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SCALAR FIELD ON NON-INTEGER-DIMENSIONAL SPACES

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Deformations of the canonical spectral triples over the n-dimensional torus are considered. These deformations have a discrete dimension spectrum consisting of non-integer values less than n. The differential algebra corresponding to these spectral triples is studied. No junk forms appear for non-vanishing deformation parameter. The action of a scalar field in these spaces is considered, leading to non-trivial extra structure compared to the integer-dimensional cases, which does not involve a loss of covariance. One-loop contributions are computed leading to finite results for non-vanishing deformation.

Keywords: Dimensional regularization; non-commutative geometry; non-integer dimensions.

1. Introduction

The dimension of a space is a basic concept of particular relevance both in nature and in mathematics. Non-commutative geometry [1–4] provides a generalization of classical geometry. In particular, it includes a definition of dimension that allows for complex non-integer values [5]. A motivation for this definition and a series of very interesting examples of geometries with non-integer dimensions has been given in relation to the study of fractal sets in this geometrical setting ([1, 6] and references therein).

The motivation for this work comes from a different subject. In the realm of quantum field theory (QFT), the widely employed dimensional regularization technique (see for example Ref. 7) provides a hint that non-integer-dimensional spaces could be of relevance there. This technique is employed in QFT as a means to regularize divergent integrals appearing in perturbation theory, being preferred in the regularization of gauge theories since it preserves gauge invariance. The technique essentially consists in considering the analytical continuation in the number of dimensions for the area of a d-dimensional sphere, a quantity that appears in the calculation of the above-mentioned integrals. The general question to be addressed in this work is whether a suitable well-defined differential geometry can be found that makes sense for non-integer dimensions and reduces to the canonical one for the integer case.[a] In the affirmative case the natural question to ask is, what does a field theory defined in such a space look like? More precisely, the idea is to take a field theory defined purely in geometrical terms and repeat the construction in the deformed case. The output of that procedure is by no means obvious since, as will be seen in subsequent sections, the differential algebra is qualitatively different between the integer and non-integer cases, and such a change reflects directly in the action of the field theory. For the case of the field theory of a scalar field considered in Sec. 6, the resulting theory is of a novel type. This theory, in spite of reflecting its non-commutative origin, does not involve a breakdown of covariance, as happens in the so-called non-commutative field theories (see for example Ref. 9).

The salient features and results of this work are summarized as follows:

- Spectral triples are considered that differ from the canonical ones only in the choice of the Dirac operator.
- The dimension spectrum of these triples consists of a discrete set of real values less than the dimension of the canonical triple.
- The differential of a zero form is not a multiplicative operator.
- There are no junk forms for a nonzero deformation parameter.
- The action of a scalar field contains derivatives of any order and involves an integration over the co-sphere.
- In spite of the "non-commutativity" of the differential algebra, there is no loss of covariance involved in the field theory mentioned above.
- The calculation of the tadpole diagram and a loop involving two free propagators, show that for nonzero deformation these diagrams give a finite result, showing neither ultraviolet nor infrared singularities. The last singularities being ruled out by the appearance of a mass term whose coefficient vanishes when the deformation parameter goes to zero.
- Ultraviolet power counting and comparison with dimensional regularization, indicate that the perturbation theory obtained from the action mentioned above leads to finite contributions for a nonzero deformation parameter.

This paper is organized as follows. Section 2 describes the spectral triple to be considered. In Sec. 3, the corresponding dimension spectrum is computed. The differential of a 0-form is considered in Sec. 4. Section 5 considers the calculation of the action for a complex scalar field. Section 6 presents the one-loop computations and Sec. 7 contains conclusions and the schematic description of further research motivated by the present work. In addition, two Appendices are included, Appendix A showing the absence of junk forms, and Appendix B, which contains the calculation of the Wodzicki residue involved in the definition of the above-mentioned action.

[a]A preliminary study of this question in the 1-dimensional case appears in [8].

2. The Dirac Operator

The differential algebra derived from the canonical spectral triple involving functions over a manifold M reduces to the usual exterior differential algebra over M. The spectral triples to be considered in this work differ from the canonical ones only in the choice of the Dirac operator. More precisely, the triples $(\mathcal{A}, \mathcal{H}, D_\alpha)$ are considered, where:

- $\mathcal{A}$ is the commutative C^*-algebra of smooth functions over the n-dimensional torus $T^n, n \in \mathbb{N}$.
- $\mathcal{H}$ is the Hilbert space of square integrable sections of a spinor bundle over T^n.
- $D_\alpha : \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear operator to be defined below.

The usual Dirac operator over a n-dimensional torus T^n is given by,

$$D = i\gamma \cdot \partial = i\gamma_\mu \partial_\mu, \quad \gamma_\mu = \gamma_\mu^\dagger, \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \quad \mu, \nu = 1, \ldots, n,$$

this operator is not positive definite. Indeed since,

$$D^2 = -\Delta = -\partial_\mu \partial_\mu$$

denoting by $\lambda \geq 0$ an eigenvalue of D^2, then $\pm\sqrt{\lambda}$ will be eigenvalues of D.

