{\zeta_p(s-1)\zeta_p(2s-1)L_p(s-1, \chi_d)L_p(2s-1, \chi_d)}{L_p(3s-2, \chi_d)}$
$F \not\subset \mathrm{SL}_2(T)$	D_1	$\zeta_p(s-2)(\zeta_p(s)\zeta_p(s-1) - p^{-s+2} k _p^{s-2}\zeta_p(2s-2)\zeta_p(2s-3))$ $\frac{\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)}{\zeta_p(3s-3)}$
	D_2	$\zeta_p(s-1)(\zeta_p(s-1)^2 - p^{-s+1} k _p^{s-1}\zeta_p(2s-2)^2)$ $\frac{\zeta_p(s-1)^2\zeta_p(2s-2)^2}{\zeta_p(3s-3)}$
	$D_d, d = 3, 4, 6$	$\zeta_p(s-1)(\zeta_p(2s-2) - p^{-s+1} k _p^{s-1}\zeta_p(4s-4))$ $\frac{\zeta_p(2s-2)\zeta_p(3s-3)\zeta_p(4s-4)}{\zeta_p(6s-6)}$

Table 1. Local factors of $\zeta_G^{p, \leq}(s)$ at $p \nmid |P|$ ($F := \mathrm{Im}(P \rightarrow \mathrm{GL}(T))$ and $k := [Z : [N, N]]$).

of the subgroup zeta functions of the plane crystallographic group E (see also [6, Section 4.1]). The computation of η was done in Lemma 2.4(2). If in addition $p \nmid |P|[Z : [N, N]]$, then $|[Z : [N, N]]|_p = 1$ and the simplification of the formula is straightforward. The functional equation follows by inspection of the formula. \square

Corollary 2.8. $\zeta_G^{G, \leq}(s)$ has abscissa of convergence in the set $\{\frac{3}{2}, 2, 3\}$ and admits a meromorphic continuation to the whole plane. The same holds for $\zeta_G^{\leq}(s)$.

Proof. According to Table 1, there is a Dirichlet series $Z(s) = \prod_p Z_p(s)$ such that

- (a) it has abscissa of convergence in the set $\{\frac{3}{2}, 2, 3\}$,
- (b) it admits a meromorphic continuation to the whole plane,
- (c) for any finite set of primes, say S , $Z(s)$ and $\prod_{p \notin S} Z_p(s)$ have the same abscissa of convergence,
- (d) $Z(s)$ coincides with $\zeta_G^{G, \leq}(s)$ except for a finite number of local factors.

Since an exceptional local factor $\zeta_{G_p}^{G_p, \leq}(s)$ is a rational function of p^{-s} with rational coefficients [4], it follows from (b) and (d) that $\zeta_G^{G, s}(s)$ also has a meromorphic continuation to the whole plane. Moreover, $\zeta_G^{G, \leq}(s)$ and $Z(s)$ have the same abscissa of convergence. Indeed, by (c) and (d), it is enough to show that the abscissa of convergence of each local factor of $\zeta_G^{G, \leq}(s)$ is strictly less than the abscissa of convergence of $\zeta_G^{G, \leq}(s)$. Now, this follows from the fact that $\zeta_G^{G, \leq}(s)$ is an Euler product of cone integrals [24, Theorem 1]. This important property of cone integrals was established in [5, Section 4], and it was a key point in obtaining analytic properties of global zeta functions.

To prove the last assertion of the corollary, we apply the first part to each of the partial zeta functions of $\zeta_G^{\leq}(s)$. \square

Remark 2.9. If $F = C_1$ in Table 1, we recover [8, Proposition 8.1]. See also [9] for a generalization.

2.2 Counting normal subgroups

We begin with some preliminaries from elementary group theory. Let G be, for the moment, any group, and let $N \triangleleft G$ be a normal subgroup. We inductively define a series of normal subgroups $\gamma_1(G, N) \supseteq \gamma_2(G, N) \supseteq \dots$ by setting

$$\gamma_1(G, N) := N \quad \text{and} \quad \gamma_i(G, N) := [\gamma_{i-1}(G, N), G] \quad \text{for } i \geq 2.$$

Lemma 2.10. *Assume that G is finite and that N is a normal Hall subgroup. If $\gamma_{c+1}(G, N) = 1$ for some c , then N has a unique complement, say C . In addition, $G = N \times C$ and $\gamma_{c+1}(G) \cap N = \{1\}$.*

Proof. By the Schur–Zassenhaus theorem, there is at least one complement of N in G , and they are all conjugate. We prove that there is only one by induction on c .

Assume that $c = 1$, and let C be a complement of N in G . Since

$$[N, C] \subseteq [N, G] = \gamma_2(G, N) = 1,$$

it follows that C is normal in $NC = G$, and therefore, C is the unique complement of N .

Assume now that $c > 1$. By the inductive hypothesis, there is a unique complement, say $C'/\gamma_c(G, N)$, of $N/\gamma_c(G, N)$ in $G/\gamma_c(G, N)$. Given a complement C of N in G , clearly $(C\gamma_c(G, N))/\gamma_c(G, N)$ is a complement of $N/\gamma_c(G, N)$ in $G/\gamma_c(G, N)$; thus $C\gamma_c(G, N) = C'$. This implies that C is a complement of $\gamma_c(G, N)$ in C' . However, $[\gamma_c(G, N), C'] \subseteq \gamma_{c+1}(G, N) = 1$, so by the case $c = 1$, there is only one possibility for C . This completes the induction.

The final part of the lemma is clear. \square

Lemma 2.11. *Assume that N is nilpotent, say of class c , and let $A \triangleleft G$ such that $AN = G$. Then $\gamma_{c+1}(G) \subseteq A$; in particular, $\gamma_{c+1}(G, N) \subseteq A \cap N$.*

Proof. Note that $B := A \cap N$ is also normal in G , and there is an identification $G/B = N/B \times A/B$. It follows that

$$\gamma_{c+1}(G/B) = \gamma_{c+1}(N/B) \times \gamma_{c+1}(A/B) = 1 \times \gamma_{c+1}(A/B) \subseteq A/B.$$

Since $\gamma_{c+1}(G/B) = (\gamma_{c+1}(G)B)/B$, we deduce that $\gamma_{c+1}(G) \subseteq A$. In particular, $\gamma_{c+1}(G, N) \subseteq \gamma_{c+1}(G) \cap N \subseteq A \cap N$. \square

We return to the setting introduced at the beginning of the section except that we do not yet assume that the nilpotency class is 2. We fix an intermediate normal subgroup $N \leq H \triangleleft G$ and consider the series $\zeta_{G_p}^{H_p, \triangleleft}(s)$. Note that $\gamma_i(H, N)$ is normal in G for all i and that $(\gamma_i(H, N))_p = \gamma_i(H_p, N_p)$ and $(\gamma_i(H))_p = \gamma_i(H_p)$ (cf. [15, Theorem 1.4]).

Proposition 2.12. *Let c be the nilpotency class of N . Set $G' := G/\gamma_{c+1}(H, N)$ and $N' := N/\gamma_{c+1}(H, N)$. If $p \nmid |H/N|$, then*

$$\zeta_{G_p}^{H_p, \triangleleft}(s) = \zeta_{G'_p}^{N'_p, \triangleleft}(s).$$

In particular, if $N'_p \cong \mathbb{Z}_p/k\mathbb{Z}_p$ for some $k \in \mathbb{Z}$, then

$$\zeta_{G'_p}^{H'_p, \triangleleft}(s) = \frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}}.$$

Proof. Let $H' = H/\gamma_{c+1}(H, N)$. By Lemma 2.11, we have

$$\zeta_{G'_p}^{H'_p, \triangleleft}(s) = \zeta_{G'_p}^{H'_p, \triangleleft}(s),$$

where the series on the right is computed with respect to $N'_p \triangleleft G'_p$ (this holds for all p). Assume now that $p \nmid |H/N|$. Given $A' \leq H'_p$ of finite index and normal in G'_p such that $A'N'_p = H'_p$, the intersection $B' := N'_p \cap A'$ is normal in G'_p and $[N'_p : B'] = [H'_p : A']$. Conversely, given $B' \leq N'_p$ of finite index and normal in G'_p , we have $\gamma_{c+1}(H'_p/B', N'_p/B') = (\gamma_{c+1}(H'_p, N'_p)B')/B' = 1$ since $\gamma_{c+1}(H'_p, N'_p) = 1$. Therefore, we can apply Lemma 2.10 to $N'_p/B' \triangleleft H'_p/B'$. It follows that there is a unique $A' \leq H'_p$ such that $A'N_p = H'_p$ and $A' \cap N'_p = B'$. By the uniqueness, A'/B' is normal in G'_p/B' , and hence A' is normal in G'_p . We deduce that

$$\zeta_{G'_p}^{H'_p, \triangleleft}(s) = \zeta_{G'_p}^{N'_p, \triangleleft}(s).$$

This proves the first part of the proposition.

If in addition $N'_p \cong \mathbb{Z}_p/k\mathbb{Z}_p$, then every finite-index subgroup of N'_p is characteristic; therefore,

$$\zeta_{G'_p}^{N'_p, \triangleleft}(s) = \zeta_{\mathbb{Z}_p/k\mathbb{Z}_p}^{\leq}(s),$$

which is clearly equal to

$$\frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}}. \quad \square$$

Corollary 2.13. $\zeta_G^{\triangleleft}(s)$ and $\zeta_G^{N, \triangleleft}(s)$ have the same abscissa of convergence.

Proof. We have to show that, for each intermediate normal subgroup $N \leq H \triangleleft G$, the abscissa of convergence of $\zeta_G^{H, \triangleleft}(s)$ is bounded by that of $\zeta_G^{N, \triangleleft}(s)$. We set

$$G' := G/\sqrt{\gamma_{c+1}(H, N)} \quad \text{and} \quad N' := N/\sqrt{\gamma_{c+1}(H, N)},$$

where

$$\sqrt{\gamma_{c+1}(H, N)} := \{x \in N : x^n \in \gamma_{c+1}(H, N) \text{ for some } n \in \mathbb{N}\}.$$

By Proposition 2.12, $\zeta_G^{H, \triangleleft}(s)$ and $\zeta_{G'}^{N', \triangleleft}(s)$ have the same but a finite number of local factors. By [24, Theorem 1], both series are Euler products of cone integrals. As explained in the proof of Corollary 2.8, this suffices to ensure that $\zeta_G^{H, \triangleleft}(s)$ and $\zeta_{G'}^{N', \triangleleft}(s)$ have the same abscissa of convergence. Finally, it is clear that the abscissa of convergence of $\zeta_{G'}^{N', \triangleleft}(s)$ is bounded by that of $\zeta_G^{N, \triangleleft}(s)$. \square

Remark 2.14. We claim that if $p \nmid |H/N|$, then

$$\gamma_{c+1}(H_p, N_p) = N_p \cap \gamma_{c+1}(H_p);$$

in particular, if H_p/N_p is nilpotent of class $\leq c$, then $\gamma_{c+1}(H_p, N_p) = \gamma_{c+1}(H_p)$. Indeed, let $B \triangleleft G_p$ be a finite-index normal subgroup such that

$$\gamma_{c+1}(H_p, N_p) \subseteq B \subseteq N_p.$$

Since N_p/B is a normal Hall subgroup of H_p/B and since

$$\gamma_{c+1}(H_p/B, N_p/B) = (\gamma_{c+1}(H_p, N_p)B)/B = 1,$$

we obtain by Lemma 2.10 that $\gamma_{c+1}(H_p/B) \cap N_p/B = 1$. It follows that

$$(\gamma_{c+1}(H_p)B) \cap N_p \subseteq B,$$

and hence $\gamma_{c+1}(H_p) \cap N_p \subseteq B$. Since $N_p/\gamma_{c+1}(H_p, N_p)$ is residually finite, the obvious inclusion $\gamma_{c+1}(H_p, N_p) \subseteq \gamma_{c+1}(H_p) \cap N_p$ must be an equality.

We now return to the case when N is a \mathfrak{T}_2 -group. Recall the notation introduced at the beginning of the section. Given $V \leq Z_p$ of finite index, we denote by $X(V) \leq T_p$ the largest subgroup such that $[T_p, X(V)] \subseteq V$. It has finite index in T_p , and if V is in addition P -invariant, then $X(V)$ is also P -invariant. Proposition 2.12 allows us to focus only on the first partial zeta function

$$\zeta_G^{N, \triangleleft}(s) = \sum_{B \triangleleft G : B \subseteq N} [N : B]^{-s},$$

the other ones being of the form $\zeta_{N'}^{G', \triangleleft}(s)$ except for a finite number of local factors, where G' is a quotient of G .

Proposition 2.15. Fix a prime p . Assume that all finite-index Λ_p -submodules of T_p are isomorphic. Assume also that, for every $V \leq_{\Lambda_p} Z_p$ and $U \leq_{\Lambda_p} T_p$ of finite index such that $U \subseteq X(V)$, there exists $B \triangleleft G_p$ of finite index such that $B \subseteq N_p$, $(BZ_p)/Z_p = U$ and $B \cap Z_p = V$. Then

$$\zeta_{G_p}^{N_p, \triangleleft}(s) = \zeta_{E_p}^{T_p, \triangleleft}(s) \left(\sum_{V \leq_{\Lambda_p} Z_p} [T_p : X(V)]^{-s} [Z_p : V]^{-s} |\text{Hom}_{\Lambda_p}(T_p, Z_p/V)| \right).$$

The above assumptions hold if $p \nmid |P|$.

Proof. The proof is similar to that of [8, Lemma 6.1], so we omit it. We just point out that, given $U \leq_{\Lambda_p} T_p$ and $V \leq_{\Lambda_p} Z_p$ of finite index with $U \subseteq X(V)$, the number of normal subgroups $B \triangleleft G_p$ such that $(BZ_p)/Z_p = U$ and $B \cap Z_p = V$ is equal to $\text{Hom}_{\Lambda_p}(U, Z_p/V)$. This uses the second assumption and Lemma 2.1. In addition, by the first assumption, $\text{Hom}_{\Lambda_p}(U, Z_p/V) = \text{Hom}_{\Lambda_p}(T_p, Z_p/V)$.

The first assumption is satisfied when $p \nmid |P|$ since in this case Λ_p is a maximal order of $\mathbb{Q}_p[P]$, and the second assumption is satisfied by Lemma 2.1. □

We now specialize to the case of Hirsch length 3.

Lemma 2.16. *Assume that N is a \mathfrak{F}_2 -group of Hirsch length 3. Given $V \leq Z_p$ of finite index, the following holds:*

- (1) *If $[T_p, T_p] \subseteq V$, then $[T_p : X(V)] = 1$.*
- (2) *If $V \subseteq [T_p, T_p]$, then $[T_p : X(V)] = |[Z : [N, N]]|_p^2 [Z_p : V]^2$.*

Proof. The first assertion is trivial since in this case $X(V) = T_p$. Assume now that $V \subseteq [T_p, T_p]$. Let $\{x, y\}$ be a basis for T_p . Given a \mathbb{Z}_p -submodule $X \leq T_p$ of finite index, say generated by $ax + by$ and $cx + dy$, we have

$$[X, T_p] = \langle a[x, y], b[x, y], c[x, y], d[x, y] \rangle.$$

Thus, $[X, T_p] \subseteq V$ if and only if $a[x, y], b[x, y], c[x, y], d[x, y] \in V$. Since $[x, y]$ generates $[T_p, T_p]$, the previous condition holds if and only if

$$a, b, c, d \in \frac{[Z_p : V]}{[Z_p : [T_p, T_p]]} \mathbb{Z}_p,$$

which in turn is equivalent to saying that

$$X \subseteq \frac{[Z_p : V]}{[Z_p : [T_p, T_p]]} T_p.$$

Thus,

$$X(V) = \frac{[Z_p : V]}{[Z_p : [T_p, T_p]]} T_p.$$

The index of this subgroup is

$$[Z_p : V]^2 |[Z_p : [T_p, T_p]]|_p^2 = [Z_p : V]^2 |[Z : [N, N]]|_p^2. \quad \square$$

Theorem 2.17. Assume that N is a \mathfrak{T}_2 -group of Hirsch length 3. Set

$$F := \text{Im}(P \rightarrow \text{GL}(T)) \quad \text{and} \quad E := T \rtimes F,$$

which is a plane crystallographic group. If $p \nmid |P|$, then

$$\zeta_{G_p}^{N_p, \triangleleft}(s) = \zeta_{E_p}^{T_p, \triangleleft}(s) \left(\frac{1 - |[Z : [N, N]]|_p^{s-\eta}}{1 - p^{-s+\eta}} + |[Z : [N, N]]|_p^{s-\eta} \zeta_p(3s - \eta) \right),$$

where $\eta := \text{rank}_{\mathbb{Z}}(C_T(P))$. In particular, if $p \nmid |P|$, then

$$\zeta_{G_p}^{N_p, \triangleleft}(s) = \zeta_{E_p}^{T_p, \triangleleft}(s) \zeta_p(3s - \eta).$$

Proof. By Lemma 2.4, $|\text{Hom}_{\Lambda_p}(T_p, Z_p/V)| = [Z_p : V]^\eta$ for any $V \leq_{\Lambda_p} Z_p$, where $\eta = \text{rank}_{\mathbb{Z}}(C_T(P))$. It follows from Proposition 2.15 and Lemma 2.16 that

$$\begin{aligned} \zeta_{G_p}^{N_p, \triangleleft}(s) &= \zeta_{E_p}^{T_p, \triangleleft}(s) \left(\sum_{[T_p, T_p] \subsetneq V \leq Z_p} [Z_p : V]^{-s+\eta} \right. \\ &\quad \left. + \sum_{V \leq [T_p, T_p]} |[Z : [N, N]]|_p^{-2s} [Z_p : V]^{-3s+\eta} \right) \\ &= \zeta_{E_p}^{T_p, \triangleleft}(s) \left(\sum_{k=0}^{v_p([T_p, T_p])-1} p^{(-s+\eta)k} \right. \\ &\quad \left. + \sum_{V \leq [T_p, T_p]} |[Z : [N, N]]|_p^{-2s} \right. \\ &\quad \left. \times [Z_p : [T_p, T_p]]^{-3s+\eta} \right. \\ &\quad \left. \times [[T_p, T_p] : V]^{-3s+\eta} \right) \\ &= \zeta_{E_p}^{T_p, \triangleleft}(s) \left(\frac{1 - |[Z : [N, N]]|_p^{s-\eta}}{1 - p^{-s+\eta}} + |[Z : [N, N]]|_p^{s-\eta} \zeta_p(3s - \eta) \right). \end{aligned}$$

If in addition $p \nmid [Z : [N, N]]$, then $|[Z : [N, N]]|_p = 1$, and the proof follows. \square

Corollary 2.18. If $p \nmid |P|$, then the series $\zeta_{G_p}^{N_p, \triangleleft}(s)$ is given as in Table 2, where $k := [Z : [N, N]]$. Therefore, if $p \nmid |P|$, then $\zeta_{G_p}^{N_p, \triangleleft}(s)$ satisfies the functional equation

$$\zeta_{G_p}^{N_p, \triangleleft}(s)|_{p \rightarrow p-1} = \begin{cases} (-1)^3 p^{-5s+3} \zeta_{G_p}^{N_p, \triangleleft}(s) & \text{if } F = C_1, \\ (-1)^3 p^{-5s+1} \zeta_{G_p}^{N_p, \triangleleft}(s) & \text{if } F \cong C_2 = D_1, \\ (-1)^3 \chi_d(p) p^{-5s} \zeta_{G_p}^{N_p, \triangleleft}(s) & \text{if } F \cong C_d, d \in \{3, 4, 6\}, \\ (-1)^3 p^{-5s} \zeta_{G_p}^{N_p, \triangleleft}(s) & \text{if } F \cong D_2, \\ p^{-5s} \zeta_{G_p}^{N_p, \triangleleft}(s) & \text{if } F \cong D_d, d \in \{3, 4, 6\}. \end{cases}$$

F	$\zeta_{G_p}^{N, <}(s)$	
$F \subset \mathrm{SL}_2(T)$	C_1	$\zeta_p(s)\zeta_p(s-1)\left(\frac{1- k _p^{s-2}}{1-p^{-s+2}} + k _p^{s-2}\zeta_p(3s-2)\right)$ $\zeta_p(s)\zeta_p(s-1)\zeta_p(3s-2)$ $\zeta_p(s)\zeta_p(s-1)\left(\frac{1- k _p^s}{1-p^{-s}} + k _p^s\zeta_p(3s)\right)$ $\zeta_p(s)\zeta_p(s-1)\zeta_p(3s)$
	C_2	$\zeta_p(s)\zeta_p(s-1)\left(\frac{1- k _p^s}{1-p^{-s}} + k _p^s\zeta_p(3s)\right)$ $\zeta_p(s)\zeta_p(s-1)\zeta_p(3s)$
	$C_d, d = 3, 4, 6$	$\zeta_p(s)L_p(s, \chi_d)\left(\frac{1- k _p^s}{1-p^{-s}} + k _p^s\zeta_p(3s)\right)$ $\zeta_p(s)L_p(s, \chi_d)\zeta_p(3s)$
$F \not\subset \mathrm{SL}_2(T)$	D_1	$\zeta_p(s)^2\left(\frac{1- k _p^{s-1}}{1-p^{-s+1}} + k _p^{s-1}\zeta_p(3s-1)\right)$ $\zeta_p(s)^2\zeta_p(3s-1)$
	D_2	$\zeta_p(s)^2\left(\frac{1- k _p^s}{1-p^{-s}} + k _p^s\zeta_p(3s)\right)$ $\zeta_p(s)^2\zeta_p(3s)$
	$D_d, d = 3, 4, 6$	$\zeta_p(2s)\left(\frac{1- k _p^s}{1-p^{-s}} + k _p^s\zeta_p(3s)\right)$ $\zeta_p(2s)\zeta_p(3s)$

Table 2. Local factors of $\zeta_G^{N, <}(s)$ at $p \nmid |P|$ ($F := \mathrm{Im}(P \rightarrow \mathrm{GL}(T))$ and $k := [Z : [N, N]]$).

Proof. This follows from Theorem 2.15 and uses the formula for $\zeta_E^{T, \triangleleft}(s)$ obtained in [13, Chapter 6] as a step in the computation of the normal zeta function of the plane crystallographic group E (see also [6, Section 4.2]). The calculation of η was done in Lemma 2.4. The local functional equation follows by inspection of the formula. \square

Corollary 2.19. $\zeta_G^{N, \triangleleft}(s)$ has abscissa of convergence in the set $\{\frac{1}{2}, 1, 2\}$ and admits a meromorphic continuation to the whole plane. The same holds for $\zeta_G^{\triangleleft}(s)$.

Proof. The assertion about $\zeta_G^{N, \triangleleft}(s)$ follows by inspection of Table 2 and the same argument used in the proof of Corollary 2.8. Next, by Corollary 2.13, $\zeta_G^{\triangleleft}(s)$ has the same abscissa of convergence as $\zeta_G^{N, \triangleleft}(s)$. To complete the proof, it is enough to show that, for each intermediate normal subgroup $N \leq H \triangleleft G$ different from N , the series $\zeta_G^{H, \triangleleft}(s)$ admits a meromorphic continuation to the whole plane. We use the notation from the proof of Corollary 2.13 with $c = 2$. Since the local factor of $\zeta_G^{H, \triangleleft}(s)$ at a prime p is a rational function in p^{-s} by [4], it is enough to show that $\zeta_{G'}^{N', \triangleleft}(s)$ admits a meromorphic continuation to the whole plane. If $N' = N$, then we are in the same situation as in the case of $\zeta_G^N(s)$, so we are done. If $N' \neq N$, then either $N' \cong \mathbb{Z}^2$, in which case the result follows from the formulae of the normal zeta functions of the plane crystallographic groups given in [13, Chapter 6], or else $N' \cong \mathbb{Z}$ or $N' = 0$, in which case the result is trivial. \square

2.3 Remark on the local functional equations

A natural problem is to decide whether local functional equations hold for $\zeta_G^{G, \leq}(s)$ and $\zeta_G^{N, \triangleleft}(s)$ when the Hirsch length of N is higher than 3. When $G = N$ (in which case the series are just $\zeta_N^{\leq}(s)$ and $\zeta_N^{\triangleleft}(s)$), this is discussed in [25]. When $G \neq N$ and N is abelian, the existence of local functional equations follows from the explicit formulae obtained in [6, Section 2]. In contrast to the situation of zeta functions of nilpotent groups, here the functional equations are not uniform in p . They depend on how p ramifies in certain number fields that arise in the decomposition of $\mathbb{Q}[G/N]$ into simple algebras. Propositions 2.2 and 2.15 enable us to linearize the problem when the nilpotency class is 2 (indeed, the linearization for $\zeta_G^{N, \triangleleft}(s)$ holds more generally by the Mal'cev correspondence). We will not go into details. Briefly, the problem can be stated as follows. Let L be a not necessarily associative ring additively isomorphic to \mathbb{Z}^h , and let P be a finite group acting on L by ring automorphisms. We consider the Dirichlet series

$$\zeta_{P \curvearrowright L}^{\leq}(s) = \sum [L : B]^{-s} |\text{Der}(P, L/B)| \quad \text{and} \quad \zeta_{P \curvearrowright L}^{\triangleleft}(s) = \sum [L : B]^{-s},$$

where the first sum runs over the P -invariant finite-index subrings, and the second one only over the P -invariant finite-index ideals. When L is an abelian Lie

ring, $\zeta_{P \curvearrowright L}^{\triangleleft}(s)$ is a Solomon zeta function [23]. In general, the series $\zeta_{P \curvearrowright L}^{\triangleleft}(s)$ are special cases of *submodule zeta functions*, as introduced by Rossmann in [19, Section 2.2]. Corollaries 2.7 and 2.18 imply that, when L is a 2-step nilpotent Lie ring of additive rank 3, $\zeta_{P \curvearrowright L}^{\leq}(s)$ and $\zeta_{P \curvearrowright L}^{\triangleleft}(s)$ satisfy local functional equations. In analogy with the results of [25], we might ask the following questions.

Question 1. Do $\zeta_{P \curvearrowright L}^{\leq}(s)$ and $\zeta_{P \curvearrowright L}^{\triangleleft}(s)$ satisfy local functional equations when L is a 2-step nilpotent Lie ring of rank greater than 3? Does $\zeta_{P \curvearrowright L}^{\leq}(s)$ satisfy local functional equations for arbitrary rings (not necessarily nilpotent) that are additively isomorphic to some \mathbb{Z}^h ?

