# LITTLEWOOD-PALEY FUNCTIONS ASSOCIATED WITH GENERAL ORNSTEIN-UHLENBECK SEMIGROUPS 

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#### Abstract

In this paper we establish $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$-boundedness properties for square functions involving time and spatial derivatives of Ornstein-Uhlenbeck semigroups. Here $\gamma_{\infty}$ denotes the invariant measure. In order to prove the strong type results for $1<p<\infty$ we use $R$-boundedness. The weak type $(1,1)$ property is established by studying separately global and local operators defined for the square Littlewood-Paley functions. By the way we prove $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$-boundedness properties for maximal and variation operators for Ornstein-Uhlenbeck semigroups.


## 1. Introduction

In this paper we establish $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$-boundedness properties of Littlewood-Paley functions associated with Ornstein-Uhlenbeck semigroups.

Suppose that $Q$ is a real, symmetric and positive definite $d \times d$ matrix and that $B$ is a real $d \times d$-matrix having all its eigenvalues with negative real parts. $Q$ and $B$ are usually named the covariance and the drift matrix, respectively. For every $t \in(0, \infty]$ we define the symmetric and positive measure matrix $Q_{t}$ by

$$
Q_{t}=\int_{0}^{t} e^{s B} Q e^{s B^{*}} d s
$$

and the normalized measure $\gamma_{t}$ in $\mathbb{R}^{d}$ by

$$
d \gamma_{t}(x)=(2 \pi)^{-\frac{d}{2}}\left(\operatorname{det} Q_{t}\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\left\langle Q_{t}^{-1} x, x\right\rangle} d x
$$

By $C_{b}\left(\mathbb{R}^{d}\right)$ we denote the space of bounded continuous functions in $\mathbb{R}^{d}$. We consider the semigroup of operators $\left\{\mathcal{H}_{t}\right\}_{t>0}$ where, for every $f \in C_{b}\left(\mathbb{R}^{d}\right)$,

$$
\mathcal{H}_{t}(f)(x)=\int_{\mathbb{R}^{d}} f\left(e^{t B} x-y\right) d \gamma_{t}(y), \quad x \in \mathbb{R}^{d} .
$$

$\left\{\mathcal{H}_{t}\right\}_{t>0}$ is called the Ornstein-Uhlenbeck semigroup defined by $Q$ and $B . \gamma_{\infty}$ is the unique invariant measure with respect to $\left\{\mathcal{H}_{t}\right\}_{t>0}$.

After some manipulations we can write, for every $f \in C_{b}\left(\mathbb{R}^{d}\right)$,

$$
\mathcal{H}_{t}(f)(x)=\int_{\mathbb{R}^{d}} h_{t}(x, y) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
\begin{equation*}
h_{t}(x, y)=\left(\frac{\operatorname{det} Q_{\infty}}{\operatorname{det} Q_{t}}\right)^{\frac{1}{2}} e^{R(x)} \exp \left[-\frac{1}{2}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle\right], \quad x, y \in \mathbb{R}^{d} \text { and } t>0 \tag{1.1}
\end{equation*}
$$

being $D_{t}=Q_{\infty} e^{-t B^{*}} Q_{\infty}^{-1}, t>0$, and $R(x)=\frac{1}{2}\left\langle Q_{\infty}^{-1} x, x\right\rangle, x \in \mathbb{R}^{d}$. When $Q=I$ and $B=-I$ the semigroup $\left\{\mathcal{H}_{t}\right\}_{t>0}$ reduces to the symmetric Ornstein-Uhlenbeck semigroup.

Let $1 \leq p<\infty$. $\left\{\mathcal{H}_{t}\right\}_{t>0}$ extends to a positivity preserving semigroup of contractions in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. By denoting by $-\mathfrak{L}_{p}$ the infinitesimal generator of $\left\{\mathcal{H}_{t}\right\}_{t>0}$ in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, the space $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ of smooth and compactly supported functions in $\mathbb{R}^{d}$ is a core for $\mathfrak{L}_{p}$ and, for every $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$,

$$
\mathfrak{L}_{p}(f)(x)=-\frac{1}{2} \operatorname{div}(Q \nabla f)(x)-\langle\nabla f(x), B x\rangle, \quad x \in \mathbb{R}^{d}
$$

[^0]([32, Chapter 5]). If $1<p<\infty\left\{\mathcal{H}_{t}\right\}_{t>0}$ extends to an analytic contraction semigroup in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ in the sector $S_{\theta_{p}}$ of angle $\theta_{p}=\frac{\pi}{2}-\varphi_{p}$ where
$$
\varphi_{p}=\arctan \frac{\sqrt{(p-2)^{2}+p^{2}\left(\tan \varphi_{\mathrm{A}}\right)^{2}}}{2 \sqrt{p-1}}
$$
and $\varphi=\arctan \left\|Q^{-\frac{1}{2}}\left(B-B^{*}\right) Q^{-\frac{1}{2}}\right\|([11$, Theorem 2 and Remark 6]).
The operator $\mathcal{L}_{p}$ is sectorial but it is not one to one being the kernel of $\mathcal{L}_{p}$ the subspace $N\left(\mathcal{L}_{p}\right)$ consisting of all the constant functions ([32, Theorem 8.1.17]). $\mathcal{L}_{p}$ is a sectorial one to one operator with dense range on $L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)=\left\{f \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right): \int_{\mathbb{R}^{d}} f d \gamma_{\infty}=0\right\}$. In [6, Theorem 10] it was established that if $1<p<\infty, \mathcal{L}_{p}$ has bounded holomorphic functional calculus whose sharp angle is $\theta_{p}$.

The study of harmonic analysis in the Ornstein-Uhlenbeck setting was began by Muckenhoupt 40 who considered the one dimensional case. In the higher dimension the situation is quite different and new arguments and ideas are needed. E. M. Stein, in his celebrated monography [46], developed a general theory for harmonic analysis associated with symmetric diffusion semigroup. The symmetric Ornstein-Uhlenbeck semigroup is a special case of Stein symmetric diffusion semigroup. Sjögren (45) extended the Muckenhoupt result proving that the maximal operator defined by $\left\{\mathcal{H}_{t}\right\}_{t>0}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ for every $d \geq 1$. Twenty years later, higher order Riesz transforms in the Ornstein-Uhlenbeck setting were studied in [19] and [20] where it is proved that they are bounded operators from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for every $1<p<\infty$. Riesz transforms are not bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself. Harmonic analysis operators associated with the symmetric Ornstein-Uhlenbeck have been studied in the last two decades. Some of these operators are the following ones: maximal operators ([16] and [38), Littlewood-Paley functions ([22], 41] and 43]), spectral multipliers ( 18 ), singular integrals ( [1] , 17] and [42]), operators with $H^{\infty}$-functional calculus ( 15$]$ ) and variation operators ([21]). All those operators are bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.

In contrast with the symmetric case, harmonic analysis operators in the general nonsymmetric setting had not been very studied. Mauceri and Noselli considered the maximal operator (37]) and Riesz transforms of the first order ([36]) where $Q=I$ and $B=-\lambda(I+R)$ being $\lambda>0$ and $R$ generates a periodic group $\left\{e^{R t}\right\}_{t>0}$ of rotations. Recently, Casarino, Ciatti and Sjögren have studied maximal operators ( 7 ] and [9]), Riesz transforms ([10]) and spectral multipliers ([8) associated with general nonsymmetric Ornstein-Uhlenbeck operators.

One of our objective in this paper is to established the $L^{p}$-boundedness properties of some Littlewood-Paley functions, also called square functions, involving time and spatial derivatives of subordinated Ornstein-Uhlenbek semigroups in the nonsymmetric setting.

Let $\nu>0$. We consider the Poisson-like integral defined by

$$
\begin{equation*}
P_{t}^{\nu}(f)(x)=\frac{t^{2 \nu}}{4^{\nu} \Gamma(\nu)} \int_{0}^{\infty} e^{-\frac{t^{2}}{4 u}} \mathcal{H}_{u}(f)(x) \frac{d u}{u^{\nu+1}}, \quad x \in \mathbb{R}^{d} \text { and } t>0 \tag{1.2}
\end{equation*}
$$

that is subordinated to the Ornstein-Uhlenbeck $\left\{\mathcal{H}_{t}\right\}_{t>0}$. Note that when $\nu=\frac{1}{2}$ we recover the Poisson semigroup generated by $-\sqrt{\mathcal{L}}$. According to [47, (1.9)], (1.2) defines a solution of the initial value problem

$$
\left.\begin{array}{rl}
\partial_{t}^{2} u+\frac{1-2 \nu}{t^{\prime}} \partial_{t} u & =\mathcal{L} u \\
u(0, x) & =f(x)
\end{array}\right\}
$$

for suitable $f$, and this partial differential equation is connected with the fractional power $\mathcal{L}^{\nu}$ of $\mathcal{L}$.
Let $k \in \mathbb{N}$ and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{d}\right) \in \mathbb{N}^{d}$ such that $k+\sum_{i=1}^{d} \alpha_{i}>0$. We define the Littlewood-Paley function $g_{k, \alpha}^{\nu}$ as follows

$$
g_{k, \alpha}^{\nu}(f)(x)=\left(\int_{0}^{\infty}\left|t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^{d}
$$

where $\widehat{\alpha}=\sum_{i=1}^{d} \alpha_{i}$ and $\partial_{x}^{\alpha}=\partial_{x_{1}}^{\alpha_{1}} \ldots \partial_{x_{d}}^{\alpha_{d}}$.
We now state one of our main result.

Theorem 1.1. Let $\nu>0, k \in \mathbb{N}$ and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{d}\right)$ such that $k+\widehat{\alpha}>0$. Then, $g_{k, \alpha}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for every $1<p<\infty$. If in addition $\widehat{\alpha}=1$ and $d>1$ or $\widehat{\alpha}=0,2$ and $d>2$, $g_{k, \alpha}^{\nu}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.

Some comments about the proof of this theorem are in order. To see that $g_{k, \alpha}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for every $1<p<\infty$, when $\widehat{\alpha}>0$, we consider the square function $G_{\alpha}$ associated with $\left\{\mathcal{H}_{t}\right\}_{t>0}$ defined by

$$
G_{\alpha}(f)(x)=\left(\int_{0}^{\infty}\left|t^{\frac{\hat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^{d}
$$

Let $1<p<\infty$ and $\alpha \in \mathbb{N}^{d} \backslash\{0\}$. The $\alpha$-order Riesz transform in this context is given by

$$
\mathcal{R}_{\alpha}(f)=\partial_{x}^{\alpha} \mathcal{L}^{-\frac{\hat{\alpha}}{2}} \Pi_{0}(f), \quad f \in L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)
$$

where $\Pi_{0}$ denotes the projection from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ to $L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ and the $-\frac{\widehat{\alpha}}{2}$ power of $\mathcal{L}$ is defined by

$$
\mathcal{L}^{-\frac{\widehat{\alpha}}{2}} g=\frac{1}{\Gamma\left(\frac{\hat{\alpha}}{2}\right)} \int_{0}^{\infty} t^{\frac{\hat{\alpha}}{2}-1} \mathcal{H}_{t}(g) d t, \quad g \in L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)
$$

Here $\partial_{x}^{\alpha}$ is understood as a distributional derivative. According to [36, Proposition 2.3], $\mathcal{R}_{\alpha}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself. We prove that the set $\left\{t^{\frac{\alpha}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}\right\}_{t>0}$ is $R$-bounded in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ and then that the square function $G_{\alpha}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself. We recall that a family $\mathcal{A}$ of bounded operators of $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself is said to be $R$-bounded in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ when there exists $C>0$ such that if $T_{k} \in \mathcal{A}$ and $f_{k} \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right), k=1, \ldots, n$, with $n \in \mathbb{N} \backslash\{0\}$, then

$$
\mathbb{E}\left\|\sum_{k=1}^{n} r_{k} T_{k} f_{k}\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \leq C \mathbb{E}\left\|\sum_{k=1}^{n} r_{k} f_{k}\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}
$$

where $\left\{r_{k}\right\}_{k=1}^{\infty}$ is a sequence of independent Rademacher variables and $\mathbb{E}$ denotes as usual the expectation. The main properties of the $R$-bounded sets of operators can be found in [23].

Finally we can conclude that $g_{k, \alpha}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself because $G_{\alpha}$ has this property.
We now fix our attention in the case $\alpha=0$ and $k \in \mathbb{N}, k \geq 1$. Let $1<p<\infty$. It was mentioned that the Ornstein-Uhlenbeck operator $\mathcal{L}_{p}$ has bounded holomorphic functional calculus with sharp angle $\theta_{p}$. Let $\theta \in\left(\theta_{p}, \frac{\pi}{2}\right)$. We denote $S_{\theta}$ the sector of angle $\theta$ and $\Psi\left(S_{\theta}\right)$ the function space that consists of all those analytic functions $m$ in $S_{\theta}$ such that for a certain $s>0$ the function $M$ defined by

$$
M(z)=\frac{(1+z)^{2 s}}{z^{s}} m(z), \quad z \in S_{\theta}
$$

is bounded in $S_{\theta}$. According to [13, Corollary 6.7], for every $m \in \Psi\left(S_{\theta}\right)$ there exists $C>0$ such that, for every $f \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$,

$$
\begin{equation*}
\left\|\left(\int_{0}^{\infty}\left|m\left(t \mathcal{L}_{p}\right) f(\cdot)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \tag{1.3}
\end{equation*}
$$

By defining $m(z)=z e^{-z}, z \in S_{\theta}$, (1.3) implies that the Littlewood-Paley function $G^{1}$ associated with $\left\{\mathcal{H}_{t}\right\}_{t>0}$ and defined by

$$
G^{1}(f)(x)=\left(\int_{0}^{\infty}\left|t \partial_{t} \mathcal{H}_{t}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^{d}
$$

is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, and then $g_{k, 0}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself (see Section 21).
We remark that in the symmetric case, that is, when $Q=I$ and $B=-I$, the $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$-boundedness of $G^{1}$ can be deduced from [46, Theorem 10, p. 111]. Furthermore, in the proof of [37, Theorem 4.2] it was proved that $G^{1}$ is bounded from $L^{2}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself.

In order to see that $g_{k, \alpha}^{\nu}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ we can not use Littlewood-Paley functions defined by $\left\{\mathcal{H}_{t}\right\}_{t>0}$ as above. As far as we know, it has not been proved that $G^{1}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ even in the symmetric case. Furthermore, $g_{0, \alpha}^{\frac{1}{2}}$ is not bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ when $\widehat{\alpha}>2$ in the symmetric setting.

We prove that $g_{k, \alpha}^{\nu}$ is of weak type $(1,1)$ with respect to $\gamma_{\infty}$ when $0 \leq \widehat{\alpha} \leq 2$ by considering two operators called the local and the global parts of $g_{k, \alpha}^{\nu}$. This local-global method appears in the first time in the seminal Muckenhoupt's paper [40] and it is usual to study $L^{p}$-boundedness properties of harmonic analysis operators in the Ornstein-Uhlenbeck context. The local part of the original operator $g_{k, \alpha}^{\nu}$ can be seen as a vector valued singular integral while the global part can be controlled by a positive operator. Our proof does not use Calderón-Zygmund theory to establish the weak type $(1,1)$ for the local part. We exploit the fact that in the local region, close to the diagonal, the Ornstein-Uhlenbeck semigroup is in some sense a nice perturbation of the classical heat semigroup. In order to establish our result we need some estimates involving the integral kernel of the Ornstein-Uhlenbeck semigroup due to Casarino, Ciatti and Sjögren ( 7 , [9] and [10]). Our comparative procedure to deal with the local operators can be applied to study Littlewood-Paley functions associated to generalized Ornstein-Uhlenbeck operators, where the norm $L^{2}((0, \infty), d t / t)$ is replaced by $L^{q}((0, \infty), d t / t), 1<q<\infty$, in Banach valued setting and also to obtain new characterizations for the uniformly convex and smooth Banach spaces (see [2], [22], 35] and 48]).

We consider the maximal operator $P_{*, k, \alpha}^{\nu}$ defined by

$$
P_{*, k, \alpha}^{\nu}(f)=\sup _{t>0}\left|t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)\right|
$$

where $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{d}$. As a consequence of Theorem 1.1 we can establish the following result that extends [10, Theorem 1.1].
Theorem 1.2. Let $\nu>0, k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{d}$. Then, $P_{*, k, \alpha}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for every $1<p<\infty$. Furthermore, $P_{*, k, \alpha}^{\nu}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ provided that $\widehat{\alpha} \leq 1$ and $d>1$ or when $\widehat{\alpha}=2$ and $d>2$.

Let $\rho>2$. Suppose that $g$ is a complex function defined in $(0, \infty)$. The $\rho$-variation $V_{\rho}(g)$ is defined

$$
V_{\rho}(g)=\sup _{\substack{0<t_{k}<t_{k-1}<\ldots<t_{1} \\ k \in \mathbb{N}}}\left(\sum_{j=1}^{k-1}\left|g\left(t_{j+1}\right)-g\left(t_{j}\right)\right|^{\rho}\right)^{\frac{1}{\rho}} .
$$

For the exponent $\rho=2$ the oscillation associated to a particular sequence $\left\{t_{j}\right\}_{j \in \mathbb{N}}$ is considered instead of the variation. Suppose that $\left\{t_{j}\right\}_{j \in \mathbb{N}}$ is a decreasing sequence in $(0, \infty)$ such that $t_{j} \rightarrow 0$, as $j \rightarrow \infty$. We define the oscillation $O\left(g,\left\{t_{j}\right\}_{j \in \mathbb{N}}\right)$ associated with $\left\{t_{j}\right\}_{j \in \mathbb{N}}$ by

$$
O\left(g,\left\{t_{j}\right\}_{j \in \mathbb{N}}\right)=\left(\sum_{j \in \mathbb{N}} \sup _{t_{j+1} \leq \varepsilon_{i+1}<\varepsilon_{i} \leq t_{j}}\left|g\left(\varepsilon_{i+1}\right)-g\left(\varepsilon_{i}\right)\right|^{2}\right)^{\frac{1}{2}}
$$

The variation and oscillation give information about the convergence properties of $g$.
Given a family of bounded operator $\mathcal{T}=\left\{T_{t}\right\}_{t>0}$ in $L^{p}(\Omega, \mu)$ with $1 \leq p<\infty$ and for a certain measure space $(\Omega, \mu)$, an important problem in analysis is the existence of the limit $\lim _{t \rightarrow t_{0}} T_{t}(f)(x)$ with $t_{0} \in[0,+\infty]$ and $x \in \Omega$ and its speed of convergence. In order to study those questions the following variation and oscillation operators can be considered

$$
V_{\rho}(\mathcal{T})(f)(x)=V_{\rho}\left(t \rightarrow T_{t}(f)(x)\right), \quad x \in \Omega
$$

where $\rho>2$, and

$$
O\left(\mathcal{T},\left\{t_{j}\right\}_{j \in \mathbb{N}}\right)(f)(x)=O\left(t \rightarrow T_{t}(f)(x),\left\{t_{j}\right\}_{j \in \mathbb{N}}\right), \quad x \in \Omega
$$

being $\left\{t_{j}\right\}_{j \in \mathbb{N}} \subset(0, \infty)$ a sequence as above.
Variation inequalities have been investigated in probability, ergodic theory and harmonic analysis. The first variation inequality is due to Lépingle ([30]) for martingales. Later Bourgain ([3]) proved a variation estimates for ergodic averages that inspired the interest of a number of authors in oscillation and variation inequalities for collections of operators in ergodic theory ( 25 ] and [26]) and harmonic analysis ([4], [5], [24], 27], [28], [33], [34], [39] and [40]).

The local-global strategy and the arguments developed in the proof of Theorem 1.1 allow us to get the following variation and oscillation inequalities in our context.
Theorem 1.3. Let $\nu>0, k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{d}$. The operators

$$
V_{\rho}\left(\left\{t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}\right\}_{t>0}\right)
$$

and

$$
O\left(\left\{t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}\right\}_{t>0},\left\{t_{j}\right\}_{j \in \mathbb{N}}\right)
$$

where $\left\{t_{j}\right\}_{j \in \mathbb{N}} \subset(0, \infty)$ is decreasing and such that $\lim _{j \rightarrow \infty} t_{j}=0$, are bounded
(1) from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for every $1<p<\infty$, when $\alpha=0$,
(2) from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ when $\widehat{\alpha} \leq 2$.

The result proved in [21, Theorem 1.2] appears as a special case of the last theorem when the symmetric Ornstein-Uhlenbeck is considered with $k=0$ and $\alpha=0$.

The paper is organized as follow. Theorem 1.1 is proved in Section 2 and Section 3, In Section 2 we are concerned with $1<p<\infty$. The case $p=1$ is considered in Section 3. Theorems 1.2 and 1.3 are proved in Section 4 and Section [5] respectively.

Throughout this paper by $c$ and $C$ we denote always positive constants that can change in each occurrence.

## 2. Proof of Theorem 1.1 for $1<p<\infty$

Assume that $1<p<\infty$. We consider firstly that $\alpha=0$ and $k \in \mathbb{N}, k \geq 1$. We have that

$$
\begin{equation*}
\partial_{t} h_{t}(x, y)=\mathcal{L}_{x} h_{t}(x, y), \quad x, y \in \mathbb{R}^{d}, x \neq y \tag{2.1}
\end{equation*}
$$

According to [10, Lemma 4.1] we can write, for every $x, y \in \mathbb{R}^{d}, t>0$, and $i, j=1, \ldots, d$,

$$
\begin{equation*}
\partial_{x_{j}} h_{t}(x, y)=h_{t}(x, y) P_{j}(t, x, y) \tag{2.2}
\end{equation*}
$$

where

$$
P_{j}(t, x, y)=\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle+\left\langle Q_{t}^{-1} e^{t B} e_{j}, y-D_{t} x\right\rangle
$$

and

$$
\begin{equation*}
\partial_{x_{i} x_{j}}^{2} h_{t}(x, y)=h_{t}(x, y)\left(P_{i}(t, x, y) P_{j}(t, x, y)+\Delta_{i, j}(t)\right) \tag{2.3}
\end{equation*}
$$

where

$$
\Delta_{i, j}(t)=-\left\langle e_{j}, e^{t B^{*}} Q_{t}^{-1} e^{t B} e_{i}\right\rangle
$$

Here, for every $j=1, \ldots, d, e_{j}=\left(e_{j \ell}\right)_{\ell=1}^{d}$ being $e_{j \ell}=0, \ell=1, \ldots, d, \ell \neq j$, and $e_{j j}=1$.
By combining [10, (2.10) and (4.5)] and (2.2) we get, for every $j=1, \ldots, d$,

$$
\begin{align*}
\left|\partial_{x_{j}} h_{t}(x, y)\right| & \leq C \frac{e^{R(x)}}{t^{\frac{d}{2}}} e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}\left(|x|+\frac{\left|y-D_{t} x\right|}{t}\right) \\
& \leq C \frac{e^{R(x)}}{t^{\frac{d}{2}}} e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}\left(|x|+\frac{1}{\sqrt{t}}\right), \quad x, y \in \mathbb{R}^{d} \text { and } 0<t<1 \tag{2.4}
\end{align*}
$$

By (2.2) and [10, (2.11), (4.5) and Lemma 2.1] it follows that, for every $j=1, \ldots, d$,

$$
\begin{align*}
\left|\partial_{x_{j}} h_{t}(x, y)\right| & \leq C e^{R(x)} e^{-c\left|D_{-t} y-x\right|^{2}}\left(e^{-c t}\left|D_{-t} y-x\right|+\left|D_{-t} y\right|\right) \\
& \leq C e^{R(x)} e^{-c\left|D_{-t} y-x\right|^{2}} e^{-c t}(1+|y|), \quad x, y \in \mathbb{R}^{d} \text { and } t \geq 1 \tag{2.5}
\end{align*}
$$

By using now (2.3) and [10, (2.10), (2.11), (4.5), (4.6) and Lemma 2.1] we obtain, for every $i, j=1, \ldots, d$,

$$
\begin{equation*}
\left|\partial_{x_{i} x_{j}}^{2} h_{t}(x, y)\right| \leq C \frac{e^{R(x)}}{t^{\frac{d}{2}}} e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}\left(|x|+\frac{1}{\sqrt{t}}\right)^{2}, \quad x, y \in \mathbb{R}^{d} \text { and } 0<t<1 \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\partial_{x_{i} x_{j}}^{2} h_{t}(x, y)\right| \leq C e^{R(x)} e^{-c\left|D_{-t} y-x\right|^{2}} e^{-c t}(1+|y|)^{2}, \quad x, y \in \mathbb{R}^{d} \text { and } t \geq 1 \tag{2.7}
\end{equation*}
$$

