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Convergence of optimal control problems governed by second kind parabolic variational inequalities

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Abstract: We consider a family of optimal control problems where the control variable is given by a boundary condition of Neumann type. This family is governed by parabolic variational inequalities of the second kind. We prove the strong convergence of the optimal control and state systems associated to this family to a similar optimal control problem. This work solves the open problem left by the authors in IFIP TC7 CSMO2011.

Keywords: Parabolic variational inequalities of the second kind; Aubin compactness arguments; Boundary control; Convergence of optimal control problems; Tresca boundary conditions, free boundary problems

1 Introduction

The motivation of this paper is to prove the strong convergence of the boundary optimal controls and state systems associated with a family of second kind parabolic variational inequalities. In this paper, we solve the open question, left in [1] and we generalize our work [2], to study the Neumann boundary optimal controls governed by second kind parabolic variational inequalities.

To illustrate the problem, we consider, for example, two free boundary problems which leads to the second kind parabolic variational inequalities.

We assume that the boundary of a multidimensional regular domain Ω is given by $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with $\mathrm{meas}(\Gamma_1) > 0$ and $\mathrm{meas}(\Gamma_3) > 0$. We consider a family of optimal control problems where the control variable is given by a boundary condition of Neumann type whose state system is governed by a free boundary problem with Tresca conditions on a portion Γ_2 of the boundary, with a flux f on Γ_3 as the control variable, given by

Problem 1

$$\dot{u} - \Delta u = g \ \text{ in } \ \Omega \times (0,T),$$
$$|\frac{\partial u}{\partial n}| < q \Rightarrow u = 0 \ \text{ on } \ \Gamma_2 \times (0,T),$$
$$|\frac{\partial u}{\partial n}| = q \Rightarrow \exists k > 0 : u = -k\frac{\partial u}{\partial n} \ \text{ on } \ \Gamma_2 \times (0,T),$$
$$u = b \ \text{ on } \ \Gamma_1 \times (0,T),$$
$$-\frac{\partial u}{\partial n} = f \ \text{ on } \ \Gamma_3 \times (0,T),$$

with the initial condition

$$u(0) = u_b \ \text{ on } \ \Omega,$$

and the compatibility condition on $\Gamma_1 \times (0,T)$

$$u_b = b \ \text{ on } \ \Gamma_1 \times (0,T),$$

where $q > 0$ is the Tresca friction coefficient on Γ_2 [3–5]. We define the spaces $\mathcal{F} = L^2((0,T) \times \Gamma_3)$, $V = H^1(\Omega)$, $V_0 = \{v \in V : v_{|\Gamma_1} = 0\}$, $H = L^2(\Omega)$, $\mathcal{H} = L^2(0,T;H)$, $\mathcal{V} = L^2(0,T;V)$ and the closed convex set $K_b = \{v \in V : v_{|\Gamma_1} = b\}$. Let

$$\begin{cases} g \in \mathcal{H}, \ \ b \in L^2(0,T,H^{\frac{1}{2}}(\Gamma_1)), \ \ f \in \mathcal{F}, \\ q \in L^2((0,T) \times \Gamma_2), \ \ q > 0, \ \ u_b \in K_b. \end{cases} \tag{1}$$

The variational formulation of Problem 1 leads to the following parabolic variational problem:

Problem 2 Let g, b, q, u_b and f be as in (1). Find $u = u_f \in \mathcal{C}(0,T,H) \cap L^2(0,T;K_b)$ with $\dot{u} \in \mathcal{H}$, such that $u(0) = u_b$, and for $t \in (0,T)$,

$$\langle \dot{u}, v-u \rangle + a(u, u-v) + \Phi(v) - \Phi(u)$$
$$\geqslant (g, v-u) - \int_{\Gamma_3} f(v-u)\mathrm{d}s, \ \ \forall v \in K_b.$$

where $(\cdot,\cdot)$ is the scalar product in H, a and Φ are defined by

$$a(u,v) = \int_\Omega \nabla u \nabla v \mathrm{d}x, \ \ \Phi(v) = \int_{\Gamma_2} q|v|\mathrm{d}s. \tag{2}$$

The functional Φ comes from the Tresca condition on Γ_2 [4–5]. We consider also the following problem where we change, in Problem 1, only the Dirichlet condition on $\Gamma_1 \times (0,T)$ by the Newton law or a Robin boundary condition, i.e.,

Problem 3

$$\dot{u} - \Delta u = g \ \text{ in } \ \Omega \times (0,T),$$
$$|\frac{\partial u}{\partial n}| < q \Rightarrow u = 0 \ \text{ on } \ \Gamma_2 \times (0,T),$$
$$|\frac{\partial u}{\partial n}| = q \Rightarrow \exists k > 0 : u = -k\frac{\partial u}{\partial n} \ \text{ on } \ \Gamma_2 \times (0,T),$$
$$-\frac{\partial u}{\partial n} = h(u-b) \ \text{ on } \ \Gamma_1 \times (0,T),$$
$$-\frac{\partial u}{\partial n} = f \ \text{ on } \ \Gamma_3 \times (0,T),$$

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with the initial condition

$$u(0) = u_b \text{ on } \Omega,$$

and the condition of compatibility on $\Gamma_1 \times (0, T)$

$$u_b = b \text{ on } \Gamma_1 \times (0, T).$$

The variational formulation of Problem 3 leads to the following parabolic variational problem.