In this work the usual Dirac operator will be replaced by D_α given below. One of the motivations for this choice is to obtain a dimension spectrum with non-integer real values. This could be done in many ways, for example, by choosing,

$$D_a = D|D^2|^{-\frac{(1-a)}{2}}, \quad a \in \mathbb{R}, \quad 1 > a > 0,$$

this operator leads to a dimension spectrum[b] which consists in a single value given by $z = \frac{n}{a}$. However, it is not well-behaved in the infrared. In order to improve its infrared properties and have the same behavior in the ultraviolet, the following operator will be considered in this work,

$$D_\alpha = D(1 + D^2)^{-\alpha}, \quad \alpha > 0,$$

the power appearing in this last equation being defined by,

$$(1 + D^2)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha - 1} e^{-\tau(1+D^2)}. \tag{2.1}$$

Thus the Dirac operator to be considered is,

$$D_\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} D(\tau), \quad D(\tau) = e^{-\tau(1+D^2)} D,$$

this operator is self-adjoint in $\mathcal{H}$, with compact resolvent, and such that the differential of any $a \in \mathcal{A}$ is bounded. This last condition is ensured by the choice $\alpha \geq 0$, as can be readily shown using the expression for the differential of Sec. 4. Therefore, the triple fulfills all the properties required for it to be a spectral triple.

[b] See the next section for the definition of dimension spectrum [5].

3. Dimension Spectrum

The definition of dimension spectrum of a spectral triple is briefly reviewed.

Definition 3.1 (Connes–Moscovici). *Discrete dimension spectrum.* A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has discrete dimension spectrum Sd if $Sd \subset \mathbb{C}$ is discrete and for any element b in the algebra[c] $\mathcal{B}$ the function,

$$\zeta_b^D(z) = \mathrm{Tr}[\pi(b)\,|D|^{-z}] \tag{3.3}$$

extends holomorphically to $\mathbb{C}/Sd$.

The interpretation of these poles is that each of them gives the dimension of a certain piece of the whole space.

In order to apply this definition to the spectral triples considered in this work, it is useful to note that,

$$\begin{aligned}|D_\alpha|^{-z} &= |D|^{-z}(1+|D|^2)^{\alpha z} = |D|^{-z}\sum_{k=0}^{\infty}\binom{\alpha z}{k}|D|^{2(\alpha z-k)}\\ &= \sum_{k=0}^{\infty}\binom{\alpha z}{k}|D|^{2((\alpha-\frac{1}{2})z-k)},\end{aligned} \tag{3.4}$$

where Newton's binomial formula has been employed. From the definitions above it is clear that,

$$\zeta_b^{D_\alpha}(z) = \sum_{k=0}^{\infty}\binom{\alpha z}{k}\zeta_b^D\left(2\left(k-\left(\alpha-\frac{1}{2}\right)z\right)\right), \tag{3.5}$$

where the binomial coefficients are given by,

$$\binom{\alpha z}{k} = \frac{\alpha z(\alpha z-1)\cdots(\alpha z-k+1)}{k!}, \quad \binom{\alpha z}{0} = 1.$$

The zeta functions appearing in the right-hand side of (3.5) are the ones corresponding to the canonical spectral triple. Thus, since for the canonical spectral triples the corresponding zeta functions have a single simple pole at its argument equal to n, then $\zeta_b^{D_\alpha}(z)$ has simple poles at,

$$z = \frac{n-2k}{1-2\alpha}, \quad k = 0, 1, 2, \ldots,$$

these values of z are therefore the dimension spectrum of the spectral triple considered in this work.

[c] The definition of the algebra $\mathcal{B}$ is the following. Let δ denote the derivation $\delta : L(\mathcal{H}) \to L(\mathcal{H})$ defined by,

$$\delta(T) = [|D|, T], \quad T \in L(\mathcal{H}). \tag{3.1}$$

The algebra $\mathcal{B}$ is generated by the elements,

$$\delta^n(\pi(a)), \quad a \subset \mathcal{A}, \quad n \geq 0\ (\delta^0(\pi(a)) = \pi(a)). \tag{3.2}$$

4. The Differential

The differential of a 0-form f is given by,

$$df = [D_\alpha, f] = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} df(\tau), \tag{4.1}$$

$$df(\tau) = [D(\tau), f] \cdot D(\tau) = U(\tau)D, \quad U(\tau) = e^{-\tau(1+D^2)}, \tag{4.2}$$

thus when applied to an element ϕ of $\mathcal{H}$, $df(\tau)$ is given by,

$$\begin{aligned} df(\tau)\phi &= [D(\tau), f]\,\phi = U(\tau)[(Df)\phi + fD\phi] - fU(\tau)D\phi \\ &= [U(\tau)(Df) + [U(\tau)f - fU(\tau)]D]\phi \\ &= U(\tau)[(Df) + [f - U(-\tau)fU(\tau)]D]\phi, \end{aligned} \tag{4.3}$$