By taking in account (2.1),

$$
\begin{align*}
\partial_{u} h_{u}(x, y) & =-\frac{1}{2} \operatorname{div}_{x}\left(Q \nabla_{x} h_{u}(x, y)\right)-\left\langle\nabla_{x} h_{u}(x, y), B x\right\rangle \\
& =\sum_{i, j=1}^{d}\left(c_{i, j} \partial_{x_{i} x_{j}}^{2} h_{u}(x, y)+d_{i, j} x_{i} \partial_{x_{j}} h_{u}(x, y)\right), \quad x, y \in \mathbb{R}^{d} \text { and } u>0 \tag{2.8}
\end{align*}
$$

for certain $c_{i, j}$ and $d_{i, j} \in \mathbb{R}, i, j=1, \ldots, d$. We deduce from (2.4) and (2.6) that

$$
\begin{equation*}
\left|\partial_{t} h_{t}(x, y)\right| \leq C \frac{e^{R(x)}}{t^{\frac{d}{2}+1}} e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}(1+|x|)^{2}, \quad x, y \in \mathbb{R}^{d} \text { and } 0<t<1 \tag{2.9}
\end{equation*}
$$

and from (2.5) and (2.7) that

$$
\begin{equation*}
\left|\partial_{t} h_{t}(x, y)\right| \leq C e^{R(x)} e^{-c\left|D_{-t} y-x\right|^{2}} e^{-c t}(1+|y|)^{2}, \quad x, y \in \mathbb{R}^{d} \text { and } t \geq 1 \tag{2.10}
\end{equation*}
$$

Let $f \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. By (2.9) it follows that

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left|\partial_{t} h_{t}(x, y)\left\|f(y) \left\lvert\, d \gamma_{\infty}(y) \leq C \frac{e^{R(x)}}{t^{\frac{d}{2}+1}}(1+|x|)^{2}\right.\right\| f \|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}, \quad x \in \mathbb{R}^{d} \text { and } 0<t<1\right. \tag{2.11}
\end{equation*}
$$

From (2.10) we get

$$
\begin{align*}
\int_{\mathbb{R}^{d}}\left|\partial_{t} h_{t}(x, y)\right||f(y)| d \gamma_{\infty}(y) & \leq C e^{R(x)} e^{-c t} \int_{\mathbb{R}^{d}}|f(y)|(1+|y|)^{2} d \gamma_{\infty}(y)  \tag{2.12}\\
& \leq C e^{R(x)} e^{-c t}\|f\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}, \quad x \in \mathbb{R}^{d} \text { and } t>1
\end{align*}
$$

On the other hand, we have that

$$
P_{t}^{\nu}(f)(x)=\frac{1}{\Gamma(\nu)} \int_{0}^{\infty} e^{-s} s^{\nu-1} \mathcal{H}_{\frac{t^{2}}{4 s}}(f)(x) d s, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

According to (2.11) and (2.12) we obtain

$$
\begin{aligned}
\int_{0}^{\infty} e^{-s} s^{\nu-1} & \left|\partial_{t} \mathcal{H}_{\frac{t^{2}}{4 s}}(f)(x)\right| d s=\frac{t}{2} \int_{0}^{\infty} e^{-s} s^{\nu-2}\left|\partial_{u} \mathcal{H}_{u}(f)(x)_{\left\lvert\, u=\frac{t^{2}}{4 s}\right.}\right| d s \\
& \leq C t\left(\int_{0}^{\frac{t^{2}}{4}} e^{-s} s^{\nu-2} e^{-c \frac{t^{2}}{s}} d s+\int_{\frac{t^{2}}{4}}^{\infty} e^{-s} s^{\nu-2}\left(\frac{s}{t^{2}}\right)^{\frac{d}{2}+1} d s\right) e^{R(x)}(1+|x|)^{2}\|f\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \\
& \leq C t\left(\frac{1}{t^{2}} \int_{0}^{\frac{t^{2}}{4}} e^{-s} s^{\nu-1} d s+\frac{1}{t^{d+2}} \int_{\frac{t^{2}}{4}}^{\infty} e^{-s} s^{\nu+\frac{d}{2}-1} d s\right) e^{R(x)}(1+|x|)^{2}\|f\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \\
& \leq C e^{R(x)} \frac{(1+|x|)^{2}}{t}\left(1+\frac{1}{t^{d}}\right)\|f\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}, \quad x \in \mathbb{R}^{d} \text { and } t>0
\end{aligned}
$$

Derivation under the integral sign is justified and we can write

$$
\partial_{t} P_{t}^{\nu}(f)(x)=\frac{1}{\Gamma(\nu)} \int_{0}^{\infty} e^{-s} s^{\nu-1} \partial_{t} \mathcal{H}_{\frac{t^{2}}{4 s}}(f)(x) d s, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

It follows that, for every $x \in \mathbb{R}^{d}$ and $t>0$,

$$
\partial_{t} P_{t}^{\nu}(f)(x)=\frac{t}{2 \Gamma(\nu)} \int_{0}^{\infty} e^{-s} s^{\nu-2} \partial_{u} \mathcal{H}_{u}(f)(x)_{\left\lvert\, u=\frac{t^{2}}{4 s}\right.} d s=\frac{t^{2 \nu-1}}{2^{2 \nu-1} \Gamma(\nu)} \int_{0}^{\infty} \frac{e^{-\frac{t^{2}}{4 u}}}{u^{\nu}} \partial_{u} \mathcal{H}_{u}(f)(x) d u
$$

As above we can justify the derivation under the integral sign and we get

$$
\begin{align*}
t^{k} \partial_{t}^{k} P_{t}^{\nu}(f)(x) & =\frac{t^{k}}{2^{2 \nu-1} \Gamma(\nu)} \int_{0}^{\infty} \frac{\partial_{t}^{k-1}\left[t^{2 \nu-1} e^{-\frac{t^{2}}{4 u}}\right]}{u^{\nu}} \partial_{u} \mathcal{H}_{u}(f)(x) d u \\
& =\frac{2}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{u} \mathcal{H}_{u}(f)(x) d u \\
& =\frac{4}{\Gamma(\nu)} \int_{0}^{\infty} s^{k-1} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\left[u \partial_{u} \mathcal{H}_{u}(f)(x)\right]_{\left\lvert\, u=\frac{t^{2}}{4 s^{2}}\right.} d s, \quad x \in \mathbb{R}^{d} \text { and } t>0 \tag{2.13}
\end{align*}
$$

where $\mathfrak{h}_{\nu}(s)=s^{2 \nu-1} e^{-s^{2}}, s>0$.
By using Minkowski inequality we get

$$
\begin{aligned}
\left(\int_{0}^{\infty}\left|t^{k} \partial_{t}^{k} P_{t}^{\nu}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}} & \leq C \int_{0}^{\infty} s^{k-1}\left|\partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right|\left\|u \partial_{u} \mathcal{H}_{u}(f)(x)_{\left\lvert\, u=\frac{t^{2}}{4 s^{2}}\right.}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} d s \\
& \leq C G^{1}(f)(x) \int_{0}^{\infty} s^{k-1}\left|\partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right| d s, \quad x \in \mathbb{R}^{d}
\end{aligned}
$$

where

$$
G^{1}(f)(x)=\left(\int_{0}^{\infty}\left|u \partial_{u} \mathcal{H}_{u}(f)(x)\right|^{2} \frac{d u}{u}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^{d}
$$

Since $\int_{0}^{\infty} s^{k-1}\left|\partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right| d s<\infty$ (note that $s^{\ell}\left|\partial_{s}^{\ell} \mathfrak{h}_{\nu}(s)\right| \leq C \mathfrak{h}_{\nu}(s / 2), s>0$ and $\ell \in \mathbb{N}$ ) we conclude that

$$
g_{k, 0}^{\nu}(f) \leq C G^{1}(f)
$$

As it was mentioned in Section 1 the square function $G^{1}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself. Then, $g_{k, 0}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself.

Suppose now that $\alpha \in \mathbb{N}^{d} \backslash\{0\}$. By iterating the formulas in [10, Lemma 4.1] and by using [10, (2.10), (2.11), (4.5), (4.6) and Lemma 2.1] we obtain

$$
\begin{equation*}
\left|\partial_{x}^{\alpha} h_{t}(x, y)\right| \leq C \sum_{n=0}^{\left[\frac{\widehat{\alpha}}{2}\right]} \frac{1}{t^{n}}\left(|x|+\frac{\left|y-D_{t} x\right|}{t}\right)^{\widehat{\alpha}-2 n} \frac{e^{R(x)}}{t^{\frac{d}{2}}} e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}, \quad x, y \in \mathbb{R}^{d} \text { and } 0<t<1 \tag{2.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\partial_{x}^{\alpha} h_{t}(x, y)\right| \leq C \sum_{n=0}^{\left[\frac{\hat{\alpha}}{2}\right]}\left(\left|D_{-t} y-x\right|+|y|\right)^{\widehat{\alpha}-2 n} e^{-c t} e^{R(x)} e^{-c\left|x-D_{-t} y\right|^{2}}, \quad x, y \in \mathbb{R}^{d} \text { and } t \geq 1 \tag{2.15}
\end{equation*}
$$

By proceeding as above we can see that the derivation under the integral sign is justified and we can write

$$
\begin{align*}
t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x) & =\frac{t^{k+\widehat{\alpha}}}{4^{\nu} \Gamma(\nu)} \int_{0}^{\infty} \partial_{t}^{k}\left(t^{2 \nu} e^{-\frac{t^{2}}{4 u}}\right) \partial_{x}^{\alpha} \mathcal{H}_{u}(f)(x) \frac{d u}{u^{\nu+1}} \\
& =\frac{2^{\widehat{\alpha}}}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+\widehat{\alpha}} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{x}^{\alpha} \mathcal{H}_{u}(f)(x) u^{\frac{\widehat{\alpha}}{2}-1} d u  \tag{2.16}\\
& =\frac{2^{\widehat{\alpha}+1}}{\Gamma(\nu)} \int_{0}^{\infty} s^{k+\widehat{\alpha}-1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\left[u^{\frac{\widehat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{u}(f)(x)\right]_{\left\lvert\, u=\frac{t^{2}}{4 s^{2}}\right.} d s, \quad x \in \mathbb{R}^{d} \text { and } t>0
\end{align*}
$$

where $\mathfrak{g}_{\nu}(s)=s \mathfrak{h}_{\nu}(s)=s^{2 \nu} e^{-s^{2}}, s>0$.
Minkowski inequality leads to

$$
\begin{aligned}
\left(\int_{0}^{\infty}\left|t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}} & \leq C \int_{0}^{\infty} s^{k+\widehat{\alpha}-1}\left|\partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right|\left\|\left[u^{\frac{\widehat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{u}(f)(x)\right]_{\left\lvert\, u=\frac{t^{2}}{4 s^{2}}\right.}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \\
& \leq C G_{\alpha}(f)(x) \int_{0}^{\infty} s^{k+\widehat{\alpha}-1}\left|\partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right| d s, \quad x \in \mathbb{R}^{d}
\end{aligned}
$$

where

$$
G_{\alpha}(f)(x)=\left(\int_{0}^{\infty}\left|u^{\frac{\hat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{u}(f)(x)\right|^{2} \frac{d u}{u}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^{d}
$$

Since $s^{\ell+\sigma}\left|\partial_{s}^{\ell} \mathfrak{g}_{\nu}(s)\right| \leq C \mathfrak{g}_{\nu}(s / 2), s>0, \ell \in \mathbb{N}$ and $\sigma \geq 0$, we have that $\int_{0}^{\infty} s^{k+\widehat{\alpha}-1}\left|\partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right| d s<\infty$ and we conclude that

$$
g_{k, \alpha}^{\nu}(f) \leq C G_{\alpha}(f)
$$

Our next objective is to see that the square function $G_{\alpha}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself.
According to [36, Proposition 2.3] the $\alpha$-Riesz transform $\mathcal{R}_{\alpha}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself. Then, the set $\left\{t^{\frac{\alpha}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}\right\}_{t>0}$ of operators is $R$-bounded in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Indeed, let $t>0$. We consider the function $\Phi_{t}(z)=z^{\frac{\alpha}{2}} e^{-t z}, z \in \mathbb{C} \backslash(-\infty, 0]$. Thus, $\Phi_{t}$ is analytic en $\mathbb{C} \backslash(-\infty, 0]$ and the function $\Psi_{t}(z)=(1+z)^{\widehat{\alpha}} z^{-\frac{\hat{\alpha}}{2}} \Phi_{t}(z)$, $z \in \mathbb{C} \backslash(-\infty, 0]$, is analytic in $\mathbb{C} \backslash(-\infty, 0]$ and bounded in $S_{\mu}=\{z \in \mathbb{C}:|\operatorname{Arg} z|<\mu\} \backslash\{0\}$, for every $\mu \in\left(0, \frac{\pi}{2}\right)$. Since $\mathcal{L}_{p}$ is a sectorial and one to one operator in $L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ with sharp angle $\theta_{p}$ we define the operator $\Phi_{t}\left(\mathcal{L}_{p}\right)$ (see [13, (2.1)]) as follows

$$
\begin{equation*}
\Phi_{t}\left(\mathcal{L}_{p}\right)=\frac{1}{2 \pi i} \int_{\Gamma_{\sigma}}\left(z I-\mathcal{L}_{p}\right)^{-1} \Phi_{t}(z) d z \tag{2.17}
\end{equation*}
$$

where $\sigma \in\left(\theta_{p}, \frac{\pi}{2}\right)$ and

$$
\Gamma_{\sigma}(s)= \begin{cases}-s e^{-i \sigma}, & -\infty<s \leq 0 \\ s e^{i \sigma}, & 0 \leq s<+\infty\end{cases}
$$

Note that integral in (2.17) converges in the Bochner sense with respect to the space of the bounded operators from $L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself.

Let $f \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Since $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ is the direct sum of $\mathcal{N}\left(\mathcal{L}_{p}\right)$, the kernel of $\mathcal{L}_{p}$, and $L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ we can write $f=f_{1}+f_{2}$ where $f_{1} \in \mathcal{N}\left(\mathcal{L}_{p}\right)$ and $f_{2} \in L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Since $f_{1}$ is constant and $\left\{\mathcal{H}_{t}\right\}_{t>0}$ is conservative we have that

$$
\partial_{x}^{\alpha} \mathcal{H}_{t}(f)=\partial_{x}^{\alpha} \mathcal{H}_{t}\left(f_{2}\right)
$$

Furthermore, since $\gamma_{\infty}$ is a invariant measure for $\left\{\mathcal{H}_{t}\right\}_{t>0}, \mathcal{H}_{t}\left(f_{2}\right) \in L_{0}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. By using [13, (ii), p. 56] it follows that

$$
\partial_{x}^{\alpha} \mathcal{H}_{t}(f)=\partial_{x}^{\alpha} \mathcal{L}_{p}^{-\frac{\widehat{\alpha}}{2}} \Phi_{t}\left(\mathcal{L}_{p}\right) f_{2}
$$

Then,

$$
t^{\frac{\hat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}(f)=\mathcal{R}_{\alpha} \Phi_{1}\left(t \mathcal{L}_{p}\right) \Pi_{0}(f)
$$

Suppose that $t_{j}>0$ and $f_{j} \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right), j=1, \ldots, n, n \in \mathbb{N}, n \geq 1$. By [36, Proposition 2.3] we deduce inspired in [12, Proposition 2.1] that

$$
\begin{aligned}
\mathbb{E}\left\|\sum_{j=1}^{n} r_{j} t^{\frac{\hat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t_{j}}\left(f_{j}\right)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} & =\mathbb{E}\left\|\mathcal{R}_{\alpha} \sum_{j=1}^{n} r_{j_{1}} \Phi_{1}\left(t_{j} \mathcal{L}_{p}\right) \Pi_{0}\left(f_{j}\right)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \\
& \leq C \mathbb{E}\left\|\sum_{j=1}^{n} r_{j} \Phi_{1}\left(t_{j} \mathcal{L}_{p}\right) \Pi_{0}\left(f_{j}\right)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}
\end{aligned}
$$

By $H^{\infty}\left(S_{\mu}\right)$ we denote the space of bounded and analytic functions in $S_{\mu}$ with $0<\mu<\pi$. On $H^{\infty}\left(S_{\mu}\right)$ we consider the norm $\|\cdot\|_{\infty, \mu}$ defined by

$$
\|f\|_{\infty, \mu}=\sup _{z \in S_{\mu}}|f(z)|, \quad f \in H^{\infty}\left(S_{\mu}\right)
$$

The set $\left\{t^{\frac{\alpha}{2}} \Phi_{t}\left(\mathcal{L}_{p}\right)\right\}_{t>0}$ is $R$-bounded in $H^{\infty}\left(S_{\mu}\right)$ for every $0<\mu<\frac{\pi}{2}$. According to [6, Theorem 10] and 23, Theorem 10.3.4] the set $\left\{t^{\frac{\hat{\alpha}}{2}} \Phi_{t}\left(\mathcal{L}_{p}\right)\right\}_{t>0}$ is $R$-bounded in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. It follows that

$$
\begin{aligned}
\mathbb{E}\left\|\sum_{j=1}^{n} r_{j} \Phi_{1}\left(t_{j} \mathcal{L}_{p}\right) \Pi_{0}\left(f_{j}\right)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} & \leq C \mathbb{E}\left\|\sum_{j=1}^{n} r_{j} \Pi_{0}\left(f_{j}\right)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \\
& \leq C \mathbb{E}\left\|\Pi_{0}\left(\sum_{j=1}^{n} r_{j} f_{j}\right)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \\
& \leq C \mathbb{E}\left\|\sum_{j=1}^{n} r_{j} f_{j}\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}
\end{aligned}
$$

We conclude that the set $\left\{t^{\frac{\alpha}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}\right\}_{t>0}$ is $R$-bounded in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.
We continue the argument adapting a procedure developed in the proof of [12, Proposition 5.1]. Let $f \in$ $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. By partial integration we get

$$
\begin{align*}
\int_{0}^{\infty}\left|t^{\widehat{\alpha}} \partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right|^{2} \frac{d t}{t} & =\int_{0}^{\infty} t^{\widehat{\alpha}-1}\left|\partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right|^{2} d t  \tag{2.18}\\
& =\frac{1}{\widehat{\alpha}}\left(\lim _{t \rightarrow+\infty} t^{\widehat{\alpha}}\left|\partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right|^{2}-2 \operatorname{Re} \int_{0}^{\infty} t^{\widehat{\alpha}} \partial_{x}^{\alpha} \partial_{t} \mathcal{H}_{t}(f)(x) \overline{\partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)} d t\right), \quad x \in \mathbb{R}^{d}
\end{align*}
$$

By (2.15) we deduce that

$$
t^{\widehat{\alpha}}\left|\partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right|^{2} \leq C t^{\widehat{\alpha}} e^{-c t} e^{R(x)}\|f\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}, \quad x \in \mathbb{R}^{d} \text { and } t>1
$$

Then,

$$
\begin{aligned}
\int_{0}^{\infty}\left|t^{\frac{\widehat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right|^{2} \frac{d t}{t} & \leq C \int_{0}^{\infty} t^{\widehat{\alpha}}\left|\partial_{x}^{\alpha} \partial_{t} \mathcal{H}_{t}(f)(x)\right|\left|\partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right| d t \\
& \leq\left(\int_{0}^{\infty}\left|t^{\frac{\alpha}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}\left(\int_{0}^{\infty}\left|t^{\frac{\alpha}{2}+1} \partial_{x}^{\alpha} \partial_{t} \mathcal{H}_{t}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^{d}
\end{aligned}
$$

By using the semigroup property it follows that

$$
\begin{aligned}
G_{\alpha}(f)(x) & \leq C\left(\int_{0}^{\infty}\left|t^{\frac{\hat{\alpha}}{2}+1} \partial_{x}^{\alpha} \partial_{t} \mathcal{H}_{t}(f)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}} \\
& \leq C\left(\int_{0}^{\infty}\left|t^{\frac{\hat{\alpha}}{2}} \partial_{x}^{\alpha} \mathcal{H}_{\frac{t}{2}}\left(t \partial_{t} \mathcal{H}_{\frac{t}{2}}(f)\right)(x)\right|^{2} \frac{d t}{t}\right)^{\frac{1}{2}}, x \in \mathbb{R}^{d}
\end{aligned}
$$

As it was seen in Section 1 the Littlewood-Paley function $G^{1}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself. Hence, for almost every $x \in \mathbb{R}^{d}$, the function $t \rightarrow t \partial_{t} \mathcal{H}_{\frac{t}{2}}(f)(x)$ is in $L^{2}\left((0, \infty), \frac{d t}{t}\right)$. Since the set $\left\{t^{\frac{\alpha}{2}} \partial_{x}^{\alpha} \mathcal{H}_{t}\right\}_{t>0}$ is $R$-bounded in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, by using [12, Lemma 2.3] we get

$$
\left\|G_{\alpha}(f)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \leq C\left\|G^{1}(f)\right\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)}
$$

We conclude that $g_{k, \alpha}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself.

## 3. Proof of Theorem 1.1 for $p=1$.

In order to prove that $g_{k, \alpha}^{\nu}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ we use the local-global technique.
Let $A>0$. We define the local region $L_{A}$ by

$$
L_{A}=\left\{(x, y) \in \mathbb{R}^{d} \times \mathbb{R}^{d}:|x-y| \leq \frac{A}{1+|x|}\right\}
$$

and the global region $G_{A}$ by $G_{A}=\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right) \backslash L_{A}$. The value of $A$ will be fixed later.
We choose a smooth function $\varphi_{\mathrm{A}}$ on $\mathbb{R}^{d} \times \mathbb{R}^{d}$ such that $0 \leq \varphi_{\mathrm{A}} \leq 1, \varphi_{\mathrm{A}}(x, y)=1,(x, y) \in L_{A}, \varphi_{\mathrm{A}}(x, y)=0$, $(x, y) \in G_{2 A}$, and there exists $C>0$ such that

$$
\sum_{i=1}^{d}\left(\left|\partial_{x_{i}} \varphi_{\mathrm{A}}(x, y)\right|+\left|\partial_{y_{i}} \varphi_{\mathrm{A}}(x, y)\right|\right) \leq \frac{C}{|x-y|}, \quad x, y \in \mathbb{R}^{d}, x \neq y
$$

Given an operator $T$ defined on $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right), 1 \leq p<\infty$, we consider the local and global operators given by $T_{\text {loc }}(f)(x)=T\left(\varphi_{\mathrm{A}}(x, \cdot) f\right)(x), x \in \mathbb{R}^{d}$, and $T_{\text {glob }}=T-T_{\text {loc }}$, respectively.

In the study of the $L^{p}$-boundedness for our operators of local type we make use of the operator $\mathscr{S}_{\eta}$, with $\eta>0$, defined by

$$
\mathscr{S}_{\eta}(g)(x)=\int_{(x, y) \in L_{\eta}} \frac{1+|x|}{|x-y|^{d-1}} g(y) d y, \quad x \in \mathbb{R}^{d}
$$

Let $\eta>0$. We observe that $\mathscr{S}_{\eta}$ is a bounded operator from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1 \leq p \leq \infty$. Indeed, for every $x \in \mathbb{R}^{d}$, we can write

$$
\int_{(x, y) \in L_{\eta}} \frac{1+|x|}{|x-y|^{d-1}} d y \leq C \int_{0}^{\frac{\eta}{1+|x|}}(1+|x|) d r \leq C
$$

Then

$$
\sup _{x \in \mathbb{R}^{d}} \int_{(x, y) \in L_{\eta}} \frac{1+|x|}{|x-y|^{d-1}} d y<\infty
$$

On the other hand, when $y \in \mathbb{R}^{d},|y| \leq 2 \eta$, it follows that

$$
\int_{(x, y) \in L_{\eta}} \frac{1+|x|}{|x-y|^{d-1}} d x \leq \int_{|x-y|<\eta} \frac{1+|x-y|+|y|}{|x-y|^{d-1}} d x \leq(3 \eta+1) \int_{|x-y|<\eta} \frac{d x}{|x-y|^{d-1}} \leq C
$$

and if $|y| \geq 2 \eta$ and $(x, y) \in L_{\eta}$, then, $|x| \geq|y|-|x-y| \geq|y|-\eta \geq|y|-\frac{|y|}{2}=\frac{|y|}{2}$. Hence, $|x-y| \leq \frac{2 \eta}{1+|y|}$ when $(x, y) \in L_{\eta}$ and $|y| \geq 2 \eta$, and we can write

$$
\int_{(x, y) \in L_{\eta}} \frac{1+|x|}{|x-y|^{d-1}} d x \leq \int_{|x-y| \leq \frac{2 \eta}{2+|y|}} \frac{1+|x-y|+|y|}{|x-y|^{d-1}} d x \leq C \int_{0}^{\frac{2 \eta}{2+|y|}}(1+\eta+|y|) d r \leq C, \quad y \in \mathbb{R}^{d},|y| \geq 2 \eta
$$

Then,

$$
\sup _{y \in \mathbb{R}^{d}} \int_{(x, y) \in L_{\eta}} \frac{1+|x|}{|x-y|^{d-1}} d x<\infty
$$

By using interpolation we conclude that $\mathscr{S}_{\eta}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1 \leq p \leq \infty$. Since $\mathscr{S}_{\eta}$ is a local operator we can deduce that $\mathscr{S}_{\eta}$ is also bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, $1 \leq p<\infty$.