Problem 4 Let g, b, q, u_b and f be as in (1). For all $h > 0$, find $u = u_{hf}$ in $\mathcal{C}(0, T, H) \cap \mathcal{V}$ with $\dot{u}$ in $\mathcal{H}$, such that $u(0) = u_b$, and for $t \in (0, T)$,

$$\begin{aligned} &\langle \dot{u}, v - u\rangle + a_h(u, u - v) + \Phi(v) - \Phi(u) \\ &\geqslant (g, v - u) - \int_{\Gamma_3} f(v - u)\mathrm{d}s + h\int_{\Gamma_1} b(v - u)\mathrm{d}s, \\ &\forall v \in V, \end{aligned}$$

where a_h is defined by

$$a_h(u, v) = a(u, v) + h\int_{\Gamma_1} uv\mathrm{d}s.$$

Moreover, from [6–9] we have that: $\exists \lambda_1 > 0$ such that

$$\lambda_h\|v\|_V^2 \leqslant a_h(v, v), \forall v \in V, \text{ with } \lambda_h = \lambda_1 \min\{1, h\},$$

that is, a_h is also a bilinear, continuous, symmetric and coercive form $V \times V$ to $\mathbb{R}$. The existence and uniqueness of the solution to each of the above Problems 2 and 4, is well known see for example [3, 10–11].

The main goal of this paper is to prove in Section 2 the existence and uniqueness of a family of optimal control problems 5 and 6 where the control variable is given by a boundary condition of Neumann type whose state system is governed by a free boundary problem with Tresca conditions on a portion Γ_2 of the boundary, with a flux f on Γ_3 as the control variable, using a regularization method to overcome the nondifferentiability of the functional Φ. Then, in Section 3, we study the convergence when $h \to +\infty$ of the state systems and optimal controls associated to Problem 6 to the corresponding state system and optimal control associated to Problem 5. In order to obtain this last result we obtain an auxiliary strong convergence by using the Aubin compactness arguments (see Lemma 2). This paper completes our previous paper [2] and solves the open problem left in [1].

Remark here that our study still valid with the bilinear form a in more general cases, provided that a must be symmetric, coercive and continuous from $V \times V$ to $\mathbb{R}$.

2 Boundary optimal control problems

Let $M > 0$ be a constant and we define the space

$$\mathcal{F}_- = \{f \in \mathcal{F} : f \leqslant 0\}.$$

We consider the following Neumannn boundary optimal control problems defined by [12–15].

Problem 5 Find the optimal control $f_{\mathrm{op}} \in \mathcal{F}_-$ such that

$$J(f_{\mathrm{op}}) = \min_{f \in \mathcal{F}_-} J(f), \tag{3}$$

where the cost functional $J : \mathcal{F}_- \to \mathbb{R}^+$ is given by

$$J(f) = \frac{1}{2}\|u_f\|_{\mathcal{H}}^2 + \frac{M}{2}\|f\|_{\mathcal{F}}^2 \ (M > 0), \tag{4}$$

and u_f is the unique solution to Problem 2 for a given $f \in \mathcal{F}_-$.

Problem 6 Find the optimal control $f_{\mathrm{op}_h} \in \mathcal{F}_-$ such that

$$J(f_{\mathrm{op}_h}) = \min_{f \in \mathcal{F}_-} J_h(f), \tag{5}$$

where the cost functional $J_h : \mathcal{F}_- \to \mathbb{R}^+$ is given by

$$J_h(f) = \frac{1}{2}\|u_{hf}\|_{\mathcal{H}}^2 + \frac{M}{2}\|f\|_{\mathcal{F}}^2 \ (M > 0,\ h > 0), \tag{6}$$

and u_{hf} is the unique solution to Problem 4 for a given $f \in \mathcal{F}_-$ and $h > 0$.

Theorem 1 Under the assumptions $g \geqslant 0$ in $\Omega \times (0, T)$, $b \geqslant 0$ on $\Gamma_1 \times (0, T)$ and $u_b \geqslant 0$ in Ω, we have the following properties:

a) the cost functional J is strictly convex on $\mathcal{F}_-$; and

b) there exists a unique optimal control $f_{\mathrm{op}} \in \mathcal{F}_-$ solution to the Neumann boundary optimal control Problem 5.