We return to the case when L is a 2-step nilpotent Lie ring additively isomorphic to \mathbb{Z}^3 . It is not difficult to check, say by inspection of Table 1 and the results of [6, Section 2], that if we forget the structure of Lie ring on L (obtaining \mathbb{Z}^3) and consider the Solomon zeta function $\zeta_{P \curvearrowright \mathbb{Z}^3}^{\leq}(s)$, then $\zeta_{P \curvearrowright L}^{\leq}(s)$ and $\zeta_{P \curvearrowright \mathbb{Z}^3}^{\leq}(s)$ satisfy the same local functional equations. By this, and again in analogy with the results of [25], we can also ask the following question.

Question 2. Let L be a ring additively isomorphic to \mathbb{Z}^h . Let P be a finite group acting on L . If $\zeta_{P \curvearrowright L}^{\leq}(s)$ satisfies local functional equations, are these equations the same as those satisfied by $\zeta_{P \curvearrowright \mathbb{Z}^h}^{\leq}(s)$?

In a forthcoming paper, we show that Question 1 and Question 2 have positive answer for various nilpotent Lie rings of rank 4.

3 Local zeta functions as p -adic integrals

Let N_p be a torsion-free finitely generated nilpotent pro- p group, and let G_p be a profinite group that includes N_p as an open normal subgroup. Fix also an intermediate normal subgroup $N_p \leq H_p \triangleleft G_p$. We review a method, developed in [8, Section 2] for \mathfrak{T} -groups and extended to virtually nilpotent groups in [24], to express $\zeta_{H_p}^{H_p, \leq}(s)$ and $\zeta_{G_p}^{H_p, \triangleleft}(s)$ as p -adic integrals. This method is used in the next sections to calculate local factors at “bad” primes in some cases.

3.1 Expressing $\zeta_{N_p}^{\leq}(s)$ and $\zeta_{N_p}^{\triangleleft}(s)$ as p -adic integrals

An additional reference for this part is [12, Chapter 15], where the notation is more adapted to ours. Fix a Mal’cev basis $\mathbf{x} = (x_1, \dots, x_h)$ for N_p . By definition, the series of subgroups

$$N_p = \overline{\langle x_1, \dots, x_h \rangle} \supset \overline{\langle x_2, \dots, x_h \rangle} \supset \dots \supset \overline{\langle x_h \rangle}$$

is central, and each $x \in N_p$ determines a unique vector $\mathbf{a} = (a_1, \dots, a_h) \in \mathbb{Z}_p^h$ such that $x = \mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_h^{a_h}$.

Let $T_h(\mathbb{Z}_p)$ denote the set of $h \times h$ -upper-triangular matrices with entries in \mathbb{Z}_p , and let $T_h^+(\mathbb{Z}_p) \subset T_h(\mathbb{Z}_p)$ denote the subset of those matrices with non-zero determinant. For $\mathbf{t} \in T_h(\mathbb{Z}_p)$, we set $B_{\mathbf{t}} := \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \dots, \mathbf{x}^{\mathbf{t}_h} \rangle} \leq N_p$, where \mathbf{t}_i denotes the i -th row of \mathbf{t} . The subgroup $B_{\mathbf{t}}$ is open if and only if $\mathbf{t} \in T_h^+(\mathbb{Z}_p)$.

Given an open subgroup $B \leq N_p$, we say that $\mathbf{t} \in T_h^+(\mathbb{Z}_p)$ represents a good basis for B (with respect to \mathbf{x}) if

$$B = B_{\mathbf{t}} \quad \text{and} \quad B = \{(\mathbf{x}^{\mathbf{t}_1})^{\lambda_1} \cdots (\mathbf{x}^{\mathbf{t}_h})^{\lambda_h} : \lambda_1, \dots, \lambda_h \in \mathbb{Z}_p\}.$$

In this case, $(\mathbf{x}^{\mathbf{t}_1}, \dots, \mathbf{x}^{\mathbf{t}_h})$ is a Mal'cev basis for B (also called a good basis for B). We set

$$\mathcal{M}(B) := \{\mathbf{t} \in T_h(\mathbb{Z}_p) : \mathbf{t} \text{ represents a good basis for } B\}$$

Note that, given $\mathbf{t} = (t_{ij}) \in \mathcal{M}(B)$, the value $|t_{ii}|_p$ depends only on B (and on the fixed Mal'cev basis \mathbf{x}). Indeed, we have $|t_{ii}|_p^{-1} = [\langle x_i, \dots, x_h \rangle : B \cap \langle x_i, \dots, x_h \rangle]$. We also see from this that

$$[N_p : B]^{-1} = \prod_{i=1}^h |t_{ii}|_p. \tag{3.1}$$

We collect results from [8, Section 2] in the next lemma. The topological group $T_h(\mathbb{Z}_p) \cong \mathbb{Z}_p^{h(h+1)/2}$ has a normalized Haar measure, which we denote by μ .

Lemma 3.1. $\mathcal{M}(B)$ is an open subset of $T_h(\mathbb{Z}_p)$, and for any $\mathbf{t} = (t_{ij}) \in \mathcal{M}(B)$,

$$\mu(\mathcal{M}(B)) = (1 - p^{-1})^h \prod_{i=1}^h |t_{ii}|_p^i. \tag{3.2}$$

Therefore, for a complex variable s ,

$$[N_p : B]^{-s} = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{M}(B)} \prod_{i=1}^h |t_{ii}|_p^{s-i} d\mu. \tag{3.3}$$

Proof. The first part of the lemma is proved in [8, Lemma 2.5]; see also [12, Lemma 15.1.1]. We recall here the proof of (3.3). We start from the right-hand side and use (3.1) and (3.2):

$$\begin{aligned} \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{M}(B)} \prod_{i=1}^h |t_{ii}|_p^{s-i} d\mu &= \frac{1}{(1 - p^{-1})^h} \left(\prod_{i=1}^h |t_{ii}|_p^{s-i} \right) \mu(\mathcal{M}(B)) \\ &= \prod_{i=1}^h |t_{ii}|_p^{s-i} \prod_{i=1}^h |t_{ii}|_p^i = [N_p : B]^{-s}. \quad \square \end{aligned}$$

We now set $\mathcal{M}_N^{\leq} := \bigcup_{B \leq N} \mathcal{M}(B)$, $\mathcal{M}_N^{\triangleleft} := \bigcup_{B \triangleleft N} \mathcal{M}(B)$, where the unions run only over open subgroups. These are open subsets of $T_h^+(\mathbb{Z}_p)$, by Lemma 3.1. They both coincide with $T_h^+(\mathbb{Z}_p)$ if N is abelian. In the general case, the following descriptions were given in [8, Lemmas 2.3 and 2.4]:

$$\begin{aligned} \mathcal{M}_{N_p}^{\leq} &= \{ \mathbf{t} \in T_h^+(\mathbb{Z}_p) : [\mathbf{x}^{t_i}, \mathbf{x}^{t_j}] \in \overline{\langle \mathbf{x}^{t_{j+1}}, \dots, \mathbf{x}^{t_h} \rangle}, 1 \leq i < j \leq h \}, \\ \mathcal{M}_{N_p}^{\triangleleft} &= \{ \mathbf{t} \in T_h^+(\mathbb{Z}_p) : [x_i, \mathbf{x}^{t_j}] \in \overline{\langle \mathbf{x}^{t_{j+1}}, \dots, \mathbf{x}^{t_h} \rangle}, 1 \leq i, j \leq h \}. \end{aligned} \tag{3.4}$$

As an immediate consequence of formula (3.3), we obtain the following corollary.

Corollary 3.2 ([8, Proposition 2.7]). *For $\ast \in \{\leq, \triangleleft\}$,*

$$\zeta_{N_p}^{\ast}(s) = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{M}_{N_p}^{\ast}} \prod_{i=1}^h |t_{ii}|_p^{s-i} d\mu.$$

3.2 Expressing $\zeta_{H_p}^{H_p, \leq}(s)$ and $\zeta_{G_p}^{H_p, \triangleleft}(s)$ as p -adic integrals

We begin with the following particular case, which is an immediate consequence of (3.3).

Corollary 3.3. *Let $\alpha_1, \dots, \alpha_r \in G_p$ be such that their classes modulo N_p generate G_p/N_p . Then*

$$\zeta_{G_p}^{N_p, \triangleleft}(s) = \frac{1}{(1 - p^{-1})^h} \int_{\substack{\mathbf{t} \in \mathcal{M}_{N_p}^{\triangleleft} \\ \alpha_j(\mathbf{x}^{t_i}) \in \overline{\langle \mathbf{x}^{t_1}, \dots, \mathbf{x}^{t_h} \rangle}, \\ i=1, \dots, h, j=1, \dots, r}} \prod_{i=1}^h |t_{ii}|_p^{s-i} d\mu.$$

It remains to consider the case $H_p \neq N_p$. We set $F := H_p/N_p$ and denote by e its identity element. The following is fixed in the rest of this section.

- A presentation

$$\langle f_1, \dots, f_r \mid R_\lambda(f_1, \dots, f_r) = e, \lambda \in J \rangle$$

for the group F ($f_j \neq e$ for all j).

- Group words $w_f(X_1, \dots, X_r)$, $f \in F$, such that $f = w_f(f_1, \dots, f_r)$. In the case $f = f_j$ ($j = 1, \dots, r$), we just set $w_{f_j} = X_j$, and in the case $f = e$, we set $w_e = e$, the empty word.
- A transversal $\{\beta_f : f \in F\}$ to the cosets of N_p in H_p such that $\beta_f N_p = f$ for all $f \in F$ and $\beta_e = 1$ (the identity of N_p). We denote $\beta_i := \beta_{f_i}$ (this is why we denoted the identity of F by e and not by 1).
- Elements $\alpha_1, \dots, \alpha_s \in G_p$ whose classes modulo N_p generate G_p/N_p . Assume that the first t of them ($t \leq s$), $\alpha_1, \dots, \alpha_t$, generate G_p modulo H_p .

Given $A \leq H_p$ of finite index such that $AN_p = H_p$, the intersection $A \cap N_p \subseteq N_p$ is open and A can be written as

$$A = (A \cap N_p) \bigcup \left(\bigcup_{f \in F \setminus \{e\}} \beta_f n_f (A \cap N_p) \right) \quad \text{for some } n_f \in N_p.$$

Now, given an open subgroup $B \leq N_p$ and $n_f \in N_p$ for $f \in F \setminus \{e\}$, Lemma 3.4 below establishes necessary and sufficient conditions on the elements n_f for the set

$$A(B, (n_f)_{f \in F \setminus \{e\}}) := B \bigcup \left(\bigcup_{f \in F \setminus \{e\}} \beta_f n_f B \right)$$

to be a subgroup of G_p . In this case, necessarily, $A(B, (n_f)_{f \in F \setminus \{e\}})N_p = H_p$.

Lemma 3.4. *Fix an open subgroup $B \leq N_p$ and $n_f \in N_p$ for each $f \in F \setminus \{e\}$. Let $n_j := n_{f_j}$. Then the set $A := A(B, (n_f)_{f \in F \setminus \{e\}})$ is a subgroup of G_p if and only if the following three conditions are satisfied.*

- (1) $\beta_j n_j B \subseteq B$ for $1 \leq j \leq r$.
- (2) $R_\lambda(\beta_1 n_1, \dots, \beta_r n_r) \in B$ for all $\lambda \in J$.
- (3) $\beta_f n_f \in w_f(\beta_1 n_1, \dots, \beta_r n_r)B$ for all $f \in F \setminus \{e, f_1, \dots, f_r\}$.

Moreover, A is a normal subgroup of G_p if and only if (2), (3) and the following four conditions are satisfied.

- (4) B is normal in N_p .
- (5) $\alpha_k B \subseteq B$ for $1 \leq k \leq s$.
- (6) $[\beta_j n_j, x_i] \in B$ for $1 \leq j \leq r$ and $1 \leq i \leq h$.
- (7) $[\alpha_k, \beta_j n_j] \in w_{[\bar{\alpha}_k, f_j]}(\beta_1 n_1, \dots, \beta_r n_r)B$ for $1 \leq k \leq t$ and $1 \leq j \leq r$, where $\bar{\alpha}_k$ denotes the class of α_k at G_p/N_p .

Proof. Assume first that A is a subgroup of G_p . Then clearly $B = A \cap N_p$, so B is normal in A ; in particular, (1) holds. Note that

$$R_\lambda(\beta_1 n_1, \dots, \beta_r n_r)N_p = R_\lambda(f_1, \dots, f_r) = e \quad (e \in F),$$

and hence $R_\lambda(\beta_1 n_1, \dots, \beta_r n_r) \in A \cap N_p = B$; thus (2) holds. Similarly, $\beta_f n_f$ and $w_f(\beta_1 n_1, \dots, \beta_r n_r)$ have the same image at F , and hence (3) also holds.

Conversely, assume that conditions (1), (2) and (3) are satisfied for the collection $(n_f)_{f \in F \setminus \{e\}}$. Condition (1) implies that $\beta_j n_j \in N_{H_p}(B)$, the normalizer of B in H_p , and then (3) implies that $\beta_f n_f \in N_{H_p}(B)$ for all $f \in F \setminus \{e\}$. It follows that $A \subseteq N_{H_p}(B)$. Let A'/B denote the subgroup of $N_{H_p}(B)/B$ generated

by A/B . By (3), A'/B is generated by the classes of $\beta_1 n_1, \dots, \beta_r n_r$, and by (2), these generators satisfy the relations defining the presentation of F . It follows that $|A'/B| \leq |F|$. On the other hand, $|A/B|$ has exactly $|F|$ elements. Thus, $A/B = A'/B$, whence $A = A'$. We conclude that A is a subgroup of G_p . This completes the proof of the first part of the lemma.

Assume now that A is a normal subgroup of G_p . According to the first part of the lemma, (2) and (3) are satisfied. Note that (4) and (5) also hold since $B = A \cap N_p$ is also normal in G_p . Now, the normality of A implies that $[\beta_j n_j, x_i] \in A$, and the normality of N_p that $[\beta_j n_j, x_i] \in N_p$; thus $[\beta_j n_j, x_i] \in B = A \cap N_p$, whence (6) is also satisfied. Finally, $(w_{[\bar{\alpha}_k, f_j]}(\beta_1 n_1, \dots, \beta_r n_r))^{-1} [\alpha_k, \beta_j n_j]$ is in A since A is normal, and it is in N_p since its image at G_p/N_p is the identity. This shows that (7) also holds.

Conversely, assume that (2) to (7) are satisfied. Note that (4) and (5) imply (1); hence, according to the first part of the lemma, A is a subgroup. They also imply that B is normal in G_p . We saw in the second paragraph of the proof that A/B is generated by the classes of the elements $\beta_j n_j$; thus, by (6), N_p/B is included in $N_{G_p/B}(A/B)$. It follows that A/B is normal in H_p/B . Finally, by (7), A/B is normal in G_p/B , and hence A is normal in G_p . This completes the proof of the lemma. □

We set more notation for the next proposition. We denote by $M_{k \times h}(\mathbb{Z}_p)$ the set of $k \times h$ -matrices with entries in \mathbb{Z}_p . Given $\mathbf{v} = (v_{ij}) \in M_{k \times h}(\mathbb{Z}_p)$, its i -th row vector $(v_{i1}, \dots, v_{ih}) \in \mathbb{Z}_p^h$ is denoted by \mathbf{v}_i . The normalized Haar measure of

$$T_h(\mathbb{Z}_p) \times M_{k \times h}(\mathbb{Z}_p) \cong \mathbb{Z}_p^{kh+h(h+1)/2}$$

is denoted by μ . Elements of $T_h(\mathbb{Z}_p) \times M_{k \times h}(\mathbb{Z}_p)$ are denoted (\mathbf{t}, \mathbf{v}) .

Proposition 3.5. *It holds*

$$\zeta_{H_p}^{H_p, \leq}(s) = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{T}_{H_p}^{H_p, \leq}} \prod_{i=1}^h |t_{ii}|_p^{s-i-r} d\mu,$$

$$\zeta_{G_p}^{H_p, <}(s) = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{T}_{G_p}^{H_p, <}} \prod_{i=1}^h |t_{ii}|_p^{s-i-r} d\mu,$$

where $\mathcal{T}_{H_p}^{H_p, \leq} \subset T_h(\mathbb{Z}_p) \times M_{r,h}(\mathbb{Z}_p)$ is the set of pairs (\mathbf{t}, \mathbf{v}) such that $\mathbf{t} \in \mathcal{M}_{N_p}^{\leq}$ and

$$\beta_j [\mathbf{x}^{\mathbf{v}_j}, \mathbf{x}^{\mathbf{t}_i}][\beta_j, \mathbf{x}^{\mathbf{t}_i}], R_\lambda(\beta_1 \mathbf{x}^{\mathbf{v}_1}, \dots, \beta_r \mathbf{x}^{\mathbf{v}_r}) \in \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \dots, \mathbf{x}^{\mathbf{t}_h} \rangle},$$

$$1 \leq i \leq h, 1 \leq j \leq r, \lambda \in J, \tag{3.5}$$

and $\mathcal{T}_{G_p}^{H_p, \triangleleft} \subset T_h(\mathbb{Z}_p) \times M_{r,h}(\mathbb{Z}_p)$ is the set of pairs (\mathbf{t}, \mathbf{v}) such that $\mathbf{t} \in \mathcal{M}_{N_p}^{\triangleleft}$ and

$$\alpha^k(\mathbf{x}^{t_i}), R_\lambda(\beta_1 \mathbf{x}^{v_1}, \dots, \beta_r \mathbf{x}^{v_r}), \beta_j [\mathbf{x}^{v_j}, x_i][\beta_j, x_i],$$

$$(w_{[\bar{\alpha}_l, f_j]}(\beta_1 \mathbf{x}^{v_1}, \dots, \beta_r \mathbf{x}^{v_r}))^{-1} [\alpha_l, \beta_j \mathbf{x}^{v_j}] \in \overline{\langle \mathbf{x}^{t_1}, \dots, \mathbf{x}^{t_h} \rangle}$$

$$1 \leq k \leq s, 1 \leq i \leq h, \lambda \in J, 1 \leq j \leq r, 1 \leq l \leq t. \tag{3.6}$$

Proof. Throughout the proof, we shall refer repeatedly to conditions (1)–(7) of Lemma 3.4. We will also use the same notation. The first part of this lemma establishes a bijection between the family $\mathcal{F}_{H_p}^{H_p, \leq}$ of open subgroups $A \leq G_p$ such that $AN_p = H_p$ and the family of sequences $(n_f B)_{f \in F}$, where B is an open subgroup of N_p and the n_f 's are elements of N with $n_e = 1$ such that (1), (2) and (3) are fulfilled. As (3) simply expresses $n_f B$ in terms of $n_1 B, \dots, n_r B$ (recall that n_j denotes n_{f_j}), it follows that $\mathcal{F}_{H_p}^{H_p, \leq}$ is in a bijection with the set, say \mathcal{T} , of sequences $(B, n_1 B, \dots, n_r B)$ for which conditions (1) and (2) are satisfied. Thus,

$$\zeta_{H_p}^{H_p, \leq}(s) := \sum_{A \in \mathcal{F}_{H_p}^{H_p, \leq}} [H_p : A]^{-s} = \sum_{(B, n_1 B, \dots, n_r B) \in \mathcal{T}} [N_p : B]^{-s}. \tag{3.7}$$

We now fix $(B, n_1 B, \dots, n_r B) \in \mathcal{T}$ and set

$$\mathcal{S}(B, n_1 B, \dots, n_r B) := \{ \mathbf{v} \in M_{r \times h}(\mathbb{Z}_p) : (B, \mathbf{x}^{v_1} B, \dots, \mathbf{x}^{v_r} B) \\ = (B, n_1 B, \dots, n_r B) \}.$$

This is an open subset of $M_{r \times h}(\mathbb{Z}_p)$ of Haar measure $[N_p : B]^{-r}$. In fact, the mapping $\mathbb{Z}_p^h \rightarrow N_p$ given by $\mathbf{a} \mapsto \mathbf{x}^{\mathbf{a}}$ is a homeomorphism that preserves the (normalized) Haar measure (cf. [24, Lemma 2.4]), and the measure of each coset $n_j B$ is $[N_p : B]^{-1}$. Combining this with Lemma 3.1 and formula (3.1), we deduce that $\mathcal{M}(B) \times \mathcal{S}(B, n_1 B, \dots, n_r B)$ is an open subset of $T_h(\mathbb{Z}_p) \times M_{r \times h}(\mathbb{Z}_p)$ and that, for any $\mathbf{t} = (t_{ij}) \in \mathcal{M}(B)$,

$$\mu(\mathcal{M}(B) \times \mathcal{S}(B, n_{f_1} B, \dots, n_{f_r} B)) = (1 - p^{-1})^h \prod_{i=1}^h |t_{ii}|_p^i \prod_{i=1}^h |t_{ii}|_p^r \\ = (1 - p^{-1})^h \prod_{i=1}^h |t_{ii}|_p^{r+i}.$$

It follows (again, using formula (3.1)) that

$$[N_p : B]^{-s} = \prod_{i=1}^h |t_{ii}|_p^s \\ = (1 - p^{-1})^{-h} \prod_{i=1}^h |t_{ii}|_p^{s-r-i} \mu(\mathcal{M}(B) \times \mathcal{S}(B, n_{f_1} B, \dots, n_{f_r} B))$$

$$= \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{M}(B) \times \mathcal{S}(B, n_{f_1} B, \dots, n_{f_r} B)} \prod_{i=1}^h |t_{ii}|_p^{s-r-i} d\mu.$$

This and (3.7) imply that

$$\zeta_{H_p}^{H_p, \leq}(s) = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{T}_{H_p}^{H_p, \leq}} \prod_{i=1}^h |t_{ii}|_p^{s-r-i} d\mu,$$

where

$$\mathcal{T}_{H_p}^{H_p, \leq} := \bigcup_{(B, n_1 B, \dots, n_r B) \in \mathcal{F}} \mathcal{M}(B) \times \mathcal{S}(B, n_1 B, \dots, n_r B).$$

Note that this is the set of pairs $(\mathbf{t}, \mathbf{v}) \in T_h(\mathbb{Z}_p) \times M_{r \times h}(\mathbb{Z}_p)$ such that

- (a) $\mathbf{t} \in \mathcal{M}_{N_p}^{\leq}$ and
- (b) the elements $n_j := \mathbf{x}^{\mathbf{v}^j}$ for $j = 1, \dots, r$ satisfy conditions (1) and (2) with $B = \langle \mathbf{x}^{\mathbf{t}^1}, \dots, \mathbf{x}^{\mathbf{t}^h} \rangle$.

These are precisely the conditions listed in (3.5) since

$$\beta_j [\mathbf{x}^{\mathbf{v}^j}, \mathbf{x}^{\mathbf{t}^i}] [\beta_j, \mathbf{x}^{\mathbf{t}^i}] = [\beta_j \mathbf{x}^{\mathbf{v}^j}, \mathbf{x}^{\mathbf{t}^i}].$$

This proves the integral expression for $\zeta_{H_p}^{H_p, \leq}(s)$.

The expression for $\zeta_{G_p}^{H_p, \triangleleft}(s)$ is obtained similarly. This time we have to consider sequences $(n_f B)_{f \in F}$ such that conditions (2) to (7) are satisfied. Working as in the previous case, we arrive at the expression

$$\zeta_{G_p}^{H_p, \triangleleft}(s) = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{T}_{G_p}^{H_p, \triangleleft}} \prod_{i=1}^h |t_{ii}|_p^{s-r-i} d\mu,$$

where $\mathcal{T}_{G_p}^{H_p, \triangleleft} = \bigcup \mathcal{M}(B) \times \mathcal{S}(B, n_1 B, \dots, n_r B)$, and the union runs over those $(B, n_1 B, \dots, n_r B)$ with B open and normal in N_p and the n_j 's satisfying (2), (5), (6), (7). Therefore, $\mathcal{T}_{G_p}^{H_p, \triangleleft}$ consists of the pairs (\mathbf{t}, \mathbf{v}) such that

- (a) $\mathbf{t} \in \mathcal{M}_{N_p}^{\triangleleft}$,
- (b) $\alpha_k(\mathbf{x}^{\mathbf{t}^i}) \in \overline{\langle \mathbf{x}^{\mathbf{t}^1}, \dots, \mathbf{x}^{\mathbf{t}^h} \rangle}$ for $i = 1, \dots, h$ and $k = 1, \dots, s$ and
- (c) the elements $n_j := \mathbf{x}^{\mathbf{v}^j}$ ($j = 1, \dots, r$) satisfy conditions (2), (6) and (7).

Clearly (b) and (c) are equivalent to (3.6) since $\beta_j [\mathbf{x}^{\mathbf{v}^j}, x_i] [\beta_j, x_i] = [\beta_j \mathbf{x}^{\mathbf{v}^j}, x_i]$. This completes the proof of the proposition. □

3.3 A method to simplify cone integrals

We end this section with an elementary observation that we will use frequently to simplify the calculation of cone integrals.

Given $m, n \in \mathbb{N}$, we consider subsets of $\mathbb{Z}_p^m \times \mathbb{Z}_p^n$ obtained as follows.

- (i) Fix a measurable subset $D_0 \subset \mathbb{Z}_p^m$ and rational functions

$$g_i(\mathbf{T}) \in \mathbb{Q}_p(\mathbf{T}) = \mathbb{Q}_p(T_1, \dots, T_m), \quad i = 1, \dots, n,$$

such that $g_i(\mathbf{t}) \in \mathbb{Z}_p$ whenever $\mathbf{t} \in D_0$.

- (ii) Given an integer i such that $1 \leq i \leq n$, assume that we have defined a measurable subset $D_{i-1} \subset \mathbb{Z}_p^m \times \mathbb{Z}_p^{i-1}$. Let

$$k_i(\mathbf{T}, V_1, \dots, V_{i-1}), \lambda_i(\mathbf{T}, V_1, \dots, V_{i-1}) \in \mathbb{Q}_p(\mathbf{T}, V_1, \dots, V_{i-1})$$

be rational functions such that

$$k_i(\mathbf{t}, v_1, \dots, v_{i-1}) \in \mathbb{Z}_p \quad \text{and} \quad \lambda_i(\mathbf{t}, v_1, \dots, v_{i-1}) \in \mathbb{Z}_p^*$$

whenever $(\mathbf{t}, v_1, \dots, v_{i-1}) \in D_{i-1}$. We define D_i in one of the following two ways:

- I: $D_i = \{(\mathbf{t}, v_1, \dots, v_i) \in \mathbb{Z}_p^m \times \mathbb{Z}_p^i : (\mathbf{t}, v_1, \dots, v_{i-1}) \in D_{i-1},$
 $g_i(\mathbf{t}) \mid k_i(\mathbf{t}, v_1, \dots, v_{i-1})$
 $+ \lambda_i(\mathbf{t}, v_1, \dots, v_{i-1})v_i\},$
- II: $D_i = \{(\mathbf{t}, v_1, \dots, v_i) \in \mathbb{Z}_p^m \times \mathbb{Z}_p^i : (\mathbf{t}, v_1, \dots, v_{i-1}) \in D_{i-1},$
 $|g_i(\mathbf{t})|_p = |k_i(\mathbf{t}, v_1, \dots, v_{i-1})$
 $+ \lambda_i(\mathbf{t}, v_1, \dots, v_{i-1})v_i|_p\}.$

- (iii) We finally set $D = D_n \subset \mathbb{Z}_p^m \times \mathbb{Z}_p^n$.