the second term in the parenthesis of the right-hand side can be expressed as:

$$e^{\tau(1+D^2)} f(x) e^{-\tau(1+D^2)} = f(x - 2\tau\partial), \tag{4.4}$$

this can be easily derived using an analogy with quantum mechanics. This is done noting that $e^{-\tau(1+D^2)}$ is, up to a constant, the imaginary time evolution operator for a free particle of mass $m = 1/2$. Thus,

$$df(\tau) = U(\tau)[(Df) - [f(x) - f(x - 2\tau\partial)]D],$$

integrating the second line in (4.3) as in (4.1) leads to,

$$df = (1 + D^2)^{-\alpha}(Df) + [(1 + D^2)^{-\alpha} f - f(1 + D^2)^{-\alpha}] i\gamma \cdot \partial,$$

which clearly shows that when $\alpha \to 0$, $df \to i\gamma \cdot \partial f$, which is the corresponding expression in the canonical case. It is worth remarking that, as the last equations indicate, this differential is a non-multiplicative operator for any value of $\alpha \neq 0$. As Appendix A shows, this fact plays an important role in showing the absence of junk forms.

5. The Scalar Field

In this section the part of this space corresponding to the highest pole will be considered, i.e. for $d = \frac{n}{1-2\alpha}$. The action for a free scalar field propagating in this space is taken to be,

$$S = \frac{1}{2}\langle d\phi, d\phi \rangle,$$

where ϕ is a 0-form and the norm in the space forms is given by,[d]

$$\langle \omega, \omega \rangle = \mathrm{tr}_\omega[\omega\omega^\dagger |D_\alpha|^{-d}], \tag{5.1}$$

thus,

$$S = -\mathrm{tr}_\omega[d\phi d\phi^* |D_\alpha|^{-d}],$$

[d]See for example Ref. 3.

where it was used that $d\phi^\dagger = -d\phi^*$ and tr_ω denotes the Diximier trace. In the evaluation of this trace it is important to note that replacing $d = \frac{n}{1-2\alpha}$ in (3.4) leads to,

$$|D_\alpha|^{-d} = |D_\alpha|^{-\frac{n}{1-2\alpha}} = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{\alpha n}{1-2\alpha} \\ k \end{pmatrix} |D|^{-n-2k}. \tag{5.2}$$

Therefore S is given by,

$$S = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{\alpha n}{1-2\alpha} \\ k \end{pmatrix} S_k,$$

$$S_k = -\mathrm{tr}_\omega[d\phi(\tau) d\phi(\tau')^* |D|^{-n-2k}].$$

Noting that,

$$d\phi = [U_\alpha D, \phi(x)], \quad U_\alpha = (1 + D^2)^{-\alpha},$$

$$d\phi^* = [U_\alpha D, \phi^*(x)]$$

leads to,

$$S_k = \mathrm{tr}_\omega\{[U_\alpha D, \phi][D U_\alpha, \phi^*]|D|^{-n-2k}\}.$$

Thus replacing the expression obtained in Appendix B for S_k leads to,

$$S = -\frac{2^{[\frac{n}{2}]} V_{S^{n-1}}}{n(2\pi)^n} \int_{T^n} \phi \left(D^2 + \frac{\alpha n}{1-2\alpha} \right) (1 + D^2)^{-2\alpha} \phi^*, \tag{5.3}$$

where $V_{s^{n-1}} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the $n-1$-dimensional sphere. It is worth noting that in spite of starting with an action involving no mass term, the fact of working on a non-integer-dimensional space effectively generates such a term as shown by (5.3), with a coefficient that vanishes in the integer case ($\alpha = 0$). In that case (5.3) reduces to the usual action of a massless complex scalar field, i.e.

$$S_{\mathrm{can}} = \lim_{\alpha \to 0} S = \frac{2^{[\frac{n}{2}]} V_{s^{n-1}}}{(2\pi)^n} \int_{T^n} \left(\frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi^*(x) \right).$$

6. One Loop Calculations

As mentioned in the Introduction, the dimensional regularization technique is a widely employed tool used to make sense of divergences in perturbative QFT. These divergences appear when calculating the contribution of Feynman diagrams involving closed loops. Having obtained a field theory in a non-integer-dimensional space, it is natural to perform the same calculations and see whether the analogous diagrams are divergent or not. This is the purpose of this section. Two simple diagrams are considered: The tadpole diagram and a loop involving two free propagators. In both cases the calculations below show that the corresponding diagrams give a finite result for $\alpha \neq 0$. The comparison of the results for these diagrams with dimensionally regularized ones, show that the location of poles in the complex α-plane coincides with the one obtained in dimensional regularization. A simple argument showing that this should be so is obtained by considering the ultraviolet behavior of the integrals involved, as shown in Subsec. 6.1. In spite of these similarities, the dimensionally regularized result and the ones in this non-integer dimensional space are different. Another important point is that the appearance of the mass term in (5.3) automatically regulates possible infrared divergences.