Proof We give some sketch of the proof, following [2], we generalize for parabolic variational inequalities of the second kind, given in Problem 2, the estimates obtained for convex combination between $u_4(\mu) = u_{\mu f_1 + (1-\mu) f_2}$, and $u_3(\mu) = \mu u_{f_1} + (1 - \mu)u_{f_2}$, for any two element f_1 and f_2 in $\mathcal{F}$. The main difficulty, to prove this result comes from the fact that the functional Φ is not differentiable. To overcome this difficulty, we use the regularization method and consider for $\varepsilon > 0$ the following approach of Φ defined by

$$\Phi_\varepsilon(v) = \int_{\Gamma_2} q\sqrt{\varepsilon^2 + |v|^2}\mathrm{d}s, \ \forall v \in V, \tag{7}$$

which is Gateaux differentiable, with

$$\langle \Phi'_\varepsilon(w), v\rangle = \int_{\Gamma_2} \frac{qwv}{\sqrt{\varepsilon^2 + |w|^2}}\mathrm{d}s, \ \forall (w, v) \in V^2.$$

We define u^ε as the unique solution to the corresponding parabolic variational inequality for all $\varepsilon > 0$. We obtain that for all $\mu \in [0, 1]$ we have $u_4^\varepsilon(\mu) \leqslant u_3^\varepsilon(\mu)$ for all $\varepsilon > 0$.

When $\varepsilon \to 0$ we have that for $i = 1, \ldots, 4$,

$$u_i^\varepsilon \to u_i \text{ strongly in } \mathcal{V} \cap L^\infty(0, T; H). \tag{8}$$

As $f \in \mathcal{F}_-$, $g \geqslant 0$ in $\Omega \times (0, T)$, $b \geqslant 0$ in $\Gamma_1 \times (0, T)$ and $u_b \geqslant 0$ in Ω, we obtain by the weak maximum principle that for all $\mu \in [0, 1]$ we have $0 \leqslant u_4(\mu)$, and so following [2], we obtain

$$0 \leqslant u_4(\mu) \leqslant u_3(\mu) \text{ in } \Omega \times [0, T], \ \forall \mu \in [0, 1]. \tag{9}$$

Then, for all $\mu \in [0, 1]$, and for all f_1, f_2 in $\mathcal{F}_-$, and by using $f_3(\mu) = \mu f_1 + (1 - \mu) f_2$, we obtain

$$\begin{aligned} &\mu J(f_1) + (1 - \mu)J(f_2) - J(f_3(\mu)) \\ &= \frac{1}{2}(\|u_3(\mu)\|_{\mathcal{H}}^2 - \|u_4(\mu)\|_{\mathcal{H}}^2) + \frac{1}{2}\mu(1-\mu)\|u_{f_1} - u_{f_2}\|_{\mathcal{H}}^2 \\ &\quad + \frac{M}{2}\mu(1 - \mu)\|f_1 - f_2\|_{\mathcal{F}}^2. \end{aligned} \tag{10}$$

Then, J is strictly convex functional on $\mathcal{F}_-$, and therefore, there exists a unique optimal $f_{\mathrm{op}} \in \mathcal{F}_-$ solution to the Neumann boundary optimal control Problem 5. This completes the proof.

Theorem 2 Under the assumptions $g \geqslant 0$ in $\Omega \times (0, T)$, $b \geqslant 0$ in $\Gamma_1 \times (0, T)$ and $u_b \geqslant 0$ in Ω, we have the following properties:

a) the cost functional J_h are strictly convex on $\mathcal{F}_-$, for all $h > 0$; and

b) there exists a unique optimal control $f_{h_{\mathrm{op}}} \in \mathcal{F}_-$ solution to the Neumann boundary optimal control problem 6, for all $h > 0$.

Proof We follow a similar method to the one developed in Theorem 1 for all $h > 0$.

3 Convergence when $h \to +\infty$

In this section, we study the convergence of the Neumann optimal control problem 6 to the optimal control problem 5 when $h \to \infty$. For a given $f \in \mathcal{F}$, we have first the following result which generalizes [2, 6–7, 16].

Lemma 1 Let u_{hf} be the unique solution to Problem 4 and u_f the unique solution to Problem 2, and then,

$$u_{hf} \to u_f \in \mathcal{V} \text{ strongly as } h \to +\infty, \ \ \forall f \in \mathcal{F}.$$

Proof Following [2], we take $v = u_f(t)$ in the variational inequality of Problem 4 where $u = u_{hf}$, and recalling that $u_f(t) = b$ on $\Gamma_1 \times [0, T]$, taking $\phi_h(t) = u_{hf}(t) - u_f(t)$ we obtain for $h > 1$, that $\|u_{hf}\|_{\mathcal{V}}$ is bounded for all $h > 1$ and for all $f \in \mathcal{F}$. Then, there exists $\eta \in \mathcal{V}$ such that (when $h \to +\infty$)

$$u_{hf} \rightharpoonup \eta \ \text{ weakly in } \ \mathcal{V},$$

and

$$u_{hf} \to b \ \text{ strongly on } \ L^2((0, T) \times \Gamma_1),$$

and so $\eta(0) = u_b$.