To see this assertion let us consider the sequence $\left\{B_{\ell}=B\left(w_{\ell}, \frac{1}{20\left(1+\left|w_{\ell}\right|\right)}\right)\right\}_{\ell \in \mathbb{N}}$ of balls given in [18, Lemma 3.1], where $w_{\ell} \in \mathbb{R}^{d}, \ell \in \mathbb{N}$. We have that, for each $\sigma>0$ there exists $C>0$ such that

$$
\frac{e^{R\left(w_{\ell}\right)}}{C} \leq e^{R(x)} \leq C e^{R\left(w_{\ell}\right)}, \quad x \in \sigma B_{\ell} \text { and } \ell \in \mathbb{N}
$$

Here $\sigma B_{\ell}=B\left(w_{\ell}, \frac{\sigma}{20\left(1+\left|w_{\ell}\right|\right)}\right), \ell \in \mathbb{N}$ and $\sigma>0$. Indeed, for $\sigma>0$ we can write

$$
\begin{align*}
e^{\left|R(x)-R\left(w_{\ell}\right)\right|} & =e^{\left.\left.\frac{1}{2}| | Q_{\infty}^{-1 / 2} x\right|^{2}-\left|Q_{\infty}^{-1 / 2} w_{\ell}\right|^{2} \right\rvert\,} \leq e^{\frac{1}{2}\left|Q_{\infty}^{-1 / 2}\left(x+w_{\ell}\right)\right|\left|Q_{\infty}^{-1 / 2}\left(x-w_{\ell}\right)\right|} \\
& \left.\leq e^{c\left|x+w_{\ell}\right|\left|x-w_{\ell}\right|} \leq e^{c\left(\left|x-w_{\ell}\right|^{2}+2\left|w_{\ell}\right|\left|x-w_{\ell}\right|\right.}\right) \leq e^{c\left(\sigma^{2}+2 \sigma\right)}=C, \quad x \in \sigma B_{\ell} \text { and } \ell \in \mathbb{N} \tag{3.1}
\end{align*}
$$

We take $\sigma=\sigma(\eta)>0$ according to [18, Lemma 3.1] in such a way that for every $\ell \in \mathbb{N}$, if $x \in B_{\ell}$ and $(x, y) \in L_{\eta}$, then $y \in \sigma B_{\ell}$. Then, we have that $\mathscr{S}_{\eta}(f)(x)=\mathscr{S}_{\eta}\left(\mathcal{X}_{\sigma B_{\ell}} f\right)(x), x \in B_{\ell}$ and $\ell \in \mathbb{N}$.

Let $1 \leq p<\infty$. According to the properties of the sequence $\left\{B_{\ell}\right\}_{\ell \in \mathbb{N}}([18$, Lemma 3.1]) we deduce that

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}\left|\mathscr{S}_{\eta}(f)(x)\right|^{p} d \gamma_{\infty}(x) & \leq C \sum_{\ell \in \mathbb{N}} \int_{B_{\ell}}\left|\mathscr{S}_{\eta}\left(\mathcal{X}_{\sigma B_{\ell}} f\right)(x)\right|^{p} e^{-R(x)} d x \leq C \sum_{\ell \in \mathbb{N}} e^{-R\left(w_{\ell}\right)} \int_{\mathbb{R}^{d}}\left|\mathscr{S}_{\eta}\left(\mathcal{X}_{\sigma B_{\ell}} f\right)(x)\right|^{p} d x \\
& \leq C \sum_{\ell \in \mathbb{N}} e^{-R\left(w_{\ell}\right)} \int_{\sigma B_{\ell}}|f(y)|^{p} d y \leq C \sum_{\ell \in \mathbb{N}} \int_{\sigma B_{\ell}}|f(y)|^{p} e^{-R(y)} d y \leq C \int_{\mathbb{R}^{d}}|f(y)|^{p} e^{-R(y)} d y
\end{aligned}
$$

We note that the reasoning given above also works for any local operator $T_{\text {loc }}$. Thus, given $1 \leq p<\infty$, if $T_{\text {loc }}$ is an operator bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, then is also bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself. Moreover, in a similar way we can establish that if $T_{\text {loc }}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L^{p, \infty}\left(\mathbb{R}^{d}, d x\right)$, then it is also bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{p, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right), 1 \leq p<\infty$. We will turn to this argument in the proofs of our results.

Along this section we also consider the function $\mathfrak{m}(z)=\min \left\{1,|z|^{-2}\right\}, z \in \mathbb{R}^{d} \backslash\{0\}$, and $\mathfrak{m}(0)=1$. We observe that $\mathfrak{m}(z) \sim(1+|z|)^{-2}, z \in \mathbb{R}^{d}$, an estimate that we will frequently use.
3.1. Assume first that $0<\widehat{\alpha} \leq 2$ and $k \in \mathbb{N}$. According to (2.16), for every $f \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right), 1 \leq p<\infty$, we can write

$$
t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x)=\int_{\mathbb{R}^{d}} P_{k, \alpha}^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
P_{k, \alpha}^{\nu}(x, y, t)=\frac{2^{\widehat{\alpha}}}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+\widehat{\alpha}} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{s=\frac{t}{2 \sqrt{u}}} \partial_{x}^{\alpha} h_{u}(x, y) u^{\frac{\widehat{\alpha}}{2}-1} d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

Here, as before, $\mathfrak{g}_{\nu}(s)=s^{2 \nu} e^{-s^{2}}, s \in(0, \infty)$.
We define the local part of $g_{k, \alpha}^{\nu}$ by

$$
g_{k, \alpha, \mathrm{loc}}^{\nu}(f)(x)=\left(\int_{0}^{\infty}\left|\int_{\mathbb{R}^{d}} P_{k, \alpha}^{\nu}(x, y, t) \varphi_{\mathrm{A}}(x, y) f(y) d \gamma_{\infty}(y)\right|^{2} \frac{d t}{t}\right)^{1 / 2}, \quad x \in \mathbb{R}^{d}
$$

and the global part of $g_{k, \alpha}^{\nu}$ by

$$
g_{k, \alpha, \text { glob }}^{\nu}(f)(x)=\left(\int_{0}^{\infty}\left|\int_{\mathbb{R}^{d}} P_{k, \alpha}^{\nu}(x, y, t)\left(1-\varphi_{\mathrm{A}}(x, y)\right) f(y) d \gamma_{\infty}(y)\right|^{2} \frac{d t}{t}\right)^{1 / 2}, \quad x \in \mathbb{R}^{d}
$$

It is clear that

$$
g_{k, \alpha}^{\nu}(f) \leq g_{k, \alpha, \operatorname{loc}}^{\nu}(f)+g_{k, \alpha, \text { glob }}^{\nu}(f), \quad f \in L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right) \quad(1 \leq p<\infty)
$$

We study $g_{k, \alpha, \text { loc }}^{\nu}$ and $g_{k, \alpha, \text { glob }}^{\nu}$ separately. We take advantage of some estimates established in [10]. We also take into account that, for each $\ell \in \mathbb{N},\left|s^{\ell} \partial_{s}^{\ell} \mathfrak{g}_{\nu}(s)\right| \leq C \mathfrak{g}_{\nu}(s / 2), s \in(0, \infty)$, from which it follows that $\left\|s^{\ell+\sigma} \partial_{s}^{\ell} \mathfrak{g}_{\nu}(s)\right\|_{L^{2}\left((0, \infty), \frac{d s}{s}\right)} \leq C, \sigma \geq 0$.
3.1.1. About the local part of $g_{k, \alpha}^{\nu}$ when $\widehat{\alpha}=1$. Assume that $j \in\{1, \ldots, d\}$ and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{d}\right) \in \mathbb{N}^{d}$ being $\alpha_{j}=1$ and $\alpha_{i}=0, i \in\{1, \ldots, d\} \backslash\{j\}$. We can write

$$
g_{k, \alpha, \operatorname{loc}}^{\nu}(f)(x)=\left\|\int_{\mathbb{R}^{d}} K_{k, j}^{\nu}(x, y, \cdot) \varphi_{\mathrm{A}}(x, y) f(y) d \gamma_{\infty}(y)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}, \quad x \in \mathbb{R}^{d}
$$

where

$$
K_{k, j}^{\nu}(x, y, t)=\frac{2}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{x_{j}} h_{u}(x, y) \frac{d u}{\sqrt{u}}, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

We consider the operator $S_{k, j}^{\nu}$ defined by

$$
\begin{equation*}
S_{k, j}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} S_{k, j}^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0 \tag{3.2}
\end{equation*}
$$

where, for every $x, y \in \mathbb{R}^{d}$,

$$
S_{k, j}^{\nu}(x, y, t)=\frac{2}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \mathbb{S}_{u}^{j}(x, y) \frac{d u}{\sqrt{u}}, \quad t>0
$$

and

$$
\begin{aligned}
\mathbb{S}_{u}^{j}(x, y) & =\left(\frac{\operatorname{det} Q_{\infty}}{u^{d} \operatorname{det} Q}\right)^{1 / 2} e^{R(x)} \partial_{x_{j}} e^{-\frac{1}{2 u}\left\langle Q^{-1}(y-x), y-x\right\rangle} \\
& =\left(\frac{\operatorname{det} Q_{\infty}}{u^{d} \operatorname{det} Q}\right)^{1 / 2} e^{R(x)} \frac{1}{u}\left\langle Q^{-1} e_{j}, y-x\right\rangle e^{-\frac{1}{2 u}\left\langle Q^{-1}(y-x), y-x\right\rangle}, \quad u>0
\end{aligned}
$$

Let us denote by $S_{k, j, \text { loc }}^{\nu}$ the corresponding local operator, that is, $S_{k, j, \text { loc }}^{\nu}(f)(x)=S_{k, j}^{\nu}\left(\varphi_{\mathrm{A}}(x, \cdot) f\right)(x), x \in \mathbb{R}^{d}$. In order to obtain the $L^{p}$-boundedness properties for $g_{k, \alpha, \text { loc }}^{\nu}$ it is sufficient to establish the following two items:
(I1) $S_{k, j, \text { loc }}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.
(I2) The operator given by $\mathcal{D}_{k, j}^{\nu}(f)(x)=g_{k, \alpha, \text { loc }}^{\nu}(f)(x)-\left\|S_{k, j, \text { loc }}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}, x \in \mathbb{R}^{d}$, is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for each $1 \leq p \leq \infty$.
First, we prove (I1). For that, we decompose $S_{k, j, \text { loc }}^{\nu}$ as follows

$$
\begin{aligned}
S_{k, j, \text { loc }}^{\nu}(f)(x, t)= & \int_{\mathbb{R}^{d}} S_{k, j}^{\nu}(x, y, t)\left[\varphi_{\mathrm{A}}(x, y)-\varphi_{\mathrm{A}}\left(Q^{-1 / 2} x, Q^{-1 / 2} y\right)\right] f(y) d \gamma_{\infty}(y) \\
& +\int_{\mathbb{R}^{d}} S_{k, j}^{\nu}(x, y, t) \varphi_{\mathrm{A}}\left(Q^{-1 / 2} x, Q^{-1 / 2} y\right) f(y) d \gamma_{\infty}(y) \\
= & S_{k, j, \text { loc }}^{\nu, 1}(f)(x, t)+S_{k, j, \text { loc }}^{\nu, 2}(f)(x, t), \quad x \in \mathbb{R}^{d} \text { and } t>0
\end{aligned}
$$

We can find $\eta>0$ such that $\frac{\eta^{-1}}{1+|x|} \leq|x-y| \leq \frac{\eta}{1+|x|}$, provided that $\varphi_{\mathrm{A}}(x, y)-\varphi_{\mathrm{A}}\left(Q^{-1 / 2} x, Q^{-1 / 2} y\right) \neq 0$. Then, we can write

$$
\left\|S_{k, j, \mathrm{loc}}^{\nu, 1}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \int_{\frac{\eta^{-1}}{1+|x|} \leq|x-y| \leq \frac{\eta}{1+|x|}}\left\|S_{k, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}|f(y)| e^{-R(y)} d y
$$

We have that

$$
\begin{align*}
R(x)-R(y) & =\frac{1}{2}\left(\left|Q_{\infty}^{-1 / 2} x\right|^{2}-\left|Q_{\infty}^{-1 / 2} y\right|^{2}\right) \leq \frac{1}{2}\left|Q_{\infty}^{-1 / 2}(x+y)\right|\left|Q_{\infty}^{-1 / 2}(x-y)\right| \\
& \leq C|x+y||x-y| \leq C \frac{2|x|+|x-y|}{1+|x|} \leq C, \quad(x, y) \in L_{\eta}, \tag{3.3}
\end{align*}
$$

and thus, when $\frac{\eta^{-1}}{1+|x|} \leq|x-y| \leq \frac{\eta}{1+|x|}$ we get

$$
\begin{aligned}
e^{-R(y)} \| S_{k, j}^{\nu} & (x, y, \cdot) \|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \\
& \leq C \int_{0}^{\infty}\left\|\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\left|\frac{1}{u}\left\langle Q^{-1} e_{j}, y-x\right\rangle e^{-\frac{1}{2 u}\left\langle Q^{-1}(y-x), y-x\right\rangle}\right| \frac{d u}{u^{\frac{d+1}{2}}} \\
& \leq C \int_{0}^{\infty}\left|Q^{-1 / 2}(y-x)\right| e^{-\frac{\left|Q^{-1 / 2}(y-x)\right|^{2}}{2 u}} \frac{d u}{u^{\frac{d+3}{2}}} \leq C \int_{0}^{\infty} \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}+1}} d u \\
& =\frac{C}{|x-y|^{d}} \leq C \frac{1+|x|}{|x-y|^{d-1}} .
\end{aligned}
$$

Observe that $\left\|\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}=\left\|s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right\|_{L^{2}\left((0, \infty), \frac{d s}{s}\right)}=C, u>0$.
Hence we conclude that

$$
\left\|S_{k, j, \mathrm{loc}}^{\nu, 1}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \mathscr{S}_{\eta}(|f|)(x), \quad x \in \mathbb{R}^{d}
$$

and then, $S_{k, j, \text { loc }}^{\nu, 1}$ is a bounded operator from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1 \leq p \leq \infty$.
We now deal with $S_{k, j, \text { loc }}^{\nu, 2}$. A straightforward change of variables allows us to write

$$
S_{k, j, \mathrm{loc}}^{\nu, 2}(f)(x, t)=c \widetilde{S}_{k, j, \mathrm{loc}}^{\nu}\left(f\left(Q^{1 / 2} \cdot\right)\right)\left(Q^{-1 / 2} x, t\right), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where the operator $\widetilde{S}_{k, j, \text { loc }}^{\nu}$ is defined by

$$
\widetilde{S}_{k, j, \mathrm{loc}}^{\nu}(g)(x, t)=\int_{\mathbb{R}^{d}} \widetilde{S}_{k, j}^{\nu}(x, y, t) \varphi_{\mathrm{A}}(x, y) g(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

and being $\widetilde{S}_{k, j}^{\nu}(x, y, t)=e^{-R\left(Q^{1 / 2} y\right)} S_{k, j}^{\nu}\left(Q^{1 / 2} x, Q^{1 / 2} y, t\right), x, y \in \mathbb{R}^{d}, t>0$. Let us establish that the operator $\widetilde{S}_{k, j, \text { loc }}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$. Consequently, we will have the same $L^{p}$-boundedness properties for $S_{k, j, \text { loc }}^{\nu, 2}$.

If $Q^{-1 / 2}=\left(c_{n \ell}\right)_{n, \ell=1}^{d}$ with $c_{n \ell} \in \mathbb{R}, n, \ell=1, \ldots, d$, we have that

$$
\left\langle Q^{-1 / 2} e_{j}, z\right\rangle=\sum_{n=1}^{d} c_{n j} z_{n}, \quad z=\left(z_{1}, \ldots, z_{d}\right) \in \mathbb{R}^{d}
$$

Then, we can write

$$
\begin{aligned}
\widetilde{S}_{k, j}^{\nu}(x, y, t) & =\frac{2}{\Gamma(\nu)}\left(\frac{\operatorname{det} Q_{\infty}}{\operatorname{det} Q}\right)^{1 / 2} e^{R\left(Q^{1 / 2} x\right)-R\left(Q^{1 / 2} y\right)} \int_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\left\langle Q^{-1 / 2} e_{j}, y-x\right\rangle e^{-\frac{|y-x|^{2}}{2 u}} \frac{d u}{u^{\frac{d+3}{2}}} \\
& =\frac{2}{\Gamma(\nu)}\left(\frac{\operatorname{det} Q_{\infty}}{\operatorname{det} Q}\right)^{1 / 2} e^{R\left(Q^{1 / 2} x\right)-R\left(Q^{1 / 2} y\right)} \sum_{n=1}^{d} c_{n j}\left(y_{n}-x_{n}\right) \int_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} e^{-\frac{|y-x|^{2}}{2 u}} \frac{d u}{u^{\frac{d+3}{2}}}
\end{aligned}
$$

and we get $\widetilde{S}_{k, j, \text { loc }}^{\nu}(f)(x, t)=c \sum_{n=1}^{d} c_{n j} T_{k, n, \text { loc }}^{\nu}(f)(x, t), x \in \mathbb{R}^{d}, t>0$, where for every $n=1, \ldots, d$, the operator $T_{k, n}^{\nu}$ is given by

$$
T_{k, n}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} T_{k, n}^{\nu}(x, y, t) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

and $T_{k, n, \text { loc }}^{\nu}$ is defined in the usual way, and where, for every $x=\left(x_{1}, \ldots, x_{d}\right), y=\left(y_{1}, \ldots, y_{d}\right) \in \mathbb{R}^{d}$ and $t>0$,

$$
T_{k, n}^{\nu}(x, y, t)=\frac{2}{\Gamma(\nu)} e^{R\left(Q^{1 / 2} x\right)-R\left(Q^{1 / 2} y\right)} \int_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{x_{n}} \mathbb{W}_{u}(x-y) \frac{d u}{\sqrt{u}}
$$

Here and in the sequel $\left\{\mathbb{W}_{u}\right\}_{u>0}$ represents the Euclidean heat semigroup in $\mathbb{R}^{d}$, that is, $\mathbb{W}_{u}(z)=(2 \pi u)^{-\frac{d}{2}} e^{-\frac{|z|^{2}}{2 u}}$, $z \in \mathbb{R}^{d}$ and $u>0$.

Let $n=1, \ldots, d$. To see that $T_{k, n, \text { loc }}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$ we introduce the operator $\mathbb{T}_{k, n}^{\nu}$ as follows

$$
\mathbb{T}_{k, n}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} \mathbb{T}_{k, n}^{\nu}(x-y, t) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
\mathbb{T}_{k, n}^{\nu}(z, t)=\left.\frac{2}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]\right|_{s=\frac{t}{2 \sqrt{u}}} \partial_{z_{n}} \mathbb{W}_{u}(z) \frac{d u}{\sqrt{u}}, \quad z=\left(z_{1}, \ldots, z_{d}\right) \in \mathbb{R}^{d} \text { and } t>0
$$

Note that $T_{k, n}^{\nu}(f)(x, t)=e^{R\left(Q^{1 / 2} x\right)} \mathbb{T}_{k, n}^{\nu}\left(e^{-R\left(Q^{1 / 2}\right)} f\right)(x, t), x \in \mathbb{R}^{d}$ and $t>0$.
We will show below that the local operator $\mathbb{T}_{k, n, \text { loc }}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$. From this fact, we can deduce the same $L^{p}$-properties for $T_{k, n, \text { loc }}^{\nu}$. Effectively, let us consider the sequence $\left\{B_{\ell}\right\}_{\ell \in \mathbb{N}}$ of balls given in the proof of (3.1). Following the same reasoning we obtain that

$$
e^{\left|R\left(Q^{1 / 2} x\right)-R\left(Q^{1 / 2} w_{\ell}\right)\right|} \leq C, \quad x \in \sigma B_{\ell} \text { and } \ell \in \mathbb{N}
$$

with $\sigma>0$. The constant $C>0$ depends on $\sigma$ but it does not depend on $\ell \in \mathbb{N}$.
We take $\sigma=\sigma(A)>0$ according to [18, Lemma 3.1] and for which we have that

$$
T_{k, n, \mathrm{loc}}^{\nu}(f)(x)=T_{k, n, \mathrm{loc}}^{\nu}\left(\mathcal{X}_{\sigma B_{\ell}} f\right)(x), \quad x \in B_{\ell} \text { and } \ell \in \mathbb{N}
$$

Let $1<p<\infty$. According to the properties of the sequence $\left\{B_{\ell}\right\}_{\ell \in \mathbb{N}}([18$, Lemma 3.1]) we deduce that

$$
\begin{aligned}
\int_{\mathbb{R}^{d}}\left\|T_{k, n, \mathrm{loc}}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p} d x & \leq C \sum_{\ell \in \mathbb{N}} \int_{B_{\ell}}\left\|T_{k, n, \mathrm{loc}}^{\nu}\left(\mathcal{X}_{\sigma B_{\ell}} f\right)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p} d x \\
& \leq C \sum_{\ell \in \mathbb{N}} e^{p R\left(Q^{1 / 2} w_{\ell}\right)} \int_{\mathbb{R}^{d}}\left\|\mathbb{T}_{k, n, \mathrm{loc}}^{\nu}\left(e^{-R\left(Q^{1 / 2} \cdot\right)} \mathcal{X}_{\sigma B_{\ell}} f\right)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p} d x \\
& \leq C \sum_{\ell \in \mathbb{N}} e^{p R\left(Q^{1 / 2} w_{\ell}\right)} \int_{\sigma B_{\ell}}\left|e^{-R\left(Q^{1 / 2} y\right)} f(y)\right|^{p} d y \\
& \leq C \sum_{\ell \in \mathbb{N}} \int_{\sigma B_{\ell}}|f(y)|^{p} d y \leq C \int_{\mathbb{R}^{d}}|f(y)|^{p} d y
\end{aligned}
$$

In a similar way we can see that

$$
\left|\left\{x \in \mathbb{R}^{d}:\left\|T_{k, n, \mathrm{loc}}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}>\lambda\right\}\right| \leq \frac{C}{\lambda}\|f\|_{L^{1}\left(\mathbb{R}^{d}, d x\right)}, \quad \lambda>0
$$

By all the estimations above, to obtain the $L^{p}$-boundedness properties for $S_{k, j, \text { loc }}^{\nu, 2}$ it remains to establish that $\mathbb{T}_{k, n, \text { loc }}^{\nu}$ is a bounded operator from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

We observe that, for every $f \in L^{p}\left(\mathbb{R}^{d}, d x\right), 1 \leq p<\infty$,

$$
\mathbb{T}_{k, n}^{\nu}(f)(x, t)=t^{k+1} \partial_{t}^{k} \partial_{x_{n}} \mathbb{P}_{t}^{\nu}(f)(x), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where, for every $t>0$,

$$
\mathbb{P}_{t}^{\nu}(f)(x)=\frac{t^{2 \nu}}{4^{\nu} \Gamma(\nu)} \int_{0}^{\infty} e^{-\frac{t^{2}}{4 u}} \mathbb{W}_{u}(f)(x) \frac{d u}{u^{\nu+1}}, \quad x \in \mathbb{R}^{d}
$$

The derivations under the integral signs can be justified by proceeding as in Section 2.
We define the Littlewood-Paley function $\mathbb{G}_{k, n}^{\nu}$ as follows

$$
\mathbb{G}_{k, n}^{\nu}(f)(x)=\left(\int_{0}^{\infty}\left|t^{k+1} \partial_{t}^{k} \partial_{x_{n}} \mathbb{P}_{t}^{\nu}(f)(x)\right|^{2} \frac{d t}{t}\right)^{1 / 2}, \quad x \in \mathbb{R}^{d}
$$

Then, it is clear that

$$
\mathbb{G}_{k, n}^{\nu}(f)(x)=\left\|\mathbb{T}_{k, n}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}, \quad x \in \mathbb{R}^{d}
$$

We are going to see that the operator $\mathbb{T}_{k, n}^{\nu}$ can be seen as a $L^{2}\left((0, \infty), \frac{d t}{t}\right)$-valued Calderón-Zygmund operator. The integral kernel $\mathbb{T}_{k, n}^{\nu}(z, \cdot)$ is a standard $L^{2}\left((0, \infty), \frac{d t}{t}\right)$-valued Calderón-Zygmund kernel. Indeed, Minkowski inequality leads to

$$
\left\|\mathbb{T}_{k, n}^{\nu}(z, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \int_{0}^{\infty}\left\|\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{s=\frac{t}{2 \sqrt{u}}}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\left|z_{n}\right| e^{-\frac{|z|^{2}}{2 u}} \frac{d u}{u^{\frac{d+3}{2}}}
$$

$$
\begin{equation*}
\leq C|z| \int_{0}^{\infty} e^{-\frac{|z|^{2}}{2 u}} \frac{d u}{u^{\frac{d+3}{2}}} \leq \frac{C}{|z|^{d}}, \quad z \neq 0 \tag{3.4}
\end{equation*}
$$