In order to compare with results of standard-dimensional regularization calculations, it is convenient to restore physical units in our calculations. Unlike common use in physics, where coordinates are assumed to have dimensions of length, up to this point in this work coordinates and fields have been taken to be dimensionless. Physical units, in the natural system of units where action and speed are measured in units of the Planck constant $\hbar$ and the velocity of light c, are restored by,

$$x_P = \frac{x}{M}, \quad \phi_P = M^{\frac{n-2}{2}}\phi,$$

where x_P and ϕ_P denote the dimensionful quantities and M is a mass scale. This can be derived recalling the basic requirement that the action should be dimensionless in natural units. To show how this works it is noted for example that,

$$\frac{\partial}{\partial x_P} = \frac{1}{M}\frac{\partial}{\partial x} \Rightarrow (1 + D^2) = \frac{1}{M^2}(M^2 + D_P^2).$$

In the subsections below it should be understood that the quantities involved are dimensionful, although the subindices P will not be explicitly written.

6.1. *The tadpole*

According to (5.3) the propagator corresponding to that action is,[e] in terms of dimensionful quantities,[f]

$$D(x-y) = \int d^n p \frac{1}{(p^2+m^2)(M^2+p^2)^{-\alpha}} e^{-ip\cdot(x-y)},$$

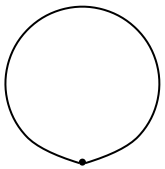
where,

$$m^2 = M^2 \frac{\alpha n}{1-2\alpha}.$$

[e] In the following expressions the discrete summation over the allowable momenta is replaced by an integral, this is justified in the limit where all the radius of the n-dimensional torus tend to infinity.

[f] In this work $p^2 > 0$ indicates the Euclidean positive norm squared of the vector p, this is different from the usual notation in field theory where after Wick rotating the Euclidean momentum is considered with $p_E^2 < 0$.

The tadpole diagram is,



it corresponds to the following integral,

$$I_\alpha^T(m) = \int d^n p \frac{1}{(p^2+m^2)(M^2+p^2)^{-\alpha}}. \tag{6.1}$$

It will be shown below that this integral converges for $\alpha \neq 0$ and $|\alpha| < 1$. A simple argument showing that this should be so, can be given comparing the ultraviolet behavior of $I_\alpha^T(m)$ and of the corresponding dimensionally regularized integral $I_d^T(m)$ given by,

$$I_d^T(m) = \int d^d p \frac{1}{(p^2+m^2)},$$

the behavior of the integrand in the ultraviolet ($p \to \infty$) is given by p^{d-1-2}, as is well-known this dimensionally regularized integral converges for any $d \neq 2, 4, 6, \dots$. Next the ultraviolet behavior of $I_\alpha^T(m)$ is considered it goes like $p^{n-1-2+2\alpha}$ which coincides with dimensionally regularized case if $d = n + 2\alpha$. The important statement being that considering $\alpha \neq 0$ in $I_\alpha^T(m)$ is equivalent, from the point of view of the ultraviolet behavior of the integrand, to considering $I_d^T(m)$ for non-integer d. Of course, the correspondence between ultraviolet behavior of the integrands does not mean equality of the corresponding integrals, as is shown by the following calculation.

In order to evaluate $I_\alpha^T(m)$, the integral representation of a power in (2.1) is recalled,

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} e^{-\tau A}, \tag{6.2}$$

this last formula is valid only for negative α, which is not the case of interest here. In what follows, only the case $\alpha < 0$ is considered. It will be shown that the final result can be analytically continued to the case $\alpha > 0$. Applying the last formula to (6.1) leads to,

$$I_\alpha^T(m) = \int d^n p \int_0^\infty da e^{-a(p^2+m^2)} \frac{1}{\Gamma(-\alpha)} \int_0^\infty db b^{-\alpha-1} e^{-b(M^2+p^2)}.$$

Next, the following change of variables is employed,

$$z = a + b, \quad x = \frac{a}{z} \Rightarrow a = zx, \quad b = z(1-x),$$

the Jacobian and limits of integration in these new variables implying that,

$$\int_0^\infty dadb = -\int_0^\infty dzz \int_0^1 dx,$$

which leads to,

$$I_\alpha^T(m) = \frac{-1}{\Gamma(-\alpha)} \int d^n p \int_0^\infty dzz \int_0^1 dx [z(1-x)]^{-\alpha-1} e^{-z[p^2+(1-x)M^2+xm^2]},$$

in order to make the p integration the following change of variables is performed,

$$p \to \tilde{p} = \sqrt{z} p \Rightarrow d^n p = d^n \tilde{p} z^{-\frac{n}{2}},$$

thus,

$$I_\alpha^T(m) = \frac{-\pi^{\frac{n}{2}}}{\Gamma(-\alpha)} \int_0^1 dx (1-x)^{-\alpha-1} \int_0^\infty dz z^{-\alpha-\frac{n}{2}} e^{-z[(1-x)M^2+xm^2]},$$

where use was made of,

$$\int d^n \tilde{p} e^{-\tilde{p}^2} = \pi^{\frac{n}{2}}.$$

Next employing (2.1),

$$I_\alpha^T(m) = \frac{-\pi^{\frac{n}{2}}}{\Gamma(-\alpha)} \Gamma\left(1 - \frac{n}{2} - \alpha\right) \int_0^1 dx (1-x)^{-\alpha-1} [(1-x)M^2 + xm^2]^{\alpha+\frac{n}{2}-1}$$

this last integral can be written in terms of the hypergeometric function ${}_2F_1(a, b, c, z)$, i.e.