Let φ be in $L^2(0, T, H_0^1(\Omega))$ and take the variational inequality of Problem 4 where $u = u_{hf}$, $v = u_{hf}(t) \pm \varphi(t)$, and then, as $\|u_{hf}\|_{\mathcal{V}}$ is bounded for all $h > 1$, we deduce that $\|\dot{u}_{hf}\|_{L^2(0,T,H^{-1}(\Omega))}$ is also bounded for all $h > 1$. Then, we conclude that

$$\begin{aligned} &u_{hf} \rightharpoonup \eta \text{ in } \mathcal{V} \text{ weak, and in } L^\infty(0, T, H) \text{ weak star,} \\ &\text{and } \dot{u}_{hf} \rightharpoonup \dot{\eta} \text{ in } L^2(0, T, H^{-1}(\Omega)) \text{ weak.} \end{aligned} \tag{11}$$

From the variational inequality of Problem 4, taking $v \in K$ so $v = b$ on Γ_1, we obtain a.e. $t \in [0, T]$

$$\begin{aligned} &\langle \dot{u}_{hf}, v - u_{hf} \rangle + a(u_{hf}, v - u_{hf}) - h \int_{\Gamma_1} |u_{hf} - b|^2 \mathrm{d}s \\ &\geqslant \Phi(u_{hf}) - \Phi(v) + (g, v - u_{hf}) - \int_{\Gamma_3} f(v - u_{hf}) \mathrm{d}s, \end{aligned}$$

for all $v \in K$, and then, as $h > 0$ we have a.e. $t \in [0, T]$,

$$\begin{aligned} &\langle \dot{u}_{hf}, v - u_{hf} \rangle + a(u_{hf}, v - u_{hf}) \\ &\geqslant \Phi(u_{hf}) - \Phi(v) + (g, v - u_{hf}) - \int_{\Gamma_3} f(v - u_{hf}) \mathrm{d}s, \\ &\quad \forall v \in K. \end{aligned} \tag{12}$$

Therefore, using (11) and passing to the limit, when $h \to +\infty$, we obtain

$$\begin{aligned} &\langle \dot{\eta}, v - \eta \rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) \\ &\geqslant (g, v - \eta) - \int_{\Gamma_3} f(v - \eta) \mathrm{d}s, \ \ \forall v \in K \ \text{ a.e. } \ t \in [0, T], \end{aligned}$$

and $\eta(0) = u_b$. Using the uniqueness of the solution to Problem 2, we obtain $\eta = u_f$.

To prove the strong convergence, we take $v = u_f(t)$ in the variational inequality of Problem 4,

$$\begin{aligned} &\langle \dot{u}_{hf}, u_f - u_{hf} \rangle + a_h(u_{hf}, u_f - u_{hf}) + \Phi(u_f) - \Phi(u_{hf}) \\ &\geqslant (g, u_f - u_{hf}) + h \int_{\Gamma_1} b(u_f - u_{hf}) \mathrm{d}s \\ &\quad - \int_{\Gamma_3} f(u_f - u_{hf}) \mathrm{d}s, \end{aligned}$$

a.e. $t \in [0, T]$, and thus as $u_f = u_b$ on $\Gamma_1 \times [0, T]$, we put $\phi_h = u_{hf} - u_f$, and so a.e. $t \in [0, T]$ we have

$$\begin{aligned} &\langle \dot{\phi}_h, \phi_h \rangle + a(\phi_h, \phi_h) + h \int_{\Gamma_1} |\phi_h|^2 \mathrm{d}s + \Phi(u_{hf}) - \Phi(u_f) \\ &\leqslant \langle \dot{u}_f, \phi_h \rangle + a(u_f, \phi_h) + (g, \phi_h) - \int_{\Gamma_3} f \phi_h \mathrm{d}s, \end{aligned}$$

and so

$$\begin{aligned} &\frac{1}{2} \|\phi_h\|^2_{L^\infty(0,T,H)} + \lambda_h \|\phi_h\|^2_{\mathcal{V}} + \Phi(u_{hf}) - \Phi(u_f) \\ &\leqslant - \int_0^T \langle \dot{u}_f(t), \phi_h(t) \rangle \mathrm{d}t - \int_0^T a(u_f(t), \phi_h(t)) \mathrm{d}t \\ &\quad + \int_0^T (g(t), \phi_h(t)) \mathrm{d}t - \int_0^T \int_{\Gamma_3} f \phi_h \mathrm{d}s \mathrm{d}t. \end{aligned}$$

Using the weak semicontinuity of Φ and the weak convergence (11), the right side of the above inequality tends to zero when $h \to +\infty$, and then, we deduce the strong convergence of $\phi_h = u_{hf} - u_f$ to 0 in $\mathcal{V} \cap L^\infty(0, T, H)$, for all $f \in \mathcal{F}_-$ and the proof holds.

We prove now the following lemma by using the Aubin compactness arguments. Lemma 2 is very important and necessary which allow us to conclude this paper. Indeed this result is needed to pass to the limit exactly in the last term of the inequality (22) in the proof of the main Theorem 3.

Lemma 2 Let $u_{hf_{\mathrm{op}_h}}$ the state system defined by the unique solution to Problem 4, where the flux f is replaced by f_{op_h}. Then, for $h \to +\infty$, we have

$$u_{hf_{\mathrm{op}_h}} \to u_f \ \text{ in } \ L^2((0, T) \times \partial\Omega), \tag{13}$$

where u_f is the the state system defined by the unique solution to Problem 2 with the flux f on Γ_3.