In a similar way we can see that

$$
\begin{equation*}
\sum_{\ell=1}^{d}\left\|\partial_{z_{\ell}} T_{k, n}^{\nu}(z, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq \frac{C}{|z|^{d+1}}, \quad z=\left(z_{1}, \ldots, z_{d}\right) \in \mathbb{R}^{d}, z \neq 0 \tag{3.5}
\end{equation*}
$$

The square function $\mathbb{G}_{k, n}^{\nu}$ is bounded from $L^{2}\left(\mathbb{R}^{d}, d x\right)$ into itself. In order to prove this property we use the Fourier transform $\mathcal{F}$ defined by

$$
\begin{equation*}
\mathcal{F}(f)(\xi)=\frac{1}{(2 \pi)^{\frac{d}{2}}} \int_{\mathbb{R}^{d}} f(x) e^{-i\langle x, \xi\rangle} d x, \quad \xi \in \mathbb{R}^{d} \tag{3.6}
\end{equation*}
$$

Let $f \in L^{2}\left(\mathbb{R}^{d}, d x\right)$. Plancherel equality allows us to get

$$
\begin{aligned}
\left\|\mathbb{G}_{k, n}^{\nu}(f)\right\|_{L^{2}\left(\mathbb{R}^{d}, d x\right)}^{2} & =\int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left|t^{k+1} \partial_{t}^{k} \partial_{x_{n}} \mathbb{P}_{t}^{\nu}(f)(x)\right|^{2} d x \frac{d t}{t}=\int_{0}^{\infty} \int_{\mathbb{R}^{d}}\left|t^{k+1} \partial_{t}^{k} \mathcal{F}\left(\partial_{x_{n}} \mathbb{P}_{t}^{\nu}(f)\right)(y)\right|^{2} d y \frac{d t}{t} \\
& \leq C \int_{0}^{\infty} \int_{\mathbb{R}^{d}}|y|^{2}\left|t^{k+1} \partial_{t}^{k} \mathcal{F}\left(\mathbb{P}_{t}^{\nu}(f)\right)(y)\right|^{2} d y \frac{d t}{t}
\end{aligned}
$$

We can write

$$
\begin{aligned}
\mathcal{F}\left(\mathbb{P}_{t}^{\nu}(f)\right)(y) & =\frac{t^{2 \nu}}{4^{\nu} \Gamma(\nu)} \int_{0}^{\infty} e^{-\frac{t^{2}}{4 u}} \mathcal{F}\left(\mathbb{W}_{u}(f)\right)(y) \frac{d u}{u^{\nu+1}} \\
& =\frac{\mathcal{F}(f)(y)}{4^{\nu} \Gamma(\nu)} \int_{0}^{\infty} t^{2 \nu} e^{-\frac{t^{2}}{4 u}} e^{-\frac{u|y|^{2}}{2}} \frac{d u}{u^{\nu+1}} \\
& =\frac{\mathcal{F}(f)(y)}{\Gamma(\nu)} \int_{0}^{\infty} \mathfrak{g}_{\nu}(s)_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} e^{-\frac{u|y|^{2}}{2}} \frac{d u}{u}, \quad y \in \mathbb{R}^{d} \text { and } t>0 .
\end{aligned}
$$

Then,

$$
t^{k+1} \partial_{t}^{k} \mathcal{F}\left(\mathbb{P}_{t}^{\nu}(f)\right)(y)=\frac{2}{\Gamma(\nu)} \mathcal{F}(f)(y) \int_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} e^{-\frac{u|y|^{2}}{2}} \frac{d u}{\sqrt{u}}
$$

By using Minkowski inequality we get

$$
\begin{align*}
\left\|\mathbb{G}_{k, n}^{\nu}(f)\right\|_{L^{2}\left(\mathbb{R}^{d}, d x\right)}^{2} & \leq C \int_{\mathbb{R}^{d}}|y|^{2}|\mathcal{F}(f)(y)|^{2}\left\|_{0}^{\infty}\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} e^{-\frac{u|y|^{2}}{2}} \frac{d u}{\sqrt{u}}\right\|_{\left.L^{2}(0, \infty), \frac{d t}{t}\right)}^{2} d y \\
& \leq C \int_{\mathbb{R}^{d}}|y|^{2}|\mathcal{F}(f)(y)|^{2}\left(\int_{0}^{\infty}\left\|\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right\|_{\left.L^{2}(0, \infty), \frac{d t}{t}\right)} e^{-\frac{u|y|^{2}}{2}} \frac{d u}{\sqrt{u}}\right)^{2} d y \\
& \leq C \int_{\mathbb{R}^{d}}|y|^{2}|\mathcal{F}(f)(y)|^{2}\left(\int_{0}^{\infty} e^{-\frac{u|y|^{2}}{2}} \frac{d u}{\sqrt{u}}\right)^{2} d y \\
& \leq C \int_{\mathbb{R}^{d}}|\mathcal{F}(f)(y)|^{2} d y=C\|f\|_{L^{2}\left(\mathbb{R}^{d}, d x\right)}^{2} \tag{3.7}
\end{align*}
$$

Let $N \in \mathbb{N}, N \geq 2$. We define $E_{N}=L^{2}\left(\left(\frac{1}{N}, N\right), \frac{d t}{t}\right)$. Assume that $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Let $x \in \mathbb{R}^{d}$. We consider the $E_{N}$-valued function $F_{x}$ defined on $\mathbb{R}^{d}$ as follows

$$
\left[F_{x}(y)\right](t)=f(y) \mathbb{T}_{k, n}^{\nu}(x-y, t), \quad t \in\left(\frac{1}{N}, N\right) \text { and } y \in \mathbb{R}^{d}
$$

The function $F_{x}$ is $E_{N}$-strongly measurable in $\mathbb{R}^{d}$ because $F_{x}$ is $E_{N}$-valued continuous in $\mathbb{R}^{d}$ and $E_{N}$ is a separable Banach space. On the other hand, from (3.4) we deduce that

$$
\int_{\mathbb{R}^{d}}\left\|\mathbb{T}_{k, n}^{\nu}(x-y, \cdot)\right\|_{E_{N}}|f(y)| d y<\infty, \quad x \notin \operatorname{supp} f
$$

We define

$$
\tau_{k, n}^{\nu}(f)(x)=\int_{\mathbb{R}^{d}} \mathbb{T}_{k, n}^{\nu}(x-y, \cdot) f(y) d y, \quad x \notin \operatorname{supp} f
$$

where the integral is understood in the $E_{N^{-}}$Bochner sense.

By using the properties of the Bochner integral, for every $g \in E_{N}$, we have that

$$
\int_{1 / N}^{N} g(t)\left[\tau_{k, n}^{\nu}(f)(x)\right](t) \frac{d t}{t}=\int_{\mathbb{R}^{d}} f(y) \int_{1 / N}^{N} \mathbb{T}_{k, n}^{\nu}(x-y, t) g(t) \frac{d t}{t} d y=\int_{1 / N}^{N} g(t) \mathbb{T}_{k, n}^{\nu}(f)(x, t) \frac{d t}{t}, \quad x \notin \operatorname{supp} f
$$

The interchange of the order of integration is justified because from (3.4) we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^{d}} \int_{1 / N}^{N}\left|f(y)\left\|\mathbb{T}_{k, n}^{\nu}(x-y, t)\right\| g(t)\right| \frac{d t}{t} d y & \leq C \int_{\mathbb{R}^{d}}|f(y)|\|g\|_{E_{N}}\left\|\mathbb{T}_{k, n}^{\nu}(x-y, \cdot)\right\|_{E_{N}} d y \\
& \leq C \int_{\operatorname{supp} f} \frac{|f(y)|}{|x-y|^{d}} d y<\infty, \quad x \notin \operatorname{supp} f
\end{aligned}
$$

It follows that, for every $x \notin \operatorname{supp} f$,

$$
\tau_{k, n}^{\nu}(f)(x)=\mathbb{T}_{k, n}^{\nu}(f)(x, \cdot), \quad \text { in } E_{N}
$$

According to the vector valued Calderón-Zygmund theory (44), $\mathbb{T}_{k, n}^{\nu}$ can be extended from $L^{p}\left(\mathbb{R}^{d}, d x\right) \cap$ $L^{2}\left(\mathbb{R}^{d}, d x\right)$ to $L^{p}\left(\mathbb{R}^{d}, d x\right)$ as a bounded operator $\mathscr{T}_{k, n}^{\nu}$ from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{E_{N}}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{E_{N}}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

Furthermore, since (3.4), (3.5) and (3.7) do not depend on $N$, we obtain, for every $1<p<\infty$,

$$
\sup _{N \in \mathbb{N}}\left\|\mathscr{T}_{k, n}^{\nu}\right\|_{L^{p}\left(\mathbb{R}^{d}, d x\right) \rightarrow L_{E_{N}}^{p}\left(\mathbb{R}^{d}, d x\right)}<\infty
$$

and

$$
\sup _{N \in \mathbb{N}}\left\|\mathscr{T}_{k, n}^{\nu}\right\|_{L^{1}\left(\mathbb{R}^{d}, d x\right) \rightarrow L_{E_{N}}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)}<\infty
$$

Let $f \in L^{p}\left(\mathbb{R}^{d}, d x\right)$ with $1 \leq p<\infty$. We have that

$$
\mathbb{T}_{k, n}^{\nu}(f)(x, t) \leq C \sup _{u>0}\left|\mathbb{W}_{u}(f)(x)\right|, \quad x \in \mathbb{R}^{d} \text { and } t \in\left(\frac{1}{N}, N\right)
$$

It follows that

$$
\left\|\mathbb{T}_{k, n}^{\nu}(f)(x, \cdot)\right\|_{E_{N}} \leq C \sup _{u>0}\left|\mathbb{W}_{u}(f)(x)\right|, \quad x \in \mathbb{R}^{d}
$$

Here $C=C(N)$ is a positive constant depending on $N$.
Suppose that, for every $r \in \mathbb{N}, f_{r} \in L^{p}\left(\mathbb{R}^{d}, d x\right) \cap L^{2}\left(\mathbb{R}^{d}, d x\right)$ and that $f_{r} \longrightarrow f$, as $r \rightarrow \infty$, in $L^{p}\left(\mathbb{R}^{d}, d x\right)$. Let $1<p<\infty$. Since the maximal operator $\mathbb{W}_{*}$ associated with $\left\{\mathbb{W}_{t}\right\}_{t>0}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself we deduce that

$$
\left\|\mathbb{T}_{k, n}^{\nu}(f)(x, \cdot)\right\|_{E_{N}}=\lim _{r \rightarrow \infty}\left\|\mathbb{T}_{k, n}^{\nu}\left(f_{r}\right)(x, \cdot)\right\|_{E_{N}}, \quad \text { in } L^{p}\left(\mathbb{R}^{d}, d x\right)
$$

Hence $\left\|\mathbb{T}_{k, n}^{\nu}(f)(x, \cdot)\right\|_{E_{N}}=\left\|\mathscr{T}_{k, n}^{\nu}(f)(x, \cdot)\right\|_{E_{N}}$, a.e. $x \in \mathbb{R}^{d}$.
By taking into account that $\mathbb{W}_{*}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$ as above we obtain that, for every $f \in L^{1}\left(\mathbb{R}^{d}, d x\right)$,

$$
\left\|\mathbb{T}_{k, n}^{\nu}(f)(x, \cdot)\right\|_{E_{N}}=\left\|\mathscr{T}_{k, n}^{\nu}(f)(x, \cdot)\right\|_{E_{N}}, \quad \text { a.e. } x \in \mathbb{R}^{d}
$$

Then, we get that, for every $1<p<\infty$,

$$
\sup _{N \in \mathbb{N}}\left\|\mathbb{T}_{k, n}^{\nu}\right\|_{L^{p}\left(\mathbb{R}^{d}, d x\right) \rightarrow L_{E_{N}}^{p}}\left(\mathbb{R}^{d}, d x\right)<\infty
$$

and

$$
\sup _{N \in \mathbb{N}}\left\|\mathbb{T}_{k, n}^{\nu}\right\|_{L^{1}\left(\mathbb{R}^{d}, d x\right) \rightarrow L_{E_{N}}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)}<\infty
$$

By using the monotone convergence theorem we conclude that the square function $\mathbb{G}_{k, n}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$, that is, $\mathbb{T}_{k, n}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{\left.L^{2}(0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

According to [18, Proposition 3.4] we deduce that the local operator $\mathbb{T}_{k, n, \text { loc }}^{\nu}$ is also bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

We conclude that $S_{k, j, \text { loc }}^{\nu, 2}$ is a bounded operator from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, for every $1<p<$ $\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$. Then, the same $L^{p}$-boundedness properties are obtained for $S_{k, j, \text { loc }}^{\nu}$ and by considering the comments after formula (3.1) the proof of (I1) can be finished.

Let us show now property (I2). Minkowski inequality leads to

$$
\left|\mathcal{D}_{k, j}^{\nu}(f)(x)\right| \leq \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)|\left\|K_{k, j}^{\nu}(x, y, \cdot)-S_{k, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} e^{-R(y)} d y, \quad x \in \mathbb{R}
$$

Our objective is to prove that

$$
\begin{equation*}
e^{-R(y)}\left\|K_{k, j}^{\nu}(x, y, \cdot)-S_{k, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.8}
\end{equation*}
$$

Thus, we obtain that $\mathcal{D}_{k, j}^{\nu}(f) \leq C \mathscr{S}_{\eta}(|f|)$ for $\eta=2 A$, and, consequently, $\mathcal{D}_{k, j}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself (and also from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself), for every $1 \leq p \leq \infty$.

Again by Minkowski inequality, it follows that

$$
\begin{aligned}
e^{-R(y)} \| K_{k, j}^{\nu}(x, y, \cdot)-S_{k, j}^{\nu} & (x, y, \cdot) \|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \\
& \leq C e^{-R(y)} \int_{0}^{\infty}\left\|\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\left|\partial_{x_{j}}\left(h_{u}(x, y)\right)-\mathbb{S}_{u}^{j}(x, y)\right| \frac{d u}{\sqrt{u}} \\
& \leq C e^{-R(y)} \int_{0}^{\infty}\left|\partial_{x_{j}}\left(h_{u}(x, y)\right)-\mathbb{S}_{u}^{j}(x, y)\right| \frac{d u}{\sqrt{u}} \\
& =C e^{-R(y)}\left(\int_{0}^{\mathfrak{m}(x)}+\int_{\mathfrak{m}(x)}^{\infty}\right)\left|\partial_{x_{j}}\left(h_{u}(x, y)\right)-\mathbb{S}_{u}^{j}(x, y)\right| \frac{d u}{\sqrt{u}} \\
& =I_{j}^{0}(x, y)+I_{j}^{\infty}(x, y), \quad x, y \in \mathbb{R}^{d}
\end{aligned}
$$

We first estimate $I_{j}^{\infty}(x, y),(x, y) \in L_{2 A}$. Here the cancellation does not play now any role.
By using (2.4) and (2.5) we get

$$
\begin{equation*}
\left|\partial_{x_{j}} h_{u}(x, y)\right| \leq C \frac{e^{R(x)}}{u^{\frac{d}{2}}}(1+|x|), \quad x, y \in \mathbb{R}^{d}, \mathfrak{m}(x) \leq u \leq 1 \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\partial_{x_{j}} h_{u}(x, y)\right| \leq C e^{R(x)} e^{-c u}(1+|y|), \quad x, y \in \mathbb{R}^{d}, u>1 . \tag{3.10}
\end{equation*}
$$

On the other hand,

$$
\left|\mathbb{S}_{u}^{j}(x, y)\right| \leq C \frac{e^{R(x)}}{u^{\frac{d}{2}}} \frac{|y-x|}{u} e^{-c \frac{|y-x|^{2}}{u}} \leq C \frac{e^{R(x)}}{u^{\frac{d+1}{2}}} \leq C \frac{e^{R(x)}}{u^{\frac{d}{2}}}(1+|x|), \quad x, y \in \mathbb{R}^{d} \text { and } \mathfrak{m}(x) \leq u
$$

Thus, by using also (3.3) with $\eta=2 A$ and taking into account that if $(x, y) \in L_{2 A}$, then $|x-y| \leq \frac{2 A}{1+|x|} \leq$ $2 A \sqrt{\mathfrak{m}(x)} \leq 2 A$ and $|y| \leq C(1+|x|)$, we obtain

$$
\begin{aligned}
I_{j}^{\infty}(x, y) & \leq C e^{R(x)-R(y)}(1+|x|)\left(\int_{\mathfrak{m}(x)}^{1} \frac{d u}{u^{\frac{d+1}{2}}}+\int_{1}^{\infty}\left(e^{-c u}+\frac{1}{u^{\frac{d}{2}}}\right) \frac{d u}{\sqrt{u}}\right) \\
& \leq C(1+|x|)\left(\int_{\mathfrak{m}(x)}^{\infty} u^{-\frac{d+1}{2}} d u+1\right) \leq C(1+|x|)\left(\frac{1}{\mathfrak{m}(x)^{\frac{d-1}{2}}}+1\right) \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A}
\end{aligned}
$$

Next we analyze the term $I_{j}^{0}(x, y),(x, y) \in L_{2 A}$. Now the cancellation property of the difference is relevant to get the bound we need. We consider

$$
\begin{equation*}
\widetilde{H}_{u}(x, y)=\left(\frac{\operatorname{det} Q_{\infty}}{u^{d} \operatorname{det} Q}\right)^{1 / 2} e^{R(x)} e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{u} x\right), y-D_{u} x\right\rangle}, \quad x, y \in \mathbb{R}^{d} \text { and } u>0 \tag{3.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{W}_{u}(x, y)=\left(\frac{\operatorname{det} Q_{\infty}}{u^{d} \operatorname{det} Q}\right)^{1 / 2} e^{R(x)} e^{-\frac{1}{2 u}\left\langle Q^{-1}(y-x), y-x\right\rangle}, \quad x, y \in \mathbb{R}^{d} \text { and } u>0 \tag{3.13}
\end{equation*}
$$

We note that $h_{u}(x, y)=\left(u^{d} \operatorname{det} Q\right)^{1 / 2}\left(\operatorname{det} Q_{u}\right)^{-1 / 2} \widetilde{H}_{u}(x, y), x, y \in \mathbb{R}^{d}$ and $u>0$ and that

$$
\begin{equation*}
\partial_{x_{j}} \widetilde{W}_{u}(x, y)=\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle \widetilde{W}_{u}(x, y)+\mathbb{S}_{u}^{j}(x, y), \quad x, y \in \mathbb{R}^{d} \text { and } u>0 \tag{3.14}
\end{equation*}
$$

Hence, we can write

$$
\begin{align*}
I_{j}^{0}(x, y) \leq & C e^{-R(y)} \int_{0}^{\mathfrak{m}(x)}\left(\left|\partial_{x_{j}}\left[h_{u}(x, y)-\widetilde{W}_{u}(x, y)\right]\right|+\left|\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle\right| \widetilde{W}_{u}(x, y)\right) \frac{d u}{\sqrt{u}} \\
\leq & C e^{-R(y)}\left(\int_{0}^{\mathfrak{m}(x)}\left(\left|\left(u^{d} \operatorname{det} Q\right)^{-1 / 2}\left(\operatorname{det} Q_{u}\right)^{1 / 2}-1\right|\left|\partial_{x_{j}} h_{u}(x, y)\right|\right) \frac{d u}{\sqrt{u}}\right. \\
& \left.+\int_{0}^{\mathfrak{m}(x)}\left|\partial_{x_{j}}\left[\widetilde{H}_{u}(x, y)-\widetilde{W}_{u}(x, y)\right]\right| \frac{d u}{\sqrt{u}}+\int_{0}^{\mathfrak{m}(x)}\left|\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle\right| \widetilde{W}_{u}(x, y) \frac{d u}{\sqrt{u}}\right) \\
= & I_{j}^{0,1}(x, y)+I_{j}^{0,2}(x, y)+I_{j}^{0,3}(x, y), \quad x, y \in \mathbb{R}^{d} . \tag{3.15}
\end{align*}
$$

From (3.3) and since $Q$ is symmetric and positive definite we deduce that

$$
\begin{equation*}
I_{j}^{0,3}(x, y) \leq C(1+|x|) \int_{0}^{\mathfrak{m}(x)} \frac{e^{-c \frac{|x-y|^{2}}{u}}}{u^{\frac{d+1}{2}}} d u \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.16}
\end{equation*}
$$

Next, we analyze $I_{j}^{0,1}(x, y),(x, y) \in L_{2 A}$. We recall that $Q_{t}=\int_{0}^{t} e^{B s} Q e^{B^{*} s} d s, t>0$. We can write

$$
\frac{1}{t} Q_{t}-Q=\frac{1}{t} \int_{0}^{t}\left(e^{B s}-I\right) Q e^{B^{*} s} d s+\frac{1}{t} \int_{0}^{t} Q\left(e^{B^{*} s}-I\right) d s, \quad t>0
$$

Then, since $\left|e^{B s}-I\right| \leq C s, s \in(0,1)$, it follows that $\lim _{t \rightarrow 0^{+}} \frac{1}{t} Q_{t}=Q$. Hence,

$$
\lim _{t \rightarrow 0^{+}} \operatorname{det}\left(\frac{1}{t} Q_{t}\right)=\lim _{t \rightarrow 0^{+}} \frac{1}{t^{d}} \operatorname{det}\left(Q_{t}\right)=\operatorname{det}(Q)
$$

Furthermore, we have that $\left|\frac{1}{t} Q_{t}-Q\right| \leq C t, t \in(0,1)$. We write $Q_{t}=\left(q_{i j}^{(t)}\right)_{i, j=1}^{d}$ and $Q=\left(q_{i j}\right)_{i, j=1}^{d}$. Then, $\frac{1}{t} q_{i j}^{(t)}=q_{i j}+\phi_{i j}(t), t \in(0,1)$, where $\left|\phi_{i j}(t)\right| \leq C t, t \in(0,1)$. We get $\frac{1}{t^{d}} \operatorname{det}\left(Q_{t}\right)=\operatorname{det}(Q)+\phi(t), t \in(0,1)$, being $|\phi(t)| \leq C t, t \in(0,1)$. According to [10, Lemma 2.2] we obtain

$$
\begin{align*}
\left|\frac{\left(\operatorname{det} Q_{u}\right)^{1 / 2}}{\left(u^{d} \operatorname{det} Q\right)^{1 / 2}}-1\right| & =\frac{\left|\left(\operatorname{det} Q_{u}\right)^{1 / 2}-u^{\frac{d}{2}}(\operatorname{det} Q)^{1 / 2}\right|}{\left(u^{d} \operatorname{det} Q\right)^{1 / 2}} \\
& =\frac{\left|\operatorname{det} Q_{u}-u^{d} \operatorname{det} Q\right|}{\left(u^{d} \operatorname{det} Q\right)^{1 / 2}\left[\left(\operatorname{det} Q_{u}\right)^{1 / 2}+u^{\frac{d}{2}}(\operatorname{det} Q)^{1 / 2}\right)} \leq C|\phi(u)| \leq C u, \quad u \in(0,1) \tag{3.17}
\end{align*}
$$

By [10, Lemma 2.3] we get

$$
\begin{equation*}
\left|y-D_{t} x\right| \geq|x-y|-\left|x-D_{t} x\right| \geq|y-x|-C t|x| \geq|y-x|-C, \quad x, y \in \mathbb{R}^{d} \text { and } 0<t<\mathfrak{m}(x) \tag{3.18}
\end{equation*}
$$

Then, by taking into account also (2.4) and (3.3) it follows that

$$
\begin{align*}
I_{j}^{0,1}(x, y) & \leq C e^{R(x)-R(y)} \int_{0}^{\mathfrak{m}(x)} e^{-c \frac{|y-x|^{2}}{u}}\left(|x|+\frac{1}{\sqrt{u}}\right) \frac{d u}{u^{\frac{d-1}{2}}} \\
& \leq C(1+|x|) \int_{0}^{\mathfrak{m}(x)} e^{-c \frac{|y-x|^{2}}{u}} \frac{d u}{u^{\frac{d}{2}}} \leq C(1+|x|) \int_{0}^{\mathfrak{m}(x)} e^{-c \frac{|y-x|^{2}}{u}} \frac{d u}{u^{\frac{d+1}{2}}} \\
& \leq C(1+|x|) \int_{0}^{\infty} e^{-c \frac{|y-x|^{2}}{u}} \frac{d u}{u^{\frac{d+1}{2}}} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.19}
\end{align*}
$$