We say that $(D_0, g_1(\mathbf{T}), \dots, g_n(\mathbf{T}))$ is the initial datum of definition of D and call (v_1, \dots, v_n) the sequence of *pivots*. A pivot v_i is said to be of type I or type II according to the way we choose to define D_i .

Proposition 3.6. *Let $f_0, g_0 \in \mathbb{Q}_p[\mathbf{T}] = \mathbb{Q}_p[T_1, \dots, T_m]$ be non-zero polynomials, and consider the integral*

$$\int_{D \subseteq \mathbb{Z}_p^m \times \mathbb{Z}_p^n} |f_0(\mathbf{t})|_p^s |g_0(\mathbf{t})|_p d\mu(\mathbf{t}), \quad s \in \mathbb{C}, \quad (3.8)$$

where $D \subseteq \mathbb{Z}_p^m \times \mathbb{Z}_p^n$ is a measurable subset defined from a datum

$$(D_0, g_1(\mathbf{T}), \dots, g_n(\mathbf{T}))$$

and pivots (v_1, \dots, v_n) . If the number of pivots of type II is r , then

$$\int_D |f_0(\mathbf{t})|_p^s |g_0(\mathbf{t})|_p d\mu(\mathbf{t}) = (1 - p^{-1})^r \int_{D_0} |f_0(\mathbf{t})|_p^s \prod_{i=0}^n |g_i(\mathbf{t})|_p d\mu(\mathbf{t}).$$

Proof. The integral (3.8) can be performed as follows. We start integrating with respect to the variable v_n . Note that

$$g_n(\mathbf{t}) \mid k_n(\mathbf{t}, v_1, \dots, v_{n-1}) + \lambda_n(v_1, \dots, v_{n-1})v_n$$

if and only if $v_n \in g_n(\mathbf{t})\mathbb{Z}_p - \frac{k_n(\mathbf{t}, v_1, \dots, v_{n-1})}{\lambda_n(\mathbf{t}, v_1, \dots, v_{n-1})}$,

and the Haar measure of this set is $|g_n(\mathbf{t})|_p$. Similarly,

$$|g_i(\mathbf{t})|_p = |k_n(\mathbf{t}, v_1, \dots, v_{n-1}) + \lambda_n(\mathbf{t}, v_1, \dots, v_{n-1})v_n|_p$$

if and only if $v_n \in g_n(\mathbf{t})\mathbb{Z}_p^* - \frac{k_n(\mathbf{t}, v_1, \dots, v_{n-1})}{\lambda_n(\mathbf{t}, v_1, \dots, v_{n-1})}$,

and the Haar measure of the this set is $(1 - p^{-1})|g_n(\mathbf{t})|_p$. To sum up, after integrating with respect to v_n , the integrand is multiplied by $|g_n(\mathbf{t})|_p$ or $(1 - p^{-1})|g_n(\mathbf{t})|_p$ according to whether v_n is a pivot of type I or II, and the new domain of integration is clearly D_{n-1} . We next integrate with respect to v_{n-1} and so on. After n steps, we arrive at the desired form of the integral. □

4 The formulae

We now present the complete formulae for the subgroup and normal zeta functions of the 3-dimensional almost-Bieberbach groups, whose definition we recall below. Proofs of these formulae will be left for the next two and last sections. For details of the next discussion, we refer to [2].

Let \mathcal{N} be a connected and simply connected nilpotent Lie group, and $\text{Aut}(\mathcal{N})$ the topological group of the Lie group automorphisms of \mathcal{N} . The semi-direct product $\mathcal{N} \rtimes \text{Aut}(\mathcal{N})$ acts on \mathcal{N} in a canonical way by

$$({}^{n,\alpha})x = n\alpha(x) \quad \text{for all } x, n \in \mathcal{N} \text{ and all } \alpha \in \text{Aut}(\mathcal{N}).$$

All maximal compact subgroups of $\text{Aut}(\mathcal{N})$ are conjugate. We fix one of them, say $\mathcal{C} \subset \text{Aut}(\mathcal{N})$. A torsion-free uniform discrete subgroup G of $\mathcal{N} \rtimes \mathcal{C}$ is called an almost-Bieberbach group (abbreviated as AB-group). Note that the quotient space $G \backslash \mathcal{N}$ is a compact manifold of the same dimension as \mathcal{N} , whose fundamental group is identified with G . The dimension of G is defined as the dimension of \mathcal{N} .

When $\mathcal{N} = \mathbb{R}^n$ and $\mathcal{C} = O(\mathbb{R}^n)$, the orthogonal group, we recover the definition of Bieberbach group. The quotient space $G \backslash \mathbb{R}^n$, with the metric induced from the Euclidean space \mathbb{R}^n , is a compact flat manifold.

The AB-groups were characterized algebraically as the finitely generated torsion-free virtually nilpotent groups (cf. [2, Theorem 3.1.3]). If G is an AB-group, then its Fitting subgroup $\text{Fitt}(G)$ (the maximal normal nilpotent subgroup of G) is indeed maximal nilpotent. It follows that any intermediate subgroup

$$\text{Fitt}(G) \leq H \leq G$$

is also an AB-group with $\text{Fitt}(H) = N$. An AB-group G is a Bieberbach group if and only if $\text{Fitt}(G)$ is abelian.

4.1 The 3-dimensional Bieberbach groups and their zeta functions

There are only ten 3-dimensional Bieberbach groups up to isomorphism (cf. [29, Chapter 3]). They are listed below with their corresponding zeta functions expressed in terms of the partial zeta functions with respect to the Fitting subgroup (which is always $\langle x_1, x_2, x_3 \rangle$). The first six are the fundamental groups of the orientable compact flat manifolds, and the last four are the fundamental groups of the non-orientable ones. We have arranged the presentation so that $\zeta_G^{\leq}(s)$ appears as the first term in the formula of $\zeta_G^{\leq}(s)$ and $[G : N]^{-s} \zeta_N^{\leq}(s)$ as the last one. The same holds for $\zeta_G^{\triangleleft}(s)$.

The group

$$\mathcal{G}_1 = \langle x_1, x_2, x_3 : [x_1, x_2] = [x_1, x_3] = [x_2, x_3] = 1 \rangle$$

has zeta functions

$$\zeta_{\mathcal{G}_1}^{\leq}(s) = \zeta_{\mathcal{G}_1}^{\triangleleft}(s) = \zeta(s)\zeta(s-1)\zeta(s-2).$$

The group

$$\mathcal{G}_2 = \langle \alpha, x_1, x_3, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \alpha^2 = x_3, \alpha x_1 = x_1^{-1}, \alpha x_2 = x_2^{-1} \rangle$$

has zeta functions

$$\begin{aligned} \zeta_{\mathcal{G}_2}^{\leq}(s) &= \zeta(s)\zeta(s-1)\zeta(s-2)(1-2^{-s}) + 2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2) \\ &= \zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{G}_2}^{\triangleleft}(s) &= (1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s})(1-2^{-s})\zeta(s) \\ &\quad + 2^{-s}\zeta(s)^2\zeta(s-1)(1 + 3 \cdot 2^{-s}). \end{aligned}$$

The group

$$\mathcal{G}_3 = \langle \alpha, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \alpha^3 = x_1, \alpha x_2 = x_3, \alpha x_3 = x_2^{-1} x_3^{-1} \rangle$$

has zeta functions

$$\begin{aligned} \zeta_{\mathcal{G}_3}^{\leq}(s) &= (1 - 3^{-s})\zeta(s)\zeta(s-1)L(s-1, \chi_3) + 3^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{G}_3}^{\triangleleft}(s) &= (1 + 3 \cdot 3^{-s})(1 - 3^{-s})\zeta(s) + 3^{-s}\zeta(s)^2 L(s, \chi_3)(1 + 2 \cdot 3^{-s}). \end{aligned}$$

The group

$$\mathcal{G}_4 = \langle \alpha, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \alpha^4 = x_1, \alpha x_2 = x_3, \alpha x_3 = x_2^{-1} \rangle$$

has zeta functions

$$\begin{aligned} \zeta_{\mathcal{G}_4}^{\leq}(s) &= \zeta(s)\zeta(s-1)L(s-1, \chi_4)(1 - 2^{-s}) \\ &\quad + 2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)(1 - 2^{-s}) \\ &\quad + 4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{G}_4}^{\triangleleft}(s) &= (1 + 2 \cdot 2^{-s})(1 - 2^{-s})\zeta(s) \\ &\quad + 2^{-s}(1 + 2 \cdot 2^{-s} + 2 \cdot 4^{-s})(1 - 2^{-s})\zeta(s) \\ &\quad + 4^{-s}\zeta(s)^2 L(s, \chi_4)(1 + 2^{-s}). \end{aligned}$$

The group

$$\mathcal{G}_5 = \langle \alpha, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \alpha^6 = x_1, \alpha x_2 = x_3, \alpha x_3 = x_2^{-1} x_3 \rangle$$

has zeta functions

$$\begin{aligned} \zeta_{\mathcal{G}_5}^{\leq}(s) &= \zeta(s)\zeta(s-1)L(s-1, \chi_3)(1 - 2^{-s})(1 - 3^{-s}) \\ &\quad + 2^{-s}\zeta(s)\zeta(s-1)L(s-1, \chi_3)(1 - 3^{-s}) \\ &\quad + 3^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)(1 - 2^{-s}) + 6^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{G}_5}^{\triangleleft}(s) &= (1 - 2^{-s})(1 - 3^{-s})\zeta(s) + 2^{-s}(1 + 3^{-s})(1 - 3^{-s})\zeta(s) \\ &\quad + 3^{-s}(1 + 4^{-s})(1 - 2^{-s})\zeta(s) + 6^{-s}\zeta(s)^2 L(s, \chi_3). \end{aligned}$$

The group

$$\begin{aligned} \mathcal{G}_6 = \langle \alpha, \beta, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \\ \alpha^2 = x_1, \alpha x_2 = x_2^{-1}, \alpha x_3 = x_3^{-1}, \\ \beta^2 = x_2, \beta x_1 = x_1^{-1}, \beta x_3 = x_3^{-1}, (\alpha\beta)^2 = x_3 \rangle \end{aligned}$$

has zeta functions

$$\begin{aligned}\zeta_{\mathcal{G}_6}^{\leq}(s) &= \zeta(s-1)^3(1-2\cdot 2^{-s})^3 \\ &\quad + 3\cdot 2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)(1-2^{-s}) \\ &\quad + 4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{G}_6}^{\triangleleft}(s) &= 1 + 3\cdot 2^{-s}(1+2\cdot 2^{-s})(1-2^{-s})\zeta(s) \\ &\quad + 4^{-s}\zeta(s)^3(1+4\cdot 2^{-s}+4^{-s}).\end{aligned}$$

The group

$$\mathcal{B}_1 = \langle \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \\ \varepsilon^2 = x_1, \varepsilon x_2 = x_2, \varepsilon x_3 = x_3^{-1} \rangle$$

has zeta functions

$$\begin{aligned}\zeta_{\mathcal{B}_1}^{\leq}(s) &= \zeta(s)\zeta(s-1)^2(1-2^{-s})(1+2^{-(s-1)}) \\ &\quad + 2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{B}_1}^{\triangleleft}(s) &= (1-2^{-s})(1+2^{-(s-2)})\zeta(s)\zeta(s-1) \\ &\quad + 2^{-s}(1+3\cdot 2^{-s})\zeta(s)^2\zeta(s-1).\end{aligned}$$

The group

$$\mathcal{B}_2 = \langle \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \\ \varepsilon^2 = x_1, \varepsilon x_2 = x_2, \varepsilon x_3 = x_1 x_2 x_3^{-1} \rangle$$

has zeta functions

$$\begin{aligned}\zeta_{\mathcal{B}_2}^{\leq}(s) &= \zeta(s)\zeta(s-1)^2(1-2^{-s})(1-2\cdot 2^{-s}+8\cdot 4^{-s}) \\ &\quad + 2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{B}_2}^{\triangleleft}(s) &= \zeta(s)\zeta(s-1)(1-2^{-s})+2^{-s}\zeta(s)^2\zeta(s-1)(1-2^{-s}+4\cdot 4^{-s}).\end{aligned}$$

The group

$$\mathcal{B}_3 = \langle \alpha, \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \\ \alpha^2 = x_1, \alpha x_2 = x_2^{-1}, \alpha x_3 = x_3^{-1}, \\ \varepsilon^2 = x_2, \varepsilon x_1 = x_1, \varepsilon x_3 = x_3^{-1}, [\varepsilon, \alpha] = x_2 \rangle$$

has zeta functions

$$\begin{aligned} \zeta_{\mathcal{B}_3}^{\leq}(s) &= \zeta(s)\zeta(s-1)^2(1-2^{-s})(1-4\cdot 4^{-s}) \\ &\quad + 2\cdot 2^{-s}\zeta(s)\zeta(s-1)^2(1-2^{-s})(1+2\cdot 2^{-s}) \\ &\quad + 2^{-s}\zeta(s)\zeta(s-2)\zeta(s-3)(1-2^{-s}) \\ &\quad + 4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{B}_3}^{\triangleleft}(s) &= \zeta(s)(1-2^{-s})(1+3\cdot 2^{-s}) \\ &\quad + 2^{-s}\zeta(s)^2(1-2^{-s})(1+2^{-s}+2\cdot 4^{-s}) \\ &\quad + 2^{-s}\zeta(s)(1-2^{-s})(1+2\cdot 2^{-s}) \\ &\quad + 2^{-s}\zeta(s)^2(1-2^{-s})(1+5\cdot 2^{-s}+2\cdot 4^{-s}) \\ &\quad + 4^{-s}\zeta(s)^3(1+4\cdot 2^{-s}+4^{-s}). \end{aligned}$$

The group

$$\begin{aligned} \mathcal{B}_4 &= \langle \alpha, \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \text{ for all } i, j, \\ &\quad \alpha^2 = x_1, \alpha x_2 = x_2^{-1}, \alpha x_3 = x_3^{-1}, \\ &\quad \varepsilon^2 = x_2, \varepsilon x_1 = x_1, \varepsilon x_3 = x_3^{-1}, [\varepsilon, \alpha] = x_2 x_3 \rangle \end{aligned}$$

has zeta functions

$$\begin{aligned} \zeta_{\mathcal{B}_4}^{\leq}(s) &= \zeta(s)\zeta(s-1)^2(1-2^{-s})(1-2^{-(s-1)})^2 \\ &\quad + 2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)(1-2^{-s}) \\ &\quad + 2\cdot 2^{-s}\zeta(s)\zeta(s-1)^2(1-2^{-s})(1+2^{-(s-1)}) \\ &\quad + 4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2), \\ \zeta_{\mathcal{B}_4}^{\triangleleft}(s) &= 4^{-s}\zeta(s)^3(1+4\cdot 2^{-s}+4^{-s})+2^{-s}\zeta(s)(1-2^{-s})(1+2\cdot 2^{-s}) \\ &\quad + 2^{-s}\zeta(s)^2(1-2^{-s})(1+2^{-s}+2\cdot 4^{-s}) \\ &\quad + 2^{-s}\zeta(s)^2(1-2^{-s})^2(1+2\cdot 2^{-s})+\zeta(s)(1-2^{-s}). \end{aligned}$$

4.2 The 3-dimensional almost-Bieberbach groups and their zeta functions

The 3-dimensional AB-groups with non-abelian Fitting subgroup were classified in [3]; see also [2, Chapter 7]. They are arranged into seven families according to the isomorphism type of $G/Z(N)$ (G is the AB-group and N the Fitting subgroup), which is one of the plane crystallographic groups **p1**, **p2**, **pg**,

p3, p4, p6, p2gg. They are listed below with their zeta functions expressed in terms of the partial zeta functions with respect to the Fitting subgroup, which is always $\langle x_1, x_2, x_3 \rangle$. This time, the normal zeta function has been arranged so that $[G : N]^{-s} \zeta_G^{N, \triangleleft}(s)$ appears as the first term in the formula and $\zeta_G^{G, \triangleleft}(s)$ as the last one.

The group

$$N_k = \langle x_1, x_2, x_3 : [x_2, x_1] = x_3^k, [x_1, x_3] = [x_2, x_3] = 1 \rangle, \quad k \in \mathbb{N},$$

has zeta functions

$$\begin{aligned} \zeta_{N_k}^{\leq}(s) &= \prod_{p \nmid k} \frac{\zeta_p(s) \zeta_p(s-1) \zeta_p(2s-2) \zeta_p(2s-3)}{\zeta_p(3s-3)} \\ &\quad \times \prod_{p|k} (\zeta_p(s-2) (\zeta_p(s) \zeta_p(s-1) \\ &\quad - p^{-s+2} |k|_p^{s-2} \zeta_p(2s-2) \zeta_p(2s-3))), \\ \zeta_{N_k}^{\triangleleft}(s) &= \prod_{p \nmid k} \zeta_p(s) \zeta_p(s-1) \zeta_p(3s-2) \\ &\quad \times \prod_{p|k} \zeta_p(s) \zeta_p(s-1) \left(\frac{1 - |k|_p^{s-2}}{1 - p^{-s+2}} + |k|_p^{s-2} \zeta_p(3s-\eta) \right). \end{aligned}$$

The group

$$\begin{aligned} G_{\mathbf{p}2,2k} &= \langle \alpha, x_1, x_2, x_3 : [x_2, x_1] = x_3^{2k}, [x_1, x_3] = [x_2, x_3] = 1, \\ &\quad \alpha^2 = x_3, \alpha x_1 = x_1^{-1}, \alpha x_2 = x_2^{-1} \rangle, \quad k \in \mathbb{N}, \end{aligned}$$

has zeta functions

$$\begin{aligned} \zeta_{G_{\mathbf{p}2,2k}}^{\leq}(s) &= \prod_{p \nmid 2k} \frac{\zeta_p(s-1) \zeta_p(s-2) \zeta_p(2s-1) \zeta_p(2s-2)}{\zeta_p(3s-3)} \\ &\quad \times \prod_{p|k, p \neq 2} \zeta_p(s) (\zeta_p(s-1) \zeta_p(s-2) \\ &\quad - p^{-s} |k|_p^s \zeta_p(2s-1) \zeta_p(2s-2)) \\ &\quad \times \zeta_2(s) \zeta_2(s-1) + 2^{-s} \zeta_{N_{2k}}^{\leq}(s), \\ \zeta_{G_{\mathbf{p}2,2k}}^{\triangleleft}(s) &= 2^{-s} \prod_{p \nmid 2k} \zeta_p(s) \zeta_p(s-1) \zeta_p(3s) \\ &\quad \times \prod_{p|k, p \neq 2} \zeta_p(s) \zeta_p(s-1) \left(\frac{1 - |k|_p^s}{1 - p^{-s}} + |k|_p^s \zeta_p(3s) \right) \\ &\quad \times \zeta_2(s) \zeta_2(s-1) \left(1 + \frac{2^{2-s} (1 - |k|_2^s)}{1 - 2^{-s}} + \zeta_2(3s) 2^{2-s} |k|_2^s \right) \\ &\quad + \prod_{p|k, p \neq 2} \left(\frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}} \right) \times (1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s}). \end{aligned}$$

The group

$$G_{\mathbf{pg},2k} = \langle \beta, x_1, x_2, x_3 : [x_2, x_1] = x_3^{2k}, [x_1, x_3] = [x_2, x_3] = 1, \beta^2 = x_2, \beta x_1 = x_1^{-1} x_3^{-k}, \beta x_3 = x_3^{-1} \rangle, \quad k \in \mathbb{N},$$

has zeta functions

$$\begin{aligned} \zeta_{G_{\mathbf{pg},2k}}^{\leq}(s) &= \prod_{p \nmid 2k} \frac{\zeta_p(s) \zeta_p(s-1) \zeta_p(2s-2) \zeta_p(2s-3)}{\zeta_p(3s-3)} \\ &\quad \times \prod_{p|k, p \neq 2} \zeta_p(s-2) (\zeta_p(s) \zeta_p(s-1) \\ &\quad - p^{-s+2} |k|_p^{s-2} \zeta_p(2s-2) \zeta_p(2s-3)) \\ &\quad \times \zeta_2(s-1) (\zeta_2(s-2) (1 - |k|_2^{s-2}) + \zeta_2(2s-3) |k|_2^{s-2}) \\ &\quad + 2^{-s} \zeta_{N_{2k}}^{\leq}(s), \end{aligned}$$

$$\begin{aligned} \zeta_{G_{\mathbf{pg},2k}}^{\triangleleft}(s) &= 2^{-s} \prod_{p \nmid 2k} \zeta_p(s)^2 \zeta_p(3s-1) \\ &\quad \times \prod_{p|k, p \neq 2} \zeta_p(s)^2 \left(\frac{1 - |k|_p^{s-1}}{1 - p^{-s+1}} + |k|_p^{s-1} \zeta_p(3s-1) \right) \\ &\quad \times \zeta_2(s)^2 (\zeta_2(s-1) (1 + 3 \cdot 2^{-s}) (1 - |k|_2^{s-1}) \\ &\quad + \zeta_2(3s-1) (1 + 2^{-s} + 6 \cdot 4^{-s} \\ &\quad - 2 \cdot 8^{-s} - 2 \cdot 16^{-s}) |k|_2^{s-1}) \\ &\quad + (1 + (4 + (-1)^k 2) \cdot 2^{-s} + 4 \cdot 4^{-s}) (1 - 2^{-s}) \zeta(s). \end{aligned}$$

The group

$$G_{\mathbf{p3},k,\epsilon} = \langle \gamma, x_1, x_2, x_3 : [x_2, x_1] = x_3^k, [x_1, x_3] = [x_2, x_3] = 1, \gamma^3 = x_3^\epsilon, \gamma x_1 = x_2, \gamma x_2 = x_1^{-1} x_2^{-1} \rangle,$$

with $k \in \mathbb{N}$ and $\epsilon \in \{1, -1\}$ verifying $k(k + \epsilon) \equiv 0 \pmod 3$, has zeta functions

$$\begin{aligned} \zeta_{G_{\mathbf{p3},k,\epsilon}}^{\leq}(s) &= \prod_{p \nmid 3k} \frac{\zeta_p(s-1) L_p(s-1, \chi_3) \zeta_p(2s-1) L_p(2s-1, \chi_3)}{L_p(3s-2, \chi_3)} \\ &\quad \times \prod_{p|k, p \neq 3} \zeta_p(s) (L_p(s-1, \chi_3) \zeta_p(s-1) \\ &\quad - p^{-s} |k|_p^s L_p(2s-1, \chi_3) \zeta_p(2s-1)) \\ &\quad \times \zeta_3(s-1) + 3^{-s} \zeta_{N_k}^{\leq}(s), \end{aligned}$$

$$\begin{aligned} \zeta_{G_{p^3, k, \epsilon}}^{\triangleleft}(s) &= 3^{-s} \prod_{p \nmid 3k} \zeta_p(s) \zeta_p(3s) L_p(s, \chi_3) \\ &\quad \times \prod_{p|k, p \neq 3} \zeta_p(s) L_p(s, \chi_3) \left(\frac{1 - |k|_p^s}{1 - p^{-s}} + |k|_p^s \zeta_p(3s) \right) \\ &\quad \times (\zeta_3(s) + 3\zeta_3(s)(3^{-s}\zeta_3(s)(1 - |k|_3^s) \\ &\quad \quad \quad + 3^{-3s}\zeta_3(3s)|k|_3^s)) \\ &\quad + \prod_{p|k, p \neq 3} \frac{1 - p^{-s}|k|_p^s}{1 - p^{-s}} \times (1 + 3 \cdot 3^{-s}). \end{aligned}$$

The group

$$\begin{aligned} G_{p^4, 2k, \epsilon} &= \langle \gamma, x_1, x_2, x_3 : [x_2, x_1] = x_3^{2k}, [x_1, x_3] = [x_2, x_3] = 1, \\ &\quad \gamma^4 = x_3^\epsilon, \gamma x_1 = x_2, \gamma x_2 = x_1^{-1} \rangle, \end{aligned}$$

with $k \in \mathbb{N}$ and $\epsilon \in \{1, -1\}$, has zeta functions

$$\begin{aligned} \zeta_{G_{p^4, 2k, \epsilon}}^{\leq}(s) &= \prod_{p \nmid 2k} \frac{\zeta_p(s-1) L_p(s-1, \chi_4) \zeta_p(2s-1) L_p(2s-1, \chi_4)}{L_p(3s-2, \chi_4)} \\ &\quad \times \prod_{p|k, p \neq 2} \zeta_p(s) (L_p(s-1, \chi_d) \zeta_p(s-1) \\ &\quad \quad \quad - p^{-s} |k|_p^s L_p(2s-1, \chi_d) \zeta_p(2s-1)) \\ &\quad \times \zeta_2(s-1) + 2^{-s} \zeta_{G_{p^2, 2k}}^{\leq}(s), \end{aligned}$$

$$\begin{aligned} \zeta_{G_{p^4, 2k, \epsilon}}^{\triangleleft}(s) &= 4^{-s} \prod_{p \nmid 2k} \zeta_p(s) \zeta_p(3s) L_p(s, \chi_4) \\ &\quad \times \prod_{p|k, p \neq 2} \zeta_p(s) L_p(s, \chi_d) \left(\frac{1 - |k|_p^s}{1 - p^{-s}} + |k|_p^s \zeta_p(3s) \right) \\ &\quad \times (\zeta_2(s) + 2\zeta_2(s)(2^{-s}\zeta_2(s)(1 - |2k|_2^s) \\ &\quad \quad \quad + 2^{-3s}\zeta_2(3s)|2k|_2^s)) \\ &\quad + 2^{-s} \prod_{p|k, p \neq 2} \frac{1 - |k|_p^s p^{-s}}{1 - p^{-s}} \times (1 + 2 \cdot 2^{-s} + 2 \cdot 4^{-s}) \\ &\quad + \prod_{p|k, p \neq 2} \frac{1 - |k|_p^s p^{-s}}{1 - p^{-s}} \times (1 + 2 \cdot 2^{-s}). \end{aligned}$$

The group

$$\begin{aligned} G_{p^6, 2k, \epsilon} &= \langle \gamma, x_1, x_2, x_3 : [x_2, x_1] = x_3^{2k}, [x_1, x_3] = [x_2, x_3] = 1, \\ &\quad \gamma^6 = x_3^\epsilon, \gamma x_1 = x_2, \gamma x_2 = x_1^{-1} x_2 \rangle, \end{aligned}$$