Proof Let consider the variational inequality of Problem 4 with $u = u_{hf_{\mathrm{op}_h}}$ and $f = f_{\mathrm{op}_h}$, i.e.,

$$\begin{aligned} &\langle \dot{u}_{hf_{\mathrm{op}_h}}, v - u_{hf_{\mathrm{op}_h}} \rangle + a_h(u_{hf_{\mathrm{op}_h}}, v - u_{hf_{\mathrm{op}_h}}) + \Phi(v) \\ &- \Phi(u_{hf_{\mathrm{op}_h}}) \\ &\geqslant (g, v - u_{hf_{\mathrm{op}_h}}) - \int_{\Gamma_3} f_{\mathrm{op}_h}(v - u_{hf_{\mathrm{op}_h}}) \mathrm{d}s \\ &\quad + h \int_{\Gamma_1} b(v - u_{hf_{\mathrm{op}_h}}) \mathrm{d}s, \ \ \forall v \in V, \end{aligned} \tag{14}$$

and let $\varphi \in L^2(0, T; H_0^1(\Omega))$, and set $v = u_{hf_{\mathrm{op}_h}}(t) \pm \varphi(t)$ in (14), we obtain

$$\langle \dot{u}_{hf_{\mathrm{op}_h}}, \varphi \rangle = (g, \varphi) - a(u_{hf_{\mathrm{op}_h}}, \varphi).$$

By integration in times for $t \in (0, T)$, we obtain

$$\int_0^T \langle \dot{u}_{hf_{\mathrm{op}_h}}, \varphi \rangle \mathrm{d}t = \int_0^T (g, \varphi) \mathrm{d}t - \int_0^T a(u_{hf_{\mathrm{op}_h}}, \varphi) \mathrm{d}t,$$

and thus, for $A = (c\|g\|_{\mathcal{H}} + \|u_{hf_{\mathrm{op}_h}}\|_{\mathcal{V}})$, we obtain

$$\Big| \int_0^T < \dot{u}_{hf_{\mathrm{op}_h}}, \varphi > \mathrm{d}t \Big| \leqslant A \|\varphi\|_{L^2(0,T;H_0^1(\Omega))},$$

where c comes from the Poincaré inequality, and as in Lemma 1 we can obtain that $u_{hf_{\mathrm{op}_h}}$ is bounded in $\mathcal{V}$, and so there exists a positive constant C such that

$$\|\dot{u}_{hf_{\mathrm{op}_h}}\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C. \tag{15}$$

Using now the Aubin compactness arguments, see for example [17] with the three Banach spaces V, $H^{\frac{2}{3}}(\Omega)$ and $H^{-1}(\Omega)$, and then,

$$u_{hf_{\mathrm{op}_h}} \to u_f \ \ L^2(0, T; H^{\frac{2}{3}}(\Omega)).$$

As the trace operator γ_0 is continuous from $H^{\frac{2}{3}}(\Omega)$ to $L^2(\partial\Omega)$, and then, the result follows. This completes the proof.

We give now, without need to use the notion of adjoint states [14, 18], the convergence result which generalizes the result obtained in [19] for a parabolic variational equalities (see also [18, 20–23]). Other optimal control problems gouverned by variational inequalities are given in [24–26].

Theorem 3 Let $u_{hf_{\mathrm{op}_h}} \in \mathcal{V}$, $f_{\mathrm{op}_h} \in \mathcal{F}_-$ and $u_{f_{\mathrm{op}}} \in \mathcal{V}$,

$f_{\mathrm{op}} \in \mathcal{F}_-$ be respectively the state systems and the optimal controls defined in Problems 4 and 2. Then,

$$\lim_{h\to+\infty} \|u_{hf_{\mathrm{op}_h}} - u_{f_{\mathrm{op}}}\|_{\mathcal{V}} = \lim_{h\to+\infty} \|u_{hf_{\mathrm{op}_h}} - u_{f_{\mathrm{op}}}\|_{L^\infty(0,T,H)} = \lim_{h\to+\infty} \|u_{hf_{\mathrm{op}_h}} - u_{f_{\mathrm{op}}}\|_{L^2((0,T)\times\Gamma_1)} = 0, \quad (16)$$

$$\lim_{h\to+\infty} \|f_{\mathrm{op}_h} - f_{\mathrm{op}}\|_{\mathcal{F}} = 0. \quad (17)$$