To deal with $I_{j}^{0,2}(x, y),(x, y) \in L_{2 A}$, we write, according to (2.2),

$$
\partial_{x_{j}} \tilde{H}_{u}(x, y)=\left(\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle+\left\langle Q_{u}^{-1} e^{u B} e_{j}, y-D_{u} x\right\rangle\right) \widetilde{H}_{u}(x, y), \quad x, y \in \mathbb{R}^{d} \text { and } u>0
$$

and consider (3.14) to obtain

$$
\begin{align*}
\left|\partial_{x_{j}}\left[\widetilde{H}_{u}(x, y)-\widetilde{W}_{u}(x, y)\right]\right| \leq & \left|\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\right|\left|\widetilde{H}_{u}(x, y)-\widetilde{W}_{u}(x, y)\right| \\
& +\left|\left\langle Q_{u}^{-1} e^{u B} e_{j}, y-D_{u} x\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\right| \widetilde{H}_{u}(x, y) \\
\leq & \left|\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\right|\left|\widetilde{H}_{u}(x, y)-\widetilde{W}_{u}(x, y)\right| \tag{3.20}
\end{align*}
$$

$$
\begin{aligned}
& +\left(\left|\left\langle Q_{u}^{-1} e^{u B} e_{j}-\frac{1}{u} Q^{-1} e_{j}, y-D_{u} x\right\rangle\right|\right. \\
& \left.+\left|\left\langle\frac{1}{u} Q^{-1} e_{j}, x-D_{u} x\right\rangle\right|\right) \widetilde{H}_{u}(x, y), \quad x, y \in \mathbb{R}^{d} \text { and } u>0 .
\end{aligned}
$$

We now establish some useful estimates.
(i) We can write

$$
\begin{aligned}
Q_{t}^{-1} e^{t B} e_{j}-\frac{1}{t} Q^{-1} e_{j} & =Q_{t}^{-1} e^{t B} e_{j}-\frac{1}{t} Q^{-1} e^{t B} e_{j}+\frac{1}{t} Q^{-1}\left(e^{t B}-I\right) e_{j} \\
& =Q_{t}^{-1}\left(t Q-Q_{t}\right) \frac{1}{t} Q^{-1} e^{t B} e_{j}+\frac{1}{t} Q^{-1}\left(e^{t B}-I\right) e_{j}, \quad t>0
\end{aligned}
$$

and

$$
e^{t B^{*}} Q_{t}^{-1} e^{t B} e_{j}-\frac{1}{t} Q^{-1} e_{j}=\left(e^{t B^{*}}-I\right) Q_{t}^{-1} e^{t B} e_{j}-\left(Q_{t}^{-1} e^{t B} e_{j}-\frac{1}{t} Q^{-1} e_{j}\right), \quad t>0
$$

By using [10, Lemma 2.2], since $\left|t Q-Q_{t}\right| \leq C t^{2}, t \in(0,1)$, we get

$$
\begin{equation*}
\left|Q_{t}^{-1} e^{t B} e_{j}-\frac{1}{t} Q^{-1} e_{j}\right|+\left|e^{t B^{*}} Q_{t}^{-1} e^{t B} e_{j}-\frac{1}{t} Q^{-1} e_{j}\right| \leq C, \quad t \in(0,1) . \tag{3.21}
\end{equation*}
$$

(ii) By [10, Lemma 2.3] we obtain

$$
\begin{equation*}
\left|y-D_{t} x\right| \leq|y-x|+C t|x|, \quad x, y \in \mathbb{R}^{d} \text { and } t \in(0,1) . \tag{3.22}
\end{equation*}
$$

(iii) For every $t>0$, since $Q_{t}$ is a symmetric positive definite matrix, $\left|Q_{t}^{1 / 2}\right|=\left|Q_{t}\right|^{1 / 2}$. We deduce that

$$
|z|=\left|Q_{t}^{1 / 2} Q_{t}^{-1 / 2} z\right| \leq\left|Q_{t}\right|^{1 / 2}\left|Q_{t}^{-1 / 2} z\right|, \quad z \in \mathbb{R}^{d} \text { and } t>0
$$

Furthermore,

$$
\left|Q_{t}\right| \leq|Q| \int_{0}^{t} e^{s\left(|B|+\left|B^{*}\right|\right)} d s \leq C t, \quad t \in(0,1) .
$$

Then, we get

$$
\begin{equation*}
\left|Q_{t}^{-1 / 2} z\right| \geq C \frac{|z|}{\sqrt{t}}, \quad z \in \mathbb{R}^{d} \text { and } t \in(0,1) . \tag{3.23}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
e^{-\frac{1}{2}\left\langle Q_{t}^{-1}\left(y-D_{t} x\right), y-D_{t} x\right\rangle}=e^{-\frac{1}{2}\left|Q_{t}^{-1 / 2}\left(y-D_{t} x\right)\right|^{2}} \leq C e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}, \quad x, y \in \mathbb{R}^{d} \text { and } t \in(0,1) . \tag{3.24}
\end{equation*}
$$

According to [10, (2.10)] we obtain

$$
\begin{equation*}
e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle} \leq C e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}, \quad x, y \in \mathbb{R}^{d} \text { and } t \in(0,1) . \tag{3.25}
\end{equation*}
$$

We can write

$$
e^{-\frac{1}{2}\left\langle Q_{t}^{-1}(y-x), y-D_{t} x\right\rangle}=e^{-\frac{1}{2}\left\langle Q_{t}^{-1}(y-x), y-x\right\rangle} e^{-\frac{1}{2}\left\langle Q_{t}^{-1}(y-x), x-D_{t} x\right\rangle}, \quad x, y \in \mathbb{R}^{d} \text { and } t>0 .
$$

By [10, Lemmas 2.2, (ii), and 2.3] we have that

$$
\begin{aligned}
-\left\langle Q_{t}^{-1}(y-x), x-D_{t} x\right\rangle & \leq\left|\left\langle Q_{t}^{-1}(y-x), x-D_{t} x\right\rangle\right| \leq\left|Q_{t}^{-1}(y-x)\right|\left|x-D_{t} x\right| \\
& \leq C|y-x||x| \leq C \frac{|x|}{1+|x|} \leq C, \quad(x, y) \in L_{2 A} \text { and } t \in(0,1) .
\end{aligned}
$$

Then, by (3.23)

$$
\begin{align*}
e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}(y-x), y-D_{t} x\right\rangle\right.} & \leq C e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}(y-x), y-x\right\rangle\right.} \leq C e^{-\frac{1}{2}\left|Q_{t}^{-1 / 2}(y-x)\right|^{2}} \\
& \leq C e^{-c \frac{|x-y|^{2}}{t}}, \quad(x, y) \in L_{2 A} \text { and } t \in(0,1) . \tag{3.26}
\end{align*}
$$

(iv) In the following three estimates we use again [10, Lemmas 2.2 and 2.3]. It follows that

$$
\begin{align*}
& \mid\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle- \\
& \text { 7) } \\
& \begin{array}{ll} 
& \left\langle Q_{t}^{-1}\left(y-D_{t} x\right), y-D_{t} x\right\rangle \mid \\
\left|\left\langle Q_{t}^{-1}\left(y-D_{t} x\right), y-D_{t} x\right\rangle-\left\langle Q_{t}^{-1}(y-x), y-D_{t} x\right\rangle \leq\right| & \left.\leq Q_{t}^{-1}\left(x-D_{t} x\right), y-D_{t} x\right\rangle \mid \\
\leq & \leq\left|Q_{t}^{-1}\left(x-D_{t} x\right)\right|\left|y-D_{t} x\right| \leq C|x|(|y-x|+t|x|), \quad x, y \in \mathbb{R}^{d} \text { and } t \in(0,1) ;
\end{array}
\end{align*}
$$

$$
\begin{align*}
\left|\left\langle Q_{t}^{-1}(y-x), y-D_{t} x\right\rangle-\left\langle Q_{t}^{-1}(y-x), y-x\right\rangle\right| & \leq\left|Q_{t}^{-1}(y-x)\right|\left|x-D_{t} x\right| \\
& \leq C|y-x||x|, \quad x, y \in \mathbb{R}^{d} \text { and } t \in(0,1) \tag{3.29}
\end{align*}
$$

and

$$
\begin{align*}
\left|\left\langle Q_{t}^{-1}(y-x), y-x\right\rangle-\frac{1}{t}\left\langle Q^{-1}(y-x), y-x\right\rangle\right| & \leq\left|\left(Q_{t}^{-1}-\frac{1}{t} Q^{-1}\right)(y-x) \| y-x\right| \\
& \leq C|y-x|^{2}, \quad x, y \in \mathbb{R}^{d} \text { and } t \in(0,1) \tag{3.30}
\end{align*}
$$

(v) We have that, for every $x, y \in \mathbb{R}^{d}$ and $t>0$,

$$
\begin{aligned}
& e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle}-e^{-\frac{1}{2 t}\left\langle Q^{-1}(y-x), y-x\right\rangle} \\
&=\left(e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle}-e^{-\frac{1}{2}\left\langle Q_{t}^{-1}\left(y-D_{t} x\right), y-D_{t} x\right\rangle}\right)+\left(e^{-\frac{1}{2}\left\langle Q_{t}^{-1}\left(y-D_{t} x\right), y-D_{t} x\right\rangle}-e^{-\frac{1}{2}\left\langle Q_{t}^{-1}(y-x), y-D_{t} x\right\rangle}\right) \\
&+\left(e^{-\frac{1}{2}\left\langle Q_{t}^{-1}(y-x), y-D_{t} x\right\rangle}-e^{-\frac{1}{2}\left\langle Q_{t}^{-1}(y-x), y-x\right\rangle}\right)+\left(e^{-\frac{1}{2}\left\langle Q_{t}^{-1}(y-x), y-x\right\rangle}-e^{-\frac{1}{2 t}\left\langle Q^{-1}(y-x), y-x\right\rangle}\right) .
\end{aligned}
$$

We now use the elementary inequality

$$
\left|e^{-a}-e^{-b}\right| \leq|b-a| e^{-\min \{a, b\}}, \quad a, b \in \mathbb{R}
$$

By estimations (3.24)- (3.30) we get

$$
\begin{aligned}
& \left|e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle}-e^{-\frac{1}{2 t}\left\langle Q^{-1}(y-x), y-x\right\rangle}\right| \\
& \leq \\
& \quad C\left[\left(|y-x|^{2}+t^{2}|x|^{2}\right) e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}+|x|(|y-x|+t|x|)\left(e^{-c \frac{\left|y-D_{t} x\right|^{2}}{t}}+e^{-c \frac{|y-x|^{2}}{t}}\right)\right. \\
& \\
& \left.\quad+\left(|y-x||x|+|y-x|^{2}\right) e^{-c \frac{|y-x|^{2}}{t}}\right], \quad(x, y) \in L_{2 A} \text { and } t \in(0,1)
\end{aligned}
$$

By (3.18) we deduce that

$$
\begin{align*}
& \left|e^{-\frac{1}{2}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle}-e^{-\frac{1}{2 t}\left\langle Q^{-1}(y-x), y-x\right\rangle}\right| \leq C e^{-c \frac{|y-x|^{2}}{t}}\left(|y-x|^{2}+|y-x||x|+t|x|^{2}\right) \\
& \quad \leq C e^{-c \frac{|y-x|^{2}}{t}}\left(|y-x|^{2}+|y-x||x|+t|x|^{2}\right) \leq C e^{-c \frac{|y-x|^{2}}{t}}\left(t+\sqrt{t}|x|+t|x|^{2}\right) \\
& \quad \leq C e^{-c \frac{|y-x|^{2}}{t}} \sqrt{t}(1+|x|), \quad(x, y) \in L_{2 A} \text { and } 0<t<\mathfrak{m}(x) \tag{3.31}
\end{align*}
$$

In the last inequality we have used that $\mathfrak{m}(x) \sim\left(1+|x|^{2}\right)^{-1}, x \in \mathbb{R}^{d}$.
Hence, by using (3.3) and (3.31) and that $\sqrt{u}|x| \leq C$, when $x \in \mathbb{R}^{d}$ and $0<u<\mathfrak{m}(x)$, we obtain that

$$
\begin{aligned}
e^{-R(y)} \mid\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle- & \left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\left|\left|\widetilde{H}_{u}(x, y)-\widetilde{W}_{u}(x, y)\right| \leq C\left(|x|+\frac{|y-x|}{u}\right) \frac{e^{R(x)-R(y)}}{u^{\frac{d}{2}}} e^{-c \frac{|y-x|^{2}}{u}} \sqrt{u}(1+|x|)\right. \\
& \leq C(\sqrt{u}|x|+1) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}(1+|x|) \leq C \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}(1+|x|), \quad(x, y) \in L_{2 A} \text { and } 0<u<\mathfrak{m}(x) .
\end{aligned}
$$

On the other hand, from (3.21), (3.25), [10, Lemma 2.3], and again (3.3) and (3.18), it follows that

$$
\begin{gathered}
e^{-R(y)}\left(\left|\left\langle Q_{u}^{-1} e^{u B} e_{j}-\frac{1}{u} Q^{-1} e_{j}, y-D_{u} x\right\rangle\right|+\left|\left\langle\frac{1}{u} Q^{-1} e_{j}, x-D_{u} x\right\rangle\right|\right) \widetilde{H}_{u}(x, y) \leq C e^{-R(y)}\left(\left|y-D_{u} x\right|+|x|\right) \widetilde{H}_{u}(x, y) \\
\leq C\left(\left|y-D_{u} x\right|+|x|\right) \frac{e^{R(x)-R(y)}}{u^{\frac{d}{2}}} e^{-c \frac{\left|y-D_{u} x\right|^{2}}{u}} \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}, \quad(x, y) \in L_{2 A} \text { and } 0<u<\mathfrak{m}(x)
\end{gathered}
$$

These estimations, jointly (3.20), lead to

$$
e^{-R(y)}\left|\partial_{x_{j}}\left[\widetilde{H}_{u}(x, y)-\widetilde{W}_{u}(x, y)\right]\right| \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}, \quad(x, y) \in L_{2 A} \text { and } 0<u<\mathfrak{m}(x)
$$

and we conclude that

$$
\begin{equation*}
I_{j}^{0,2}(x, y) \leq C(1+|x|) \int_{0}^{\mathfrak{m}(x)} \frac{e^{-c \frac{|x-y|^{2}}{u}}}{u^{\frac{d+1}{2}}} d u \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.32}
\end{equation*}
$$

By considering (3.11), (3.15), (3.16), (3.19) and(3.32) we get the estimation (3.8) and property (I2) is established.
3.1.2. About the local part of $g_{k, \alpha}^{\nu}: \widehat{\alpha}=2$. Assume that $i, j \in\{1, \ldots, d\}$ and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{d}\right)$ being $\alpha_{i}=\alpha_{j}=1$ and $\alpha_{\ell}=0, \ell \in\{1, \ldots, d\} \backslash\{i, j\}$, when $i \neq j$, and $\alpha_{i}=2$ and $\alpha_{\ell}=0, \ell \in\{1, \ldots, d\} \backslash\{i\}$, when $i=j$.

In this case we have

$$
g_{k, \alpha, \mathrm{loc}}^{\nu}(f)(x)=\left\|\int_{\mathbb{R}^{d}} K_{k, i, j}^{\nu}(x, y, \cdot) \varphi_{\mathrm{A}}(x, y) f(y) d \gamma_{\infty}(y)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}, \quad x \in \mathbb{R}^{d}
$$

where

$$
K_{k, i, j}^{\nu}(x, y, t)=\frac{4}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+2} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{x_{i} x_{j}}^{2} h_{u}(x, y) d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

We consider the operator $S_{k, i, j}^{\nu}$ defined by

$$
S_{k, i, j}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} S_{k, i, j}^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y) \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
S_{k, i, j}^{\nu}(x, y, t)=\frac{4}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+2} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \mathbb{S}_{u}^{i, j}(x, y) d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

and

$$
\begin{equation*}
\mathbb{S}_{u}^{i, j}(x, y)=\left(\frac{\operatorname{det} Q_{\infty}}{u^{d} \operatorname{det} Q}\right)^{1 / 2} e^{R(x)} \partial_{x_{i} x_{j}}^{2}\left[e^{-\frac{1}{2 u}\left\langle Q^{-1}(y-x), y-x\right\rangle}\right], \quad x, y \in \mathbb{R}^{d} \text { and } u>0 \tag{3.33}
\end{equation*}
$$

We also introduce the operator $\widetilde{S}_{k, i, j}^{\nu}$ by

$$
\widetilde{S}_{k, i, j}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} \widetilde{S}_{k, i, j}^{\nu}(x, y, t) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

being

$$
\widetilde{S}_{k, i, j}^{\nu}(x, y, t)=e^{-R\left(Q^{1 / 2} y\right)} S_{k, i, j}^{\nu}\left(Q^{1 / 2} x, Q^{1 / 2} y, t\right), \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

We define, for every $\ell, m=1, \ldots, d$, the operators

$$
T_{k, \ell, m}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} e^{R\left(Q^{1 / 2} x\right)-R\left(Q^{1 / 2} y\right)} \mathbb{T}_{k, \ell, m}^{\nu}(x-y, t) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

and

$$
\mathbb{T}_{k, \ell, m}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} \mathbb{T}_{k, \ell, m}^{\nu}(x-y, t) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
\mathbb{T}_{k, \ell, m}^{\nu}(z, t)=\frac{4}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+2} \partial_{s}^{k}\left(\mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{z_{\ell} z_{m}}^{2} \mathbb{W}_{u}(z) d u, \quad z=\left(z_{1}, \ldots, z_{d}\right) \in \mathbb{R}^{d} \text { and } t>0\right.
$$

Let $\ell, m \in\{1, \ldots, d\}$. For every $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ we have that

$$
\mathbb{T}_{k, \ell, m}^{\nu}(f)(x, t)=t^{k+2} \partial_{t}^{k} \partial_{x_{\ell} x_{m}}^{2} \mathbb{P}_{t}^{\nu}(f)(x), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

We consider the square function $\mathbb{G}_{k, \ell, m}^{\nu}$ defined by

$$
\mathbb{G}_{k, \ell, m}^{\nu}(f)(x)=\left(\int_{0}^{\infty}\left|\mathbb{T}_{k, \ell, m}^{\nu}(f)(x, t)\right|^{2} \frac{d t}{t}\right)^{1 / 2}, \quad x \in \mathbb{R}^{d}
$$

We are going to see the $L^{p}$-boundedness properties of $\mathbb{G}_{k, \ell, m}^{\nu}$ by using vector valued Calderón-Zygmund theory. The arguments are similar to the ones used for the studying of $\mathbb{G}_{k, n}^{\nu}$ in the Section 3.1.1. We sketch the proof in this case.

The operator $\mathbb{T}_{k, \ell, m}^{\nu}$ can be seen as a $L^{2}\left((0, \infty), \frac{d t}{t}\right)$-singular integral having as integral kernel $t \rightarrow \mathbb{T}_{k, \ell, m}^{\nu}(x-$ $y, t), t \in(0, \infty)$, for every $x, y \in \mathbb{R}^{d}$. Minkowski inequality leads to

$$
\begin{aligned}
\left\|\mathbb{T}_{k, \ell, m}^{\nu}(z, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} & \leq C \int_{0}^{\infty} \|\left[s^{k+2} \partial_{s}^{k}\left(\mathfrak{g}_{\nu}(s)\right]_{s=\frac{t}{2 \sqrt{u}}} \|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\left|\partial_{z_{\ell} z_{m}}^{2}\left(e^{-\frac{|z|^{2}}{2 u}}\right)\right| \frac{d u}{u^{\frac{d}{2}}}\right. \\
& \leq C \int_{0}^{\infty} \frac{|z|^{2}}{u^{\frac{d}{2}+2}} e^{-\frac{|z|^{2}}{4 u}} d u \leq C \int_{0}^{\infty} \frac{e^{-c \frac{|z|^{2}}{4 u}}}{u^{\frac{d}{2}+1}} d u \leq \frac{C}{|z|^{d}}, \quad z \in \mathbb{R}^{d}, z \neq 0
\end{aligned}
$$

In a similar way we can see that,

$$
\sum_{i=1}^{d}\left\|\partial_{z_{i}} \mathbb{T}_{k, \ell, m}^{\nu}(z, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq \frac{C}{|z|^{d+1}}, \quad z \in \mathbb{R}^{d}, z \neq 0
$$

By using Fourier transform we can see that the square function $\mathbb{G}_{k, \ell, m}^{\nu}$ is bounded from $L^{2}\left(\mathbb{R}^{d}, d x\right)$ into itself.
Arguing as in the Section 3.1.1 for $\mathbb{G}_{k, n}^{\nu}$ and by using Calderón-Zygmund theory for vector valued singular integrals, we can conclude that $\mathbb{G}_{k, \ell, m}^{\nu}$ is a bounded operator from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$. According to [18, Proposition 3.4] the local operator $\mathbb{T}_{k, \ell, m, \text { loc }}^{\nu}$ defined in the usual way is a bounded operator from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, d x\right)$, when $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

We can now follow the proof of (I1) in Section 3.1.1 to establish that $S_{k, i, j, \text { loc }}^{\nu}$ is a bounded operator $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, when $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. For that, we have to take into account that $T_{k, \ell, m}^{\nu}(f)(x, t)=e^{R\left(Q^{1 / 2} x\right)} \mathbb{T}_{k, \ell, m}^{\nu}(f)(x, t), x \in \mathbb{R}^{d}$ and $t>0$, and also that $\widetilde{S}_{k, i, j, \text { loc }}^{\nu}(f)=\sum_{\ell, m=1}^{d} a_{\ell, m}^{i, j} T_{k, \ell, m, \text { loc }}^{\nu}$, for certain constants $a_{\ell, m}^{i, j} \in \mathbb{R}$. To see this last property it is sufficient to note that

$$
\begin{aligned}
\partial_{x_{i} x_{j}}^{2} e^{-\frac{1}{2 t}\left\langle Q^{-1}(y-x), y-x\right\rangle}= & \left(\left\langle\frac{1}{t} Q^{-1} e_{j}, y-x\right\rangle\left\langle\frac{1}{t} Q^{-1} e_{i}, y-x\right\rangle-\left\langle e_{j}, \frac{1}{t} Q^{-1} e_{i}\right\rangle\right) e^{-\frac{1}{2 t}\left\langle Q^{-1}(y-x), y-x\right\rangle} \\
34) & \left(\left\langle\frac{1}{t} Q^{-1 / 2} e_{j}, Q^{-1 / 2}(y-x)\right\rangle\left\langle\frac{1}{t} Q^{-1 / 2} e_{i}, Q^{-1 / 2}(y-x)\right\rangle-\left\langle Q^{-1 / 2} e_{j}, \frac{1}{t} Q^{-1 / 2} e_{i}\right\rangle\right) \\
& \times e^{-\frac{1}{2 t}\left\langle Q^{-1 / 2}(y-x), Q^{-1 / 2}(y-x)\right\rangle}, \quad x, y \in \mathbb{R}^{d} \text { and } t>0 .
\end{aligned}
$$

Suppose that $Q^{-1 / 2}=\left(c_{\ell m}\right)_{\ell, m=1}^{d}$ with $c_{\ell m} \in \mathbb{R}, \ell, m=1, \ldots, d$. It follows that

$$
\begin{aligned}
\left\langle Q^{-1 / 2} e_{j}, y-x\right\rangle\left\langle Q^{-1 / 2} e_{i}, y-x\right\rangle & =\left\langle\left(c_{\ell j}\right)_{\ell=1}^{d}, y-x\right\rangle\left\langle\left(c_{\ell i}\right)_{\ell=1}^{d}, y-x\right\rangle=\left(\sum_{\ell=1}^{d} c_{\ell j}\left(y_{\ell}-x_{\ell}\right)\right)\left(\sum_{\ell=1}^{d} c_{\ell i}\left(y_{\ell}-x_{\ell}\right)\right) \\
& =\sum_{\ell=1}^{d} c_{\ell j} c_{\ell i}\left(y_{\ell}-x_{\ell}\right)^{2}+\sum_{\substack{\ell, m=1 \\
\ell \neq m}}^{d} c_{\ell j} c_{m i}\left(y_{\ell}-x_{\ell}\right)\left(y_{m}-x_{m}\right)
\end{aligned}
$$

for every $x=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d}, y=\left(y_{1}, \ldots, y_{d}\right)$, and

$$
\left\langle Q^{-1 / 2} e_{j}, Q^{-1 / 2} e_{i}\right\rangle=\left\langle\left(c_{\ell j}\right)_{\ell=1}^{d},\left(c_{\ell i}\right)_{\ell=1}^{d}\right\rangle=\sum_{\ell=1}^{d} c_{\ell j} c_{\ell i}
$$

We get

$$
\begin{aligned}
\left(\left\langle\frac{1}{t} Q^{-1 / 2} e_{j}, y-x\right\rangle\right. & \left.\left\langle\frac{1}{t} Q^{-1 / 2} e_{i}, y-x\right\rangle-\left\langle Q^{-1 / 2} e_{j}, \frac{1}{t} Q^{-1 / 2} e_{i}\right\rangle\right) e^{-\frac{|y-x|^{2}}{2 t}} \\
& =\left(\frac{1}{t^{2}} \sum_{\substack{\ell, m=1 \\
l \neq m}}^{d} c_{\ell j} c_{m i}\left(y_{\ell}-x_{\ell}\right)\left(y_{m}-x_{m}\right)+\sum_{\ell=1}^{d} c_{\ell j} c_{\ell i}\left(\frac{\left(y_{\ell}-x_{\ell}\right)^{2}}{t^{2}}-1\right)\right) e^{-\frac{|y-x|^{2}}{2 t}} \\
& =\sum_{\substack{\ell, m=1 \\
\ell \neq m}}^{d} c_{\ell j} c_{m i} \partial_{x_{\ell} x_{m}}^{2} e^{-\frac{|y-x|^{2}}{2 t}}+\sum_{\ell=1}^{d} c_{\ell j} c_{\ell i} \partial_{x_{\ell} x_{\ell}}^{2} e^{-\frac{|y-x|^{2}}{2 t}}, \quad x, y \in \mathbb{R}^{d} \text { and } t>0 .
\end{aligned}
$$

As in the Section 3.1.1 in order to finish the proof it remains to establish that the operator $\mathcal{D}_{k, i, j}^{\nu}(f)(x)=$ $g_{k, \alpha, \text { loc }}^{\nu}(f)(x)-\left\|S_{k, i, j, \text { loc }}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}, x \in \mathbb{R}^{d}$, is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for each $1 \leq p \leq \infty$.