$$I_\alpha^T(m) = \frac{-\pi^{\frac{n}{2}} \Gamma(1 - \frac{n}{2} - \alpha)}{\Gamma(-\alpha)\alpha} (M^2)^{\alpha+\frac{n}{2}-1} {}_2F_1\left(1, 1-\alpha-\frac{n}{2}, 1-\alpha, 1-\frac{m^2}{M^2}\right).$$

For illustrative purposes, let us replace $m^2 = M^2 \frac{\alpha n}{1-2\alpha}$ by,

$$\tilde{m}^2 = m^2 + m_0^2,$$

where m_0^2 is a constant additional mass. Taking the limit $\alpha \to 0$ of $I_\alpha^T(\tilde{m})$ gives,

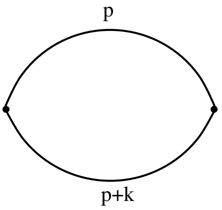
$$\lim_{\alpha \to 0} I_\alpha^T(\tilde{m}) = \pi^{\frac{n}{2}} \Gamma\left(1 - \frac{n}{2}\right) (m_0^2)^{\frac{n}{2}-1}$$

which, upon replacing n by a complex number, is the dimensionally regularized result of this diagram for a scalar field of mass m_0. It is important to note that the analytical properties of both results are the same, that is so because,

- $\Gamma(-\alpha)\alpha$ is an analytic function of α for $|\alpha| < 1$.
- The hypergeometric function ${}_2F_1(1, 1-\alpha-\frac{n}{2}, 1-\alpha, 1-\frac{m^2}{M^2})$ is an analytic function of α whenever its third argument is not equal to $0, -1, -2, \ldots$, i.e. for α not a positive integer.

6.2. *A loop involving two free propagators*

The contribution of the closed loop in the following Feynman diagram,



is given by,

$$I^L_\alpha(k,m) = \int d^np \frac{1}{(p^2+m^2)(M^2+p^2)^{-\alpha}((p+k)^2+m^2)(M^2+(p+k)^2)^{-\alpha}}, \tag{6.3}$$

using Eq. (6.2) the integral $I^L_\alpha(k,m)$ can be written as follows,

$$I^L_\alpha(k,m) = \int d^np \frac{1}{\Gamma(-\alpha)^2}\int_0^\infty dadbd\tilde{a}d\tilde{b}\tilde{a}^{-\alpha-1}\tilde{b}^{-\alpha-1}e^{E(p,k,a,b,\tilde{a},\tilde{b})}$$

where,

$$\begin{aligned}
&E(p,k,a,b,\tilde{a},\tilde{b})\\
&\quad = -[a(p^2+m^2)+b((p+k)^2+m^2)+\tilde{a}(M^2+p^2)+\tilde{b}(M^2+(p+k)^2)]\\
&\quad = -[(a+b+\tilde{a}+\tilde{b})p^2+(b+\tilde{b})(k^2+2p\cdot k)+(\tilde{a}+\tilde{b})M^2+(a+b)m^2]\\
&\quad = -z[(p+(1-x)k)^2+[(1-x)-(1-x)^2]k^2+yM^2+(1-y)m^2],
\end{aligned}$$

where in the last equality, the following change of variables has been used,

$$z = (a+b+\tilde{a}+\tilde{b}), \quad x = \frac{a+\tilde{a}}{z},$$
$$y = \frac{\tilde{a}+\tilde{b}}{z}, \quad w = \frac{\tilde{b}}{z},$$

the Jacobian and limits of integration in these new variables implying that,

$$\int_0^\infty dadbd\tilde{a}d\tilde{b} = \int_0^\infty dz \int_0^1 dx \int_0^x dy \int_0^y (-z^3),$$

in terms of the variable $p' = \sqrt{z}(p+(1-x)k)$ the quantity $I^L_\alpha(k,m)$ is written as follows,

$$\begin{aligned}
I^L_\alpha(k,m) = &-\int d^np' e^{-p'^2}\int_0^\infty dz z^{3-\frac{n}{2}-2-2\alpha}\\
&\times\int_0^1 dx\int_0^x dy\int_0^y dw[w(y-w)]^{-(1+\alpha)}e^{-z[x(1-x)k^2+yM^2+(1-y)m^2]}
\end{aligned}$$

defining,[g]