Proof We have first

$$J_h(f_{\mathrm{op}_h}) = \frac{1}{2}\|u_{hf_{\mathrm{op}_h}}\|^2_{\mathcal{H}} + \frac{M}{2}\|f_{\mathrm{op}_h}\|^2_{\mathcal{F}} \leqslant \frac{1}{2}\|u_{hf}\|^2_{\mathcal{H}} + \frac{M}{2}\|f\|^2_{\mathcal{F}},$$

for all $f \in \mathcal{F}_-$, and then, for $f = 0 \in \mathcal{F}_-$ we obtain

$$J_h(f_{\mathrm{op}_h}) = \frac{1}{2}\|u_{hf_{\mathrm{op}_h}}\|^2_{\mathcal{H}} + \frac{M}{2}\|f_{\mathrm{op}_h}\|^2_{\mathcal{F}} \leqslant \frac{1}{2}\|u_{h0}\|^2_{\mathcal{H}}, \quad (18)$$

where $u_{h0} \in \mathcal{V}$ is the solution to the following parabolic variational inequality:

$$\langle \dot{u}_{h0}, v - u_{h0}\rangle + a_h(u_{h0}, v - u_{h0}) + \Phi(v) - \Phi(u_{h0}) \geqslant \int_\Omega g(v - u_{h0})\mathrm{d}x + h\int_{\Gamma_1} b(v - u_{h0})\mathrm{d}s, \text{ a.e. } t \in [0,T]$$

for all $v \in V$ and $u_{h0}(0) = u_b$.

Taking $v = u_b \in K_b$ we obtain that $\|u_{h0} - u_b\|_{\mathcal{V}}$ is bounded independently of h, and then, $\|u_{h0}\|_{\mathcal{H}}$ is bounded independently of h. Therefore, we deduce with (18) that $\|u_{hf_{\mathrm{op}_h}}\|_{\mathcal{H}}$ and $\|f_{\mathrm{op}_h}\|_{\mathcal{F}}$ are also bounded independently of h. Therefore, there exist $\tilde{f} \in \mathcal{F}_-$ and η in $\mathcal{H}$ such that

$$f_{\mathrm{op}_h} \rightharpoonup \tilde{f} \text{ in } \mathcal{F}_- \text{ and } u_{hf_{\mathrm{op}_h}} \rightharpoonup \eta \text{ in } \mathcal{H} \text{ (weakly)}. \quad (19)$$

Taking now $v = u_{f_{\mathrm{op}}}(t) \in K_b$ in Problem (4), for $t \in [0,T]$, with $u = u_{hf_{\mathrm{op}_h}}$ and $f = f_{\mathrm{op}_h}$, we obtain

$$\langle \dot{u}_{hf_{\mathrm{op}_h}}, u_{f_{\mathrm{op}}} - u_{hf_{\mathrm{op}_h}}\rangle + a_1(u_{hf_{\mathrm{op}_h}}, u_{f_{\mathrm{op}}} - u_{hf_{\mathrm{op}_h}}) + (h-1)\int_{\Gamma_1} u_{hf_{\mathrm{op}_h}}(u_{f_{\mathrm{op}}} - u_{hf_{\mathrm{op}_h}})\mathrm{d}s + \Phi(u_{f_{\mathrm{op}}}) - \Phi(u_{hf_{\mathrm{op}_h}})$$
$$\geqslant (g, u_{f_{\mathrm{op}}} - u_{hf_{\mathrm{op}_h}}) + h\int_{\Gamma_1} b(u_{f_{\mathrm{op}}} - u_{hf_{\mathrm{op}_h}})\mathrm{d}s - \int_{\Gamma_3} f_{\mathrm{op}_h}(u_{f_{\mathrm{op}}} - u_{hf_{\mathrm{op}_h}})\mathrm{d}s, \text{ a.e. } t \in [0,T].$$

As $u_{f_{\mathrm{op}}} = b$ on $\Gamma_1 \times [0,T]$, taking $\phi_h = u_{f_{\mathrm{op}}} - u_{hf_{\mathrm{op}_h}}$, we obtain

$$\frac{1}{2}\|\phi_h\|^2_{L^\infty(0,T;H)} + \lambda_1\|\phi_h\|^2_{\mathcal{V}} + (h-1)\int_0^T\int_{\Gamma_1} |\phi_h(t)|^2\mathrm{d}s\mathrm{d}t$$
$$\leqslant \int_0^T\int_{\Gamma_3} f_{\mathrm{op}_h}\phi_h\mathrm{d}s\mathrm{d}t - \int_0^T (g(t), \phi_h(t))\mathrm{d}t + \int_0^T\int_{\Gamma_2} q|\phi_h(t)|\mathrm{d}s\mathrm{d}t + \int_0^T \langle \dot{u}_{f_{\mathrm{op}}}(t)\phi_h(t)\rangle\mathrm{d}t + \int_0^T a(u_{f_{\mathrm{op}}}(t), \phi_h(t))\mathrm{d}t.$$

As f_{op_h} is bounded in $\mathcal{F}_-$, from (15) $\dot{u}_{f_{\mathrm{op}}}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, and $u_{hf_{\mathrm{op}_h}}$ is also bounded in $\mathcal{V}$, all independently on h, and so there exists a positive constant C which does not depend on h such that

$$\|\phi_h\|_{\mathcal{V}} = \|u_{hf_{\mathrm{op}_h}} - u_{f_{\mathrm{op}}}\|_{\mathcal{V}} \leqslant C, \quad \|\phi_h\|_{L^\infty(0,T,H)} \leqslant C$$
$$(h-1)\int_0^T\int_{\Gamma_1} |u_{hf_{\mathrm{op}_h}} - b|^2\mathrm{d}s\mathrm{d}t \leqslant C,$$

and then, $\eta \in \mathcal{V}$ and

$$u_{hf_{\mathrm{op}_h}} \rightharpoonup \eta \text{ in } \mathcal{V} \text{ and in } L^\infty(0,T,H) \text{ weak star}, \quad (20)$$
$$u_{hf_{\mathrm{op}_h}} \to b \text{ in } L^2((0,T)\times\Gamma_1) \text{ strong}, \quad (21)$$

and so $\eta(t) \in K_b$ for all $t \in [0,T]$.