Minkowski inequality leads to

$$
e^{-R(y)}\left|\mathcal{D}_{k, i, j}^{\nu}(f)(x)\right| \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)|\left\|K_{k, i, j}^{\nu}(x, y, \cdot)-S_{k, i, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} e^{-R(y)} d y, \quad x \in \mathbb{R}^{d}
$$

Thus, it is sufficient to prove that

$$
\begin{equation*}
e^{-R(y)}\left\|K_{k, i, j}^{\nu}(x, y, \cdot)-S_{k, i, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.35}
\end{equation*}
$$

Again, by Minkowski inequality we can write

$$
\begin{aligned}
e^{-R(y)} \| & K_{k, i, j}^{\nu}(x, y, \cdot)-S_{k, i, j}^{\nu}(x, y, \cdot) \|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \\
& \leq C e^{-R(y)} \int_{0}^{\infty}\left\|\left[s^{k+2} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\left|\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y)\right| d u \\
& \leq C e^{-R(y)}\left(\int_{0}^{\mathfrak{m}(x)}+\int_{\mathfrak{m}(x)}^{\infty}\right)\left|\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y)\right| d u \\
& =I_{i, j}^{0}(x, y)+I_{i, j}^{\infty}(x, y), \quad x, y \in \mathbb{R}^{d} .
\end{aligned}
$$

First, we observe that from (3.3) and (3.34) we obtain that

$$
\begin{aligned}
e^{-R(y)}\left|\mathbb{S}_{u}^{i, j}(x, y)\right| & =C \frac{e^{R(x)-R(y)}}{u^{\frac{d}{2}}}\left|\partial_{x_{i} x_{j}}^{2}\left(e^{-\frac{1}{2 u}\left\langle Q^{-1}(y-x), y-x\right\rangle}\right)\right| \leq C \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}\left(\frac{|y-x|^{2}}{u^{2}}+\frac{1}{u}\right) \\
& \leq \frac{C}{u^{\frac{d}{2}+1}}, \quad(x, y) \in L_{2 A} \text { and } u>0 .
\end{aligned}
$$

Then, since $\mathfrak{m}(x) \sim(1+|x|)^{-2}, x \in \mathbb{R}^{d}$,

$$
\begin{equation*}
\int_{\mathfrak{m}(x)}^{\infty} e^{-R(y)}\left|\mathbb{S}_{u}^{i, j}(x, y)\right| d u \leq C \int_{\mathfrak{m}(x)}^{\infty} \frac{d u}{u^{\frac{d}{2}+1}} \leq \frac{C}{\mathfrak{m}(x)^{\frac{d}{2}}} \leq C(1+|x|)^{d} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.36}
\end{equation*}
$$

On the other hand, by (2.6), (2.7) and (3.3) we get, for every $(x, y) \in L_{2 A}$,

$$
e^{-R(y)}\left|\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)\right| \leq C \frac{e^{R(x)-R(y)}}{u^{\frac{d}{2}}}(1+|x|)^{2} \leq C \frac{(1+|x|)^{2}}{u^{\frac{d}{2}}}, \quad \text { when } \mathfrak{m}(x)<u<1
$$

and

$$
e^{-R(y)}\left|\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)\right| \leq C e^{R(x)-R(y)} e^{-c u}(1+|y|)^{2} \leq C e^{-c u}(1+|x|)^{2}, \quad u \geq 1
$$

By assuming that $d>2$ and using that $\mathfrak{m}(x) \sim(1+|x|)^{-2}, x \in \mathbb{R}^{d}$, it follows that

$$
\begin{align*}
\int_{\mathfrak{m}(x)}^{\infty} e^{-R(y)}\left|\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)\right| d u & \leq C(1+|x|)^{2}\left(\int_{\mathfrak{m}(x)}^{\infty} \frac{d u}{u^{\frac{d}{2}}}+\int_{1}^{\infty} e^{-c u} d u\right) \leq C(1+|x|)^{2}\left(\frac{1}{\mathfrak{m}(x)^{\frac{d}{2}-1}}+1\right) \\
& \leq C(1+|x|)^{d} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.37}
\end{align*}
$$

By using together (3.36), and (3.37) we obtain that

$$
\begin{equation*}
I_{i, j}^{\infty}(x, y) \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.38}
\end{equation*}
$$

We now deal with $I_{i, j}^{0}(x, y),(x, y) \in L_{2 A}$. Let us consider again $\widetilde{H}_{u}(x, y)$ and $\widetilde{W}_{u}(x, y), x, y \in \mathbb{R}^{d}$ and $u>0$, as in (3.12) and (3.13), respectively. We write, for each $x, y \in \mathbb{R}^{d}$ and $u>0$,

$$
\begin{align*}
\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y) & =\partial_{x_{i} x_{j}}^{2}\left[h_{u}(x, y)-\widetilde{H}_{u}(x, y)\right]+\partial_{x_{i} x_{j}}^{2} \widetilde{H}_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y) \\
& =\left(1-\frac{(\operatorname{det} Q u)^{1 / 2}}{\left(u^{d} \operatorname{det} Q\right)^{1 / 2}}\right) \partial_{x_{i} x_{j}}^{2} h_{u}(x, y)+\partial_{x_{i} x_{j}}^{2} \widetilde{H}_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y) \tag{3.39}
\end{align*}
$$

where again we use that $h_{u}(x, y)=\left(u^{d} \operatorname{det} Q\right)^{1 / 2}\left(\operatorname{det} Q_{u}\right)^{-1 / 2} \widetilde{H}_{u}(x, y), x, y \in \mathbb{R}^{d}$ and $u>0$.
By considering (2.6), (3.3), (3.17) and (3.18) it follows that

$$
\begin{align*}
& e^{-R(y)}\left|\left(1-\frac{\left(\operatorname{det} Q_{u}\right)^{1 / 2}}{\left(u^{d} \operatorname{det} Q\right)^{1 / 2}}\right) \partial_{x_{i} x_{j}}^{2} h_{u}(x, y)\right| \leq C u e^{R(x)-R(y)} \frac{e^{-c \frac{\left|y-D_{u} x\right|^{2}}{u}}}{u^{\frac{d}{2}}}\left(|x|+\frac{1}{\sqrt{u}}\right)^{2} \\
& 0) \quad \leq C \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}(1+|x|)^{2} \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}}, \quad(x, y) \in L_{2 A} \text { and } 0<u<\mathfrak{m}(x) \tag{3.40}
\end{align*}
$$

On the other hand, according to (2.3), we can write

$$
\begin{aligned}
\partial_{x_{i} x_{j}}^{2} \widetilde{H}_{u}(x, y)= & \widetilde{H}_{u}(x, y)\left(P_{i}(u, x, y) P_{j}(u, x, y)+\Delta_{i, j}(u)\right) \\
= & \widetilde{H}_{u}(x, y)\left(\left\langle Q_{\infty}^{-1} x, e_{i}\right\rangle P_{j}(u, x, y)+\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle\left\langle Q_{\infty}^{-1} x, e_{j}\right\rangle\right) \\
& +\widetilde{H}_{u}(x, y)\left(\left\langle Q_{u}^{-1} e^{u B} e_{j}, y-D_{u} x\right\rangle\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle+\Delta_{i j}\right) \\
= & Z_{i, j}^{1}(x, y, u)+Z_{i, j}^{2}(x, y, u), \quad x, y \in \mathbb{R}^{d} \text { and } u>0
\end{aligned}
$$

Also, by considering (3.34) we can see that

$$
\mathbb{S}_{u}^{i, j}(x, y)=\widetilde{W}_{u}(x, y)\left(\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\left\langle\frac{1}{u} Q^{-1} e_{i}, y-x\right\rangle-\left\langle e_{j}, \frac{1}{u} Q^{-1} e_{i}\right\rangle\right), \quad x, y \in \mathbb{R}^{d} \text { and } u>0
$$

Then,

$$
\begin{aligned}
&\left|\partial_{x_{i} x_{j}}^{2} \widetilde{H}_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y)\right| \leq\left|Z_{i, j}^{1}(x, y, u)\right|+\left|Z_{i, j}^{2}(x, y, u)-\mathbb{S}_{u}^{i, j}(x, y)\right| \\
& \leq\left|Z_{i, j}^{1}(x, y, u)\right|+\left|\widetilde{H}_{u}(x, y)-\widetilde{W}_{u}(x, y)\right|\left|\left\langle Q_{u}^{-1} e^{u B} e_{j}, y-D_{u} x\right\rangle\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle+\Delta_{i j}\right| \\
&+\widetilde{W}_{u}(x, y)\left|\Delta_{i j}+\left\langle e_{j}, \frac{1}{u} Q^{-1} e_{i}\right\rangle\right| \\
& \quad+\widetilde{W}_{u}(x, y)\left|\left\langle Q_{u}^{-1} e^{u B} e_{j}, y-D_{u} x\right\rangle\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\left\langle\frac{1}{u} Q^{-1} e_{i}, y-x\right\rangle\right| \\
& \quad= \sum_{\ell=1}^{4} D_{i, j}^{\ell}(x, y, u), \quad x, y \in \mathbb{R}^{d} \text { and } u>0 .
\end{aligned}
$$

We take into account (3.3), (3.18), (3.25) and [10, (4.5) and Lemma 2.2] to see that, for every $(x, y) \in L_{2 A}$ and $0<u<\mathfrak{m}(x)$,

$$
\begin{equation*}
e^{-R(y)} D_{i, j}^{1}(x, y, u) \leq C|x| \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}\left(|x|+\frac{1}{\sqrt{u}}\right) \leq C|x| \frac{e^{-c \frac{|y-x|^{2}}{t}}}{u^{\frac{d+1}{2}}} \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}} \tag{3.41}
\end{equation*}
$$

Now, by using (3.3), (3.22), (3.31) and [10, (4.6) and Lemma 2.2] we get

$$
\begin{align*}
e^{-R(y)} D_{i, j}^{2}(x, y, u) & \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d-1}{2}}}\left(\frac{\left|y-D_{u} x\right|^{2}}{u^{2}}+\frac{1}{u}\right) \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{t}}}{u^{\frac{d-1}{2}}}\left(|x|^{2}+\frac{1}{u}\right) \\
& \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}}, \quad(x, y) \in L_{2 A} \text { and } 0<u<\mathfrak{m}(x) \tag{3.42}
\end{align*}
$$

On the other hand, (3.3), (3.21) lead to

$$
\begin{align*}
e^{-R(y)} D_{i, j}^{3}(x, y, u) & =e^{-R(y)} \widetilde{W}_{u}(x, y)\left|\left\langle e_{j}, \frac{1}{u} Q^{-1} e_{i}-e^{u B^{*}} Q_{u}^{-1} e^{u B} e_{i}\right\rangle\right| \leq C \frac{e^{-\frac{\left|Q^{-1 / 2}(y-x)\right|^{2}}{2 u}}}{u^{\frac{d}{2}}} \\
& \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}}, \quad(x, y) \in L_{2 A} \text { and } 0<u<1 \tag{3.43}
\end{align*}
$$

Finally, we estimate the term $e^{-R(y)} D_{i, j}^{4}(x, y, u)$, for $(x, y) \in L_{2 A}$ and $0<u<\mathfrak{m}(x)$. We write

$$
\begin{aligned}
D_{i, j}^{4}(x, y, u)= & \widetilde{W}_{u}(x, y) \left\lvert\,\left(\left\langle Q_{u}^{-1} e^{u B} e_{j}, y-D_{u} x\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\right)\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle\right. \\
& \left.+\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\left(\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{i}, y-x\right\rangle\right) \right\rvert\,, \quad x, y \in \mathbb{R}^{d} \text { and } u>0
\end{aligned}
$$

From (3.21), (3.22) and [10, Lemmas 2.2 and 2.3] it follows that, for every $x, y \in \mathbb{R}^{d}$ and $u \in(0,1)$,

$$
\begin{aligned}
\mid\left(\left\langleQ_{u}^{-1} e^{u B} e_{j},\right.\right. & \left.\left.y-D_{u} x\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\right)\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle \mid \\
& \leq\left(\left|\left\langle Q_{u}^{-1} e^{u B} e_{j}-\frac{1}{u} Q^{-1} e_{j}, y-D_{u} x\right\rangle\right|+\left|\left\langle\frac{1}{u} Q^{-1} e_{j}, x-D_{u} x\right\rangle\right|\right)\left|\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle\right| \\
& \leq C\left(\left|y-D_{u} x\right|+|x|\right) \frac{\left|y-D_{u} x\right|}{u} \leq C(|y-x|+|x|)\left(\frac{|y-x|}{u}+|x|\right)
\end{aligned}
$$

and, in the same way,

$$
\left|\left\langle\frac{1}{u} Q^{-1} e_{j}, y-x\right\rangle\left(\left\langle Q_{u}^{-1} e^{u B} e_{i}, y-D_{u} x\right\rangle-\left\langle\frac{1}{u} Q^{-1} e_{i}, y-x\right\rangle\right)\right| \leq C \frac{|y-x|}{u}(|y-x|+|x|) .
$$

Thus, by taking into account again (3.3), we obtain that

$$
\begin{align*}
e^{-R(y)} D_{i, j}^{4}(x, y, u) & \leq C \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}(|y-x|+|x|)\left(\frac{|y-x|}{u}+|x|\right) \leq C \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}(1+|x|)\left(\frac{1}{\sqrt{u}}+|x|\right) \\
& \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}}, \quad(x, y) \in L_{2 A} \text { and } 0<u<\mathfrak{m}(x) \tag{3.44}
\end{align*}
$$

From (3.41), (3.42), (3.43) and (3.44) we deduce that

$$
e^{-R(y)}\left|\partial_{x_{i} x_{j}}^{2} \widetilde{H}_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y)\right| \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}}, \quad(x, y) \in L_{2 A} \text { and } 0<u<\mathfrak{m}(x)
$$

that, jointly (3.39) and (3.40), allows us to conclude that

$$
\begin{equation*}
I_{i, j}^{0}(x, y) \leq C(1+|x|) \int_{0}^{\mathfrak{m}(x)} \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}} d u \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.45}
\end{equation*}
$$

By (3.38) and (3.45) we get (3.35) and then, we obtain that $\mathcal{D}_{k, i, j}^{\nu}(f) \leq C \mathscr{S}_{2 A}(f)$. As consequence we obtain that $\mathcal{D}_{k, i, j}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for each $1 \leq p \leq \infty$.

We conclude that the local square function $g_{k, \alpha, \text { loc }}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.
3.1.3. About the global part of $g_{k, \alpha}^{\nu}: 0<\widehat{\alpha} \leq 2$. In this subsection we choose $A$ large enough as in 10, Section 9]. By using Minkowski inequality we get

$$
\begin{align*}
g_{k, \alpha, \text { glob }}^{\nu}(f)(x) & \leq C \int_{\mathbb{R}^{d}}\left(1-\varphi_{\mathrm{A}}(x, y)\right)|f(y)|\left\|P_{k, \alpha}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} d \gamma_{\infty}(y) \\
& \leq C \int_{\mathbb{R}^{d}}\left(1-\varphi_{\mathrm{A}}(x, y)\right)|f(y)| \int_{0}^{\infty}\left\|\left.\left[s^{k+\widehat{\alpha}} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]\right|_{s=\frac{t}{2 \sqrt{u}}}\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\left|\partial_{x}^{\alpha} h_{u}(x, y)\right| u^{\frac{\widehat{\alpha}}{2}-1} d u d \gamma_{\infty}(y) \\
3.46) & \leq C \int_{\mathbb{R}^{d}}\left(1-\varphi_{\mathrm{A}}(x, y)\right)|f(y)| \int_{0}^{\infty}\left|\partial_{x}^{\alpha} h_{u}(x, y)\right| u^{\frac{\alpha,}{2}-1} d u d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \tag{3.46}
\end{align*}
$$

We recall that $\mathfrak{g}_{\nu}(s)=s^{2 \nu} e^{-s^{2}}, s>0$. By using [10, Lemma 4.1, Corollary 5.3 and Propositions 7.1 and 9.1] and by taking into account that in the global operators the cancellation property of the integral kernels does not play any role, we can deduce from (3.46) that the global square functions $g_{k, \alpha, \text { glob }}^{\nu}$ are bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.
3.2. We consider $\alpha=0$ and $k \geq 1$. Let $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. According to (2.13) we can write

$$
t^{k} \partial_{t}^{k} P_{t}^{\nu}(f)(x)=\int_{\mathbb{R}^{d}} P_{k}^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
P_{k}^{\nu}(x, y, t)=\frac{2}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{u} h_{u}(x, y) d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

with $\mathfrak{h}_{\nu}(s)=s^{2 \nu-1} e^{-s^{2}}, s>0$. We observe that $\mathfrak{h}_{\nu}(s)=s^{-1} \mathfrak{g}_{\nu}(s), s>0$. A simple computation shows that, for every $\ell \in \mathbb{N}$, we have that $\left|s^{\ell+1} \partial_{s}^{\ell} \mathfrak{h}_{\nu}(s)\right| \leq C g_{\nu}(s / 2), s>0$. Then, $\left\|s^{\ell+1} \partial_{s}^{\ell} \mathfrak{h}_{\nu}(s)\right\|_{L^{2}\left((0, \infty), \frac{d s}{s}\right)} \leq C$, for each $\ell \in \mathbb{N}$.

According to (2.8)

$$
\partial_{u} h_{u}(x, y) \sum_{i, j=1}^{d}\left(c_{i, j} \partial_{x_{i} x_{j}}^{2} h_{u}(x, y)+d_{i, j} x_{i} \partial_{x_{j}} h_{u}(x, y)\right), \quad x, y \in \mathbb{R}^{d} \text { and } u>0
$$

for certain $c_{i, j}$ and $d_{i, j} \in \mathbb{R}, i, j=1, \ldots, d$.

We define, for every $i, j=1, \ldots, d$,

$$
\mathcal{K}_{k, i, j}^{\nu}(x, y, t)=\int_{0}^{\infty}\left[s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{x_{i} x_{j}}^{2} h_{u}(x, y) d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

and

$$
\mathcal{R}_{k, i, j}^{\nu}(x, y, t)=\int_{0}^{\infty}\left[s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} x_{i} \partial_{x_{j}} h_{u}(x, y) d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

and consider the functions given by

$$
\mathscr{K}_{k, i, j}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} \mathcal{K}_{k, i, j}^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

and

$$
\mathscr{R}_{k, i, j}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} \mathcal{R}_{k, i, j}^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

It is clear that

$$
g_{k, 0}^{\nu}(f)(x) \leq C \sum_{i, j=1}^{d}\left(\left\|\mathscr{K}_{k, i, j}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}+\left\|\mathscr{R}_{k, i, j}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}\right), \quad x \in \mathbb{R}^{d}
$$

We define the local and the global parts of $g_{k, 0}^{\nu}, \mathscr{K}_{k, i, j}^{\nu}$ and $\mathscr{R}_{k, i, j}^{\nu}$ in the usual way.
3.2.1. About the local part of $g_{k, 0}^{\nu}$. Let $i, j=1, \ldots, d$. We firstly study $\mathscr{R}_{k, i, j, \text { loc }}^{\nu}$. Minkowski inequality leads to

$$
\left\|\mathscr{R}_{k, i, j, \mathrm{loc}}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)|\left\|\mathcal{R}_{k, i, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} e^{-R(y)} d y
$$

We assert that, when $d>2$,

$$
\begin{equation*}
e^{-R(y)}\left\|\mathcal{R}_{k, i, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{3.47}
\end{equation*}
$$

In this way, we obtain that $\left\|\mathscr{R}_{k, i, j, \mathrm{loc}}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \mathscr{S}_{\eta}(|f|)(x), x \in \mathbb{R}^{d}$, with $\eta=2 A$, and we conclude that $\mathscr{R}_{k, i, j, \text { loc }}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, for every $1 \leq p \leq \infty$.

Let us establish (3.47). Since

$$
\left\|s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right\|_{L^{2}\left((0, \infty), \frac{d s}{s}\right)}=\left\|s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right\|_{L^{2}\left((0, \infty), \frac{d s}{s}\right)}, \quad t, u>0
$$

by using again Minkowski inequality and (2.4), (2.5), (3.3) and (3.18) we can write, when $d>2$,

$$
\begin{aligned}
e^{-R(y)} \| \mathcal{R}_{k, i, j}^{\nu} & (x, y, \cdot) \|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C|x| e^{-R(y)} \int_{0}^{\infty}\left|\partial_{x_{j}} h_{u}(x, y)\right| d u \\
& \leq C|x| e^{R(x)-R(y)}\left(\int_{0}^{\mathfrak{m}(x)} \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d}{2}}}\left(|x|+\frac{1}{\sqrt{u}}\right) d u+\int_{\mathfrak{m}(x)}^{1}\left(|x|+\frac{1}{\sqrt{u}}\right) \frac{d u}{u^{\frac{d}{2}}}+(1+|y|) \int_{1}^{\infty} e^{-c u} d u\right) \\
& \leq C|x|\left(\int_{0}^{\infty} \frac{e^{-\frac{c|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}} d u+(1+|x|) \int_{\mathfrak{m}(x)}^{\infty} \frac{d u}{u^{\frac{d}{2}}}+1+|x|\right) \\
& \leq C|x|\left(\frac{1}{|x-y|^{d-1}}+\frac{1+|x|}{\mathfrak{m}(x)^{\frac{d}{2}-1}}+1+|x|\right) \leq C|x|\left(\frac{1}{|x-y|^{d-1}}+(1+|x|)^{d-1}\right) \\
& \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} .
\end{aligned}
$$

In the third inequality we have taken into account that $|y| \leq C(1+|x|)$, when $(x, y) \in L_{\eta}, \eta>0$, and that $\mathfrak{m}(x) \sim(1+|x|)^{-2}$ which leads to the following estimates

$$
|x|+\frac{1}{\sqrt{u}} \leq C\left\{\begin{array}{ll}
\frac{1}{\sqrt{u}}, & 0<u \leq \mathfrak{m}(x), \\
1+|x|, & u \geq \mathfrak{m}(x)
\end{array} \quad x \in \mathbb{R}^{d}\right.
$$

In order to study $\mathscr{K}_{k, i, j, \text { loc }}^{\nu}$ we proceed as in Section 3.1.2 We define the operator

$$
\mathscr{S}_{k, i, j}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} \mathscr{S}_{k, i, j}^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
\mathscr{S}_{k, i, j}^{\nu}(x, y, t)=\int_{0}^{\infty}\left[s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \mathbb{S}_{u}^{i, j}(x, y) d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

Here $\mathbb{S}_{u}^{i, j}(x, y), x, y \in \mathbb{R}^{d}, u>0$, is given by (3.33).
By using Minkowski inequality it follows that

$$
\left\|\mathcal{K}_{k, i, j}^{\nu}(x, y, \cdot)-\mathscr{S}_{k, i, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \int_{0}^{\infty}\left|\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)-\mathbb{S}_{u}^{i, j}(x, y)\right| d u, \quad x, y \in \mathbb{R}^{d}
$$

Then, defining the local operator $\mathscr{S}_{k, i, j, \text { loc }}^{\nu}$ in the usual way, from the proof of estimate (3.35) we get that

$$
\left\|\mathscr{K}_{k, i, j, \mathrm{loc}}^{\nu}(f)(x, \cdot)-\mathscr{S}_{k, i, j, \mathrm{loc}}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \mathscr{S}_{2 A}(f)(x), \quad x \in \mathbb{R}^{d}
$$

On the other hand, consider the square function $\mathscr{G}_{k, i, j}^{\nu}$ defined by

$$
\mathscr{G}_{k, i, j}^{\nu}(f)(x)=\left\|\mathscr{T}_{k, i, j}^{\nu}(f)(x, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}, \quad x \in \mathbb{R}^{d}
$$

where

$$
\mathscr{T}_{k, i, j}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} \mathcal{T}_{k, i, j}^{\nu}(x-y, t) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

and

$$
\mathcal{T}_{k, i, j}^{\nu}(z, t)=\int_{0}^{\infty}\left[s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{x_{i} x_{j}}^{2} \mathbb{W}_{u}(z) d u, \quad z \in \mathbb{R}^{d} \text { and } t>0
$$

By arguing as in the analysis of $\mathbb{G}_{k, \ell, m}^{\nu}$ (see Section 3.1.2) we can prove that the operator $\mathscr{G}_{k, i, j}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$ and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$ and that $\mathscr{S}_{k, i, j, \text { loc }}^{\nu}$ is a bounded operator $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, when $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L_{L^{2}\left((0, \infty), \frac{d t}{t}\right)}^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.