$$\frac{d}{2} = \frac{n}{2} - 2\alpha$$

and using (2.1) implies that,

$$\begin{aligned} &I^L_\alpha(k,m) \\ &\quad = -\frac{1}{\Gamma(-\alpha)^2}\int d^n p' e^{-p'^2}\Gamma\left(2-\frac{d}{2}\right)\times\int_0^1 dx\int_0^x dy\int_0^y dw[w(y-w)]^{-(1+\alpha)} \\ &\qquad \times [x(1-x)k^2 + yM^2 + (1-y)m^2]^{\frac{d}{2}-2} \end{aligned}$$

noting that,

$$\int d^n p' e^{-p'^2} = \pi^{\frac{n}{2}}$$

$$\int_0^y dw[w(y-w)]^{-(1+\alpha)} = \frac{2^{1+2\alpha}}{\sqrt{\pi}}\cos(\alpha\pi)\Gamma(-\alpha)\Gamma\left(\frac{1}{2}+\alpha\right)y^{-1-2\alpha}$$

and,

$$\begin{aligned} &\int_0^x dy y^{-1-2\alpha}[x(1-x)k^2 + yM^2 + (1-y)m^2]^{\frac{d}{2}-2} \\ &\quad = -\frac{x^{-2\alpha}}{2\alpha}(m^2 + k^2 x(1-x))^{\frac{d}{2}-2} \\ &\qquad \times {}_2F_1\left(-2\alpha, 2-4\alpha-\frac{d}{2}, 1-2\alpha, \frac{(m^2-M^2)x}{m^2+k^2(1-x)x}\right), \end{aligned}$$

where ${}_2F_1(a,b,c;z)$ denotes the hypergeometric function, leads to the following expression for $I^L_\alpha(k,m)$,

$$\begin{aligned} I^L_\alpha(k,m) = {} & \pi^{\frac{n}{2}}\Gamma\left(2-\frac{d}{2}\right)\frac{1}{\Gamma(-\alpha)}\int_0^1 dx\,\frac{2^{1+2\alpha}}{\sqrt{\pi}}\cos(\alpha\pi)\Gamma\left(\frac{1}{2}+\alpha\right) \\ & \times \frac{x^{-2\alpha}}{2\alpha}(m^2 + k^2 x(1-x))^{\frac{d}{2}-2} \\ & \times {}_2F_1\left(-2\alpha, 2-\frac{d}{2}, 1-2\alpha, \frac{(m^2-M^2)x}{m^2+k^2(1-x)x}\right). \end{aligned} \tag{6.4}$$

For illustrative purposes, let us replace $m^2 = M^2\frac{\alpha n}{1-2\alpha}$ by,

$$\tilde{m}^2 = m^2 + m_0^2,$$

where m_0^2 is a constant additional mass. Taking the limit $\alpha \to 0$ of $I^L_\alpha(k,\tilde{m})$ gives,

$$\lim_{\alpha\to 0} I^L_\alpha(k,\tilde{m}) = -\pi^{\frac{n}{2}}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx(m_0^2 + k^2 x(1-x))^{\frac{n}{2}},$$

[g] As in the case of the tadpole, this definition of d can be obtained by matching the ultraviolet behavior of the integrands of $I^L_\alpha(k,m)$ and $I^L_d(k,m)$ (the corresponding dimensionally regularized integral).

which, upon replacing n by a complex number, is the dimensionally regularized result of this diagram for a scalar field of mass m_0.

A closed analytical result for the integral over x in (6.4) was not found. However, it can be shown numerically that $I_\alpha(k)$ is finite for all α such that $|\alpha| < 1$, as in the case of the tadpole diagram.

7. Conclusions and Outlook

Conclusions and further research motivated by this work are summarized in the series of remarks given below:

- It is useful to compare the approach presented in this work to the usual dimensional regularization technique. In dimensional regularization, each divergent integral is dealt with separately. There is no known way of writing a Lagrangian leading to a perturbation theory where all contributions turn out to be dimensionally regularized. As a consequence it is not simple to make general statements about the dimensionally regularized perturbation theory. Although a general proof has not been given here, as regards ultraviolet power counting and comparison with dimensional regularization, the perturbation theory obtained taking as free Lagrangian the expression (5.3), leads to finite contributions if $\alpha \neq 0$. It would be misleading to say that the present work gives a Lagrangian formulation of dimensional regularization. A more precise statement is that it provides a regularization scheme that is implementable at the Lagrangian level, and presents the same singularity structure as dimensional regularization.
- There is another related question which is considered of interest. Are these theories only regularized theories? Do they make sense as physical theories? At the level of perturbation theory this question can be rephrased as follows: do they lead to quantum theories described by a unitary S-matrix?
- A very important feature of this approach is the fact that it is based on a well-defined differential geometry. This allows to consider the generalization of any field theory defined in differential geometric terms to these deformed spaces. This includes gauge theories and gravity theories. Of course the resulting theories deserve to be studied in detail.
- It is remarked that the approach presented in this work differs significantly from the so-called non-commutative field theory [9] (NCFT). No non-commutativity of the coordinates is assumed. On the contrary, non-commutativity enters at the level of the differential algebra through the deformed choice of the Dirac operator. This difference implies that this non-commutativity does not spoil the covariance of adequately chosen field theories on these spaces. Furthermore, as shown by the one loop calculations of the last section, the corresponding contributions are finite, which is not in general the case for the NCFTs.