$k \in \mathbb{N}$ and $\epsilon \in \{1, -1\}$ verifying $k(\epsilon + k) \equiv 0 \pmod 3$, has zeta functions

$$\zeta_{G_{p6,2k,\epsilon}}^{\leq}(s) = \prod_{p \nmid 6k} \frac{\zeta_p(s-1)\zeta_p(2s-1)L_p(s-1, \chi_3)L_p(2s-1, \chi_3)}{L_p(3s-2, \chi_3)} \\ \times \prod_{p|k, p \nmid 6} \zeta_p(s)(L_p(s-1, \chi_6)\zeta_p(s-1) - p^{-s}|k|_p^s L_p(2s-1, \chi_6)\zeta_p(2s-1)) \\ \times \zeta_3(s-1)\zeta_2(2s-2) \\ + 2^{-s}\zeta_{G_{p3,2k,\epsilon}}^{G_{p3,2k,\epsilon}, \leq}(s) + 3^{-s}\zeta_{G_{p2,2k}}^{G_{p2,2k}, \leq}(s) + 6^{-s}\zeta_{N_{2k}}^{\leq}(s),$$

$$\zeta_{G_{p6,2k,\epsilon}}^{\triangleleft}(s) = 6^{-s} \prod_{p \nmid 6k} \zeta_p(s)\zeta_p(3s)L_p(s, \chi_3) \\ \times \prod_{p|2k, p \neq 3} \zeta_p(s)L_p(s, \chi_3) \left(\frac{1 - |2k|_p^s}{1 - p^{-s}} + |2k|_p^s \zeta_p(3s) \right) \\ \times (\zeta_3(s) + \zeta_3(s)(3^{-s}\zeta_3(s)(1 - |k|_3^s) + 3^{-3s}\zeta_3(3s)|k|_3^s)) \\ + 3^{-s} \prod_{p|k, p \neq 2} \frac{1 - p^{-s}|k|_p^s}{1 - p^{-s}} \times (1 + 4^{-s}) \\ + 2^{-s} \prod_{p|k, p \neq 3} \frac{1 - p^{-s}|2k|_p^s}{1 - p^{-s}} \times (1 + 3^{-s}) \\ + \prod_{p|k, p \nmid 6} \frac{1 - p^{-s}|k|_p^s}{1 - p^{-s}}.$$

The group

$$G_{p2gg,4k} = \langle \alpha, \beta, x_1, x_2, x_3 : [x_2, x_1] = x_3^{4k}, [x_1, x_3] = [x_2, x_3] = 1, \\ \alpha^2 = x_3, \alpha x_1 = x_1^{-1}x_3^{2k}, \alpha x_2 = x_2^{-1}x_3^{-2k}, \\ \beta^2 = x_1, \beta x_2 = x_2^{-1}x_3^{2k}, \beta x_3 = x_3^{-1}, \\ (\beta\alpha)^2 = x_2 \rangle, \quad k \in \mathbb{N},$$

has zeta functions

$$\zeta_{G_{p2gg,4k}}^{\leq}(s) = \prod_{p \nmid 2k} \frac{\zeta_p(s-1)^2\zeta_p(2s-2)^2}{\zeta_p(3s-3)} \\ \times \prod_{p|k, p \neq 2} \zeta_p(s-1)((\zeta_p(s-1))^2 - p^{-s+1}|k|_p^{s-1}(\zeta_p(2s-2))^2) \\ + 2 \cdot 2^{-s}\zeta_{G_{pg,4k}}^{G_{pg,4k}, \leq}(s) + 2^{-s}\zeta_{G_{p2,4k}}^{G_{p2,4k}, \leq}(s) + 4^{-s}\zeta_{N_{4k}}^{\leq}(s),$$

$$\begin{aligned} \zeta_{G_{p^{2\text{gg}}, 4k}}^{\triangleleft}(s) &= 4^{-s} \prod_{p \nmid 2k} \zeta_p(s)^2 \zeta_p(3s) \\ &\quad \times \prod_{p|k, p \neq 2} \zeta_p(s)^2 \left(\frac{1 - |k|_p^s}{1 - p^{-s}} + |k|_p^s \zeta_p(3s) \right) \\ &\quad \times \zeta_2(s)^2 (\zeta_2(s)(1 + 4 \cdot 2^{-s} + 4^{-s})(1 - |k|_2^s) \\ &\quad \quad + \zeta_2(3s)(1 + 5 \cdot 2^{-s} + 2 \cdot 4^{-s} \\ &\quad \quad \quad + 8^{-s} - 16^{-s} - 2 \cdot 32^{-s}) |k|_2^s) \\ &\quad + 2^{-s} \prod_{p|k, p \neq 2} \frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}} \times (1 + 2 \cdot 2^{-s}) \\ &\quad + 2 \cdot 2^{-s} (1 + 2 \cdot 2^{-s})(1 - 2^{-s}) \zeta(s) + 1. \end{aligned}$$

5 Computing the zeta functions of the 3-dimensional Bieberbach groups

We prove the formulae presented in Section 4.1. Throughout this section, we denote by G the Bieberbach group under consideration and by N the Fitting subgroup. We keep the notation introduced at the beginning of Section 2 except that here N is not a \mathfrak{T}_2 -group but rather an abelian group (and it is in fact a $\mathbb{Z}[P]$ -module, where $P = G/N$). Instead, we denote

$$Z = C_N(P), \quad E = G/Z, \quad T = N/Z.$$

The formula for the zeta functions of \mathcal{G}_1 are already known (see the introduction), so we focus here on the other groups. We mentioned that an intermediate subgroup $N \leq H \leq G$ is again a Bieberbach group with Fitting subgroup N . Therefore, to prove the formula for $\zeta_G^{\leq}(s)$, it is enough to prove the one for $\zeta_G^{G, \leq}(s)$. The isomorphism classes of the intermediate subgroups $N \subsetneq H \subsetneq G$ (if there are any) will be identified when computing the partial zeta functions $\zeta_G^{H, \triangleleft}(s)$. The following will be useful. If $p \nmid |G/N|$, then

$$\begin{aligned} \zeta_{G_p}^{G, \leq}(s) &= \zeta_{E_p}^{E, \leq}(s) \zeta_{Z_p}^{\leq}(s) = \zeta_{E_p}^{T, \triangleleft}(s-1) \zeta_{Z_p}^{\leq}(s), \tag{5.1} \\ \zeta_{G_p}^{N, \triangleleft}(s) &= \zeta_{E_p}^{T, \triangleleft}(s) \zeta_{Z_p}^{\leq}(s), \\ \zeta_{G_p}^{H, \triangleleft}(s) &= \zeta_{E_p/Z_p}^{H/Z_p, \triangleleft}(s) \zeta_{Z_p}^{\leq}(s). \end{aligned}$$

These follow immediately from the analysis in [6, Section 2]. When the rank of Z is 1, then E is a plane crystallographic group, and we will be able to apply the results of [13] (summarized in [6, Section 4]).

When computing local factors at primes $p \mid [G : N]$, we will sometimes use the method of p -adic integration with respect to the Mal'cev basis $\{x_1, x_2, x_3\}$; specifically, Corollary 3.3 and Proposition 3.5. In this case, $\mathcal{M}_p^{\leq} = \mathcal{M}_p^{\triangleleft} = T_3^+(\mathbb{Z}_p)$. The following lemma, whose verification is straightforward, will be used to translate the conditions defining the domains of integration into *cone conditions*.

Lemma 5.1. *An element $\mathbf{x}^v \in N_p$ is in $\overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}$ if and only if all of the following hold.*

- (1) $t_{11} \mid v_1$,
- (2) $t_{22} \mid -\frac{v_1}{t_{11}}t_{12} + v_2$,
- (3) $t_{33} \mid -\frac{-(v_1/t_{11})t_{12}+v_2}{t_{22}}t_{23} - \frac{v_1}{t_{11}}t_{13} + v_3$.

5.1 Computing zeta functions of a family of groups including \mathcal{G}_2 and \mathcal{B}_1

Given positive integers a, b , we shall compute the zeta functions of

$$G = \langle \alpha, x_1, \dots, x_a, x_{a+1}, \dots, x_{a+b} : [x_i, x_j] = 1 \text{ for all } i, j, \alpha^2 = x_{a+b}, \\ \alpha x_i = x_i^{-1} \text{ for } i = 1, \dots, a, \\ \alpha x_i = x_i \text{ for } i = a + 1, \dots, a + b \rangle.$$

This, of course, includes the cases $G = \mathcal{G}_2, G = \mathcal{B}_1$. Here $N = \langle x_1, \dots, x_{a+b} \rangle, P \cong C_2$ is generated by the class of α , and $Z = C_N(P) = \langle x_{a+1}, \dots, x_{a+b} \rangle$. Note that if $U \leq Z \leq V \leq N$, then U and V are P -submodules. We first prove a series of lemmas. When working inside T or Z , we shall use additive notation.

Lemma 5.2. *For $V \leq Z$ of finite index, $\text{Hom}_P(T, Z/V) = \text{Hom}(C_2^a, Z/V)$, and this set has $[Z : 2Z + V]$ elements.*

Proof. A group homomorphism $\varphi: T \rightarrow Z/V$ is of P -modules if and only if $\varphi(-x) = \varphi(x)$ for all $x \in T$, that is, $\varphi(2x) = 0$ for all $x \in T$, or $\varphi(2T) = 0$. This implies the first equality. Next, note that

$$\text{Hom}(C_2^a, Z/V) = \text{Hom}\left(C_2^a, \left(\left(\frac{1}{2}V\right) \cap Z\right)/V\right),$$

whose size is $[(\frac{1}{2}V) \cap Z : V]^a$. Thus, the second equality follows from the following calculation:

$$\begin{aligned} \left[\left(\frac{1}{2}V\right) \cap Z : V\right] &= [V \cap 2Z : 2V] \\ &= \frac{[V : 2V]}{[V : V \cap 2Z]} = \frac{[Z : 2Z]}{[(V + 2Z) : 2Z]} \quad (5.2) \\ &= [Z : V + 2Z]. \quad \square \end{aligned}$$

Lemma 5.3. *Let $A \leq G$ be of finite index such that $AN = G$. Then the subgroup $B := A \cap N$ is normal of finite index in G and $x_{a+b} \in (B \cap Z) + 2Z$. If in addition A is normal, then also $x_1^2, \dots, x_a^2 \in B$.*

Conversely, given $B \leq N$ of finite index and normal in G such that

$$x_{a+b} \in (B \cap Z) + 2Z,$$

the set $\{A \leq G : AN = G, A \cap N = B\}$ has $[N : BZ][Z : 2Z + (B \cap Z)]$ elements. If in addition $x_1^2, \dots, x_a^2 \in B$, then every A in the latter set is normal in G .

Proof. A subgroup $A \leq G$ such that $AN = G$ must contain αn for some $n \in N$. Then $(\alpha n)^2 \in A$. It is easy to check that $(\alpha n)^2 = x_{a+b} + 2z$ for some $z \in Z$; thus $x_{a+b} + 2z \in B \cap Z$, where $B = A \cap N$, or $x_{a+b} \in (B \cap Z) + 2Z$. Note that B is clearly normal, and if A is also normal, then $[N, G] = \langle x_1^2, \dots, x_a^2 \rangle$ is included in B by Lemma 2.11. This proves the first part of the lemma.

We now fix $B \leq N$ of finite index and normal in G and assume that

$$x_{a+b} + 2z \in B \cap Z \quad \text{for some } z \in Z,$$

which will be fixed. Let $\{w_1, \dots, w_s\} \subset \langle x_1, \dots, x_a \rangle$ and $\{z_1, \dots, z_t\} \subset Z$ be transversals to the cosets of BZ in N and to the cosets of $B \cap Z$ in Z , respectively. Then

$$\{w_i z_j : 1 \leq i \leq s, 1 \leq j \leq t\}$$

is a transversal to the cosets of B in N . A subgroup $A \leq G$ such that $AN = G$ and $A \cap N = B$ is of the form $A = B \cup \alpha w_i z_j B$ for uniquely determined w_i and z_j . Now $A = B \cup \alpha w_i z_j B$ is a subgroup if and only if $(\alpha w_i z_j)^2 \in B$, that is, $x_{a+b} z_j^2 \in B$, or $x_{a+b} + 2z_j \in B \cap Z$ in additive notation. In turn, this is equivalent to $2z_j - 2z \in B \cap Z$, that is, $z_j - z \in (\frac{1}{2}(B \cap Z)) \cap Z$. Therefore, only $[(\frac{1}{2}(B \cap Z)) \cap Z : B \cap Z]$ of the z_j 's are allowed. Since there is no restriction on the w_i 's, the number of possibilities for A is $[N : BZ][(\frac{1}{2}(B \cap Z)) \cap Z : B \cap Z]$, which is equal to

$$[N : BZ][Z : 2Z + (B \cap Z)]$$

by (5.2). If in addition $[N, G] = \langle x_1^2, \dots, x_a^2 \rangle$ is included in B , then G/B is abelian and hence A is normal. This completes the proof of the lemma. \square

Lemma 5.4. *Fix $U \leq T$ and $V \leq Z$ of finite index. Then*

$$|\{B \triangleleft G : (BZ)/Z = U, B \cap Z = V\}| = [Z : V + 2Z]^a.$$

If in addition $2T \subseteq U$, then

$$|\{B \triangleleft G : (BZ)/Z = U, B \cap Z = V, x_1^2, \dots, x_a^2 \in B\}| = [Z : V + 2Z]^d,$$

where $d = \dim_{\mathbb{F}_2}(U/2T)$.

Proof. Let \tilde{U} be the pre-image of U in N . We are interested in the number of P -invariant complements of Z/V in \tilde{U}/V . One of them is clearly $(U' + V)/V$, where U' is the projection of \tilde{U} onto $\langle x_1, \dots, x_a \rangle$, and hence, by Lemma 2.1, the number of them is

$$|\text{Hom}_P(\tilde{U}/Z, Z/V)| = |\text{Hom}_P(U, Z/V)| = |\text{Hom}_P(T, Z/V)|.$$

This, in turn, is equal to $[Z : V + 2Z]^a$ by Lemma 5.2. This proves the first part of the lemma.

To show the second part, we use additive notation in \tilde{U}/V . There is no loss of generality if we assume that $U/2T$ is generated by the classes of x_1, \dots, x_d . Let \bar{x}_i denote the class of x_i modulo V , $i = 1, \dots, a$. We are interested in the number of P -invariant complements of Z/V in \tilde{U}/V that contain $2\bar{x}_1, \dots, 2\bar{x}_a$. One of them is B_0/V , where $B_0 = \langle \bar{x}_1, \dots, \bar{x}_d, 2\bar{x}_{d+1}, \dots, 2\bar{x}_a \rangle$. To count how many there are, we follow the proof of Lemma 2.1. A P -invariant complement is of the form

$$B_\varphi/V = \{x - \varphi(x) : x \in B_0/V\} = \langle \bar{x}_1 - \varphi(\bar{x}_1), \dots, \bar{x}_d - \varphi(\bar{x}_d), \\ 2\bar{x}_{d+1} - \varphi(2\bar{x}_{d+1}), \dots, 2\bar{x}_a - \varphi(2\bar{x}_a) \rangle$$

for a uniquely determined

$$\varphi \in \text{Hom}_P(B_0/V, Z/V) = \text{Hom}(B_0/(2B_0 + V), Z/V).$$

The last equality uses Lemma 5.2, which is possible since clearly $B_0/V \cong T$ as P -modules. Given φ as above, we have

$$2\bar{x}_i = 2\bar{x}_i - 2\varphi(\bar{x}_i) \in B_\varphi/V \quad \text{for } i = 1, \dots, d.$$

If it is also required that $2\bar{x}_{d+1}, \dots, 2\bar{x}_a \in B_\varphi/V$, we need $\varphi(2\bar{x}_i) = 0$ for all $i = d + 1, \dots, a$. Thus, the set of P -invariant complements of Z/V in \tilde{U}/V that contain $2\bar{x}_1, \dots, 2\bar{x}_a$ is in a bijection with

$$\text{Hom}(B_0/(2B_0 + \langle 2x_{d+1}, \dots, 2x_a \rangle + V), Z/V),$$

that is, with

$$\text{Hom}\left(\frac{\langle x_1, \dots, x_d \rangle}{2\langle x_1, \dots, x_d \rangle}, Z/V\right).$$

By Lemma 5.2, this set has $[Z : V + 2Z]^d$ elements. This completes the proof of the lemma. □

We now show a combinatorial lemma. Let M be a finite-dimensional vector space over \mathbb{F}_2 . For a flag of subspaces $\beta : 0 = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_l = M$, the length l is denoted by $l(\beta)$. The set of all flags is denoted by $\mathcal{F}(M)$.

Lemma 5.5. *Let M be a non-zero finite-dimensional vector space over \mathbb{F}_2 . Then*

$$\sum_{W \leq M} \sum_{\beta \in \mathcal{F}(M/W)} (-1)^{l(\beta)} = 0. \quad (5.3)$$

Let $k \geq 0$, and let M be a k -dimensional vector space over \mathbb{F}_2 . Then

$$\sum_{\beta \in \mathcal{F}(M)} (-1)^{l(\beta)} = (-1)^k 2 \binom{k}{2}. \quad (5.4)$$

Proof. If M has positive dimension, then any flag of M of length l produces a flag of length $l - 1$ in M/W for some $0 \neq W \leq M$, and conversely, any flag of M/W of length $l - 1$ extends to a flag of M of length l . This implies that $\sum_{\beta \in \mathcal{F}(M)} (-1)^{l(\beta)} = -\sum_{0 \neq W \leq M} \sum_{\beta \in \mathcal{F}(M/W)} (-1)^{l(\beta)}$, which yields (5.3).

We now prove (5.4) by induction on k . The case $k = 0$ is obvious, so we assume that $k > 0$. Let A_k be the left-hand side and B_k the right-hand side. Note that (5.3) implies that $\sum_{i=0}^k A_i \binom{k}{i}_2 = 0$. Here $\binom{k}{i}_2$ denotes the 2-binomial coefficient, which expresses the number of i -dimensional subspaces in a k -dimensional vector space over \mathbb{F}_2 . On the other hand, if we replace t by -1 in the 2-binomial theorem

$$(1+t)(1+2t)\cdots(1+2^{k-1}t) = \sum_{i=0}^k 2 \binom{i}{2} \binom{k}{i}_2 t^i \quad (5.5)$$

we obtain that

$$0 = \sum_{i=0}^k (-1)^i 2 \binom{i}{2} \binom{k}{i}_2 = \sum_{i=0}^k B_k \binom{k}{i}_2.$$

Since $A_i = B_i$ for $i < k$ by the inductive hypothesis, and since $\binom{k}{k}_2 = 1$, it follows that $A_k = B_k$. This completes the induction. \square

Lemma 5.6. *Let Z_1 be a free abelian group of rank b , and let $Z_0 \leq Z_1$ be a subgroup such that $2Z_1 \subseteq Z_0$. Then*

$$\sum_{\substack{V \leq Z_1 \\ V+Z_0=Z_1}} [Z_1 : V]^{-s} = \left(\prod_{i=1}^b \zeta(s-i+1) \right) \left(\prod_{k=0}^{d-1} (1-2^{k-s}) \right),$$

where $d = \dim_{\mathbb{F}_2} Z_1/Z_0$. (5.6)

Proof. By (5.3), the left-hand side of (5.6) becomes

$$\begin{aligned} \sum_{V \leq Z_1} [Z_1 : V]^{-s} \left(\sum_{Z_0 + V \leq W \leq Z_1} \sum_{\beta \in \mathcal{F}(Z_1/W)} (-1)^{l(\beta)} \right) \\ = \sum_{Z_0 \leq W \leq Z_1} \sum_{V \leq W} [Z : W]^{-s} [W : V]^{-s} \left(\sum_{\beta \in \mathcal{F}(Z_1/W)} (-1)^{l(\beta)} \right). \end{aligned}$$

By (5.4), the latter becomes

$$\begin{aligned} \prod_{i=1}^b \zeta(s - i + 1) \left(\sum_{Z_0 \leq W \leq Z_1} [Z : W]^{-s} (-1)^{\dim Z_1/W} 2^{\binom{\dim Z_1/W}{2}} \right) \\ = \prod_{i=1}^b \zeta(s - i + 1) \left(\sum_{j=0}^d 2^{-js} \binom{d}{j}_2 (-1)^j 2^{\binom{j}{2}} \right). \end{aligned}$$

Finally, by using (5.5) with $k = d$ and $t = -2^{-s}$, the last expression becomes the right-hand side of (5.6). \square

Proposition 5.7. *The partial zeta functions of G with respect to N are as follows:*

$$\begin{aligned} \zeta_G^{G, \leq}(s) &= \prod_{i=1}^a \zeta(s - i) \times \prod_{j=1}^b \zeta(s - j + 1) \\ &\quad \times \left(\sum_{k=1}^b 2^{(b-k)(a+1-s)} \binom{b-1}{k-1}_2 (1 - 2^{-s}) \cdots (1 - 2^{k-1-s}) \right), \\ \zeta_G^{G, \triangleleft}(s) &= \prod_{i=1}^a \zeta(s - i + 1) \times \prod_{j=1}^b \zeta(s - j + 1) \\ &\quad \times \left(\sum_{k=0}^b 2^{(b-k)(s-a)} \binom{b}{k}_2 (1 - 2^{-s}) \cdots (1 - 2^{k-1-s}) \right), \\ \zeta_G^{N, \triangleleft}(s) &= \prod_{j=1}^b \zeta(s - j + 1) \\ &\quad \times \left(\sum_{i=0}^a 2^{(a-i)(1-s)} \binom{a}{i}_2 \left(\sum_{k=1}^b 2^{(b-k)(i+1-s)} \binom{b-1}{k-1}_2 \right. \right. \\ &\quad \left. \left. \times (1 - 2^{-s}) \cdots (1 - 2^{k-1-s}) \right) \right). \end{aligned}$$

Proof. The first equality below uses Lemma 5.3, the second one Lemma 5.4, and the last one Lemma 5.6:

$$\zeta_G^{G, \leq}(s) = \sum_{\substack{B \triangleleft G : B \subseteq N \\ x_{a+b} \in (B \cap Z) + 2Z}} [N : B]^{-s} [N : BZ] [Z : (B \cap Z) + 2Z]$$

$$\begin{aligned}
 &= \sum_{\substack{U \leq T, V \leq Z: \\ x_{a+b} \in V+2Z}} [T : U]^{-s} [Z : V]^{-s} [Z : V + 2Z]^a [T : U][Z : V + 2Z] \\
 &= \zeta_T^{\leq}(s-1) \left(\sum_{2Z + \langle x_{a+b} \rangle \leq Z_1 \leq Z} [Z : Z_1]^{a+1-s} \sum_{\substack{V \leq Z_1 \\ V+2Z=Z_1}} [Z_1 : V]^{-s} \right) \\
 &= \prod_{i=1}^a \zeta(s-i) \sum_{k=1}^b \left(\sum_{\substack{2Z + \langle x_{a+b} \rangle \leq Z_1 \leq Z \\ [Z : Z_1] = 2^{b-k}}} 2^{(b-k)(a+1-s)} \times \sum_{\substack{V \leq Z_1 \\ V+2Z=Z_1}} [Z_1 : V]^{-s} \right) \\
 &= \prod_{i=1}^a \zeta(s-i) \times \prod_{j=1}^b \zeta(s-j+1) \\
 &\quad \times \left(\sum_{k=1}^b 2^{(b-1-k)(a+1-s)} \binom{b-1}{k-1}_2 (1-2^{-s}) \dots (1-2^{k-1-s}) \right).
 \end{aligned}$$

The proofs of the expressions for $\zeta_G^{N, \triangleleft}(s)$ and $\zeta_G^{G, \triangleleft}(s)$ are similar. In the first case, we do not have to add the condition $x_{a+b} \in V + 2Z$ or $[T : U][Z : V + 2Z]$ in the second line. In the second case, we have to add the condition $2T \leq U$ and replace $[Z : V + 2Z]^a$ by $[Z : V + 2Z]^{\dim_{\mathbb{F}_2}(U/T)}$ in the second line. \square

5.2 Computing the zeta functions of $\mathcal{G}_3, \mathcal{G}_4$ and \mathcal{G}_5

Let G be any of the groups $\mathcal{G}_3, \mathcal{G}_4$ or \mathcal{G}_5 . We set $d = |G/N|$. Note that $Z = \langle x_1 \rangle$ and that E is the plane crystallographic group **pd**.

5.2.1 Local factors of $\zeta_G^{G, \leq}(s)$

If $p \nmid d$, then by (5.1), we have

$$\zeta_G^{G, \leq}(s) = \zeta_{E_p}^{E_p, \leq}(s) \zeta_{Z_p}^{\leq}(s),$$

and according to [13, Sections 5.10, 5.13, 5.16], the latter is equal to

$$\begin{cases} \zeta_p(s-1)L_p(s-1, \chi_d)\zeta_p(s) & \text{if } d \in \{3, 4\}, \\ \zeta_p(s-1)L_p(s-1, \chi_3)\zeta_p(s) & \text{if } d = 6. \end{cases}$$

Assume now that $p \mid d$. We claim that if $A \leq G_p$ is open and satisfies $AN_p = G_p$, then $Z_p \subset A$. Indeed, A contains $\alpha \mathbf{x}^v$ for some $\mathbf{v} \in \mathbb{Z}_p^3$. Then

$$(\alpha \mathbf{x}^v)^d = x_1^{dv_1+1} \in A,$$

and hence $x_1 \in A$ since $dv_1 + 1 \in \mathbb{Z}_p^*$. This proves the claim. It follows now that $\zeta_G^{G, \leq}(s) = \zeta_{E_p}^{E_p, \leq}(s)$. According to [13, Sections 5.10, 5.13, 5.16], this series is equal to $\zeta_p(s-1)L_p(s-1, \chi_q)$, where $q = d$ if $d \in \{3, 4\}$ and $q = 3$ if $d = 6$.

5.2.2 Local factors of $\zeta_G^{N, \triangleleft}(s)$

If $p \nmid d$, then by (5.1), we have $\zeta_{G_p}^{N_p, \triangleleft}(s) = \zeta_p(s) \zeta_{E_p}^{T_p, \triangleleft}(s)$, and according to [13, Sections 6.10, 6.13, 6.16], the latter is equal to $\zeta_p(s) L_p(s, \chi_d) \zeta_p(s)$ if $d \in \{3, 4\}$ and to $\zeta_p(s) L_p(s, \chi_3) \zeta_p(s)$ if $d = 6$.