As in previous sections we can conclude that $g_{k, 0, \text { loc }}^{\nu}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, when $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.
3.2.2. About the global part of $g_{k, 0}^{\nu}$. We assume that $A$ is large enough as in [10, Section 9$]$. Let $i, j=1, \ldots, d$. We can write

$$
\left\|\mathcal{K}_{k, i, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \int_{0}^{\infty}\left|\partial_{x_{i} x_{j}}^{2} h_{u}(x, y)\right| d u, \quad x, y \in \mathbb{R}^{d}
$$

and

$$
\left\|\mathcal{R}_{k, i, j}^{\nu}(x, y, \cdot)\right\|_{L^{2}\left((0, \infty), \frac{d t}{t}\right)} \leq C \int_{0}^{\infty}\left|x_{i} \| \partial_{x_{j}} h_{u}(x, y)\right| d u, \quad x, y \in \mathbb{R}^{d}
$$

According to [10, Lemma 4.1 and Corollary 5.3] we get

$$
\begin{equation*}
\left|x_{i}\right| \int_{1}^{\infty}\left|\partial_{x_{j}} h_{u}(x, y)\right| d u+\int_{1}^{\infty}\left|\partial_{x_{i}, x_{j}}^{2} h_{u}(x, y)\right| d u \leq C(1+|x|) e^{R(x)}, \quad x, y \in \mathbb{R}^{d} \tag{3.48}
\end{equation*}
$$

On the other hand, according to (2.4) and (2.6) we obtain

$$
\begin{equation*}
\left|x_{i}\right|\left|\partial_{x_{j}} h_{u}(x, y)\right|+\left|\partial_{x_{i}, x_{j}}^{2} h_{u}(x, y)\right| \leq C \frac{e^{R(x)}}{u^{\frac{d}{2}}} e^{-c \frac{\left|y-D_{u} x\right|^{2}}{u}}\left(|x|+\frac{1}{\sqrt{u}}\right)^{2}, \quad x, y \in \mathbb{R}^{d} \text { and } u \in(0,1) \tag{3.49}
\end{equation*}
$$

The estimates in (3.48) and (3.49) allow us to proceed as in [10, Propositions 7.1 and 9.1] to establish that the global operator $g_{k, 0, \text { glob }}^{\nu}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.

## 4. Proof of Theorem 1.2

Let $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. According to (2.16) we have that

$$
t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x)=\frac{2^{\widehat{\alpha}}}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+\widehat{\alpha}} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \int_{\mathbb{R}^{d}} f(y) \partial_{x}^{\alpha} h_{u}(x, y) d \gamma_{\infty}(y) u^{\frac{\widehat{\alpha}}{2}-1} d u, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

Suppose that $\alpha \neq 0$. By taking into account that $\left|s^{k} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right| \leq C \mathfrak{g}_{\nu}(s / 2), s>0$, and using estimates (2.14) and (2.15) we obtain that

$$
\begin{aligned}
& \left|t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x)\right| \leq C \int_{0}^{\infty}\left[s^{\widehat{\alpha}} \mathfrak{g}_{\nu}\left(\frac{s}{2}\right)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \int_{\operatorname{supp} f}\left|\partial_{x}^{\alpha} h_{u}(x, y)\right| d y u^{\frac{\widehat{\alpha}}{2}-1} d u \\
& \quad \leq C t^{\widehat{\alpha}} e^{R(x)} \sum_{n=0}^{\left[\frac{\widehat{\alpha}}{2}\right]}\left(\int_{0}^{1} e^{-c \frac{t^{2}}{u}}\left(|x|+\frac{1}{\sqrt{u}}\right)^{\widehat{\alpha}-2 n} u^{-n-\frac{d}{2}-1} d u+\int_{1}^{\infty} e^{-c \frac{t^{2}}{u}} e^{-c u} \int_{\operatorname{supp} f}(1+|y|)^{\widehat{\alpha}-2 n} d y \frac{d u}{u}\right) \\
& \quad \leq C(x) t^{\widehat{\alpha}}\left(\int_{0}^{1} e^{-c \frac{t^{2}}{u}} u^{-\frac{\widehat{\alpha}}{2}-\frac{d}{2}-1} d u+t^{-\widehat{\alpha}-d} \int_{1}^{\infty} u^{\frac{\widehat{\alpha}}{2}+\frac{d}{2}} e^{-c u} d u\right) \leq \frac{C(x)}{t^{d}}, \quad x \in \mathbb{R}^{d} \text { and } t>0 .
\end{aligned}
$$

Thus,

$$
\lim _{t \rightarrow+\infty} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x)=0, \quad x \in \mathbb{R}^{d}
$$

We can write

$$
\partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}(f)(x)=-\int_{t}^{\infty} \partial_{u}^{k+1} \partial_{x}^{\alpha} P_{u}^{\nu}(f)(x) d u, \quad x \in \mathbb{R}^{d}
$$

Then, by proceeding as in [31, p. 478] we obtain that

$$
P_{*, k, \alpha}^{\nu}(f)(x) \leq C g_{k+1, \alpha}^{\nu}(f)(x), \quad x \in \mathbb{R}^{d}
$$

and Theorem 1.2 follows from Theorem 1.1 .
On the other hand, we can see that

$$
P_{*, k, 0}^{\nu}(f)(x) \leq C \sup _{t>0}\left|\mathcal{H}_{t}(f)(x)\right|, \quad x \in \mathbb{R}^{d}
$$

By using [7, Theorem 1.1 and Corollary 1.2] we can establish Theorem 1.2 for $\alpha=0$.

## 5. Proof of Theorem 1.3

We show Theorem 1.3 for the variation operator $V_{\rho}\left(\left\{t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} P_{t}^{\nu}\right\}_{t>0}\right)$. The result for the oscillation operator can be proved in a similar way.

We consider the operator $\mathbb{T}_{k, \alpha}^{\nu}$ defined by

$$
\mathbb{T}_{k, \alpha}^{\nu}(f)(x, t)=t^{k+\widehat{\alpha}} \partial_{t}^{k} \partial_{x}^{\alpha} \mathbb{P}_{t}^{\nu}(f)(x), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

We recall that

$$
\mathbb{P}_{t}^{\nu}(f)(x)=\frac{t^{2 \nu}}{4^{\nu} \Gamma(\nu)} \int_{\mathbb{R}^{d}} e^{-\frac{t^{2}}{4 u}} \mathbb{W}_{u}(f)(x) \frac{d u}{u^{1+\nu}}, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
\mathbb{W}_{u}(f)(x)=\int_{\mathbb{R}^{d}} \mathbb{W}_{u}(x-y) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } u>0
$$

and $\mathbb{W}_{u}(z)=(2 \pi u)^{-\frac{d}{2}} e^{-\frac{|z|^{2}}{2 u}}, z \in \mathbb{R}^{d}$ and $u>0$.
Let $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. As in (2.16) we can write

$$
\begin{aligned}
\mathbb{T}_{k, \alpha}^{\nu}(f)(x, t) & =\frac{2^{\widehat{\alpha}}}{\Gamma(\nu)} \int_{0}^{\infty}\left[s^{k+\widehat{\alpha}} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \partial_{x}^{\alpha} \mathbb{W}_{u}(f)(x) u^{\frac{\widehat{\alpha}}{2}-1} d u \\
& =\frac{2^{\widehat{\alpha}+1}}{\Gamma(\nu)} \int_{0}^{\infty} s^{k+\widehat{\alpha}-1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\left[u^{\frac{\widehat{\alpha}}{2}} \partial_{x}^{\alpha} \mathbb{W}_{u}(f)(x)\right]_{\left\lvert\, u=\frac{t^{2}}{4 s^{2}}\right.} d s, \quad x \in \mathbb{R}^{d} \text { and } t>0
\end{aligned}
$$

It follows that

$$
\begin{align*}
V_{\rho}\left(t \rightarrow \mathbb{T}_{k, \alpha}^{\nu}(f)(x, t)\right) & \leq C V_{\rho}\left(t \rightarrow t^{\frac{\hat{\alpha}}{2}} \partial_{x}^{\alpha} \mathbb{W}_{t}(f)(x)\right) \int_{0}^{\infty} s^{k+\widehat{\alpha}-1} \partial_{s}^{k}\left[\mathfrak{g}_{\nu}(s)\right] d s  \tag{5.1}\\
& \leq C V_{\rho}\left(t \rightarrow t^{\frac{\hat{\alpha}}{2}} \partial_{x}^{\alpha} \mathbb{W}_{t}(f)(x)\right), \quad x \in \mathbb{R}^{d}
\end{align*}
$$

Firstly we deal with the case $\widehat{\alpha}=1$. Let $j=1, \ldots, d$. By considering the Fourier transform $\mathcal{F}$ in $\mathbb{R}^{d}$ given in (3.6) we have that

$$
\mathcal{F}\left(\partial_{z_{j}} \mathbb{W}_{t}(f)\right)(y)=i y_{j} e^{-\frac{1}{2} t|y|^{2}} \mathcal{F}(f)(y)=-|y| e^{-\frac{1}{2} t|y|^{2}} \mathcal{F}\left(\mathcal{R}_{j} f\right)(y), \quad y \in \mathbb{R}^{d} \text { and } t>0
$$

where $\mathcal{R}_{j}$ represents the $j$-th Euclidean Riesz transform.
We define $\phi=-\mathcal{F}^{-1}\left\{(2 \pi)^{-\frac{d}{2}}|y| e^{-\frac{|y|^{2}}{2}}\right\}$. Then $\phi$ is a radial and smooth function in $\mathbb{R}^{d}$ and satisfies

$$
\frac{1}{t^{\frac{d}{2}}} \phi\left(\frac{x}{\sqrt{t}}\right)=-\mathcal{F}^{-1}\left\{(2 \pi)^{-\frac{d}{2}} \sqrt{t}|y| e^{-\frac{1}{2} t|y|^{2}}\right\}(x), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

If we denote by $\phi_{s}(x)=s^{-d} \phi(x / s), x \in \mathbb{R}^{d}$ and $s>0$, then we have that

$$
\mathcal{F}\left(\sqrt{t} \partial_{z_{j}} \mathbb{W}_{t}(f)\right)(y)=(2 \pi)^{\frac{d}{2}} \mathcal{F}\left(\phi_{\sqrt{t}}\right)(y) \mathcal{F}\left(\mathcal{R}_{j} f\right)(y), \quad y \in \mathbb{R}^{d} \text { and } t>0
$$

that is,

$$
\sqrt{t} \partial_{z_{j}} \mathbb{W}_{t}(f)(z)=\left(\phi_{\sqrt{t}} *\left(\mathcal{R}_{j} f\right)\right)(z), \quad z \in \mathbb{R}^{d} \text { and } t>0
$$

Here $*$ denotes the usual convolution in $\mathbb{R}^{d}$.
We represent by $h_{\mu}$ the $\mu$-Hankel transform defined by

$$
h_{\mu}(\Psi)(u)=\int_{0}^{\infty}(u v)^{-\mu} J_{\mu}(u v) \Psi(v) v^{2 \mu+1} d v, \quad u>0
$$

We have that

$$
\phi(x)=-(2 \pi)^{-\frac{d}{2}} h_{\frac{d-2}{2}}\left(v e^{-\frac{v^{2}}{2}}\right)(|x|), \quad x \in \mathbb{R}^{d}
$$

According to [14, p. 30, (14)], we get

$$
\begin{aligned}
h_{\frac{d-2}{2}}\left(v e^{-\frac{v^{2}}{2}}\right)(u) & =\int_{0}^{\infty}(u v)^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(u v) v e^{-\frac{v^{2}}{2}} v^{d-1} d v=u^{\frac{1-d}{2}} \int_{0}^{\infty} \sqrt{u v} J_{\frac{d-2}{2}}(u v) e^{-\frac{v^{2}}{2}} v^{\frac{d+1}{2}} d v \\
& =\frac{\sqrt{2} \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{ }_{1} F_{1}\left(\frac{d+1}{2}, \frac{d}{2} ;-\frac{u^{2}}{2}\right), \quad u>0
\end{aligned}
$$

where ${ }_{1} F_{1}$ denotes the Kummer's confluent hypergeometric function.
According to [29, (9.9.4), (9.11.2) and (9.12.8)] (note that in [29, (9.11.2)] there must be a minus sign in the argument), we obtain

$$
\lim _{u \rightarrow \infty}{ }_{1} F_{1}\left(\frac{d+1}{2}, \frac{d}{2} ;-\frac{u^{2}}{2}\right)=0
$$

and

$$
\begin{aligned}
\int_{0}^{\infty}\left|\partial_{u_{1}} F_{1}\left(\frac{d+1}{2}, \frac{d}{2} ;-\frac{u^{2}}{2}\right)\right| u^{d} d u & =\frac{d+1}{d} \int_{0}^{\infty}\left|{ }_{1} F_{1}\left(\frac{d+3}{2}, \frac{d+2}{2} ;-\frac{u^{2}}{2}\right)\right| u^{d+1} d u \\
& \leq C\left(\int_{0}^{1} d u+\int_{1}^{\infty} \frac{d u}{u^{2}}\right)<\infty
\end{aligned}
$$

By using [4, Lemma 2.4], the variation operator defined by the family $\left\{T_{t}\right\}_{t>0}$, where for every $t>0, T_{t}(f)=$ $\phi_{\sqrt{t}} * f$, is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

Since $\mathcal{R}_{j}$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$, we conclude that $V_{\rho}\left(\left\{t^{1 / 2} \partial_{x} \mathbb{W}_{t}\right\}_{t>0}\right)$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$.

In order to see that $V_{\rho}\left(\left\{t^{1 / 2} \partial_{x} \mathbb{W}_{t}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$ we use the CalderónZygmund theory for vector valued singular integrals.

We consider the space $E_{\rho}$ that consists of complex functions $g$ defined on $(0, \infty)$ such that $V_{\rho}(g)<\infty$. It is clear that $V_{\rho}(g)=0$ if and only if $g$ is constant in $(0, \infty)$. $V_{\rho}$ defines a norm on the corresponding quotient space. Thus $\left(E_{\rho}, V_{\rho}\right)$ can be seen as a Banach space.

Suppose that $g$ is a derivable complex function on $(0, \infty)$. If $0<t_{n}<t_{n-1}<\cdots<t_{1}$ we get

$$
\left(\sum_{i=1}^{n-1}\left|g\left(t_{i+1}\right)-g\left(t_{i}\right)\right|^{\rho}\right)^{1 / \rho}=\sum_{i=1}^{n-1}\left|\int_{t_{i+1}}^{t_{i}} g^{\prime}(s) d s\right| \leq \int_{0}^{\infty}\left|g^{\prime}(t)\right| d t
$$

Then

$$
\begin{equation*}
V_{\rho}(g) \leq \int_{0}^{\infty}\left|g^{\prime}(t)\right| d t \tag{5.2}
\end{equation*}
$$

We consider the operator $\tau_{j}$ defined by

$$
\tau_{j}(f)(x, t)=\sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(f)(x)=\int_{\mathbb{R}^{d}} \sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

Let $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. For every $x \in \mathbb{R}^{d}$, the function $t \rightarrow \tau_{j}(f)(x, t)$ is continuous in $(0, \infty)$. Then,

$$
V_{\rho}\left(\left\{\tau_{j}(f)(x, t)\right\}_{t>0}\right)=\sup _{\substack{0<t_{n}<\cdots<t_{1} \\ t_{i} \in \mathbb{Q}, i=1, \ldots, n}}\left(\sum_{i=1}^{n-1}\left|\tau_{j}(f)\left(x, t_{i+1}\right)-\tau_{j}(f)\left(x, t_{i}\right)\right|^{\rho}\right)^{1 / \rho}, \quad x \in \mathbb{R}^{d}
$$

Hence, $V_{\rho}\left(\left\{\tau_{j}(f)(x, t)\right\}_{t>0}\right)$ is a Lebesgue measurable function on $\mathbb{R}^{d}$.
By (5.2) we obtain

$$
\begin{equation*}
V_{\rho}\left(\left\{\sqrt{t} \partial_{z_{j}} \mathbb{W}_{t}(z)\right\}_{t>0}\right) \leq \int_{0}^{\infty}\left|\partial_{t}\left[\sqrt{t} \partial_{z_{j}} \mathbb{W}_{t}(z)\right]\right| d t \leq C \int_{0}^{\infty} \frac{e^{-c \frac{|z|^{2}}{t}}}{t^{\frac{d}{2}+1}} d t \leq \frac{C}{|z|^{d}}, \quad z \in \mathbb{R}^{d}, z \neq 0 \tag{5.3}
\end{equation*}
$$

and, for every $i=1, \ldots, d$,

$$
\begin{equation*}
V_{\rho}\left(\left\{\sqrt{t} \partial_{z_{j}, z_{i}}^{2} \mathbb{W}_{t}(z)\right\}_{t>0}\right) \leq \frac{C}{|z|^{d+1}}, \quad z \in \mathbb{R}^{d}, z \neq 0 \tag{5.4}
\end{equation*}
$$

Let $N \in \mathbb{N}, N \geq 2$. We define $E_{\rho, N}$ the space consisting of all those complex functions $g$ defined in $[1 / N, N]$ such that $V_{\rho}\left(\{g(t)\}_{t \in[1 / N, N]}\right)<\infty$, where this variation is defined in the natural way. $\left(E_{\rho, N}, V_{\rho}\right)$ can be seen as a Banach space by identifying the functions differing by a constant.

Let $a \in(1 / N, N) \backslash\{1\}$. We define $L_{a} g=g(a)-g(1), g \in E_{\rho, N}$. It is clear that $L_{a} \in\left(E_{\rho}\right)^{\prime}$, the dual space of $E_{\rho}$.

Let $x \in \mathbb{R}^{d}$. We define $F_{x}: \mathbb{R}^{d} \rightarrow E_{\rho, N}$ such that, for $y \in \mathbb{R}^{d}$,

$$
\begin{aligned}
F_{x}(y):\left[\frac{1}{N}, N\right] & \longrightarrow \mathbb{C} \\
t & \longmapsto\left[F_{x}(y)\right](t)=\sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y) f(y)
\end{aligned}
$$

The function $F_{x}$ is continuous. Indeed, let $y_{0} \in \mathbb{R}^{d}$. By (5.2) we have that

$$
\begin{aligned}
V_{\rho}\left(F_{x}(y)-F_{x}\left(y_{0}\right)\right) & \leq \int_{1 / N}^{N}\left|\partial_{t}\left(\left[F_{x}(y)\right](t)-\left[F_{x}\left(y_{0}\right)\right](t)\right)\right| d t \\
& =\int_{1 / N}^{N}\left|\partial_{t}\left(\sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}\left(x-y_{0}\right)-\sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y)\right)\right| d t
\end{aligned}
$$

Since the function $\partial_{t}\left[\sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y)\right]$ is uniformly continuous in $(t, y) \in[1 / N, N] \times B\left(y_{0}, 1\right)$, we conclude that

$$
\lim _{y \rightarrow y_{0}} V_{\rho}\left(F_{x}(y)-F_{x}\left(y_{0}\right)\right)=0
$$

Since $F_{x}$ is continuous, $F_{x}$ is $E_{\rho, N}$-strongly measurable.
By using (5.3) we get

$$
\int_{\mathbb{R}^{d}} V_{\rho}\left(F_{x}(y)\right) d y<\infty, \quad x \notin \operatorname{supp}(f)
$$

We define

$$
\mathbb{T}_{j}(f)(x)(t)=\int_{\mathbb{R}^{d}} \sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y) f(y) d y, \quad x \in \mathbb{R}^{d}
$$

where the integral is understood in the $E_{\rho, N}$-Bochner sense.
Let $a \in[1 / N, N] \backslash\{1\}$. Well known properties of the Bochner integral allow us to obtain

$$
\begin{aligned}
L_{a}\left(\mathbb{T}_{j}(f)(x)\right) & =\left.\int_{\mathbb{R}^{d}} \sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y)\right|_{t=a} f(y) d y-\left.\int_{\mathbb{R}^{d}} \sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y)\right|_{t=1} f(y) d y \\
& =\left(\int_{\mathbb{R}^{d}} \sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y) f(y) d y\right)(a)-\left(\int_{\mathbb{R}^{d}} \sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y) f(y) d y\right)(1)
\end{aligned}
$$

Then, $\mathbb{T}_{j}(f)(x)=\tau_{j}(f)(x, \cdot)$ in $E_{\rho, N}$.
By using the vector valued Calderón-Zygmund theory we conclude that $\mathbb{T}_{j}$ is bounded from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L_{E_{\rho, N}}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$. Furthermore,

$$
\sup _{N \in \mathbb{N}, N>2}\left\|\mathbb{T}_{j}\right\|_{L^{1}\left(\mathbb{R}^{d}, d x\right) \longrightarrow L_{E_{\rho, N}}^{1, \infty}\left(\mathbb{R}^{d}, d x\right)}<\infty
$$

The monotone convergence theorem leads to see that the operator $V_{\rho}\left(\left\{\sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

By proceeding as in Section 3 we can prove that the local variation operator $V_{\rho}\left(\left\{\mathbb{T}_{j, \operatorname{loc}}(f)(x, \cdot)\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Here, the local operator $\mathbb{T}_{j, \text { loc }}$ is defined by

$$
\mathbb{T}_{j, \operatorname{loc}}(f)(x, t)=\int_{\mathbb{R}^{d}} \sqrt{t} \partial_{x_{j}} \mathbb{W}_{t}(x-y) \varphi_{\mathrm{A}}(x, y) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

The estimate (5.1) allows us to conclude that the local variation operator $V_{\rho}\left(\left\{\mathbb{T}_{k, \alpha, \text { loc }}^{\nu}(f)(x, t)\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, where $\alpha=\left(\alpha_{1}, \ldots, \alpha_{d}\right)$ with $\alpha_{i}=0$ for $i=1, \ldots, d, i \neq j, \alpha_{j}=1$ and

$$
\mathbb{T}_{k, \alpha, \mathrm{loc}}^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} t^{k+\widehat{\alpha}} \mathbb{P}_{k, \alpha}^{\nu}(x-y, t) \varphi_{\mathrm{A}}(x, y) f(y) d y, \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

being

$$
\mathbb{P}_{k, \alpha}^{\nu}(z, t)=\frac{1}{4^{\nu} \Gamma(\nu)} \int_{0}^{\infty} \partial_{t}^{k}\left[t^{2 \nu} e^{-\frac{t^{2}}{4 u}}\right] \partial_{x}^{\alpha} \mathbb{W}_{u}(z) \frac{d u}{u^{\nu+1}}, \quad z \in \mathbb{R}^{d} \text { and } t>0
$$

We consider the operator $S_{k, j}^{\nu}$ and the kernel $K_{k, j}^{\nu}$ as in Section 3.1.1. We also define

$$
V_{\rho, \operatorname{loc}}\left(\left\{t^{k+1} \partial_{t}^{k} \partial_{x_{j}} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right)=V_{\rho}\left(\left\{\int_{\mathbb{R}^{d}} K_{k, j}^{\nu}(x, y, t) \varphi_{\mathrm{A}}(x, y) f(y) d \gamma_{\infty}(y)\right\}_{t>0}\right), \quad x \in \mathbb{R}^{d}
$$

We have that

$$
\begin{aligned}
\mid V_{\rho, \mathrm{loc}} & \left.\left\{t^{k+1} \partial_{t}^{k} \partial_{x_{j}} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right)-V_{\rho}\left(\left\{S_{k, j, \mathrm{loc}}^{\nu}(f)(x, t)\right\}_{t>0}\right) \mid \\
& \leq V_{\rho}\left(\left\{\int_{\mathbb{R}^{d}} K_{k, j}^{\nu}(x, y, t) \varphi_{\mathrm{A}}(x, y) f(y) d y-S_{k, j, \mathrm{loc}}^{\nu}(f)(x, t)\right\}_{t>0}\right) \\
& \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)| e^{-R(y)} \int_{0}^{\infty} \int_{0}^{\infty}\left|\partial_{t}\left(\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right)\right| d t\left|\partial_{x_{j}} h_{u}(x, y)-\mathbb{S}_{u}^{j}(x, y)\right| \frac{d u}{\sqrt{u}} d y \\
& \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)| e^{-R(y)} \int_{0}^{\infty}\left|\partial_{x_{j}} h_{u}(x, y)-\mathbb{S}_{u}^{j}(x, y)\right| \frac{d u}{\sqrt{u}} d y, \quad x \in \mathbb{R}^{d} .
\end{aligned}
$$

In the last inequality we have used that

$$
\left|\partial_{t}\left(\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right)\right| \leq \frac{C}{\sqrt{u}}\left(\left|s^{k} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right|+\left|s^{k+1} \partial_{s}^{k+1} \mathfrak{g}_{\nu}(s)\right|\right)_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.} \leq \frac{C}{\sqrt{u}} \mathfrak{g}_{\nu}\left(\frac{s}{2}\right)_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}, \quad t, u>0 .
$$

The arguments in Section 3.1.1 allow us to establish that $V_{\rho, \operatorname{loc}}\left(\left\{t^{k+1} \partial_{t}^{k} \partial_{x_{j}} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right)$ is a bounded operator from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.