All the above remarks indicate that, from the point of view of physics, further investigation of these theories is worth pursuing.

Appendix A. Junk Forms

In this Appendix it is shown that there are no junk forms for a nonzero deformation parameter. To show this it is noted that a generic 1-form can be written as,

$$\omega^{(1)} = \sum_{I,J} \alpha_{JI} f_J df_I,$$

where the summation is over a complete basis $B = \{f_I\}$ for $\mathcal{A}$ and the α_{IJ} are numerical coefficients. Replacing (4.1) in the last equation leads to,

$$\omega^{(1)} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} \sum_{I,J} \alpha_{JI} f_J [U(\tau)(Df_I) + (U(\tau)f_I - f_I U(\tau))D]. \qquad \text{(A.1)}$$

Junk 2-forms $\omega^{(2)}$are such that they can be written as the differential of a vanishing 1-form, i.e.

$$\omega^{(2)} = d\omega^{(1)}, \quad \omega^{(1)} = 0,$$

thus the general expression for a vanishing 1-form is looked for. From Eq. (A.1) this leads to the operatorial equation,

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} \sum_{I,J} \alpha_{JI} f_J [U(\tau)(Df_I) + (U(\tau)f_I - f_I U(\tau))D].$$

This equation when applied to a constant spinor leads to,

$$0 = \sum_{I,J} \alpha_{JI} f_J Df_I, \qquad \text{(A.2)}$$

which is the same relation that appears for the $\alpha = 0$ case. The general solution is given by,

$$\omega^{(1)} = \sum_{I,J} \beta_{JI} f_J (2 f_I df_I - d(f_I^2)) \qquad \text{(A.3)}$$

for arbitrary numerical coefficients β_{JI}. Using (4.3) gives,

$$d(f_I^2) = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} d(f_I^2)(\tau), \quad d(f_I^2)(\tau) = U(\tau) 2 f_I (Df_I) - [U(\tau), f_I^2] D,$$

$$2 f_I df_I = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} d(f_I^2)(\tau), \quad 2 f_I df_I(\tau) = 2 f_I \{U(\tau)(Df_I) - [U(\tau), f_I] D\}$$

thus,

$$2 f_I df_I - d(f_I^2)(\tau) = [U(\tau), f_I^2] D - 2 f_I [U(\tau), f_I] D - 2[U(\tau), f_I](Df_I).$$

Applying Eq. (A.3) to a constant spinor ψ_0 shows that in that case only the last term in the previous equation contributes, therefore the equation $\omega^{(1)} = 0$ leads to,

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} \sum_{I,J} \beta_{JI} f_J [U(\tau), f_I](Df_I) \psi_0$$

the linear independence of the basis $B = \{f_J\}$ implying that,

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} \sum_I \beta_{JI} [U(\tau), f_I](Df_I)\psi_0, \forall\, J. \tag{A.4}$$

Next the expansion of the quantity $[U(\tau), f_I](Df_I)$ in the basis B is considered, i.e.

$$[U(\tau), f_I](Df_I) = \sum_K \alpha_K^I(\tau) f_K,$$

the coefficients α_K^I being given by,

$$\alpha_K^I(\tau) = \int_x f_K^* [U(\tau), f_I](Df_I),$$

at this stage it is convenient to use the Fourier basis $f_I = e^{iI\cdot x}, I \in \mathbb{Z}^n$, which are eigenstates of D^2. Noting that,

$$e^{iI\cdot(x-2\tau\partial)} = e^{iI\cdot x} e^{-i2\tau I\cdot\partial} e^{\tau I\cdot I}$$

leads to,

$$\begin{aligned} \alpha_K^I(\tau) &= \int_x e^{-iK\cdot x}(e^{iI\cdot(x-2\tau\partial)} - 1)U(\tau)(De^{iI\cdot x}) \\ &= \int_x e^{-iK\cdot x}(e^{iI\cdot(x-2\tau\partial)} - 1)e^{iI\cdot x} e^{-\tau(1+I^2)}(-\gamma\cdot I) \\ &= \delta(K - 2I)C(\tau, I), \end{aligned} \tag{A.5}$$

where the matrix $C(\tau, I)$ is given by,

$$C(\tau, I) = (e^{-3\tau I^2} - 1)e^{-\tau(1+I^2)}(-\gamma\cdot I)$$

replacing in (A.4) leads to,

$$0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} \sum_{I,K} \beta_{JI} \alpha_K^I(\tau) f_K \psi_0, \forall\, j,$$

which taken into account the linear independence of the basis B and replacing (A.5), implies that,

$$\begin{aligned} 0 &= \frac{1}{\Gamma(\alpha)} \int_0^\infty d\tau \tau^{\alpha-1} \beta_{JI} C(\tau, I), \forall\, I, J \\ &= \beta_{JI}[(1 + 4I^2)^\alpha - (1 + I^2)^\alpha](-\gamma\cdot I) \end{aligned}$$

which is solved by,

$$\beta_{IJ} = 0, \quad \forall\, J, I \neq 0$$

the case $\beta_{0J} \neq 0$ is trivial since anyhow in that case $\omega^{(1)} = 0$.