Now, taking $v \in K$ in Problem 4 where $u = u_{hf_{\mathrm{op}_h}}$ and $f = f_{\mathrm{op}_h}$ so

$$\langle \dot{u}_{hf_{\mathrm{op}_h}}, v - u_{hf_{\mathrm{op}_h}}\rangle + a_h(u_{hf_{\mathrm{op}_h}}, v - u_{hf_{\mathrm{op}_h}}) + \Phi(v) - \Phi(u_{hf_{\mathrm{op}_h}})$$
$$\geqslant (f_{\mathrm{op}_h}, v - u_{hf_{\mathrm{op}_h}}) + h\int_{\Gamma_1} b(v - u_{hf_{\mathrm{op}_h}})\mathrm{d}s - \int_{\Gamma_3} f_{\mathrm{op}_h}(v - u_{hf_{\mathrm{op}_h}})\mathrm{d}s, \text{ a.e. } t \in [0,T]$$

as $v \in K_b$ so $v = b$ on Γ_1, and thus, we have

$$\langle \dot{u}_{hf_{\mathrm{op}_h}}, u_{hf_{\mathrm{op}_h}} - v\rangle + a(u_{hf_{\mathrm{op}_h}}, u_{hf_{\mathrm{op}_h}} - v) + h\int_{\Gamma_1} |u_{hf_{\mathrm{op}_h}} - b|^2\mathrm{d}s + \Phi(u_{hf_{\mathrm{op}_h}}) - \Phi(v) - (g, v - u_{hf_{\mathrm{op}_h}})$$
$$\leqslant \int_{\Gamma_3} f_{\mathrm{op}_h}(v - u_{hf_{\mathrm{op}_h}})\mathrm{d}s \text{ a.e. } t \in [0,T].$$

Thus,

$$\langle \dot{u}_{hf_{\mathrm{op}_h}}, u_{hf_{\mathrm{op}_h}} - v\rangle + a(u_{hf_{\mathrm{op}_h}}, u_{hf_{\mathrm{op}_h}} - v) + \Phi(u_{hf_{\mathrm{op}_h}}) - \Phi(v)$$
$$\leqslant -(g, v - u_{hf_{\mathrm{op}_h}}) - \int_{\Gamma_3} f_{\mathrm{op}_h}(v - u_{hf_{\mathrm{op}_h}})\mathrm{d}s \text{ a.e. } t \in [0,T]. \quad (22)$$

Using Lemma 2, (19) and (20), we deduce that [3, 27]

$$\langle \dot{\eta}, v - \eta\rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) \geqslant (f, v-\eta) - \int_{\Gamma_3} \tilde{f}(v-\eta))\mathrm{d}s, \quad \forall v \in K, \text{ a.e. } t \in [0,T],$$

so also by the uniqueness of the solution to Problem 2, we obtain

$$u_{\tilde{f}} = \eta. \quad (23)$$

We prove that $\tilde{f} = f_{\mathrm{op}}$. Indeed, we have

$$J(\tilde{f}) = \frac{1}{2}\|\eta\|^2_{\mathcal{H}} + \frac{M}{2}\|\tilde{f}\|^2_{\mathcal{F}} \leqslant \liminf_{h\to+\infty}\{\frac{1}{2}\|u_{hf_{\mathrm{op}_h}}\|^2_{\mathcal{H}} + \frac{M}{2}\|f_{\mathrm{op}_h}\|^2_{\mathcal{F}}\} = \liminf_{h\to+\infty} J_h(f_{\mathrm{op}_h})$$
$$\leqslant \liminf_{h\to+\infty} J_h(f) = \liminf_{h\to+\infty}\{\frac{1}{2}\|u_{hf}\|^2_{\mathcal{H}} + \frac{M}{2}\|f\|^2_{\mathcal{F}}\},$$

and so using now the strong convergence $u_{hf} \to u_f$ as $h \to +\infty, \forall f \in \mathcal{F}_-$ (see Lemma 1), we obtain

$$J(\tilde{f}) \leqslant \liminf_{h\to+\infty} J_h(f_{\mathrm{op}_h}) \leqslant \frac{1}{2}\|u_f\|^2_{\mathcal{H}} + \frac{M}{2}\|f\|^2_{\mathcal{F}} = J(f), \quad \forall f \in \mathcal{F}_-, \quad (24)$$