Assume now that $p \mid d$. By Corollary 3.3, we have

$$\zeta_{G_p}^{N_p, \triangleleft}(s) = \frac{1}{(1 - p^{-1})^3} \int_{\mathcal{T}} |t_{11}|_p^{s-1} |t_{22}|_p^{s-2} |t_{33}|_p^{s-3} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_p)$ is defined by the following equivalent conditions:

$$\begin{aligned} \alpha(\mathbf{x}^{t_1}) \mathbf{x}^{-t_1}, \alpha(\mathbf{x}^{t_2}), \alpha(\mathbf{x}^{t_3}) &\in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ \iff x_2^{-t_{13}-t_{12}} x_3^{t_{12}-(r+1)t_{13}}, x_2^{-t_{23}} x_3^{t_{22}-rt_{23}} &\in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}, \end{aligned}$$

where $r = 1$ if $d = 3$, $r = 0$ if $d = 4$, and $r = -1$ if $d = 6$. Therefore, by Lemma 5.1, the conditions defining \mathcal{T} are

- (1) $t_{22} \mid -t_{12} - t_{13}$,
- (2) $t_{33} \mid \frac{t_{12}+t_{13}}{t_{22}} t_{23} + t_{12} - (r + 1)t_{13}$,
- (3) $t_{22} \mid t_{23}$,
- (4) $t_{33} \mid \frac{t_{23}}{t_{22}} t_{23} + t_{22} - r t_{23}$,
- (5) $t_{22} \mid -t_{33}$,
- (6) $t_{33} \mid \frac{t_{33}}{t_{22}} t_{23}$.

Note that (3) implies (6), and (4) can be written as

$$\frac{t_{33}}{t_{22}} \mid \left(\frac{t_{23}}{t_{22}} \right)^2 - r \frac{t_{23}}{t_{22}} + 1.$$

Assume first that $d \in \{3, 4\}$. One easily checks that $T^2 - rT + 1 \equiv 0 \pmod p$, in \mathbb{Z}_p , has -1 as a unique solution modulo p , and there are no solutions in \mathbb{Z}_p for $T^2 - rT + 1 \equiv 0 \pmod{p^2}$. Therefore, (4) splits the domain of integration \mathcal{T} as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases

- (4a) $|t_{22}|_p = |t_{33}|_p$ and
- (4b) $|pt_{22}|_p = |t_{33}|_p$ and $p \mid \frac{t_{23}}{t_{22}} + 1$.

If we assume (4a), then (1) and (3) reduce (2) to $t_{33} \mid t_{12} - (r + 1)t_{13}$, and then (1) reduces (2) again to $t_{33} \mid (r + 2)t_{13}$. Note that $r + 2 = p$. The conditions

defining \mathcal{T}^a are, therefore, $t_{22} \mid -t_{12} - t_{13}$, $t_{22} \mid pt_{13}$, $t_{22} \mid t_{23}$ and $|t_{22}|_p = |t_{33}|_p$, and the integral over \mathcal{T}^a becomes

$$\begin{aligned} & \frac{1}{(1 - p^{-1})^3} \int_{\mathcal{T}^a} |t_{11}|_p^{s-1} |t_{22}|_p^{2s-5} d\mu \\ &= \frac{\zeta_p(s)}{1 - p^{-1}} \int_{t_{22}|_p t_{13}} |t_{22}|_p^{2s-2} d\mu = \zeta_p(s)(1 + p^{1-2s}\zeta_p(2s)), \end{aligned}$$

where in the first equality we integrated with respect to t_{11} and applied Proposition 3.6 with the pivots t_{12} , t_{23} , t_{33} .

If we now assume (4b), then (3) and (5) are redundant, and since $r + 2 = p$, we see that (2) can be written as

$$t_{33} \mid \left(\frac{t_{23}}{t_{22}} + 1\right)t_{12} + \left(\frac{t_{23}}{t_{22}} + 1 - p\right)t_{13}.$$

Using (1) and the assumption that $|t_{33}|_p = |pt_{22}|_p$ and $p \mid \frac{t_{23}}{t_{22}} + 1$, we can reduce (2) to $t_{33} \mid pt_{13}$, which is equivalent to $t_{22} \mid t_{13}$. Thus, \mathcal{T}^b is defined by the conditions $t_{22} \mid -t_{12} - t_{13}$, $t_{22} \mid t_{13}$, $pt_{22} \mid t_{23} + t_{22}$ and $|pt_{22}|_p = |t_{33}|_p$, and the integral over \mathcal{T}^b becomes

$$\begin{aligned} & \frac{1}{(1 - p^{-1})^3} \int_{\mathcal{T}^b} |t_{11}|_p^{s-1} |t_{22}|_p^{2s-5} p^{3-s} d\mu \\ &= \frac{1}{(1 - p^{-1})^2} \int |t_{11}|_p^{s-1} |t_{22}|_p^{2s-1} p^{1-s} d\mu = p^{1-s}\zeta_p(s)\zeta_p(2s), \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots t_{12} , t_{13} , t_{23} , t_{33} .

We conclude that

$$\begin{aligned} \zeta_{G_p}^{N_p, \triangleleft}(s) &= \zeta_p(s)(1 + p^{1-2s}\zeta_p(2s) + p^{1-s}\zeta_p(2s)) \\ &= \zeta_p^2(s)(1 + (p - 1)p^{-s}) \quad \text{for } d \in \{3, 4\} \text{ and } p \mid d. \end{aligned}$$

Assume now that $d = 6$ (and hence $r = -1$). If $p = 3$, then the analysis is the same as above except that the condition $t_{22} \mid 3t_{13}$ in the description of \mathcal{T}^a must be replaced by $t_{22} \mid t_{13}$ since this time $r + 2 = 1$, and condition (4b) is now $|3t_{22}|_3 = |t_{33}|_3$ and $3 \mid \frac{t_{23}}{t_{33}} - 1$ since $1 \in \mathbb{Z}_3$ is the unique solution modulo 3 of $T^2 + T + 1 \equiv 0 \pmod 3$. Thus, \mathcal{T}^a is defined by $t_{22} \mid -t_{12} - t_{13}$, $t_{22} \mid t_{13}$, $t_{22} \mid t_{23}$ and $|t_{22}|_3 = |t_{33}|_3$, and the integral over \mathcal{T}^a becomes

$$\begin{aligned} & \frac{1}{(1 - 3^{-1})^3} \int_{\mathcal{T}^a} |t_{11}|_3^{s-1} |t_{22}|_3^{2s-5} d\mu \\ &= \frac{1}{(1 - 3^{-1})^2} \int |t_{11}|_3^{s-1} |t_{22}|_3^{2s-1} d\mu = \zeta_3(s)\zeta_3(2s), \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots t_{12} , t_{13} , t_{23} , t_{33} .

If we assume (4b), then (3) and (6) are redundant, and (2) can be written as

$$t_{33} \mid \left(\frac{t_{23}}{t_{22}} + 1 \right) t_{12} + \frac{t_{23}}{t_{22}} t_{13}.$$

Since $\frac{t_{23}}{t_{22}}$ is a unit, we can use the previous condition and $t_{22} \mid t_{33}$ to reduce (1) to $t_{22} \mid t_{12}$. Thus, the conditions defining \mathcal{T}^b are $t_{22} \mid t_{12}, 3t_{22} \mid \left(\frac{t_{23}}{t_{22}} + 1 \right) t_{12} + \frac{t_{23}}{t_{22}} t_{13}, 3t_{33} \mid t_{23} - t_{33}$ and $|3t_{22}|_3 = |t_{33}|_3$, and the integral over \mathcal{T}^b becomes

$$\begin{aligned} & \frac{1}{(1 - 3^{-1})^3} \int_{\mathcal{T}^b} |t_{11}|_3^{s-1} |t_{22}|_3^{2s-5} 3^{3-s} d\mu \\ &= \frac{1}{(1 - 3^{-1})^2} \int |t_{11}|_3^{s-1} |t_{22}|_3^{2s-1} 3^{-s} d\mu = 3^{1-s} \zeta_3(s) \zeta_3(2s), \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots $t_{12}, t_{23}, t_{13}, t_{33}$. We conclude that

$$\zeta_{G_3}^{N_3, \triangleleft}(s) = \zeta_3(s) \zeta_3(2s) + 3^{-s} \zeta_3(s) \zeta_3(2s) = \zeta_3(s)^2.$$

Assume finally that $p = 2$. Since there are no solutions in \mathbb{Z}_2 for

$$T^2 + T + 1 \equiv 0 \pmod{2},$$

(4) is equivalent to $|t_{33}|_2 = |t_{22}|_2$ and (2) is reduced to $t_{22} \mid t_{12}$. Thus, the conditions defining \mathcal{T} are $t_{22} \mid -t_{12} - t_{13}, t_{22} \mid t_{12}, t_{22} \mid t_{23}$ and $|t_{22}|_2 = |t_{33}|_2$, and the integral becomes

$$\begin{aligned} \zeta_{G_2}^{N_2, \triangleleft}(s) &= \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-1} |t_{22}|_2^{2s-5} d\mu \\ &= \frac{1}{(1 - 2^{-1})^2} \int |t_{11}|_2^{s-1} |t_{22}|_2^{2s-1} d\mu = \zeta_2(s) \zeta_2(2s), \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots $t_{13}, t_{12}, t_{23}, t_{33}$.

5.2.3 Local factors of $\zeta_G^{G, \triangleleft}(s)$

If $p \nmid d$, then

$$[N_p, G_p] = \overline{\langle x_2^{-1} x_3, x_2^{-1} x_3^{-r-1} \rangle} = \overline{\langle x_2^{-1} x_3, x_3^{-r-2} \rangle} = \overline{\langle x_2, x_3 \rangle},$$

where the last equality holds since $r + 2 \in \mathbb{Z}_p^*$. As $N_p / \overline{\langle x_2, x_3 \rangle} \cong \mathbb{Z}_p$, we have $\zeta_{G_p}^{G_p, \triangleleft}(s) = \zeta_p(s)$ by Proposition 2.12.

Assume now that $p \mid d$. Arguing as in the computation of $\zeta_{G_p}^{G_p, \leq}(s)$ (with $p \mid d$), we find that

$$\zeta_{G_p}^{G_p, \triangleleft}(s) = \zeta_{E_p}^{E_p, \triangleleft}(s),$$

and according to the results of [13, Sections 6.10, 6.13 and 6.16], this is equal to $1 + p \cdot p^{-s}$ if $d = 3$ or 4 , and to 1 if $d = 6$.

5.2.4 Local factors of $\zeta_G^{H, \triangleleft}(s)$ for $d = 4$ and $H = \langle \alpha^2, x_1, x_2, x_3 \rangle$

Note that $H = \mathcal{G}_2$. If $p \neq 2$, then

$$[N_p, H_p] = \overline{\langle x_2^2, x_2^2 \rangle} = \overline{\langle x_2, x_2 \rangle},$$

and hence $N_p / \overline{\langle x_2, x_2 \rangle} \cong \mathbb{Z}_p$. Therefore, by Proposition 2.12, $\zeta_{G_p}^{H_p, \triangleleft}(s) = \zeta_p(s)$.

Assume that $p = 2$. An open subgroup $A \triangleleft G_2$ such that $AN_2 = H_2$ must include Z_2 . Indeed, A contains $\alpha^2 \mathbf{x}^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}_2^3$, and hence

$$(\alpha^2 \mathbf{x}^{\mathbf{v}})^2 = x_1^{2v_1+1} \in A,$$

or $x_1 \in A$ since $2t_1 + 1 \in \mathbb{Z}_2^*$. It follows that $\zeta_{G_2}^{H_2, \triangleleft}(s) = \zeta_{E_2}^{H_2/Z_2, \triangleleft}(s)$, and this is equal to $1 + 2 \cdot 2^{-s} + 2 \cdot 4^{-s}$ according to the results of [13, Section 6.10].

5.2.5 Local factors of $\zeta_G^{H, \triangleleft}(s)$ for $d = 6$ and $H = \langle \alpha^3, x_1, x_2, x_3 \rangle$

Note that $H = \mathcal{G}_2$. Arguing as in Section 5.2.4, we get $\zeta_{G_p}^{H_p, \triangleleft}(s) = \zeta_p(s)$ if $p \neq 2$, and $\zeta_{G_2}^{H_2, \triangleleft}(s) = \zeta_{E_2}^{H_2/Z_2, \triangleleft}(s)$, which is equal to $1 + 4^{-s}$ according to the results of [13, Section 6.16].

5.2.6 Local factors of $\zeta_G^{K, \triangleleft}(s)$ for $d = 6$ and $K = \langle \alpha^2, x_1, x_2, x_3 \rangle$

Note that $K = \mathcal{G}_3$. Arguing as in the case $d = 3$ and $K = G$, we obtain that $\zeta_{G_p}^{K_p, \triangleleft}(s) = \zeta_p(s)$ if $p \neq 3$, and that $\zeta_{G_3}^{K_3, \triangleleft}(s) = \zeta_{E_3}^{K_3/Z_3, \triangleleft}(s)$, which is equal to $1 + 3^{-s}$ according to the results of [13, Section 6.16].

5.3 Computing the zeta functions of $G = \mathcal{G}_6$

Note that $Z = 0$ and that $N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \langle x_3 \rangle$ is a decomposition of N into a direct sum of irreducible $\mathbb{Z}[P]$ -modules that are pairwise non-isomorphic.

5.3.1 Local factors of $\zeta_G^{G, \leq}(s)$

If $p \neq 2$, then by using the above decomposition of N and the results of [6, Section 2], we obtain that

$$\zeta_{G_p}^{G, \leq}(s) = \zeta_p(s-1)\zeta_p(s-1)\zeta_p(s-1).$$

We now compute $\zeta_{G_2}^{G_2, \leq}(s)$. An open subgroup $A \leq G_2$ such that $AN_2 = G_2$ must contain $\alpha \mathbf{x}^{\mathbf{u}}$, $\beta \mathbf{x}^{\mathbf{v}}$ and $\alpha\beta \mathbf{x}^{\mathbf{w}}$ for some $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{Z}_2^3$, whence A contains the squares of these elements, which are $x_1^{2u_1+1}$, $x_2^{2v_2+1}$ and $x_3^{2w_3+1}$. Since

$$2u_1 + 1, 2v_2 + 1, 2w_3 + 1 \in \mathbb{Z}_2^*,$$

it follows that A includes x_1, x_2, x_3 and hence also N_2 . This implies that $A = G_2$. We conclude that $\zeta_{G_2}^{G_2, \leq}(s) = 1$.

5.3.2 Local factors of $\zeta_G^{N, \leq}(s)$

If $p \neq 2$, arguing as in the calculation of $\zeta_{G_p}^{G, \leq}(s)$, we find that $\zeta_G^{N, \leq}(s) = \zeta_p(s)^3$.

Assume now that $p = 2$. By Corollary 3.3, we have

$$\zeta_{G_2}^{N_2, \leq}(s) = \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_2)$ is defined by the following equivalent conditions:

$$\begin{aligned} & \alpha(\mathbf{x}^{t_1})\mathbf{x}^{-t_1}, \alpha(\mathbf{x}^{t_2}), \alpha(\mathbf{x}^{t_3}), \beta(\mathbf{x}^{t_1})\mathbf{x}^{t_1}, \beta(\mathbf{x}^{t_2})\mathbf{x}^{-t_2}, \beta(\mathbf{x}^{t_3}) \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ \iff & x_2^{-2t_{12}} x_3^{-2t_{13}}, \mathbf{x}^{-t_2}, \mathbf{x}^{-t_3}, x_2^{2t_{12}}, x_3^{-2t_{23}}, \mathbf{x}^{-t_3} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ \iff & x_3^{-2t_{13}}, x_2^{2t_{12}}, x_3^{-2t_{23}} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}. \end{aligned}$$

According to Lemma 5.1, these conditions are equivalent to

- (1) $t_{22} \mid 2t_{12}$,
- (2) $t_{33} \mid -\frac{2t_{12}}{t_{22}}t_{23}$,
- (3) $t_{33} \mid 2t_{13}$,
- (4) $t_{33} \mid 2t_{23}$.

We express \mathcal{T} as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases

- (4a) $t_{33} \mid t_{23}$ and
- (4b) $|t_{33}|_2 = |2t_{23}|_2$.

In the first case, (2) is redundant, and hence \mathcal{T}^a is defined by the conditions $t_{22} \mid 2t_{12}, t_{33} \mid 2t_{13}, t_{33} \mid t_{23}$. Thus,

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^a} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ &= \frac{\zeta_2(s)}{(1-2^{-1})^2} \int_{\substack{t_{22} \mid 2t_{12} \\ t_{33} \mid 2t_{13}}} |t_{22}|_2^{s-2} |t_{33}|_2^{s-2} d\mu = \zeta_2(s)(1+2^{1-s}\zeta_2(s))^2. \end{aligned}$$

Assume now (4b). Note that (2) becomes equivalent to $t_{22} \mid t_{12}$, whence (1) is redundant. Note also that (3) is equivalent to $t_{23} \mid t_{13}$. Therefore, the conditions defining \mathcal{T}^b are $t_{22} \mid t_{12}, t_{23} \mid t_{13}$ and $|t_{33}|_2 = |2t_{23}|_2$, and the integral over \mathcal{T}^b becomes

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^b} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} 2^{3-s} |t_{23}|_2^{s-3} d\mu \\ &= \frac{\zeta_2(s)2^{1-s}}{(1-2^{-1})^2} \int |t_{22}|_2^{s-1} |t_{23}|_2^{s-1} d\mu = 2^{1-s}\zeta_2(s)^3, \end{aligned}$$

where in the second equality we used Proposition 3.6 with the pivots t_{12}, t_{13}, t_{33} .

We conclude that

$$\begin{aligned} \zeta_{G_2}^{N_2, \triangleleft}(s) &= \zeta_2(s)((1+2^{1-s}\zeta_2(s))^2 + 2^{1-s}\zeta_2(s)^2) \\ &= (1+4 \cdot 2^{-s} + 4^{-s})\zeta_2(s)^3. \end{aligned}$$

5.3.3 Local factors of $\zeta_G^{G, \triangleleft}(s)$

If $p \neq 2$, then

$$[N_p, G_p] = \overline{\langle x_1^2, x_2^2, x_3^2 \rangle} = \overline{\langle x_1, x_2, x_3 \rangle},$$

whence $\zeta_{G_p}^{G_p, \triangleleft}(s) = 1$ by Proposition 2.12. If $p = 2$, then $\zeta_{G_2}^{G_2, \triangleleft}(s) = 1$ since $\zeta_{G_2}^{G_2, \leq}(s) = 1$, as shown in Section 5.3.1.

5.3.4 Local factors of $\zeta_G^{H, \triangleleft}(s)$ for $H = \langle \alpha, x_1, x_2, x_3 \rangle$

Note that $H \cong \mathcal{G}_2$. Arguing as in Section 5.2.4, we find that

$$\zeta_{G_p}^{H_p, \triangleleft}(s) = \zeta_p(s) \quad \text{and} \quad \zeta_{G_2}^{H_2, \triangleleft}(s) = \zeta_{G_2/\langle x_1 \rangle}^{H_2/\langle x_1 \rangle, \triangleleft}(s).$$

One easily checks that $G/\langle x_1 \rangle \cong \mathbf{p2gg}$; therefore, the latter series is equal to $1 + 2 \cdot 2^{-s}$ according to the results of [13, Section 6.7].

5.3.5 Calculation of $\zeta_G^{K, \triangleleft}(s)$ and $\zeta_G^{L, \triangleleft}(s)$ for $K = \langle \beta, x_1, x_2, x_3 \rangle$ and $L = \langle \alpha\beta, x_1, x_2, x_3 \rangle$

It is easy to check that $(\alpha, \beta, x_1, x_2, x_3) \mapsto (\beta, \alpha\beta, x_2, x_3, x_1^{-1})$ defines an automorphism of G that sends H onto K , and K onto L . Thus,

$$\zeta_G^{K, \triangleleft}(s) = \zeta_G^{L, \triangleleft}(s) = \zeta_G^{H, \triangleleft}(s).$$

This series was computed in in Section 5.3.4.

5.4 Computing the zeta functions of $G = \mathcal{B}_2$

We call $y_1 = x_3, y_2 = x_1$ and $y_3 = x_1x_2$ so that now the relations are $\varepsilon^2 = y_2, \varepsilon y_3 = y_3$ and $\varepsilon y_1 = y_1^{-1}y_3$. We will use this presentation. Note that $Z = \langle y_2, y_3 \rangle$, and hence $T \cong \mathbb{Z}$.

5.4.1 Local factors of $\zeta_G^{G, \leq}(s)$

If $p \neq 2$, then by (5.1), we have $\zeta_G^{G, \leq}(s) = \zeta_{\mathbb{Z}_p}^{\leq}(s)\zeta_p(s-1) = \zeta_p(s)\zeta_p(s-1)^2$.

Assume now that $p = 2$. By Proposition 3.5 with $H = G$, we have

$$\zeta_G^{G, \leq}(s) = \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-2} |t_{22}|_2^{s-3} |t_{33}|_2^{s-4} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_2) \times \mathbb{Z}_2^3$ is the set of pairs (\mathbf{t}, \mathbf{v}) satisfying the following equivalent conditions:

$$\begin{aligned} \varepsilon(\mathbf{y}^{t_1})\mathbf{y}^{t_1}, \varepsilon(\mathbf{y}^{t_2}), \varepsilon(\mathbf{y}^{t_3}), (\varepsilon\mathbf{y}^{\mathbf{v}})^2 &\in \overline{\langle \mathbf{y}^{t_1}, \mathbf{y}^{t_2}, \mathbf{y}^{t_3} \rangle} \\ \iff y_2^{2t_{12}} y_3^{t_{11}+2t_{13}}, y_2^{2v_2+1} y_3^{2v_3+v_1} &\in \overline{\langle \mathbf{y}^{t_1}, \mathbf{y}^{t_2}, \mathbf{y}^{t_3} \rangle}. \end{aligned}$$

According to Lemma 5.1, these conditions are equivalent to

- (1) $t_{22} \mid 2t_{12}$,
- (2) $t_{33} \mid -\frac{2t_{12}}{t_{22}}t_{23} + t_{11} + 2t_{13}$,
- (3) $t_{22} \mid 2v_2 + 1$,
- (4) $t_{33} \mid -\frac{2v_1+1}{t_{22}}t_{23} + 2v_3 + v_1$.

Note that (3) is equivalent to $t_{22} \in \mathbb{Z}_2^*$, whence (1) is redundant. We now express \mathcal{T} as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases

- (a) $t_{33} \in \mathbb{Z}_2^*$ and
- (b) $t_{33} \in 2\mathbb{Z}_2$.

Note that \mathcal{T}^a is defined by $t_{22}, t_{33} \in \mathbb{Z}_p^*$, and hence

$$\frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}^a} |t_{11}|_2^{s-2} |t_{22}|_2^{s-3} |t_{33}|_2^{s-4} d\mu = \zeta_2(s - 1).$$

In case (b), (2) can be written as $\frac{t_{33}}{2} \mid -\frac{t_{12}}{t_{22}}t_{23} + \frac{t_{11}}{2} + t_{13}$, and it implies that $t_{11} \in 2\mathbb{Z}_2$. By using Proposition 3.6 with the pivots t_{13}, v_1 , the integral over \mathcal{T}^b becomes

$$\begin{aligned} & \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}^b} |t_{11}|_2^{s-2} |t_{33}|_2^{s-4} d\mu \\ &= \frac{1}{(1 - 2^{-1})^2} \int_{t_{11}, t_{33} \in 2\mathbb{Z}_2} |t_{11}|_2^{s-2} 2 |t_{33}|_2^{s-2} d\mu = 2^{3-2s} \zeta_2(s - 1)^2. \end{aligned}$$

We conclude that

$$\begin{aligned} \zeta_{G_2}^{G_2, \leq}(s) &= \zeta_2(s - 1)(1 + 2^{3-2s} \zeta_2(s - 1)) \\ &= \zeta_2(s - 1)^2(1 - 2^{-s} + 8 \cdot 4^{-s}). \end{aligned}$$

5.4.2 Local factors of $\zeta_G^{N, \triangleleft}(s)$

If $p \neq 2$, then by (5.1), we have $\zeta_{G_p}^{N_p, \triangleleft}(s) = \zeta_{\mathbb{Z}_p}^{\leq}(s) \zeta_p(s) = \zeta_p(s)^2 \zeta_p(s - 1)$.

Assume now that $p = 2$. By Corollary 3.3, we have

$$\zeta_{G_2}^{N_2, \triangleleft}(s) = \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu,$$

where

$$\begin{aligned} \mathcal{T} &= \{ \mathbf{t} \in T_3^+(\mathbb{Z}_2) : \varepsilon(\mathbf{y}^{t_1}) \mathbf{y}^{t_1}, \varepsilon(\mathbf{y}^{t_2}), \varepsilon(\mathbf{y}^{t_3}) \in \overline{\langle \mathbf{y}^{t_1}, \mathbf{y}^{t_2}, \mathbf{y}^{t_3} \rangle} \} \\ &= \{ \mathbf{t} \in T_3^+(\mathbb{Z}_2) : y_2^{2t_{12}} y_3^{t_{11} + 2t_{13}} \in \overline{\langle \mathbf{y}^{t_1}, \mathbf{y}^{t_2}, \mathbf{y}^{t_3} \rangle} \}. \end{aligned}$$

By Lemma 5.1, the conditions defining \mathcal{T} are equivalent to

- (1) $t_{22} \mid 2t_{12}$,
- (2) $t_{33} \mid -\frac{2t_{12}}{t_{22}}t_{23} + t_{11} + 2t_{13}$.

We can express \mathcal{T} as a disjoint union $\mathcal{T} = \mathcal{T}^a \cup \mathcal{T}^b$ according to the cases

- (1a) $|t_{22}|_2 = |2t_{12}|_2$ and
- (1b) $t_{22} \mid t_{12}$.

In the first case, the coefficient of t_{23} in (2) is a unit, so the integral over \mathcal{T}^a becomes

$$\begin{aligned} & \frac{2^{2-s}}{(1-2^{-1})^3} \int_{\mathcal{T}^a} |t_{11}|_2^{s-1} |t_{12}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ &= \frac{2^{2-s}}{(1-2^{-1})^2} \int |t_{11}|_2^{s-1} 2^{-2} |t_{12}|_2^{s-1} |t_{33}|_2^{s-2} d\mu \\ &= 2^{-s} \zeta_2(s)^2 \zeta_2(s-1), \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots t_{22}, t_{23} .