We consider the global variation operator defined by

$$
V_{\rho, \mathrm{glob}}\left(\left\{t^{k+1} \partial_{t}^{k} \partial_{x_{j}} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right)=V_{\rho}\left(\left\{\int_{\mathbb{R}^{d}} P_{k, \alpha}^{\nu}(x, y, t)\left(1-\varphi_{\mathrm{A}}(x, y)\right) f(y) d \gamma_{\infty}(y)\right\}_{t>0}\right), \quad x \in \mathbb{R}^{d}
$$

Since

$$
\int_{0}^{\infty}\left|\partial_{t}\left(\left[s^{k+1} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right)\right| d t \leq C, \quad u>0
$$

we get

$$
V_{\rho, \mathrm{glob}}\left(\left\{t^{k+1} \partial_{t}^{k} \partial_{x_{j}} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right) \leq C \int_{\mathbb{R}^{d}}\left(1-\varphi_{\mathrm{A}}(x, y)\right)|f(y)| e^{-R(y)} \int_{0}^{\infty}\left|\partial_{x_{j}} h_{u}(x, y)\right| \frac{d u}{\sqrt{u}} d y, \quad x \in \mathbb{R}^{d}
$$

As in Section 3.1.3 we conclude that the global variation operator $V_{\rho, \text { glob }}\left(\left\{t^{k+1} \partial_{t}^{k} \partial_{x_{j}} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Thus, we establish Theorem 1.3 when $\widehat{\alpha}=1$.

We consider the case $\widehat{\alpha}=2$. We can proceed as above by taking into account the properties that we are going to mark.

We denote by $\phi=\mathcal{F}^{-1}\left\{|y|^{2} e^{-|y|^{2}}\right\}$. We have that

$$
\phi(x)=h_{\frac{d-2}{2}}\left(v^{2} e^{-v^{2}}\right)(|x|), \quad x \in \mathbb{R}^{d}
$$

According to [14, p. 30, (13)] we get

$$
\psi(u)=h_{\frac{d-2}{2}}\left(v^{2} e^{-v^{2}}\right)(u)=2 e^{-\frac{u^{2}}{2}} L_{1}^{\frac{d-2}{2}}\left(\frac{u^{2}}{2}\right), \quad u>0
$$

where $L_{n}^{\sigma}$ denotes the $n$-th $\sigma$-Laguerre polynomial. Note that $\lim _{u \rightarrow \infty} \psi(u)=0$ and $\int_{0}^{\infty}\left|\psi^{\prime}(u)\right| u^{d} d u<\infty$.
On the other hand, we have that

$$
\int_{0}^{\infty}\left|\partial_{t}\left[s^{k+2} \partial_{s}^{k} \mathfrak{g}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right| d t \leq C, \quad u>0
$$

Our above arguments can be used now to prove Theorem 1.3 when $\widehat{\alpha}=2$.
To finish we study the case $\alpha=0$. As in (5.1) we can obtain

$$
V_{\rho}\left(\left\{t^{k} \partial_{t}^{k} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right) \leq C V_{\rho}\left(\left\{\mathcal{H}_{t}(f)(x)\right\}_{t>0}\right), \quad x \in \mathbb{R}^{d}
$$

The semigroup $\left\{\mathcal{H}_{t}\right\}_{t>0}$ is contractive and analytic in $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, for $1<p<\infty$ (see [11, Theorem 2]). Furthermore, $\left\{\mathcal{H}_{t}\right\}_{t>0}$ is contractive in $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ and $L^{\infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Then, $\left\{\mathcal{H}_{t}\right\}_{t>0}$ is contractively regular and, according to [28, Corollary 4.5], the variation operator $V_{\rho}\left(\left\{\mathcal{H}_{t}\right\}_{t>0}\right)$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for $1<p<\infty$. We conclude that the operator $V_{\rho}\left(\left\{t^{k} \partial_{t}^{k} P_{t}^{\nu}\right\}_{t>0}\right)$ is bounded from $L^{p}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself, for $1<p<\infty$.

As far as we know, it has not been proved that the variation operator $V_{\rho}\left(\left\{\mathcal{H}_{t}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Then, we need to proceed in a different way to prove that the operator $V_{\rho}\left(\left\{t^{k} \partial_{t}^{k} P_{t}^{\nu}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.

Suppose that $k \in \mathbb{N}$ and $k \geq 1$ and consider the elements given in Section 3.2 Let $f \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. We can write

$$
V_{\rho}\left(\left\{t^{k} \partial_{t}^{k} P_{t}^{\nu}(f)(x)\right\}_{t>0}\right) \leq C \sum_{i, j=1}^{d} V_{\rho}\left(\left\{\mathscr{K}_{k, i, j}^{\nu}(f)(x, t)\right\}_{t>0}\right)+V_{\rho}\left(\left\{\mathscr{R}_{k, i, j}^{\nu}(f)(x, t)\right\}_{t>0}\right), \quad x \in \mathbb{R}^{d}
$$

To proceed as in Section 3.2 we need the following estimation

$$
\int_{0}^{\infty}\left|\partial_{t}\left(\left[s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right]_{\left\lvert\, s=\frac{t}{2 \sqrt{u}}\right.}\right)\right| d t=\int_{0}^{\infty}\left|\partial_{s}\left(s^{k} \partial_{s}^{k-1} \mathfrak{h}_{\nu}(s)\right)\right| d s \leq C \int_{0}^{\infty} \mathfrak{g}_{\nu}\left(\frac{s}{2}\right) \frac{d s}{s} \leq C, \quad u>0
$$

Here, we have taken into account that $\left|s^{\ell} \partial_{s}^{\ell} \mathfrak{h}_{\nu}(s)\right| \leq C s^{-1} \mathfrak{g}_{\nu}\left(\frac{s}{2}\right), s>0, \ell \in \mathbb{N}$.
On the other hand, using the above arguments for $\widehat{\alpha}=2$ and proceeding as in the case of $\widehat{\alpha}=1$ we get that $V_{\rho, \text { loc }}\left(\left\{\mathscr{T}_{k, i, j}^{\nu}(f)(x)\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$.

We can apply the reasoning in Section 3.2 to establish Theorem 1.3 for $\alpha=0$ and $k \geq 1$.
Suppose now that $k=0$. We are going to see that the operator $V_{\rho}\left(\left\{P_{t}^{\nu}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$. We first study the global variation operator

$$
V_{\rho, \text { glob }}\left(\left\{P_{t}^{\nu}\right\}_{t>0}\right)(f)(x)=V_{\rho}\left(\left\{P_{t, \text { glob }}^{\nu}(f)(x)\right\}_{t>0}\right), \quad x \in \mathbb{R}^{d}
$$

We write

$$
P_{t}^{\nu}(x, y)=\frac{1}{\Gamma(\nu)} \int_{0}^{\infty} e^{-u} u^{\nu-1} h_{\frac{t^{2}}{4 u}}(x, y) d u, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

It follows that

$$
\begin{aligned}
V_{\rho, \text { glob }}\left(\left\{P_{t}^{\nu}\right\}_{t>0}\right)(f)(x) & \leq C \int_{\mathbb{R}^{d}}\left(1-\varphi_{\mathrm{A}}(x, y)\right)|f(y)| \int_{0}^{\infty} e^{-u} u^{\nu-1} \int_{0}^{\infty}\left|\partial_{t} h_{\frac{t^{2}}{4 u}}(x, y)\right| d t d u d \gamma_{\infty}(y) \\
& \leq C \int_{\mathbb{R}^{d}}\left(1-\varphi_{\mathrm{A}}(x, y)\right)|f(y)| \int_{0}^{\infty} e^{-u} u^{\nu-1} \int_{0}^{\infty}\left|\partial_{s} h_{s}(x, y)\right| d s d u d \gamma_{\infty}(y) \\
& \leq C \int_{\mathbb{R}^{d}}\left(1-\varphi_{\mathrm{A}}(x, y)\right)|f(y)| \int_{0}^{\infty}\left|\partial_{s} h_{s}(x, y)\right| d s d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d}
\end{aligned}
$$

By taking into account (2.8) and proceeding as in Section 3.2.2 we obtain that $V_{\rho, \mathrm{glob}}\left(\left\{P_{t}^{\nu}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$, provided that $A$ is large enough.

To deal with the the local operator

$$
V_{\rho, \text { loc }}\left(\left\{P_{t}^{\nu}\right\}_{t>0}\right)(f)(x)=V_{\rho}\left(\left\{P_{t, \text { loc }}^{\nu}(f)(x)\right\}_{t>0}\right), \quad x \in \mathbb{R}^{d}
$$

we consider the operator defined by

$$
S^{\nu}(f)(x, t)=\int_{\mathbb{R}^{d}} S^{\nu}(x, y, t) f(y) d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d} \text { and } t>0
$$

where

$$
S^{\nu}(x, y, t)=\frac{1}{\Gamma(\nu)} \int_{0}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right) \widetilde{W}_{u}(x, y) \frac{d u}{u}, \quad x, y \in \mathbb{R}^{d} \text { and } t>0
$$

where $\widetilde{W}_{u}(x, y), x, y \in \mathbb{R}^{d}, u>0$ is the kernel given by (3.13).
Our next arguments are inspired by some ideas developed in [21, Section 4] where the symmetric case is studied. We consider the local operator $S_{\mathrm{loc}}^{\nu}$ in the usual way and write $D_{\mathrm{loc}}^{\nu}(f)(x, t)=P_{t, \text { loc }}^{\nu}(f)(x)-S_{\mathrm{loc}}^{\nu}(f)(x, t), x \in \mathbb{R}^{d}$, $t>0$. We have that

$$
\begin{aligned}
\mathcal{D}_{\mathrm{loc}}^{\nu}(f)(x, t)= & \frac{1}{\Gamma(\nu)} \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y) f(y) \int_{0}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\left[h_{u}(x, y)-\widetilde{W}_{u}(x, y)\right] \frac{d u}{u} d \gamma_{\infty}(y) \\
= & \frac{1}{\Gamma(\nu)} \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y) f(y) \int_{0}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\left[h_{u}(x, y)-\mathcal{X}_{(1, \infty)}(u)-\widetilde{W}_{u}(x, y)\right] \frac{d u}{u} d \gamma_{\infty}(y) \\
& +\frac{1}{\Gamma(\nu)} \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y) f(y) \int_{1}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right) \frac{d u}{u} d \gamma_{\infty}(y) \\
= & \mathcal{D}_{\mathrm{loc}}^{\nu, 1}(f)(x, t)+\mathcal{D}_{\mathrm{loc}}^{\nu, 2}(f)(x, t), \quad x \in \mathbb{R}^{d} \text { and } t>0 .
\end{aligned}
$$

Since $\int_{0}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right) \frac{d u}{u}=\Gamma(\nu), t>0$, we have that $\frac{d}{d t}\left[\int_{0}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right) \frac{d u}{u}\right]=0$, and then

$$
\frac{d}{d t} \int_{1}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right) \frac{d u}{u}=-\frac{d}{d t} \int_{0}^{1} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right) \frac{d u}{u}, \quad t>0
$$

Thus, by using also that $s \mathfrak{g}_{\nu}^{\prime}(s) \leq C \mathfrak{g}_{\nu}\left(\frac{s}{2}\right), s>0$, we can write,

$$
\begin{aligned}
& V_{\rho}\left(\left\{\mathcal{D}_{\text {loc }}^{\nu, 2}(f)(x, t)\right\}_{t>0}\right) \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)| \int_{0}^{\infty}\left|\frac{d}{d t} \int_{1}^{\infty} \mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right) \frac{d u}{u}\right| d t d \gamma_{\infty}(y) \\
& \quad \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)|\left(\int_{0}^{1} \int_{1}^{\infty}\left|\partial_{t}\left[\mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\right]\right| \frac{d u}{u} d t+\int_{1}^{\infty} \int_{0}^{1}\left|\partial_{t}\left[\mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\right]\right| \frac{d u}{u} d t\right) d \gamma_{\infty}(y) \\
& \quad \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)|\left(\int_{0}^{1} \int_{1}^{\infty}+\int_{1}^{\infty} \int_{0}^{1}\right) t^{2 \nu-1} e^{-c \frac{t^{2}}{u}} \frac{d u}{u^{\nu+1}} d t d \gamma_{\infty}(y) \\
& \quad \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)|\left(\int_{0}^{1} \int_{1}^{\infty} \frac{t^{2 \nu-1}}{u^{\nu+1}} d u d t+\int_{1}^{\infty} \int_{0}^{1} \frac{u^{\delta-1}}{t^{\delta+1}} d u d t\right) d \gamma_{\infty}(y) \\
& \quad \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)| d \gamma_{\infty}(y) \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)| \frac{1+|x|}{|x-y|^{d-1}} d y=C \mathscr{S}_{2 A}(|f|)(x), \quad x \in \mathbb{R}^{d}
\end{aligned}
$$

Here $\delta$ can be any positive real number. In the last inequality we have taken into account that $|x-y|^{d-1} \leq 1 \leq$ $1+|x|$, when $(x, y) \in L_{2 A}$.

On the other hand, by considering that $\int_{0}^{\infty}\left|\partial_{t}\left[\mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\right]\right| d t \leq C, u>0$, it follows that

$$
\begin{aligned}
& V_{\rho}\left(\left\{\mathcal{D}_{\mathrm{loc}}^{\nu, 1}(f)(x, t)\right\}_{t>0}\right) \\
& \quad \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)| \int_{0}^{\infty} \int_{0}^{\infty}\left|\partial_{t}\left[\mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\right]\right|\left|h_{u}(x, y)-\mathcal{X}_{(1, \infty)}(u)-\widetilde{W}_{u}(x, y)\right| d t \frac{d u}{u} d \gamma_{\infty}(y) \\
& \quad \leq C \int_{\mathbb{R}^{d}} \varphi_{\mathrm{A}}(x, y)|f(y)| \int_{0}^{\infty}\left|h_{u}(x, y)-\mathcal{X}_{(1, \infty)}(u)-\widetilde{W}_{u}(x, y)\right| \frac{d u}{u} d \gamma_{\infty}(y), \quad x \in \mathbb{R}^{d}
\end{aligned}
$$

We now consider

$$
R(x, y)=e^{-R(y)} \int_{0}^{\infty}\left|h_{u}(x, y)-\mathcal{X}_{(1, \infty)}(u)-\widetilde{W}_{u}(x, y)\right| \frac{d u}{u}, \quad x, y \in \mathbb{R}^{d}
$$

Our objective is to establish that

$$
\begin{equation*}
|R(x, y)| \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{5.5}
\end{equation*}
$$

We decompose $R(x, y)=R_{0}(x, y)+R_{\infty}(x, y), x, y \in \mathbb{R}^{d}$, where

$$
R_{0}(x, y)=e^{-R(y)} \int_{0}^{1}\left|h_{u}(x, y)-\mathcal{X}_{(1, \infty)}(u)-\widetilde{W}_{u}(x, y)\right| \frac{d u}{u}, \quad x, y \in \mathbb{R}^{d}
$$

According to (3.3), (3.17), (3.18), (3.25), (3.31) and 10, Lemma 2.2] we get, for $(x, y) \in L_{2 A}$ and $0<u<\mathfrak{m}(x)$,

$$
\begin{aligned}
e^{-R(y)}\left|h_{u}(x, y)-\widetilde{W}_{u}(x, y)\right| \leq & C e^{R(x)-R(y)}\left(\left|\frac{1}{\left(\operatorname{det} Q_{u}\right)^{1 / 2}}-\frac{1}{\left(u^{d} \operatorname{det} Q\right)^{1 / 2}}\right| e^{-\frac{1}{2}\left\langle\left(Q_{u}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{u} x\right), y-D_{u} x\right\rangle}\right. \\
& \left.+\frac{1}{u^{\frac{d}{2}}}\left|e^{-\frac{1}{2}\left\langle\left(Q_{u}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{u} x\right), y-D_{u} x\right\rangle}-e^{-\frac{1}{2 u}\left\langle Q^{-1}(y-x), y-x\right\rangle}\right|\right) \\
\leq & C \frac{e^{R(x)-R(y)}}{u^{\frac{d}{2}}} e^{-c \frac{|y-x|^{2}}{u}}(u+(1+|x|) \sqrt{u}) \leq C(1+|x|) \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d-1}{2}}} .
\end{aligned}
$$

Then,

$$
\begin{equation*}
e^{-R(y)} \int_{0}^{\mathfrak{m}(x)}\left|h_{u}(x, y)-\widetilde{W}_{u}(x, y)\right| \frac{d u}{u} \leq C(1+|x|) \int_{0}^{\mathfrak{m}(x)} \frac{e^{-c \frac{|y-x|^{2}}{u}}}{u^{\frac{d+1}{2}}} d u \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{5.6}
\end{equation*}
$$

By using [10, (2.10)] and (3.3) we get

$$
\begin{align*}
e^{-R(y)} \int_{\mathfrak{m}(x)}^{1}\left|h_{u}(x, y)-\widetilde{W}_{u}(x, y)\right| \frac{d u}{u} & \leq C e^{R(x)-R(y)} \int_{\mathfrak{m}(x)}^{\infty} \frac{d u}{u^{\frac{d}{2}+1}} \leq \frac{C}{\mathfrak{m}(x)^{\frac{d}{2}}} \\
& \leq C(1+|x|)^{d} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{5.7}
\end{align*}
$$

From (5.6) and (5.7) we get that

$$
\begin{equation*}
0 \leq R_{0}(x, y) \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{5.8}
\end{equation*}
$$

We now write

$$
R_{\infty}(x, y) \leq C e^{-R(y)}\left(\int_{1}^{\infty}\left|h_{u}(x, y)-1\right| \frac{d u}{u}+\int_{1}^{\infty} \widetilde{W}_{u}(x, y) \frac{d u}{u}\right), \quad x, y \in \mathbb{R}^{d}
$$

We observe that

$$
\begin{equation*}
e^{-R(y)} \int_{1}^{\infty} \widetilde{W}_{u}(x, y) \frac{d u}{u} \leq C e^{R(x)-R(y)} \int_{1}^{\infty} \frac{d u}{u^{\frac{d}{2}+1}} \leq C \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{5.9}
\end{equation*}
$$

On the other hand, we have that $\lim _{u \rightarrow \infty} h_{u}(x, y)=1, x, y \in \mathbb{R}^{d}$. Indeed, since $D_{u}^{-1}=D_{-u}, u>0$, as in the proof of [7, Proposition 3.3], we get

$$
\begin{aligned}
\left\langle\left(Q_{u}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{u} x\right), y-\right. & \left.D_{u} x\right\rangle=\left\langle\left(Q_{u}^{-1}-Q_{\infty}^{-1}\right) D_{u}\left(D_{-u} y-x\right), D_{u}\left(D_{-u} y-x\right)\right\rangle \\
& =\left\langle Q_{\infty}^{-1}\left(D_{-u} y-x\right), D_{-u} y-x\right\rangle+\left|Q_{u}^{-1 / 2} e^{u B}\left(D_{-u} y-x\right)\right|^{2}, \quad x, y \in \mathbb{R}^{d} \text { and } u>0
\end{aligned}
$$

Then, by taking into account that $\left|D_{-u} y-x\right| \simeq\left|D_{-u} y\right|, x, y \in \mathbb{R}^{d}, u>0$, (see the proof of Lemma 5.1 in [9]) and considering [10, Lemmas 2.1 and 2.2] we obtain

$$
\lim _{t \rightarrow \infty}\left\langle\left(Q_{t}^{-1}-Q_{\infty}^{-1}\right)\left(y-D_{t} x\right), y-D_{t} x\right\rangle=\left\langle Q_{\infty}^{-1} x, x\right\rangle, x, y \in \mathbb{R}^{d}
$$

Then, we conclude that $\lim _{u \rightarrow \infty} h_{u}(x, y)=1, x, y \in \mathbb{R}^{d}$.
Thus, by considering also (2.10), we can write

$$
\begin{aligned}
\left|h_{u}(x, y)-1\right| & \leq \int_{u}^{\infty}\left|\partial_{s} h_{s}(x, y)\right| d s \leq C e^{R(x)}(1+|y|)^{2} \int_{u}^{\infty} e^{-c s} d s \\
& =C e^{R(x)}(1+|y|)^{2} e^{-c u}, \quad x, y \in \mathbb{R}^{d} \text { and } u>0
\end{aligned}
$$

We use again (3.3) and that $|y| \leq 1+|x|$, when $(x, y) \in L_{2 A}$, to get

$$
e^{-R(y)} \int_{1}^{\infty}\left|h_{u}(x, y)-1\right| \frac{d u}{u} \leq C e^{R(x)-R(y)}(1+|x|)^{2} \leq C(1+|x|)^{d} \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A}
$$

which, jointly with (5.9), leads to

$$
\begin{equation*}
0 \leq R_{\infty}(x, y) \leq C \frac{1+|x|}{|x-y|^{d-1}}, \quad(x, y) \in L_{2 A} \tag{5.10}
\end{equation*}
$$

From (5.8) and (5.10) we obtain (5.5).
All the above estimations allow us to conclude that

$$
V_{\rho}\left(t \mapsto P_{t, \mathrm{loc}}^{\nu}(f)(x)-S_{\mathrm{loc}}^{\nu}(f)(x, t)\right) \leq C \mathscr{S}_{2 A}(f)(x), \quad x \in \mathbb{R}^{d}
$$

and therefore, that $V_{\rho}\left(\left\{\mathcal{D}_{t, \text { loc }}^{\nu}\right\}_{t>0}\right)$ defined by

$$
V_{\rho}\left(\left\{\mathcal{D}_{t, \mathrm{loc}}^{\nu}\right\}_{t>0}\right)(f)(x)=V_{\rho}\left(t \mapsto P_{t, \mathrm{loc}}^{\nu}(f)(x)-S_{\mathrm{loc}}^{\nu}(f)(x, t)\right), \quad x \in \mathbb{R}^{d}
$$

is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into itself.
To conclude that the local variation operator $V_{\rho, \text { loc }}\left(\left\{P_{t}^{\nu}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ we can proceed now as in Section 3.1.1. We need the estimation $\int_{0}^{\infty}\left|\partial_{t}\left[\mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\right]\right| d t \leq C, u>0$, and the appropriate $L^{p}$-boundedness properties for the operator $V_{\rho}\left(\left\{\mathbb{P}_{t}^{\nu}\right\}_{t>0}\right)$.

According to [4, Corollary 2.7] the variation operator $V_{\rho}\left(\left\{\mathbb{W}_{t}\right\}_{t>0}\right)$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$. Then, arguing as in (5.1), $V_{\rho}\left(\left\{\mathbb{P}_{t}^{\nu}\right\}_{t>0}\right)$ is bounded from $L^{p}\left(\mathbb{R}^{d}, d x\right)$ into itself, for every $1<p<\infty$, and from $L^{1}\left(\mathbb{R}^{d}, d x\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, d x\right)$.

We also have that

$$
V_{\rho}\left(\left\{\mathbb{P}_{t}^{\nu}(z)\right\}_{t>0}\right) \leq C \int_{0}^{\infty} \int_{0}^{\infty}\left|\partial_{t}\left[\mathfrak{g}_{\nu}\left(\frac{t}{2 \sqrt{u}}\right)\right]\right| \mathbb{W}_{u}(z) \frac{d u}{u} d t \leq C \int_{0}^{\infty} \frac{e^{-\frac{|z|^{2}}{2 u}}}{u^{\frac{d}{2}+1}} d u \leq \frac{C}{|z|^{d}}, \quad z \in \mathbb{R}^{d} \backslash\{0\}
$$

and, in a similar way, we get

$$
V_{\rho}\left(\left\{\partial_{z_{k}} \mathbb{P}_{t}^{\nu}(z)\right\}_{t>0}\right) \leq \frac{C}{|z|^{d+1}}, \quad z \in \mathbb{R}^{d} \backslash\{0\} \text { and } k=1, \ldots, d
$$

We have all the ingredients to argue as in Section 3.1.1 and to conclude that the operator $V_{\rho, \text { loc }}\left(\left\{P_{t}^{\nu}\right\}_{t>0}\right)$ is bounded from $L^{1}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$ into $L^{1, \infty}\left(\mathbb{R}^{d}, \gamma_{\infty}\right)$. Theorem 1.3 for $\alpha=0$ is then established.

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