Appendix B. Evaluation of the Diximiers Trace

As shown in Sec. 5 the actions to be evaluated are,

$$\begin{aligned} S_k &= \mathrm{tr}_\omega\{[U_\alpha D, \phi][DU_\alpha, \phi^*]|D|^{-n-2k}\}, \quad U_\alpha = (1+D^2)^{-\alpha} \\ &= \mathrm{tr}_\omega\{(2\phi DU_\alpha\phi^* U_\alpha D - \phi\phi^*(U_\alpha D)^2 - \phi(U_\alpha D)^2\phi^*)|D|^{-n-2k}\} = \mathrm{tr}_\omega\{A_k\}, \end{aligned}$$

these Diximier traces will be evaluated using their expression as Wodzicki residues,

$$S_k = \frac{1}{n(2\pi)^n}\int_{S^*T^n} \mathrm{tr}\,\sigma^{A_k}_{-n}(x,\xi),$$

where $\sigma^{A_k}_{-n}(x,\xi)$ denotes the term of order $-n$ of the symbol of the operator A_k, (x,ξ) denote coordinates over the unit co-sphere on the cotangent bundle of T^n, so that $\int_{S^*T^n} = \int_x\int_\xi d\Omega_{n-1}$ where $d\Omega_{n-1}$ is the volume element in the sphere S_{n-1}. The trace is taken over the spin space and the symbol is defined by,

$$\sigma^{A_k}(x,\xi) = e^{-ix\cdot\xi}A_k e^{ix\cdot\xi}$$

so that,

$$\begin{aligned} \sigma^{A_k}(x,\xi) = {}& (2e^{-ix\cdot\xi}\phi DU_\alpha\phi^* e^{ix\cdot\xi}(-\gamma\cdot\xi)(1+|\xi|^2)^{-\alpha} \\ & - \phi\phi^*(1+|\xi|^2)^{-\alpha}|\xi|^2 - \phi(DU_\alpha)^2\phi^*)|\xi|^{-n-2k} \end{aligned}$$

using that,

$$e^{-ix\cdot\xi}DU_\alpha e^{ix\cdot\xi} = (D-\gamma\cdot\xi)(1+(D-\gamma\cdot\xi)^2)^{-\alpha},$$

ignoring terms that vanish when integrating over $|\xi| = 1$ and evaluating the trace, leads to,

$$\begin{aligned} \mathrm{tr}\,\sigma^{A_k}(x,\xi) = {}& 2^{[\frac{n}{2}]}(2\phi(1+(D-\xi)^2)^{-\alpha}\phi^*(1+|\xi|^2)^{-\alpha}|\xi|^{-n-2k+2} \\ & - \phi\phi^*(1+|\xi|^2)^{-2\alpha}|\xi|^{-n-2k+2} \\ & - \phi[D^2+\xi^2](1+(D-\xi)^2)^{-2\alpha}\phi^*|\xi|^{-n-2k}) \end{aligned}$$

for $\alpha \neq 0$ the first two terms do not contribute to the term of order $-n$ of the symbol because they include the factor,

$$(1+|\xi|^2)^{-\alpha} = \sum_{k=0}^{\infty}\binom{-\alpha}{k}|\xi|^{-2(\alpha+k)},$$

which decrease the order of the corresponding terms by at least -2α. The last term gives two non-vanishing contributions. One coming from the term D^2 inside the square bracket, which contributes to the term of order $-n$ of the symbol only when $k = 0$. And the other coming from the ξ^2 inside the square bracket, which contributes to the term of order $-n$ of the symbol only when $k = 1$. Thus,

$$\sigma_{-n}^{A_0}(x,1) = -\phi D^2(1+D^2)^{-2\alpha}\phi^*,$$
$$\sigma_{-n}^{A_1}(x,1) = -\phi(1+D^2)^{-2\alpha}\phi^*,$$

hence,

$$S_0 = -\frac{2^{[\frac{n}{2}]}V_{S^{n-1}}}{n(2\pi)^n}\int_{T^n}\phi D^2(1+D^2)^{-2\alpha}\phi^*,$$
$$S_1 = -\frac{2^{[\frac{n}{2}]}V_{S^{n-1}}}{n(2\pi)^n}\frac{\alpha n}{1-2\alpha}\int_{T^n}\phi(1+D^2)^{-2\alpha}\phi^*,$$

leading to,

$$S = -\frac{2^{[\frac{n}{2}]}V_{S^{n-1}}}{n(2\pi)^n}\int_{T^n}\phi\left(D^2+\frac{\alpha n}{1-2\alpha}\right)(1+D^2)^{-2\alpha}\phi^*,$$

where $V_{S^{n-1}}$ denotes the surface of the sphere S^{n-1} given by,

$$V_{S^{n-1}} = \int d\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

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