Assume now (1b). We express \mathcal{T}^b as a disjoint union $\mathcal{T}^b = \mathcal{T}^{b,i} \cup \mathcal{T}^{b,ii}$ according to the sub-cases (i) $t_{33} \in \mathbb{Z}_2^*$ and (ii) $t_{33} \in 2\mathbb{Z}_2$. Note that $\mathcal{T}^{b,i}$ is just defined by $t_{22} \mid t_{12}$ and $t_{33} \in \mathbb{Z}_2^*$, so the integral over $\mathcal{T}^{b,i}$ becomes

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^{b,i}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} d\mu \\ &= \frac{1}{(1-2^{-1})^2} \int |t_{11}|_2^{s-1} |t_{22}|_2^{s-1} d\mu = \zeta_2(s)^2. \end{aligned}$$

In sub-case (ii), condition (2) implies that $t_{11} \in 2\mathbb{Z}_2$, whence $\mathcal{T}^{b,ii}$ is defined by the conditions

$$2 \mid t_{11}, \quad 2 \mid t_{33}, \quad t_{22} \mid t_{12} \quad \text{and} \quad \frac{t_{33}}{2} \mid -\frac{t_{12}}{t_{22}} t_{23} + \frac{t_{11}}{2} + t_{13}.$$

By using Proposition 3.6 with the pivot t_{13} , the integral over $\mathcal{T}^{b,ii}$ becomes

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^{b,ii}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ &= \frac{1}{(1-2^{-1})^3} \int_{t_{11}, t_{33} \in 2\mathbb{Z}_2} |t_{11}|_2^{s-1} |t_{22}|_2^{s-1} 2 |t_{33}|_2^{s-2} d\mu \\ &= 2^{2-2s} \zeta_2(s)^2 \zeta_2(s-1). \end{aligned}$$

We conclude that

$$\begin{aligned} \zeta_{G_2}^{N_2, \triangleleft}(s) &= \zeta_2(s)^2 (2^{-s} \zeta_2(s-1) + 1 + 2^{2-2s} \zeta_2(s-1)) \\ &= \zeta_2(s)^2 \zeta_2(s-1) (1 - 2^{-s} + 4 \cdot 4^{-s}). \end{aligned}$$

5.4.3 Calculation of $\zeta_G^{G, \triangleleft}(s)$

Note that $[N, G] = y_1^{-2} y_3$ and $G/[N, G] \cong \mathbb{Z}^2$. By Proposition 2.12, we have

$$\zeta_G^{G, \triangleleft}(s) = \zeta_{G/[N, G]}^{G/[N, G], \triangleleft}(s) \quad (\text{with respect to } N/[N, G]).$$

Applying Lemma 5.6 with $Z_1 = G/[N, G]$ and $Z_0 = N/[N, G]$, we obtain that $\zeta_G^{G, \triangleleft}(s) = \zeta(s)\zeta(s-1)(1-2^{-s})$.

5.5 Computing the zeta functions of $G = \mathcal{B}_3$

Note that $Z = \langle x_1 \rangle$. One can easily check that $E = G/Z$ is the plane crystallographic group $\mathbf{p2mg}$.

5.5.1 Local factors of $\zeta_G^{G, \leq}(s)$

If $p \neq 2$, then by (5.1), we have

$$\zeta_{G_p}^{G_p, \leq}(s) = \zeta_p(s)\zeta_{E_2}^{E_2, \leq}(s),$$

which is equal to $\zeta_p(s)\zeta_p(s-1)^2$ according to the results of [13, Section 5.6].

We now compute $\zeta_{G_2}^{G_2, \leq}(s)$. An open subgroup $A \triangleleft G_2$ such that $AN_2 = G_2$ must contain $\alpha \mathbf{x}^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}_2^3$, and hence it contains $(\alpha \mathbf{x}^{\mathbf{v}})^2 = x_1^{2v_1+1}$. Since $2v_1+1 \in \mathbb{Z}_2^*$, we deduce that $Z_2 \subset A$. Therefore, $\zeta_{G_2}^{G_2, \leq}(s) = \zeta_{E_2}^{E_2, \leq}(s)$, which is equal to $(1+2^{1-s})\zeta_2(s-1)$ according to [13, Section 5.6].

5.5.2 Local factors of $\zeta_G^{N, \triangleleft}(s)$

If $p \neq 2$, then by (5.1), we have

$$\zeta_{G_p}^{N_p, \triangleleft}(s) = \zeta_p(s)\zeta_{E_2}^{T_2, \triangleleft}(s),$$

which is equal to $\zeta_p(s)^3$ according to [13, Section 6.6].

Assume now that $p = 2$. By Corollary 3.3,

$$\zeta_{G_2}^{N_2, \triangleleft}(s) = \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu,$$

where $\mathcal{T} \subset T_3(\mathbb{Z}_2)$ is defined by the following equivalent conditions:

$$\begin{aligned} & \alpha(\mathbf{x}^{\mathbf{t}_1})\mathbf{x}^{-\mathbf{t}_1}, \alpha(\mathbf{x}^{\mathbf{t}_2}), \alpha(\mathbf{x}^{\mathbf{t}_3}), \varepsilon(\mathbf{x}^{\mathbf{t}_1})\mathbf{x}^{-\mathbf{t}_1}, \varepsilon(\mathbf{x}^{\mathbf{t}_2})\mathbf{x}^{-\mathbf{t}_2}, \varepsilon(\mathbf{x}^{\mathbf{t}_3}) \in \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \mathbf{x}^{\mathbf{t}_2}, \mathbf{x}^{\mathbf{t}_3} \rangle} \\ & \iff x_2^{2t_{12}}x_3^{2t_{13}}, \mathbf{x}^{-\mathbf{t}_2}, \mathbf{x}^{-\mathbf{t}_3}, x_3^{2t_{13}}, x_3^{2t_{23}}, \mathbf{x}^{-\mathbf{t}_3} \in \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \mathbf{x}^{\mathbf{t}_2}, \mathbf{x}^{\mathbf{t}_3} \rangle} \\ & \iff x_2^{2t_{12}}, x_3^{2t_{13}}, x_3^{2t_{23}} \in \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \mathbf{x}^{\mathbf{t}_2}, \mathbf{x}^{\mathbf{t}_3} \rangle}. \end{aligned}$$

This is the same integral that showed up in Section 5.3.2 when $p = 2$, so the result is $(1+4 \cdot 2^{-s}+4^{-s})\zeta_2(s)^3$.

5.5.3 Local factors of $\zeta_G^{\mathcal{G}, \triangleleft}(s)$

If $p \neq 2$, then

$$[N_p, G_p] = \overline{\langle x_2^2, x_3^2 \rangle} = \overline{\langle x_2, x_3 \rangle},$$

whence $N_p/[N_p, G_p] \cong \mathbb{Z}_p$. Thus, by Proposition 2.12, $\zeta_{G_p}^{\mathcal{G}, \triangleleft}(s) = \zeta_p(s)$.

Assume now that $p = 2$. By the same argument used in the calculation of $\zeta_{G_2}^{\mathcal{G}_2, \leq}(s)$ in Section 5.5.1, we have

$$\zeta_{G_2}^{\mathcal{G}_2, \triangleleft}(s) = \zeta_{E_2}^{E_2, \triangleleft}(s),$$

which is equal to $1 + 4 \cdot 2^{-s}$ according to [13, Section 6.6].

5.5.4 Local factors of $\zeta_H^{\mathcal{H}, \triangleleft}(s)$ for $H = \langle \alpha, x_1, x_2, x_3 \rangle$

Note that $H = \mathcal{G}_2$. Arguing as in Section 5.2.4, we find that $\zeta_{G_p}^{\mathcal{H}, \triangleleft}(s)$ if $p \neq 2$, and that $\zeta_{G_2}^{\mathcal{H}_2, \triangleleft}(s) = \zeta_{E_2}^{\mathcal{H}_2/Z_2, \triangleleft}(s)$. The latter series is equal to $1 + 2^{1-s}$ according to [13, Section 6.6].

5.5.5 Local factors of $\zeta_K^{\mathcal{K}, \triangleleft}(s)$ for $K = \langle \varepsilon, x_1, x_2, x_3 \rangle$

Note that $K = \mathcal{B}_2$. If $p \neq 2$, then by (5.1), we have

$$\zeta_{G_p}^{\mathcal{K}, \triangleleft}(s) = \zeta_{Z_p}^{\leq}(s) \zeta_{E_p}^{\mathcal{K}_p/Z_p, \triangleleft}(s),$$

which is equal to $\zeta_p(s)\zeta_p(s)$ according to [13, Section 6.6].

Assume that $p = 2$. We claim that if $A \triangleleft G_2$ is open and satisfies $AN_2 = K_2$, then it must include $\overline{\langle x_2 \rangle}$. Indeed, A includes $[K_2, K_2] = \overline{\langle x_3^2 \rangle}$ by Lemma 2.11. It also contains $\varepsilon \mathbf{x}^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}_2^3$, whence it contains $[\alpha, \varepsilon \mathbf{x}^{\mathbf{v}}] = x_2^{2v_2+1} x_3^{2v_3}$. It follows that $x_2^{2v_2+1} \in A$ and hence $x_2 \in A$. Therefore,

$$\zeta_{G_2}^{\mathcal{K}_2, \triangleleft}(s) = \zeta_{G_2/\overline{\langle x_2 \rangle}}^{\mathcal{K}_2/\overline{\langle x_2 \rangle}, \triangleleft}(s).$$

We claim that this series is equal to

$$\zeta_{K_2/\overline{\langle x_2 \rangle}}^{\mathcal{K}_2/\overline{\langle x_2 \rangle}, \triangleleft}(s).$$

Indeed, α and ε commute in $G_2/\overline{\langle x_2 \rangle}$ and their actions on $N_2/\overline{\langle x_2 \rangle}$ are the same; thus a normal subgroup of $K_2/\overline{\langle x_2 \rangle}$ is already normal in $G_2/\overline{\langle x_2 \rangle}$. Finally, observe that $K/\overline{\langle x_2 \rangle}$ is isomorphic to the plane crystallographic group **pm**. According to the results of [13, Section 6.3], we obtain that

$$\zeta_{G_2}^{\mathcal{K}_2, \triangleleft}(s) = (1 + 5 \cdot 2^{-s} + 2 \cdot 4^{-s})\zeta_2(s).$$

5.5.6 Local factors of $\zeta_G^{L, \triangleleft}(s)$ for $L = \langle \varepsilon \alpha, x_1, x_2, x_3 \rangle$

Note that $L = \mathcal{B}_2$. If $p \neq 2$, then by (5.1), we have

$$\zeta_G^{L, \triangleleft}(s) = \zeta_p(s) \zeta_{E_p}^{L_p / Z_{p, \triangleleft}}(s).$$

This is equal to $\zeta_p(s) \zeta_p(s)$ according to the results of [13, Section 6.6].

Assume now that $p = 2$. By Proposition 3.5 (with $H = L$), we have

$$\zeta_G^{L, \triangleleft}(s) = \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-2} |t_{22}|_2^{s-3} |t_{33}|_2^{s-4} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_2) \times \mathbb{Z}_2^3$ is the set of pairs (\mathbf{t}, \mathbf{v}) satisfying the following equivalent conditions:

$$\begin{aligned} & \alpha(\mathbf{x}^{t_1}) \mathbf{x}^{-t_1}, \alpha(\mathbf{x}^{t_2}), \alpha(\mathbf{x}^{t_3}), \varepsilon(\mathbf{x}^{t_1}) \mathbf{x}^{-t_1}, \varepsilon(\mathbf{x}^{t_2}) \mathbf{x}^{-t_2}, \varepsilon(\mathbf{x}^{t_3}) \\ & \quad (\varepsilon \alpha \mathbf{x}^{\mathbf{v}})^2, [\varepsilon \alpha, x_1], [\varepsilon \alpha, x_2], [\varepsilon \alpha, x_3], [\alpha, \varepsilon \alpha \mathbf{x}^{\mathbf{v}}] \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}, \\ \iff & \quad x_2^{-2t_{12}} x_3^{-2t_{13}}, \mathbf{x}^{-t_2}, \mathbf{x}^{-t_3}, x_3^{-2t_{13}}, x_3^{-2t_{23}}, \mathbf{x}^{-t_3}, \\ & \quad x_1^{2v_1+1} x_3^{2v_3}, x_2^2, x_2^{2v_2+1} x_3^{-2v_3} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}, \\ \iff & \quad x_3^{2t_{13}}, x_3^{2t_{23}}, x_1^{2v_1+1} x_3^{2v_3}, x_2^2, x_2 x_3^{-2v_3} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}. \end{aligned}$$

By Lemma 5.1, these conditions are equivalent to

- (1) $t_{33} \mid 2t_{13}$,
- (2) $t_{33} \mid 2t_{23}$,
- (3) $t_{11} \mid 2v_1 + 1$,
- (4) $t_{22} \mid -\frac{2v_1+1}{t_{11}} t_{12}$,
- (5) $t_{33} \mid \frac{2v_1+1}{t_{11}t_{22}} t_{12}t_{23} - \frac{2v_1+1}{t_{11}} t_{13} + 2v_3$,
- (6) $t_{22} \mid 2$,
- (7) $t_{33} \mid \frac{2}{t_{22}} t_{23}$,
- (8) $t_{22} \mid 1$,
- (9) $t_{33} \mid -\frac{1}{t_{22}} t_{23} - 2v_3$.

Note that (3) and (8) are equivalent to $t_{11}, t_{22} \in \mathbb{Z}_2^*$, and hence (4) and (6) are redundant. Notice also that (2) implies (7). To sum up, \mathcal{T} is defined by (1), (2), (5), (9) and $t_{11}, t_{22} \in \mathbb{Z}_2^*$. We express \mathcal{T} as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases

$$(2a) \quad t_{33} \mid t_{23} \text{ and}$$

$$(2b) \quad |t_{33}|_2 = |2t_{23}|_2.$$

If we assume (2a), then (9) can be replaced by $t_{33} \mid 2v_3$, and then (5) can be replaced by $t_{33} \mid t_{13}$, which makes (1) redundant. Thus, the conditions defining \mathcal{T}^a are $t_{11}, t_{22} \in \mathbb{Z}_2^*$, $t_{33} \mid t_{13}$, $t_{33} \mid t_{23}$ and $t_{33} \mid 2v_3$, and the integral over \mathcal{T}^a becomes

$$\frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}^a} |t_{33}|_2^{s-4} d\mu = \frac{1}{1 - 2^{-1}} \int_{t_{33}|2v_3} |t_{33}|_2^{s-2} d\mu = 1 + 2^{1-s} \zeta_2(s).$$

If we assume (2b), then (9) implies $2 \mid t_{23}$. Note also that (1) and (9) can be replaced by

(1') $t_{23} \mid t_{13}$ and

(9') $t_{23} \mid -\frac{t_{23}}{2t_{22}} - v_3$,

and the latter implies that $t_{23} \mid 2v_3$. This enables us to replace (5) by

(5') $2 \mid \frac{2v_1+1}{t_{11}t_{22}}t_{12} - \frac{2v_1+1}{t_{11}}\frac{t_{13}}{t_{23}} - \frac{2v_3}{t_{23}}$.

The conditions defining \mathcal{T}^b are, therefore, $t_{11}, t_{22} \in \mathbb{Z}_2^*$, $2 \mid t_{23}$, $|t_{33}|_2 = |2t_{23}|_2$, (1'), (5') and (9'), and the integral over \mathcal{T}^b becomes

$$\begin{aligned} \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}^b} |2t_{23}|_2^{s-4} d\mu &= \frac{1}{1 - 2^{-1}} \int_{2|t_{23}} 2^{4-s} |t_{23}|_2^{s-1} 2^{-3} d\mu \\ &= 2^{1-2s} \zeta_2(s), \end{aligned}$$

where in the first equality we first applied Proposition 3.6 with the pivots $t_{11}, t_{22}, t_{13}, t_{12}, t_{33}, v_3$.

Therefore,

$$\zeta_{G_2}^{L_2, \triangleleft}(s) = 1 + 2^{1-s} \zeta_2(s) + 2^{1-2s} \zeta_2(s) = \zeta_2(s)(1 + 2^{-s} + 2 \cdot 4^{-s}).$$

5.6 Computing the zeta functions of $G = \mathcal{B}_4$

Note that $Z = \langle x_1 \rangle$. One can easily check that $E = G/Z$ is the plane crystallographic group $\mathbf{p2gg}$.

5.6.1 Local factors of $\zeta_{G_2}^{G, \leq}(s)$

If $p \neq 2$, then by (5.1), we have $\zeta_{G_p}^{G, \leq}(s) = \zeta_p(s) \zeta_{E_p}^{E, \leq}(s)$, and according to the results of [13, Section 5.7], the latter is equal to $\zeta_p(s) \zeta_p(s - 1)^2$.

Assume that $p = 2$. An open subgroup $A \leq G_2$ such that $AN_2 = G_2$ contains $\alpha \mathbf{x}^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}_2^3$, whence $(\alpha \mathbf{x}^{\mathbf{v}})^2 = x_1^{2v_1+1} \in A$. Since $2v_1 + 1 \in \mathbb{Z}_2^*$, we conclude that $Z_2 \subseteq A$. It follows that $\zeta_{G_2}^{G_2, \leq}(s) = \zeta_{E_2}^{E_2, \leq}(s)$, which is 1 according to [13, Section 5.7].

5.6.2 Calculus of $\zeta_G^{N, \triangleleft}(s)$

Note that N , as $\mathbb{Z}[P]$ -module, is isomorphic to the analogue module in the case $G = \mathcal{B}_3$. Therefore, $\zeta_G^{N, \triangleleft}(s) = \zeta(s)^3(1 + 4 \cdot 2^{-s} + 4^{-s})$ according to the result of Section 5.5.2.

5.6.3 Local factors of $\zeta_G^{G, \triangleleft}(s)$

If $p \neq 2$, then

$$[N_p, G_p] = \overline{\langle x_2^2, x_3^2 \rangle} = \overline{\langle x_2, x_3 \rangle},$$

whence $N_p/[N_p, G_p] \cong \mathbb{Z}_p$. Thus, by Proposition 2.12, $\zeta_{G_p}^{G_p, \triangleleft}(s) = \zeta_p(s)$.

Assume now that $p = 2$. If $A \triangleleft G_2$ is open and satisfies $AN_2 = G_2$, then A contains $\alpha \mathbf{x}^{\mathbf{v}}$ and $\varepsilon \mathbf{x}^{\mathbf{w}}$ for some $\mathbf{v}, \mathbf{w} \in \mathbb{Z}_2^3$, whence it contains both

$$(\alpha \mathbf{x}^{\mathbf{v}})^2 = x_1^{2v_1+1} \quad \text{and} \quad (\beta \mathbf{x}^{\mathbf{w}})^2 = x_1^{2w_1} x_2^{2w_2+1}.$$

Since $2v_2 + 1, 2w_2 + 1 \in \mathbb{Z}_2^*$, we conclude that $x_1, x_2 \in A$. In addition, we have $[G_2, G_2] \subseteq A$ by Lemma 2.11; in particular $x_2 x_3 \in A$. It follows that $x_1, x_2, x_3 \in A$ and hence $A = G_2$. Thus, $\zeta_{G_2}^{G_2, \triangleleft}(s) = 1$.

5.6.4 The local factors of $\zeta_G^{H, \triangleleft}(s)$ for $H = \langle \alpha, x_1, x_2, x_3 \rangle$

Note that $H \cong \mathcal{G}_2$. Arguing as in Section 5.2.4, we find that

$$\zeta_{G_p}^{H_p, \triangleleft}(s) = \zeta_p(s) \quad \text{and} \quad \zeta_{G_2}^{H_2, \triangleleft}(s) = \zeta_{E_2}^{H_2/Z_2, \triangleleft}(s).$$

The latter series is equal to $1 + 2 \cdot 2^{-s}$ according to the results of [13, Section 6.7].

5.6.5 Local factors of $\zeta_G^{K, \triangleleft}(s)$ for $K = \langle \varepsilon, x_1, x_2, x_3 \rangle$

Note that $K = \mathcal{B}_2$. If $p \neq 2$, then by (5.1), we have

$$\zeta_{G_p}^{K_p, \triangleleft}(s) = \zeta_p(s) \zeta_{E_p}^{K_p/Z_p, \triangleleft}(s),$$

which is equal to $\zeta_p(s) \zeta_p(s)$ according to the results of [13, Section 6.7].

Assume now that $p = 2$. By Proposition 3.5 with $H = K$, we have

$$\zeta_{G_2}^{K_2, \triangleleft}(s) = \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-2} |t_{22}|_2^{s-3} |t_{33}|_2^{s-4} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_2) \times \mathbb{Z}_2^3$ is the set of pairs (\mathbf{t}, \mathbf{v}) satisfying the following equivalent conditions:

$$\begin{aligned} & \alpha(\mathbf{x}^{t_1})\mathbf{x}^{-t_1}, \alpha(\mathbf{x}^{t_2}), \alpha(\mathbf{x}^{t_3}), \varepsilon(\mathbf{x}^{t_1})\mathbf{x}^{-t_1}, \varepsilon(\mathbf{x}^{t_2})\mathbf{x}^{-t_2}, \varepsilon(\mathbf{x}^{t_3}) \\ & \quad (\varepsilon\mathbf{x}^{\mathbf{v}})^2, [\varepsilon, x_1], [\varepsilon, x_2], [\varepsilon, x_3], [\alpha, \varepsilon\mathbf{x}^{\mathbf{v}}] \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ \iff & x_2^{-2t_{12}}x_3^{-2t_{13}}, \mathbf{x}^{-t_2}, \mathbf{x}^{-t_3}, x_3^{-2t_{13}}, x_3^{-2t_{23}}, \mathbf{x}^{-t_3}, \\ & \quad x_1^{2v_1}x_2^{2v_2+1}, x_3^2, x_2^{-1-2v_2}x_3^{-1+2v_3} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ \iff & x_2^{2t_{12}}, x_1^{2v_1}x_3^{-1}, x_3^2, x_2^{-1-2v_2}x_3^{-1} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}. \end{aligned}$$

By Lemma 5.1, these conditions are equivalent to

- (1) $t_{22} \mid 2t_{12}$,
- (2) $t_{33} \mid \frac{2t_{12}}{t_{22}}t_{23}$,
- (3) $t_{11} \mid 2v_1$,
- (4) $t_{22} \mid -\frac{2v_1}{t_{11}}t_{12}$,
- (5) $t_{33} \mid \frac{2v_1}{t_{11}}(\frac{t_{12}}{t_{22}}t_{23} - t_{13}) - 1$,
- (6) $t_{33} \mid 2$,
- (7) $t_{22} \mid -1 - 2v_2$,
- (8) $t_{33} \mid \frac{1+2v_2}{t_{22}}t_{23} - 1$.

Note that (7) is equivalent to $t_{22} \in \mathbb{Z}_2^*$, whence (1) and (4) are redundant and (6) implies (2). We express \mathcal{T} as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases

- (3a) $t_{11} \mid v_1$ and
- (3b) $|t_{11}|_2 = |2v_1|_2$.

In case (3a), (6) reduces (5) to the condition $t_{33} \mid -1$, that is, $t_{33} \in \mathbb{Z}_2^*$; hence \mathcal{T}^a is defined by $t_{11} \mid v_1, t_{22}, t_{33} \in \mathbb{Z}_2^*$. The integral over \mathcal{T}^a becomes

$$\frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^a} |t_{11}|_2^{s-2} d\mu = \frac{1}{1-2^{-1}} \int |t_{11}|_2^{s-1} d\mu = \zeta_2(s).$$

We now assume (3b). Note that \mathcal{T}^b is defined by $|t_{11}|_2 = |2v_1|_2, t_{22} \in \mathbb{Z}_2^*, (5), (6)$ and (8), and the integral over \mathcal{T}^b becomes

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^b} |2v_1|_2^{s-2} |t_{33}|_2^{s-4} d\mu \\ & = \frac{1}{(1-2^{-1})^2} \int_{|t_{33}|_2} 2^{2-s} |v|_2^{s-2} 2^{-2} |t_{33}|_2^{s-2} d\mu \\ & = (1 + 2 \cdot 2^{-s}) 2^{-s} \zeta_2(s), \end{aligned}$$

where in the second equality we applied Proposition 3.6 with the pivots $t_{11}, t_{22}, t_{13}, t_{23}$.

We conclude that

$$\zeta_{G_2}^{K_2, \triangleleft}(s) = \zeta_2(s) + (1 + 2^{-(s-1)})2^{-s}\zeta_2(s) = (1 + 2^{-s} + 2 \cdot 4^{-s})\zeta_2(s).$$

5.6.6 Local factors of $\zeta_G^{L, \triangleleft}(s)$ for $L = \langle \varepsilon\alpha, x_1, x_2, x_3 \rangle$

Note that $L \cong \mathcal{B}_2$. If $p \neq 2$, then by (5.1), we have

$$\zeta_{G_p}^{L_p, \triangleleft}(s) = \zeta_p(s)\zeta_{E_p^{L_p/Z_p, \triangleleft}}(s),$$

which is equal to $\zeta_p(s)\zeta_p(s)$ according to the results of [13, Section 6.7].

Assume now that $p = 2$. We claim that any $A \triangleleft G_2$ that is open and satisfies $AN_2 = L_2$ must include $\langle x_1x_2, x_2x_3, x_2^2 \rangle$. Indeed, A includes $[G_2, N_2] = \langle x_2^2 \rangle$ by Lemma 2.11. It also contains $\varepsilon\alpha\mathbf{x}^v$ for some $\mathbf{v} \in \mathbb{Z}_2^3$, whence it contains

$$(\varepsilon\alpha\mathbf{x}^v)^2 = x_1^{2v_1+1}x_3^{2v_3+1} \quad \text{and} \quad [\alpha, \varepsilon\alpha\mathbf{x}^v] = x_2^{2v_2-1}x_3^{-2v_3-1}.$$

We deduce that $x_1^{2v_1+1}x_2^{2v_2-1} \in A$, and using the fact that $x_2^2 \in A$, we obtain that $(x_1x_2)^{2v_1+1} \in A$. It follows that $x_1x_2 \in A$ since $2v_1 + 1$ is a unit. With a similar argument, we find that $x_2x_3 \in A$. This proves the claim. Since $[\varepsilon, \alpha] = x_2x_3$, the quotient group

$$G' := G_2 / \overline{\langle x_1x_2, x_2x_3, x_2^2 \rangle}$$

is abelian, and therefore, $\zeta_{G_2}^{L_2, \triangleleft}(s) = \zeta_{L'}^{L', \triangleleft}(s)$, where L' is the image of L_2 at G' , and the partial zeta function on the right is computed with respect to N' , the image of N_2 at G' . Note that $L' \cong C_2 \times C_2$ and N' has index 2. Thus,

$$\zeta_{L'}^{L', \triangleleft}(s) = 1 + 2 \cdot 2^{-s}.$$

6 Computing the zeta functions of the 3-dimensional AB-groups

In this final section, we prove the formulae presented in Section 4.2. We always denote by G the AB-group under consideration and by N the Fitting subgroup (which is generated by x_1, x_2, x_3). We keep the notation introduced at the beginning of Section 2. In particular,

$$Z = Z(N), \quad T = N/Z, \quad E = G/Z, \quad P = G/N.$$

The formulae for the zeta functions of

$$G = N = \langle x_1, x_2, x_3 : [x_2, x_1] = x_3^k, [x_1, x_3] = [x_2, x_3] = 1 \rangle,$$

where $k > 0$, are already computed; see Tables 1 and 2 with $F = C_1$. We focus here on the cases $G \neq N$. If $N \leq H \leq G$ is an intermediate subgroup, then H is also a 3-dimensional AB-group with Fitting subgroup N . Therefore, in order to compute $\zeta_G^{\leq}(s)$, it will be enough to calculate $\zeta_G^{G, \leq}(s)$ since the other partial zeta functions will have been computed in previous calculations. The isomorphism classes of the intermediate subgroups $N \subsetneq H \subsetneq G$ (if there are any) will be identified when computing the partial zeta functions $\zeta_G^{H, \triangleleft}(s)$.