and then, by the uniqueness of the optimal control problem 2, we obtain

$$\tilde{f} = f_{\mathrm{op}}. \quad (25)$$

Now, we prove the strong convergence of $u_{hf_{\mathrm{op}_h}}$ to $\eta = u_f$ in $\mathcal{V} \cap L^\infty(0,T;H) \cap L^2(0,T;L^2(\Gamma_1))$, indeed taking $v = \eta$ in Problem 4 where $u = u_{hf_{\mathrm{op}_h}}$ and $f = f_{\mathrm{op}_h}$, as $\eta(t) \in K$ for $t \in [0,T]$, and so $\eta = b$ on Γ_1, we obtain

$$\frac{1}{2}\|u_{hf_{\mathrm{op}_h}} - \eta\|^2_{L^\infty(0,T;H)} + \lambda_1\|u_{hf_{\mathrm{op}_h}} - \eta\|^2_{\mathcal{V}}$$

$$+\int_0^T\{\Phi(u_{hf_{\mathrm{op}_h}})-\Phi(\eta)\}\mathrm{d}t+\tilde{h}\|u_{hf_{\mathrm{op}_h}}-\eta\|^2_{L^2((0,T)\times\Gamma_1)}$$
$$\leqslant\int_0^T(g,u_{hf_{\mathrm{op}_h}}(t)-\eta(t))\mathrm{d}t-\int_0^T\langle\dot{\eta},u_{hf_{\mathrm{op}_h}}-\eta\rangle\mathrm{d}t$$
$$+\int_0^T a(\eta(t),\eta(t)-u_{hf_{\mathrm{op}_h}}(t))$$
$$-\int_{\Gamma_3}f_{\mathrm{op}_h}(u_{hf_{\mathrm{op}_h}}-\eta))\mathrm{d}s\mathrm{d}t,$$

where $\tilde{h}=h-1$.

Using (20) and the weak semicontinuity of Φ, we deduce that

$$\lim_{h\to+\infty}\|u_{hf_{\mathrm{op}_h}}-\eta\|_{L^\infty(0,T;H)}=\lim_{h\to+\infty}\|u_{hf_{\mathrm{op}_h}}-\eta\|_{\mathcal{V}}=\|u_{hf_{\mathrm{op}_h}}-\eta\|_{L^2((0,T)\times\Gamma_1)}=0,$$

and with (23) and (25) we deduce (16). Then, from (24) and (25), we can write

$$J(f_{\mathrm{op}})=\frac{1}{2}\|u_{f_{\mathrm{op}}}\|^2_{\mathcal{H}}+\frac{M}{2}\|f_{\mathrm{op}}\|^2_{\mathcal{F}}\leqslant\liminf_{h\to+\infty}J_h(f_{\mathrm{op}_h})$$
$$=\liminf_{h\to+\infty}\{\frac{1}{2}\|u_{hf_{\mathrm{op}_h}}\|^2_{\mathcal{H}}+\frac{M}{2}\|f_{\mathrm{op}_h}\|^2_{\mathcal{F}}\}$$
$$\leqslant\lim_{h\to+\infty}J_h(f_{\mathrm{op}})=J(f_{\mathrm{op}}),\tag{26}$$

and using the strong convergence (16), we obtain

$$\lim_{h\to+\infty}\|f_{\mathrm{op}_h}\|_{\mathcal{F}}=\|f_{\mathrm{op}}\|_{\mathcal{F}}.\tag{27}$$

Finally as

$$\|f_{\mathrm{op}_h}-f_{\mathrm{op}}\|^2_{\mathcal{F}}=\|f_{\mathrm{op}_h}\|^2_{\mathcal{F}}+\|f_{\mathrm{op}}\|^2_{\mathcal{F}}-2(f_{\mathrm{op}_h},f_{\mathrm{op}}),\tag{28}$$

and by the first part of (19) we have

$$\lim_{h\to+\infty}(f_{\mathrm{op}_h},f_{\mathrm{op}})=\|f_{\mathrm{op}}\|^2_{\mathcal{F}},$$

and so from (27) and (28) we obtain (17). This completes the proof.

Corollary 1 Let $u_{hf_{\mathrm{op}_h}}$ in $\mathcal{V}$, f_{op_h} in $\mathcal{F}_-$, $u_{f_{\mathrm{op}}}$ in $\mathcal{V}$ and f_{op} in $\mathcal{F}_-$ be respectively the state systems and the optimal controls defined in Problems 4 and 2. Then,

$$\lim_{h\to+\infty}|J_h(f_{\mathrm{op}_h})-J(f_{\mathrm{op}})|=0.$$

Proof It follows from the definitions (3) and (4), and the convergences (16) and (17).

4 Conclusions

The main difference between this paper and our previous work [2] where the control variable was the function g, is that we consider here as a control variable the function f given by the Neumann boundary condition on Γ_3. This change induce in the variational problems 2 and 4, and also in the proofs of Lemma 1 and Theorem 3, a new integral term on Γ_3. The main difficulty here is in Section 3, and the question is exactly how to pass to the limit for $h\to+\infty$ in the last integral term on Γ_3 in (22). To overcome this main difficulty we have introduced the new lemma 2, which is the key of our problem. The idea of Lemma 1 and Theorem 3 and their proofs are indeed similar to those of our work [2] with the differences and difficulties mentioned just above.

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