When dealing with local factors at primes $p \mid [G : N]$, we will sometimes use the method of p -adic integration described in Section 3; specifically Corollary 3.3 and Proposition 3.5. The following lemma collects some information about the \mathfrak{T} -group N .

Lemma 6.1. *Let $N = \langle x_1, x_2, x_3 : [x_2, x_1] = x_3^k, [x_1, x_3] = [x_2, x_3] = 1 \rangle$, with $k = [Z : [N, N]] > 0$.*

(1) For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_p^3$ and $r \in \mathbb{Z}_p$,

$$\begin{aligned} \mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}} &= x_1^{a_1+b_1} x_2^{a_2+b_2} x_3^{a_3+b_3+kb_1a_2}, \\ (\mathbf{x}^{\mathbf{a}})^r &= x_1^{ra_1} x_2^{ra_2} x_3^{ra_3+\binom{r}{2}ka_1a_2}, \\ [\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}] &= x_3^{k(b_1a_2-a_1b_2)}. \end{aligned}$$

(2) For $\mathbf{t} \in T_3^+(\mathbb{Z}_p)$, the following holds.

- (a) $\mathbf{t} \in \mathcal{M}_{N_p}^{\leq}$ if and only if $t_{33} \mid kt_{11}t_{22}$.
- (b) $\mathbf{t} \in \mathcal{M}_{N_p}^{\triangleleft}$ if and only if $t_{33} \mid kt_{11}, t_{33} \mid kt_{12}$ and $t_{33} \mid kt_{22}$.

(3) Fix $\mathbf{t} \in \mathcal{M}_{N_p}^{\leq}$. Given $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}_p^3$, $\mathbf{x}^{\mathbf{a}} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}$ if and only if all of the following hold.

- (a) $t_{11} \mid a_1$.
- (b) $t_{22} \mid -\frac{a_1}{t_{11}}t_{12} + a_2$.
- (c) $t_{33} \mid -\frac{(a_1/t_{11})t_{12}+a_2}{t_{22}}t_{23} - \frac{a_1}{t_{11}}t_{13} - \frac{1}{2}k\frac{a_1}{t_{11}}(\frac{a_1}{t_{11}} - 1)t_{11}t_{12} + a_3$.

In the case that $\mathbf{t} \in \mathcal{N}_{N_p}^{\triangleleft}$, (c) can be replaced by

$$t_{33} \mid -\frac{-\frac{a_1}{t_{11}}t_{12} + a_2}{t_{22}}t_{23} - \frac{a_1}{t_{11}}t_{13} + a_3.$$

Proof. The verification of (1) is straightforward, and (2) is a special case of (3.4). As for (3), given that $\{\mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3}\}$ is a good basis for $\overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}$, the relation $\mathbf{x}^{\mathbf{a}} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}$ holds if and only if $\mathbf{x}^{\mathbf{a}} = (\mathbf{x}^{t_1})^{\lambda_1} (\mathbf{x}^{t_2})^{\lambda_2} (\mathbf{x}^{t_3})^{\lambda_3}$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_p$. Using (1), we find that

$$(\mathbf{x}^{t_1})^{\lambda_1} (\mathbf{x}^{t_2})^{\lambda_2} (\mathbf{x}^{t_3})^{\lambda_3} = x_1^{t_{11}\lambda_1} x_2^{t_{12}\lambda_1+t_{22}\lambda_2} x_3^{t_{13}\lambda_1+t_{23}\lambda_2+k\binom{\lambda_1}{2}t_{11}t_{12}+t_{33}\lambda_3}.$$

The first part of (3) is now clear. In the case that $\mathbf{t} \in \mathcal{M}_{N_p}^{\triangleleft}$ (and hence $\overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}$ is normal), (2) implies that $t_{33} \mid kt_{11}$. Therefore, the term

$$\frac{1}{2}k \frac{a_1}{t_{11}} \left(\frac{a_1}{t_{11}} - 1 \right) t_{11} t_{12} = \left(\frac{a_1}{2} \right) k t_{11} t_{12}$$

can be deleted from (c). This completes the proof. □

Finally, Table 3 collects the results of some p -adic integrals that will appear in our calculations. The verification of these formulae is straightforward. The notation is as follows: k denotes an integer; the letters s, t and u are complex variables; integration takes place over \mathbb{Z}_p^n ($n = 2, 3$ or 4), where μ denotes the Haar additive measure normalized so that $\mu(\mathbb{Z}_p^n) = 1$; and θ_p denotes $\frac{1}{1-p^{-1}}$.

6.1 Computing the zeta functions of $G = G_{p^2, 2k}, k \in \mathbb{N}$

6.1.1 Local factors of $\zeta_G^{G, \leq}(s)$

If $p \neq 2$, then $\zeta_{G_p}^{G, \leq}(s)$ was given in Table 1 (the case $F = C_2$ and $F \subset \text{SL}(T)$).

Assume that $p = 2$. If $A \leq G_2$ is open and satisfies $AN_2 = G_2$, then A contains $\alpha \mathbf{x}^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}_2$, whence $(\alpha \mathbf{x}^{\mathbf{v}})^2 = x_3^{2v_3+1} \in A$. Since $2v_3 + 1 \in \mathbb{Z}_2^*$, we obtain that $Z_p \subset A$. This implies that $\zeta_{G_2}^{G, \leq}(s) = \zeta_{E_2}^{E_2, \leq}(s)$, which in turn is equal to $\zeta_2(s-1)\zeta_2(s-2)$ according to the results of [13, Section 5.2].

6.1.2 Local factors of $\zeta_G^{N, \triangleleft}(s)$

If $p \neq 2$, then $\zeta_{G_p}^{N, \triangleleft}(s)$ was given in Table 2 (the case $F = C_2$ and $F \subset \text{SL}(T)$).

Assume now that $p = 2$. We show that the assumptions of Proposition 2.15 are satisfied. Firstly, since the class of α at $P = G_2/N_2$ acts on T_2 as $-\text{id}_{T_2}$, any finite-index subgroup of T_2 is a $\mathbb{Z}_2[P]$ -submodule, and they are all isomorphic. Fix $U \leq T_2$ and $V \leq Z_2$ of finite index such that $[U, T_2] \subseteq V$. Let \tilde{U} be the pre-image of U in N_2 . Note that \tilde{U} has a good basis of the form $\{x_1^{t_{11}} x_2^{t_{12}}, x_2^{t_{22}}, x_3\}$, and that $V = \overline{\langle x_3^{t_{33}} \rangle}$ for some non-zero t_{33} . It is easy to check that the condition $[U, T_2] \subseteq V$ is translated into $t_{33} \mid 2kt_{11}, t_{33} \mid 2kt_{12}, t_{33} \mid 2kt_{22}$. Therefore, by Lemma 6.1 (2), $\{x_1^{t_{11}} x_2^{t_{12}}, x_2^{t_{22}}, x_3^{t_{33}}\}$ represents a good basis for a normal subgroup $B \triangleleft N_2$. Clearly, $(BZ_2)/Z = U$ and $B \cap Z_2 = V$. We now show that B is normal in G_2 . Indeed,

$$\begin{aligned} \alpha(x_1^{t_{11}} x_2^{t_{12}}) &= x_1^{-t_{11}} x_2^{-t_{12}} = (x_1^{t_{11}} x_2^{t_{12}})^{-1} x_3^{-2kt_{11}t_{12}} \in B, \\ \alpha(x_2^{t_{22}}) &= x_2^{-t_{22}} \in B. \end{aligned}$$

p -adic integral	result
$A_{p,k}(s, t) := \theta_p^2 \int_{x kx} x _p^{s-1} y _p^{t-1} d\mu$	$\zeta_p(t) \left(\frac{1 - k _p^s}{1 - p^{-s}} + \frac{ k _p^s}{1 - p^{-(s+t)}} \right)$
$B_{p,k}(s, t) := \theta_p^2 \int_{p x ky} x _p^{s-1} y _p^{t-1} d\mu$	$\zeta_p(t) \left(\frac{p^{-s}(1 - k _p^s)}{1 - p^{-s}} + \frac{p^{-(s+t)} k _p^s}{1 - p^{-(s+t)}} \right)$
$C_{p,k}(s, t, u) := \theta_p^3 \int_{z kx} \int_{z ky} x _p^{s-1} y _p^{t-1} z _p^{u-1} d\mu$	$\zeta_p(s) \zeta_p(t) \left(\frac{1 - k _p^u}{1 - p^{-u}} + \frac{ k _p^u}{1 - p^{-(s+t+u)}} \right)$
$D_{p,k}(s, t, u) := \theta_p^3 \int_{z kx} \int_{z pkx} x _p^{s-1} y _p^{t-1} z _p^{u-1} d\mu$	$\zeta_p(s) \zeta_p(t) \left(\frac{1 - k _p^u}{1 - p^{-u}} + \frac{(1 + p^{-(u+s)} - p^{-(u+s+t)}) k _p^u}{1 - p^{-(s+t+u)}} \right)$
$E_{p,k}(s, t, u) := \theta_p^3 \int_{z kx} \int_{z pkx} \int_{z pw} x _p^{s-1} y _p^{t-1} z _p^{u-1} d\mu$	$\zeta_p(s) \zeta_p(t) \left(\frac{(1 + p^{-u} - p^{-(u+1)})(1 - k _p^{u+1})}{1 - p^{-(u+1)}} + \frac{(1 + p^{-(u+s)} - p^{-(s+t+u+1)}) k _p^{u+1}}{1 - p^{-(s+t+u+1)}} \right)$

Table 3. List of p -adic integrals

We are now in position to apply Proposition 2.15. For computing $[T_2 : X(V)]$, we use Lemma 2.16, which tells us that

$$[T_2 : X(V)] = \begin{cases} 1 & \text{if } [Z_2 : V] < 2^{v_2(2k)}, \\ |2k|_2^2 [Z_2 : V]^2 & \text{if } V \leq \langle x_3^{2k} \rangle. \end{cases}$$

For the computation of $|\text{Hom}_{\mathbb{Z}_2[P]}(T_2, Z_2/V)|$, we use Lemma 5.2, which tells us that $|\text{Hom}_{\mathbb{Z}_2[P]}(T_2, Z_2/V)| = 1$ if $V = Z_2$ and is 2^2 otherwise. Thus, by Proposition 2.15 and the results of [13, Section 6.2],

$$\begin{aligned} \zeta_{G_2}^{N_2, \triangleleft}(s) &= \zeta_{E_2}^{T_2, \triangleleft}(s) \left(1 + \sum_{i=1}^{v_2(2k)-1} 2^{-is} 2^2 \right. \\ &\quad \left. + \sum_{V \leq 2k\mathbb{Z}_2} |2k|_2^{-2s} [Z_2 : V]^{-2s} [Z_2 : V]^{-s} 2^2 \right) \\ &= \zeta_2(s) \zeta_2(s-1) \left(1 + 2^{2-s} \frac{1 - |k|_2^s}{1 - 2^{-s}} + |2k|_2^s 2^2 \zeta_2(2s) \right). \end{aligned}$$

6.1.3 Local factors of $\zeta_G^{\triangleleft}(s)$

If $p \neq 2$, then

$$\gamma_3(G_p, N_p) = \overline{\langle x_1^4, x_2^4, x_3^{4k} \rangle} = \overline{\langle x_1, x_2, x_3^k \rangle},$$

whence $N_p/\gamma_3(G_p, N_p) \cong \mathbb{Z}_p/k\mathbb{Z}_p$. Thus, by Proposition 2.12,

$$\zeta_{G_p}^{G_p}(s) = \frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}}.$$

Assume now that $p = 2$. As shown in the calculation of $\zeta_{G_2}^{G_2, \leq}(s)$, any $A \triangleleft G_2$ that is open and satisfies $AN_2 = G_2$ must include Z_2 . Thus,

$$\zeta_{G_2}^{G_2, \triangleleft}(s) = \zeta_{E_2}^{E_2, \triangleleft}(s),$$

which is equal to $1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s}$ according to [13, Section 6.2].

6.2 Computing the zeta functions of $G = G_{\text{pg}, 2k}$

6.2.1 Local factors of $\zeta_G^{\leq}(s)$

If $p \neq 2$, then $\zeta_{G_p}^{G_p, \leq}(s)$ was given in Table 1 (the case $F = D_1$ and $F \not\subset \text{SL}(T)$).

Assume now that $p = 2$. By Proposition 3.5 and Lemma 6.1 (2), we have

$$\zeta_{G_2}^{G_2, \leq}(s) = \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-2} |t_{22}|_2^{s-3} |t_{33}|_2^{s-4} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_2) \times \mathbb{Z}_2^3$ is the set of pairs (\mathbf{t}, \mathbf{v}) satisfying $t_{33} \mid 2kt_{11}t_{22}$ and the following equivalent conditions:

$$\begin{aligned} & \mathbf{x}^{t_1} \cdot \beta^{\mathbf{x}^v}(\mathbf{x}^{t_1}), [\beta^{\mathbf{x}^v}, \mathbf{x}^{t_2}], \beta^{\mathbf{x}^v}(\mathbf{x}^{t_3}), (\beta^{\mathbf{x}^v})^2 \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ \iff & x_2^{2t_{12}} x_3^{-kt_{11}(1+2v_2)+2kv_1t_{12}-2kt_{11}t_{12}}, \\ & x_3^{-2(t_{23}+kt_{22}v_1)}, x_2^{2v_2+1} x_3^{kv_1(2v_2+1)} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}. \end{aligned}$$

By Lemma 6.1 (3), the conditions defining \mathcal{T} are equivalent to

- (1) $t_{33} \mid 2kt_{11}t_{22}$,
- (2) $t_{22} \mid 2t_{12}$,
- (3) $t_{33} \mid \frac{2t_{12}}{t_{22}}(-t_{23} + kv_1t_{22}) - kt_{11}(1 + 2v_2) - 2kt_{11}t_{12}$,
- (4) $t_{33} \mid 2(-t_{23} + kt_{22}v_1)$,
- (5) $t_{22} \mid 2v_2 + 1$,
- (6) $t_{33} \mid \frac{2v_1+1}{t_{22}}(-t_{23} + kv_1t_{22})$.

Note that (5) is equivalent to $t_{22} \in \mathbb{Z}_2^*$, so (2) is redundant. Next, (6) implies (4), and (6) and (1) reduce (3) to the simpler condition $t_{33} \mid kt_{11}$. This makes (1) redundant. To sum up, the conditions defining \mathcal{T} are

$$t_{22} \in \mathbb{Z}_2^*, \quad t_{33} \mid kt_{11}, \quad t_{33} \mid -t_{23} + kv_1t_{22}.$$

Thus,

$$\begin{aligned} \zeta_{G_2}^{G_2, \leq}(s) &= \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-2} |t_{33}|_2^{s-4} d\mu \\ &= \frac{1}{(1 - 2^{-1})^2} \int_{t_{33} \mid kt_{11}} |t_{11}|_2^{s-2} |t_{33}|_2^{s-3} d\mu = A_{2,k}(s - 2, s - 1), \end{aligned}$$

where we used Proposition 3.6 with the pivots t_{22}, t_{23} .

6.2.2 Local factors of $\zeta_G^{N, \triangleleft}(s)$

If $p \neq 2$, then $\zeta_{G_p}^{N, \triangleleft}(s)$ was given in Table 2 (the case $F = D_1$ and $F \notin \text{SL}(T)$).

Assume now that $p = 2$. By Corollary 3.3 and Lemma 6.1 (2),

$$\zeta_{G_2}^{N_2, \triangleleft}(s) = \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_2)$ is defined by $t_{33} \mid 2kt_{11}, t_{33} \mid 2kt_{12}, t_{33} \mid 2kt_{22}$ and the following equivalent conditions:

$$\begin{aligned} & \mathbf{x}^{t_1} \cdot \beta(\mathbf{x}^{t_1}), [\beta, \mathbf{x}^{t_2}], \mathbf{x}^{t_3} \cdot \beta(\mathbf{x}^{t_3}) \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ & \iff x_2^{2t_{12}} x_3^{-kt_{11}-2kt_{11}t_{12}}, x_3^{2t_{23}} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}. \end{aligned}$$

Therefore, by Lemma 6.1 (3), the conditions defining \mathcal{T} are equivalent to

- (1) $t_{33} \mid 2kt_{11}$,
- (2) $t_{33} \mid 2kt_{12}$,
- (3) $t_{33} \mid 2kt_{22}$,
- (4) $t_{22} \mid 2t_{12}$,
- (5) $t_{33} \mid \frac{2t_{12}}{t_{22}}t_{23} + kt_{11}$,
- (6) $t_{33} \mid 2t_{23}$.

We express \mathcal{T} as a disjoint union $\mathcal{T}^{a,a} \cup \mathcal{T}^{a,b} \cup \mathcal{T}^b$ according to the following three cases:

- (1a, 4a) $t_{33} \mid kt_{11}$ and $t_{22} \mid t_{12}$,
- (1a, 4b) $t_{33} \mid kt_{11}$ and $|t_{22}|_2 = |2t_{12}|_2$,
- (1b) $|t_{33}|_2 = |2kt_{11}|_2$.

In the first case, conditions (2) and (5) are redundant, and hence the conditions defining $\mathcal{T}^{a,a}$ are $t_{33} \mid kt_{11}, t_{33} \mid 2kt_{22}, t_{22} \mid t_{12}$ and $t_{33} \mid 2t_{23}$, and the integral over $\mathcal{T}^{a,a}$ becomes

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^{a,a}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ & = \frac{1}{(1-2^{-1})^3} \int_{\substack{t_{33} \mid kt_{11} \\ t_{33} \mid 2kt_{22} \\ t_{33} \mid 2t_{23}}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-1} |t_{33}|_2^{s-3} d\mu = E_{2,k}(s, s, s-2). \end{aligned}$$

In case (1a, 4b), condition (5) can be replaced by $t_{33} \mid t_{23}$, and hence the conditions defining $\mathcal{T}^{a,b}$ are $t_{33} \mid kt_{11}, t_{33} \mid 2kt_{12}, |t_{22}|_2 = |2t_{12}|_2, t_{33} \mid t_{23}$. The integral over $\mathcal{T}^{a,b}$ becomes

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^{a,b}} |t_{11}|_2^{s-1} |2t_{12}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ & = \frac{2^{-s}}{(1-2^{-1})^3} \int_{\substack{t_{33} \mid kt_{11} \\ t_{33} \mid 2kt_{12}}} |t_{11}|_2^{s-1} |t_{12}|_2^{s-1} |t_{33}|_2^{s-2} d\mu \\ & = 2^{-s} D_{2,k}(s, s, s-1), \end{aligned}$$

where in the second equality we used Proposition 3.6 with the pivots t_{23}, t_{22} .

We now assume (1b). Since in this case $t_{33} \nmid kt_{11}$, conditions (5) and (6) imply that $|t_{22}|_2 = |2t_{12}|_2$. This, (1b) and (5) imply that $|t_{23}|_2 = |kt_{11}|_2$. It is now easy to see that \mathcal{T}^b is defined by the conditions $|t_{33}|_2 = |2kt_{11}|_2$, $|t_{22}|_2 = |2t_{12}|_2$, $t_{33} \mid 2kt_{12}$, $|t_{23}|_2 = |kt_{11}|_2$, or equivalently, $|t_{33}|_2 = |2kt_{11}|_2$, $|t_{22}|_2 = |2t_{12}|_2$, $t_{11} \mid t_{12}$, $|t_{23}|_2 = |kt_{11}|_2$. Thus, the integral over \mathcal{T}^b becomes

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^b} |t_{11}|_2^{s-1} |2t_{12}|_2^{s-2} |2kt_{11}|_2^{s-3} d\mu \\ &= \frac{2^{1-2s} |k|_2^{s-1}}{(1-2^{-1})^2} \int_{t_{11}|t_{12}} |t_{11}|_2^s |t_{12}|_2^{s-1} |t_{11}|_2^{s-1} d\mu \\ &= 2^{1-2s} |k|_2^{s-1} A_{2,1}(2s-1, s), \end{aligned}$$

where in the first equality we applied Proposition 3.6 with the pivots t_{22}, t_{23}, t_{33} .

It follows from Table 3 that

$$\begin{aligned} \zeta_{G_2}^{N_2, \triangleleft}(s) &= \zeta_2(s)^2 (\zeta_2(s-1)(1+3 \cdot 2^{-s})(1-|k|_2^{s-1}) \\ &\quad + \zeta_2(3s-1)(1+2^{-s}+3 \cdot 2^{1-2s}-2^{1-3s}-2^{1-4s})|k|_2^{s-1}). \end{aligned}$$

6.2.3 Local factors of $\zeta_G^{G, \triangleleft}(s)$

If $p \neq 2$, then $\gamma_3(G_p, N_p) = \langle x_1^4, x_3^4 \rangle = \overline{\langle x_1, x_3 \rangle}$, whence $N_p/\gamma_3(G_p, N_p) \cong \mathbb{Z}_p$. Thus, by Proposition 2.12,

$$\zeta_{G_p}^{G_p, \triangleleft}(s) = \zeta_p(s).$$

For the computation of $\zeta_{G_2}^{G_2, \triangleleft}(s)$, we consider the cases k even or odd separately.

Assume that k is even. We claim that if $A \triangleleft G_2$ is open and satisfies $AN_2 = G_2$, then A includes $\langle x_1^2, x_2, x_3^2 \rangle$. In fact, any such A contains $\beta \mathbf{x}^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}_2^3$, whence A contains

$$\begin{aligned} [\beta \mathbf{x}^{\mathbf{v}}, x_3] &= [\beta, x_3] = x_3^{-2}, \\ [\beta \mathbf{x}^{\mathbf{v}}, x_1] &= \beta [\mathbf{x}^{\mathbf{v}}, x_1][\beta, x_1] = x_1^{-2} x_3^{-2kv_2} x_3^{-k}, \\ (\beta \mathbf{x}^{\mathbf{v}})^2 &= x_2^{2v_2+1} x_3^{(2v_2+1)v_1k}. \end{aligned}$$

From the first two contentions, we deduce that $x_1^{-2} \in A$ since k is even. From the first and third contentions and the fact that k is even, we also deduce that $x_2^{2v_2+1} \in A$, and hence $x_2 \in A$ since $2v_2+1 \in \mathbb{Z}_2^*$. This proves the claim. It follows that $\zeta_{G_2}^{G_2, \triangleleft}(s) = \zeta_{G'}^{G', \triangleleft}(s)$, where G' is the quotient of G_2 by $\langle x_1^2, x_2, x_3^2 \rangle$, and $\zeta_{G'}^{G', \triangleleft}(s)$ is computed with respect to N' , the image of N_2 at G' . Note that $G' := (\mathbb{Z}/2\mathbb{Z})^3$, the classes of β, x_1 and x_3 being identified with $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. The series $\zeta_{G'}^{G'}(s)$ enumerates the subgroups of G' that are not included in N' . An easy calculation gives $\zeta_{G'}^{G', \triangleleft}(s) = 1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s}$.

Assume now that k is odd. With a similar analysis to that in the previous case, we find that if $A \triangleleft G_2$ is open and satisfies $AN_2 = G_2$, then A includes

$$\overline{\langle x_1^2 x_3, x_2^2, x_3^2 \rangle}.$$

It follows that $\zeta_{G_2}^{G_2, \triangleleft} = \zeta_{G'}^{G', \triangleleft}(s)$, where G' is the quotient of G_2 by $\overline{\langle x_1^2 x_3, x_2^2, x_3^2 \rangle}$, and $\zeta_{G'}^{G', \triangleleft}(s)$ is computed with respect to N' , the image of N at G' . Note that $G' \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, where the classes of β and x_1 are identified with $(1, 0)$ and $(0, 1)$, respectively. Thus, the series $\zeta_{G'}^{G', \triangleleft}(s)$ enumerates the subgroups of $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ that are not included in $2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. A simple inspection shows that the only such subgroups are G' , $\langle(1, 0)\rangle$, $\langle(0, 2)\rangle$, $\langle(1, 1)\rangle$, $\langle(0, 2)\rangle$, $\langle(1, 0)\rangle$, $\langle(1, 2)\rangle$, $\langle(1, 3)\rangle$ and $\langle(1, 4)\rangle$; thus $\zeta_{G'}^{G', \triangleleft}(s) = 1 + 2 \cdot 2^{-s} + 4 \cdot 4^{-s}$.

6.3 Computing the zeta functions of $G_{p3,k,\epsilon}$, $G_{p4,2k,\epsilon}$, $G_{p6,2k,\epsilon}$

The general form of these groups is

$$G_{q,\delta k,\epsilon} = \langle \gamma, x_1, x_2, x_3 : [x_2, x_1] = x_3^{\delta k}, [x_1, x_3] = [x_2, x_3] = 1, \gamma^q = x_3^\epsilon, \gamma x_1 = x_2, \gamma x_2 = x_1^{-1} x_2^{-r} \rangle,$$

where $k \in \mathbb{N}$, $(d, \delta, r) \in \{(3, 1, 1), (4, 2, 0), (6, 2, -1)\}$ and

$$\epsilon = \begin{cases} 1 \text{ or } -1 & \text{if } k \equiv 0 \pmod{\frac{q}{\delta}}, \\ -1 & \text{if } k \equiv 1 \pmod{\frac{q}{\delta}}, \\ 1 & \text{otherwise.} \end{cases}$$

We let G be one of these groups.

6.3.1 Local factors of $\zeta_G^{G, \leq}(s)$

If $p \nmid d$, then $\zeta_{G_p}^{G_p, \leq}(s)$ was given in Table 1 (the case $F = C_d$; note that if $p \nmid 6$, then $\chi_6(p) = \chi_3(p)$).

Assume now $p \mid d$. We claim that if $A \leq G_p$ is open and satisfies $AN_p = G_p$, then $Z_p \subset A_p$. Indeed, A contains $\gamma \mathbf{x}^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}_p^3$, whence

$$(\gamma \mathbf{x}^{\mathbf{v}})^d = x_3^{d v_3 + \epsilon - (r^2 - r + 1)\delta^2 k \frac{(v_1 + v_2)(v_1 + v_2 + r)}{2}} \in A.$$

The claim will follow from the assertion that

$$d v_3 + \epsilon - (r^2 - r + 1)\delta^2 k \frac{(v_1 + v_2)(v_1 + v_2 + r)}{2} \in \mathbb{Z}_p^*.$$

If $d = 4$, then $p = 2$ and $\delta = 2$, and the assertion is clear. The same argument works for $d = 6$ and $p = 2$. Assume that $d = 6$ and $p = 3$. Since $r = -1$, we have $r^2 - r + 1 = 3$; hence the assertion is true in this case too. We finally assume that $d = 3$ and $p = 3$. In this case, $r = 1$, whence $r^2 - r + 1 = 1$. Therefore, the assertion is that $\epsilon - k \frac{(v_1+v_2)(v_1+v_2+1)}{2} \in \mathbb{Z}_3^*$. If one of the elements $v_1 + v_2$, $v_1 + v_2 + 1$ is divisible by 3, the assertion is clear. Otherwise, necessarily,

$$\frac{(v_1 + v_2)(v_1 + v_2 + 1)}{2} \equiv 1 \pmod{3},$$

so the assertion in this case is equivalent to $\epsilon - k \in \mathbb{Z}_3^*$, and this follows from the definition of ϵ in terms of k . This completes the proof of the claim. It follows that $\zeta_{G_p}^{G_p, \leq}(s) = \zeta_{E_p}^{E_p, \leq}(s)$. According to [13, Sections 5.10, 5.13, 5.16], the latter series is equal to $\zeta_p(s - 1)L_p(s - 1, \chi_d)$ if $d \in \{3, 4\}$ and to $\zeta_p(s - 1)L_p(s - 1, \chi_3)$ if $d = 6$.

6.3.2 Local factors of $\zeta_G^{N, \triangleleft}(s)$

If $p \nmid d$, then $\zeta_{G_p}^{N, \triangleleft}(s)$ was given in Table 2 (the case $F = C_d$).

Assume now that $p \mid d$. By Corollary 3.3 and Lemma 6.1 (2), we have

$$\zeta_{G_p}^{N, \triangleleft}(s) = \frac{1}{(1 - p^{-1})^3} \int_{\mathcal{T}} |t_{11}|_p^{s-1} |t_{22}|_p^{s-2} |t_{33}|_p^{s-3} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_p)$ is defined by $t_{33} \mid \delta k t_{11}$, $t_{33} \mid \delta k t_{12}$, $t_{33} \mid \delta k t_{22}$ and the following equivalent conditions:

$$\begin{aligned} \gamma(\mathbf{x}^{t_1}), \gamma(\mathbf{x}^{t_2}), \gamma(\mathbf{x}^{t_3}) &\in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle} \\ \iff x_1^{-t_{12}} x_2^{t_{11} - r t_{12}} x_3^{-k \delta t_{11} t_{12} + \binom{t_{12}}{2} \delta k r + t_{13}}, \\ &x_1^{-t_{22}} x_2^{-r t_{22}} x_3^{\binom{t_{22}}{2} \delta k r + t_{23}} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}. \end{aligned}$$

By Lemma 6.1 (3), the conditions defining \mathcal{T} are equivalent to

- (1) $t_{33} \mid \delta k t_{11}, t_{33} \mid \delta k t_{12}, t_{33} \mid \delta k t_{22}$,
- (2) $t_{11} \mid t_{12}$,
- (3) $t_{22} \mid \frac{t_{12}^2}{t_{11}} + t_{11} - r t_{12}$,
- (4) $t_{33} \mid \frac{t_{12}}{t_{11}} t_{13} - \frac{(t_{12}^2/t_{11}) + t_{11} - r t_{12}}{t_{22}} t_{23} + \binom{t_{12}}{2} \delta k r + t_{13}$,
- (5) $t_{11} \mid -t_{22}$,
- (6) $t_{22} \mid \frac{t_{22} t_{12}}{t_{11}} - r t_{22}$,
- (7) $t_{33} \mid \frac{t_{22}}{t_{11}} t_{13} - \left(\frac{t_{12}}{t_{11}} - r\right) t_{23} + \binom{t_{22}}{2} \delta k r + t_{23}$.

Condition (3) can be written as $\frac{t_{22}}{t_{11}} \mid \left(\frac{t_{12}}{t_{11}}\right)^2 - r\frac{t_{12}}{t_{11}} + 1$. Condition (6) is implied by (2), and the second and third conditions in (1) are implied by the first one, (2) and (5). Note that (4) and (7) have the form

$$(4) \quad t_{33} \mid P_1 t_{13} + Q_1 t_{23} + \binom{t_{12}}{2} \delta k r,$$

$$(7) \quad t_{33} \mid P_2 t_{13} + Q_2 t_{23} + \binom{t_{22}}{2} \delta k r,$$

where P_1, P_2, Q_1 and Q_2 are rational functions that do not involve t_{13} or t_{23} and take values in \mathbb{Z}_p when restricted to \mathcal{T} . Note also that

$$\begin{aligned} \det \begin{pmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{pmatrix} &= -\frac{t_{12} + t_{11}}{t_{11}} \left(\frac{t_{12}}{t_{11}} - r - 1 \right) + \frac{t_{22} \frac{t_{12}^2}{t_{11}} + t_{11} - r t_{12}}{t_{22}} \\ &= -\left(\frac{t_{12}}{t_{11}} \right)^2 - \frac{t_{12}}{t_{11}} + r \frac{t_{12}}{t_{11}} + r + \frac{t_{12}}{t_{11}} \\ &\quad + 1 + \left(\frac{t_{12}}{t_{11}} \right)^2 + 1 - r \frac{t_{12}}{t_{11}} = 2 + r. \end{aligned} \tag{6.1}$$

We first assume that $(d, p) \in \{(3, 3), (4, 2), (6, 3)\}$. The terms $\binom{t_{12}}{2} \delta k r$ and $\binom{t_{22}}{2} \delta k r$ in conditions (4) and (7) can be deleted. In fact, they are already zero when $d = 4$ since in this case $r = 0$, and if $d \in \{3, 6\}$ and $p = 3$, then (1) implies that these two terms are divisible by t_{33} . Now, one easily checks that the equation $T^2 - rT + 1 \equiv 0 \pmod p$ has $1 - r - r^2$ as unique solution modulo p in \mathbb{Z}_p , and that the equation $T^2 - rT + 1 \equiv 0 \pmod{p^2}$ has no solutions in \mathbb{Z}_p . Therefore, by (3), we can express \mathcal{T} as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases

$$(3a) \quad |t_{11}|_p = |t_{22}|_p \text{ and}$$

$$(3b) \quad |p t_{11}|_p = |t_{22}|_p \text{ and } p t_{11} \mid t_{12} - (1 - r - r^2) t_{11}.$$

We describe \mathcal{T}^a . By (3a), the coefficient P_2 in (7) is a unit; therefore, by (6.1), we can replace (4) by $t_{33} \mid (2 + r)t_{23}$. We conclude that \mathcal{T}^a is defined by the conditions $t_{33} \mid \delta k t_{22}, |t_{11}|_{p_0} = |t_{22}|_{p_0}, t_{22} \mid t_{12}, t_{33} \mid (2 + r)t_{23}$ and (7), and the integral over \mathcal{T}^a becomes

$$\begin{aligned} &\frac{1}{(1 - p^{-1})^3} \int_{\mathcal{T}^a} |t_{22}|_p^{2s-3} |t_{33}|_p^{s-3} d\mu \\ &= \frac{1}{(1 - p^{-1})^2} \int_{\substack{t_{33} \mid \delta k t_{22} \\ t_{33} \mid (2+r)t_{23}}} |t_{22}|_p^{2s-1} |t_{33}|_p^{s-2} d\mu \\ &= \zeta_p(2s) + \frac{1}{(1 - p^{-1})^2} \int_{\substack{p \mid t_{33} \\ t_{33} \mid \delta k t_{22} \\ \frac{t_{33}}{(2+r)} \mid t_{23}}} |t_{22}|_p^{2s-1} |t_{33}|_p^{s-2} d\mu \end{aligned}$$

$$= \zeta_p(2s) + \frac{p^{v_p(r+2)}}{(1-p^{-1})^2} \int_{p|t_{33}|\delta k t_{22}} |t_{22}|_p^{2s-1} |t_{33}|_p^{s-1} d\mu,$$

where in the first equality we used Proposition 3.6 with the pivots t_{12}, t_{11}, t_{13} , and in the third one we used this proposition with the pivot t_{23} .

We now describe \mathcal{T}^b . Condition (3b) implies that $|t_{11}|_p = |t_{12}|_p$. This makes condition (2) redundant. Next, note that

$$Q_1 = -\frac{\left(\frac{t_{12}}{t_{11}}\right)^2 + 1 - r \frac{t_{12}}{t_{11}}}{\frac{t_{22}}{t_{11}}}$$

is a unit of \mathbb{Z}_p since we are assuming that $p \mid \frac{t_{22}}{t_{11}}$ and since $T^2 - rT + 1 \equiv 0 \pmod{p^2}$ has no solutions in \mathbb{Z}_p . Therefore, by (6.1), condition (7) can be replaced by the condition $t_{33} \mid (2+r)t_{13}$. We conclude that the conditions defining \mathcal{T}^b are $t_{33} \mid \delta k t_{11}, |p t_{11}|_p = |t_{22}|_p, p t_{11} \mid t_{12} - (1-r-r^2)t_{11}, t_{33} \mid (2+r)t_{13}$ and (4). The integral over \mathcal{T}^b becomes

$$\begin{aligned} & \frac{1}{(1-p^{-1})^3} \int_{\mathcal{T}^b} |t_{11}|_p^{s-1} |p t_{11}|_p^{s-2} |t_{33}|_p^{s-3} d\mu \\ &= \frac{1}{(1-p^{-1})^2} p^{-s} \int_{\substack{t_{33}|\delta k t_{11}, \\ t_{33}|(2+r)t_{13}}} |t_{11}|_p^{2s-1} |t_{33}|_p^{s-2} d\mu, \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots t_{22}, t_{12}, t_{23} .

We conclude that

$$\begin{aligned} \zeta_{G_p}^{N_{p,\triangleleft}(s)} &= \frac{1+p^{-s}}{(1-p^{-1})^2} \int_{\substack{t_{33}|\delta k t_{11}, \\ t_{33}|(2+r)t_{13}}} |t_{11}|_p^{2s-1} |t_{33}|_p^{s-2} d\mu \\ &= \zeta_p(s) + \frac{1+p^{-s}}{(1-p^{-1})^2} \int_{\substack{p|t_{33} \\ t_{33}|\delta k t_{22} \\ \frac{t_{33}}{(2+r)}|t_{23}}} |t_{22}|_p^{2s-1} |t_{33}|_p^{s-2} d\mu \\ &= \zeta_p(s) + |2+r|_p^{-1} \frac{1+p^{-s}}{(1-p^{-1})^2} \int_{p|t_{33}|\delta k t_{22}} |t_{22}|_p^{2s-1} |t_{33}|_p^{s-1} d\mu \\ &= \zeta_p(s) + |2+r|_p^{-1} (1+p^{-s}) B_{p,\delta k}(s, 2s). \end{aligned}$$

We now assume that $(d, p) = (6, 2)$. The equation $T^2 - rT + 1 \equiv 0 \pmod{2}$ has no solutions in \mathbb{Z}_2 , and hence (3) and (5) can be replaced by $|t_{11}|_2 = |t_{22}|_2$. It follows that the coefficient P_2 in condition (7) is a unit. Since the value of the determinant (6.1) is $2+r = -1$, condition (4) can be replaced by one of the form $t_{33} \mid t_{23} + R_1$, where R_1 is a rational function that does not involve t_{13} or t_{23} and takes values in \mathbb{Z}_2 when restricted to \mathcal{T} . The conditions defining \mathcal{T} are therefore

$t_{33} \mid 2kt_{11}, t_{11} \mid t_{12}, |t_{11}|_2 = |t_{22}|_2, t_{33} \mid t_{23} + R_1$ and (7). The integral over \mathcal{T} becomes

$$\begin{aligned} \zeta_{G_2}^{N_2, \triangleleft}(s) &= \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_2^{2s-3} |t_{33}|_2^{s-3} d\mu \\ &= \frac{1}{(1 - 2^{-1})^2} \int_{t_{33} \mid 2kt_{11}} |t_{11}|_2^{2s-1} |t_{33}|_2^{s-1} d\mu = A_{2,2k}(s, 2s), \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots $t_{12}, t_{22}, t_{23}, t_{13}$.

6.3.3 Local factors of $\zeta_G^{G, \triangleleft}(s)$

If $p \nmid d$, then

$$\gamma_3(G_p, N_p) = \overline{\langle x_1^{r+2}, x_2^{r+2}, x_3^{\delta k} \rangle} = \overline{\langle x_1, x_2, x_3^k \rangle},$$

where the last equality holds since $r + 2 \in \mathbb{Z}_p^*$. Thus, $N_p/\gamma_3(G_p, N_p) \cong \mathbb{Z}_p/k\mathbb{Z}_p$, and by Proposition 2.12, we have

$$\zeta_{G_p}^{G_p, \triangleleft}(s) = \frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}}.$$

Assume now that $p \mid d$. Arguing as in the calculation of $\zeta_{G_p}^{G_p, \leq}(s)$, we obtain that $\zeta_{G_p}^{G_p, \triangleleft}(s) = \zeta_{E_p}^{E_p, \triangleleft}(s)$. According to the results of [13, Sections 6.10, 6.13, 6.16], this series is equal to $1 + p \cdot p^{-s}$ if $d = 3$ or 4 , and is 1 if $d = 6$.

6.3.4 Local factors of $\zeta_G^{H, \triangleleft}(s)$ for $d = 4$ and $H = \langle \gamma^2, x_1, x_2, x_3 \rangle$

Note that $H = G_{p^2, 2k}$. If $p \neq 2$, then

$$\gamma_3(H_p, N_p) = \overline{\langle x_1^4, x_2^4, x_3^{4k} \rangle} = \overline{\langle x_1, x_2, x_3^k \rangle},$$

and hence $N_p/\gamma_3(H_p, N_p) \cong \mathbb{Z}_p/k\mathbb{Z}_p$. By Proposition 2.12, we obtain

$$\zeta_{G_p}^{H_p, \triangleleft}(s) = \frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}}.$$

Assume now that $p = 2$. Arguing as in the calculation of $\zeta_{G_{p^2, 2k}}^{G_{p^2, 2k}, \triangleleft}(s)$ in Section 6.1.3, we obtain that

$$\zeta_{G_2}^{H_2, \triangleleft}(s) = \zeta_{E_2}^{H_2/Z_2, \triangleleft}(s),$$

which is equal to $1 + 2 \cdot 2^{-s} + 2 \cdot 4^{-s}$ according to [13, Section 6.10].

6.3.5 Local factors of $\zeta_G^{H, \triangleleft}(s)$ for $d = 6$ and $H := \langle \gamma^3, x_1, x_2, x_3 \rangle$

Note that $H = G_{\mathbf{p}2,2k}$. Arguing as in the previous paragraph, we obtain that

$$\zeta_{G_p}^{H_p, \triangleleft}(s) = \frac{1 - p^{-s} |k|_p^s}{1 - p^{-s}}$$

if $p \neq 2$, and that $\zeta_{G_2}^{H_2, \triangleleft}(s) = \zeta_{E_2}^{H_2/Z_2, \triangleleft}(s)$. The latter is equal to $1 + 4^{-s}$ according to the results of [13, Section 6.16].

6.3.6 Local factors of $\zeta_G^{K, \triangleleft}(s)$ for $d = 6$ and $K = \langle \gamma^2, x_1, x_2, x_3 \rangle$

Note that $K = G_{\mathbf{p}3,2k,\eta}$, where $\eta = \epsilon$ if $3 \mid k$, and $\eta = -\epsilon$ if $3 \nmid k$. If $p \neq 3$, then arguing as in Section 6.3.3 with $G = G_{\mathbf{p}3,2k,\eta}$, we find that

$$\zeta_{G_p}^{K_p, \triangleleft}(s) = \frac{1 - p^{-s} |2k|_p^s}{1 - p^{-s}} \quad \text{and} \quad \zeta_{G_3}^{K_3, \triangleleft}(s) = \zeta_{E_3}^{K_3/Z_3, \triangleleft}(s),$$

which is equal to $1 + 3^{-s}$ according to the results of [13, Section 6.16].

6.4 Computing the zeta functions of $G = G_{\mathbf{p}2\mathbf{g}\mathbf{g},4k}$

6.4.1 Local factors of $\zeta_G^{G, \leq}(s)$

If $p \neq 2$, then $\zeta_{G_p}^{G_p, \leq}(s)$ was given in Table 1 (the case $F = D_2$).

Assume now that $p = 2$. Note that $\langle \alpha, x_1, x_2, x_3 \rangle = G_{\mathbf{p}2,4k}$. Arguing as in Section 6.1.1 for $G = G_{\mathbf{p}2,4k}$, we find that

$$\zeta_{G_2}^{G_2, \leq}(s) = \zeta_{E_2}^{E_2, \leq}(s).$$

The latter is equal to 1 according to the results of [13, Section 5.7].

6.4.2 Local factors of $\zeta_G^{N, \triangleleft}(s)$

If $p \neq 2$, then $\zeta_{G_p}^{N_p, \triangleleft}(s)$ was computed in Table 2 (the case $F = D_2$).

Assume now that $p = 2$. By Corollary 3.3 and Lemma 6.1 (2), we have

$$\zeta_{G_2}^{N_2, \triangleleft}(s) = \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}} |t_{11}|_p^{s-1} |t_{22}|_p^{s-2} |t_{33}|_p^{s-3} d\mu,$$

where $\mathcal{T} \subset T_3^+(\mathbb{Z}_2)$ is defined by $t_{33} \mid 4k t_{11}, t_{33} \mid 4k t_{12}, t_{33} \mid 4k t_{22}$ and the following equivalent conditions:

$$\begin{aligned} & \alpha(\mathbf{x}^{t_i}) \cdot \mathbf{x}^{t_i}, [\mathbf{x}^{t_i}, \beta] \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}, \quad i = 1, 2, 3 \\ & \iff x_3^{2t_{13} + 2k(t_{11} - t_{12}) - 4k t_{11} t_{12}}, x_3^{2t_{23} - 4k t_{22}}, \\ & \quad x_2^{-2t_{12}} x_3^{-2t_{13} + 2k t_{12}}, x_3^{2k t_{22}} \in \overline{\langle \mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \mathbf{x}^{t_3} \rangle}. \end{aligned}$$

Therefore, by Lemma 6.1 (3), the conditions defining \mathcal{T} are equivalent to

- (1) $t_{33} \mid 4kt_{11}$,
- (2) $t_{33} \mid 4kt_{12}$,
- (3) $t_{33} \mid 2t_{13} + 2k(t_{11} - t_{12})$,
- (4) $t_{33} \mid 2t_{23}$,
- (5) $t_{22} \mid 2t_{12}$,
- (6) $t_{33} \mid \frac{2t_{12}}{t_{22}}t_{23} - 2t_{13} + 2kt_{12}$,
- (7) $t_{33} \mid 2kt_{22}$.

Note that (2) follows from (5) and (7). Next, we express \mathcal{T} as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the following cases:

- (a) $t_{33} \in \mathbb{Z}_2^*$ and
- (b) $2 \mid t_{33}$.

In case (a), all the conditions but (5) are redundant, so

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^a} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ &= \frac{\zeta_2(s)}{1-2^{-1}} \int_{t_{22}|2t_{12}} |t_{22}|_2^{s-2} d\mu = \zeta_2(s)(1+2^{1-s}\zeta_2(s)). \end{aligned}$$

In case (b), we split \mathcal{T}^b as a union $\mathcal{T}^{b,a} \cup \mathcal{T}^{b,b}$ according to the following cases:

- (5a) $t_{22} \mid t_{12}$ and
- (5b) $|t_{22}|_2 = |2t_{12}|_2$.

Assume (5a). We sum the right-hand sides of (3) and (6) and obtain

$$t_{33} \mid \frac{2t_{12}}{t_{22}}t_{23} + 2kt_{11}.$$

This and (4) imply that $t_{33} \mid 2kt_{11}$. Thus, (1) can be replaced by $t_{33} \mid 2kt_{11}$, and then (3) and (4) imply (6) clearly. To sum up, $\mathcal{T}^{b,a}$ is defined by the conditions $2 \mid t_{33}$, $\frac{t_{33}}{2} \mid kt_{11}$, $\frac{t_{33}}{2} \mid kt_{22}$, $t_{22} \mid t_{12}$, $\frac{t_{33}}{2} \mid t_{13} - kt_{12}$, $\frac{t_{33}}{2} \mid t_{23}$. Therefore,

$$\begin{aligned} & \frac{1}{(1-2^{-1})^3} \int_{\mathcal{T}^{b,a}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ &= \frac{2^{2-s}}{(1-2^{-1})^3} \int_{\substack{u \mid kt_{11} \\ u \mid kt_{22}}} |t_{11}|_2^{s-1} |t_{22}|_2^{s-1} |u|_2^{s-1} d\mu = 2^{2-s} C_{2,k}(s, s, s), \end{aligned}$$

where in the first equality we used Proposition 3.6 with the pivots t_{12}, t_{13}, t_{23} and performed the change of variables $t_{33} = 2u$. Assume now (5b). Observe that (2) and (7) are equivalent. Note also that (6) implies that

$$t_{33} \mid 2\left(\frac{2t_{12}}{t_{22}}\right)t_{23} - 4t_{13} + 4kt_{12},$$

and then that (1), (2) and (3) imply that $t_{33} \mid 4t_{13}$. It follows that (1), (2), (3) and (6) imply (4). Summarizing, the conditions defining $\mathcal{T}^{b,b}$ are $2 \mid t_{33}$, (1), (2), $\frac{t_{33}}{2} \mid t_{13} + 2k(t_{11} - t_{12})$, $|t_{22}|_2 = |2t_{12}|_2$ and (6). Therefore, the integral over $\mathcal{T}^{b,b}$ becomes

$$\begin{aligned} & \frac{1}{(1 - 2^{-1})^3} \int_{\mathcal{T}^{b,b}} |t_{11}|_2^{s-1} |2t_{12}|_2^{s-2} |t_{33}|_2^{s-3} d\mu \\ &= \frac{2^{1-2s}}{(1 - 2^{-1})^3} \int_{\substack{u|2kt_{11} \\ u|2kt_{12}}} |t_{11}|_2^{s-1} |t_{12}|_2^{s-1} |u|_2^{s-1} d\mu \\ &= 2^{1-2s} \left(\zeta_2(s)^2 + \frac{2^{-s}}{(1 - 2^{-1})^3} \int_{\substack{v|kt_{11} \\ v|kt_{12}}} |t_{11}|_2^{s-1} |t_{12}|_2^{s-1} |v|_2^{s-1} d\mu \right) \\ &= 2^{1-2s} (\zeta_2(s)^2 + 2^{-s} C_{2,2k}(s, s, s)). \end{aligned}$$

In the first equality, we used Proposition 3.6 with the pivots t_{22}, t_{13}, t_{23} and applied the change of variables $t_{33} = 2u$.

We conclude that

$$\begin{aligned} \zeta_{G_2}^{N_2, \triangleleft}(s) &= \zeta_2(s)(1 + 2^{1-s}\zeta_2(s)) + 2^{1-2s}\zeta_2(s)^2 \\ &\quad + (2^{2-s} + 2^{1-3s})C_{2,k}(s, s, s). \end{aligned}$$

6.4.3 Local factors of $\zeta_G^{\mathcal{G}, \triangleleft}(s)$

If $p \neq 2$, then one can easily check that $\gamma_3(G_p, N_p) = \overline{\langle x_1, x_2, x_3 \rangle}$, and hence $\zeta_{G_p}^{\mathcal{G}_p, \triangleleft}(s) = 1$. If $p = 2$, then $\zeta_{G_2}^{\mathcal{G}_2, \triangleleft}(s) = 1$ since already $\zeta_{G_2}^{\mathcal{G}_2, \leq}(s) = 1$.

6.4.4 Local factors of $\zeta_G^{H, \triangleleft}(s)$ for $H = \langle \alpha, x_1, x_2, x_3 \rangle$

Note that $H = G_{\mathbf{p}2,4k}$. Arguing as in Section 6.1.3 with $G = G_{\mathbf{p}2,4k}$, we find that

$$\zeta_{G_p}^{H_p, \triangleleft}(s) = \frac{1 - p^{-s}|k|_p^s}{1 - p^{-s}} \quad \text{for } p \neq 2,$$

and that $\zeta_{G_2}^{H_2, \triangleleft}(s) = \zeta_{E_2/Z_2}^{H_2, \triangleleft}(s)$. The latter is equal to $1 + 2 \cdot 2^{-s}$ according to the results of [13, Section 6.7].

6.4.5 Local factors of $\zeta_G^{K, \triangleleft}(s)$ for $K = \langle \beta, x_1, x_2, x_3 \rangle$

Note that $K = G_{\text{pg},4k}$. If $p \neq 2$, then $\gamma_3(H_p, N_p) = \langle x_1^4, x_3^4 \rangle = \overline{\langle x_1, x_3 \rangle}$, and hence $N_p/\gamma_3(G_p, N_p) \cong \mathbb{Z}_p$. Thus, by Proposition 2.12, we have

$$\zeta_{G_p}^{K_p, \triangleleft}(s) = \zeta_p(s).$$

Assume now that $p = 2$. Arguing as in Section 6.2.3 with $G = G_{\text{pg},4k}$, we find that

$$\zeta_{G_2}^{K_2, \triangleleft}(s) = \zeta_{G'}^{K', \triangleleft}(s),$$

where G' is the quotient of G_2 by $\overline{\langle x_1, x_2^2, x_3^2 \rangle}$, K' is the image of K , and the new partial zeta function is computed with respect to N' , the image of N . Note that $K' \cong (\mathbb{Z}/2\mathbb{Z})^3$, the classes of β , x_2 and x_3 being identified with $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. The action of the class of α on K' is given by $(a, b, c) \mapsto (a, a + b, a + c)$. The series $\zeta_{G'}^{K', \triangleleft}(s)$ enumerates the subgroups of G' that are not included in N' and are invariant under $(a, b, c) \mapsto (a, a + b, a + c)$. It is easy to check that $\zeta_{G'}^{K', \triangleleft}(s) = 1 + 2 \cdot 2^{-s}$.

6.4.6 Local factors of $\zeta_G^{L, \triangleleft}(s)$ for $L = \langle \alpha\beta, x_1, x_2, x_3 \rangle$

One can easily check that the assignment $\alpha \mapsto \alpha^{-1}$, $\beta \mapsto \beta\alpha$, $x_1 \mapsto x_2$, $x_2 \mapsto x_1$, $x_3 \mapsto x_3^{-1}$ extends to an automorphism of G mapping K onto L . Therefore,

$$\zeta_G^{L, \triangleleft}(s) = \zeta_G^{K, \triangleleft}(s).$$

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