Lorentz gauge-invariant variables in torsion-based theories of gravity

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General relativity dynamics can be derived from different actions—which depart from the Einstein-Hilbert action in boundary terms—and for different choices of the dynamical variables. Among them, the *teleparallel equivalent of general relativity* is a torsion-based theory for the tetrad field. More general torsion-based theories have been built in the last years, intending to supersede general relativity. There are two current ways to formulate such theories; one includes a spin connection and the other does not. We discuss the notion of Lorentz gauge invariance in such theories, and give a simple but important proof that both formulations are physically equivalent.

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I. INTRODUCTION

Fernparallelismus or teleparallelism is firstly known as an attempt by Einstein to base a unified theory of the electromagnetic and gravitational fields on the mathematical structure of absolute teleparallelism. In this framework, spacetime is characterized by a curvatureless connection together with a metric tensor field, both defined in terms of a dynamical *tetrad* or *vierbein* field (a field of orthonormal tangent-space bases). This was an episode in Einstein's research that lasted for three years from 1928 to 1931. Although the pertinent mathematical structures had been developed before by Cartan and Weitzenböck, they were introduced just as mathematical concepts, and the search for field equations for the tetrads was the novelty, efforts which eventually were abandoned [1].

The concepts of teleparallelism were not further explored until three decades later, when Møller [2] revived Einstein's original idea, but in order to find a tensorial complex for the gravitational energy-momentum density. After this, the Lagrangian formulation for teleparallel gravity was written by Pellegrini and Plebianski [3]. Later, Hayashi and Shirafuji [4] proposed the new general relativity (NGR), a teleparallel theory based on the curvatureless Weitzenböck connection (which is linear in first derivatives of the tetrad). The NGR Lagrangian combines the three quadratic invariants emerging from the decomposition of the Weitzenböck torsion in its irreducible parts. Three parameters enter the NGR Lagrangian, which can be fixed to render NGR the *teleparallel equivalent* of general relativity (TEGR). The equivalence between TEGR and GR lies in the fact that TEGR dynamics for the tetrad yields the GR dynamics for the metric whenever the tetrad is thought as an *orthonormal* basis in the tangent space, so linking the tetrad to the metric. The orthonormality property is clearly invariant under *local*-Lorentz transformations of the tetrad. The equivalence between TEGR and GR would suggest that the TEGR Lagrangian should exhibit such a *gauge invariance*. Actually, TEGR Lagrangian is not invariant but *pseudoinvariant* [5]: a local-Lorentz transformation of the tetrad produces a boundary term in the TEGR action, which does not affect the dynamics.

In the middle of the 2000's, Ferraro and Fiorini proposed a *nonlinear* extension of TEGR theory that consisted in replacing the *torsion scalar* $\mathbb{T}$ in the Lagrangian with a convenient function of it. For a Born-Infeld-like Lagrangian, they have shown that the inflationary early Universe is a solution for ordinary sources such as radiation or dust [6]. This opened the study of the so-called $f(\mathbb{T})$ theories of modified teleparallel gravity [7–9]. We should notice that, although pseudoinvariance works like a full invariance at the level of the TEGR equations of motion, this feature is lost when passing to $f(\mathbb{T})$ gravity. The divergence associated with the boundary term now remains encapsulated within the argument of the function $f(\mathbb{T})$ in the Lagrangian. Thus the equations of motion are no longer

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invariant under local-Lorentz transformations of the tetrad (however, they remain invariant under global Lorentz transformations). The loss of this gauge freedom implies that $f(\mathbb{T})$ theories contain some degrees of freedom (d.o.f.) associated with the local orientation of the tetrad field (i.e., with the *parallelization* of the spacetime). How many extra d.o.f. are involved in $f(\mathbb{T})$ and other modified teleparallel gravities is a subject under study; but given the nonlinear background-dependent nature of its constraints, it can be expected that the extra d.o.f. will depend on the point in the phase space [10–15]. As long as the matter couples just to the metric, or even the Levi-Civita connection, the sole possibility of observing these elusive extra d.o.f. would be via changes of gravitational dynamics, but not at the level of the motion of matter in a given spacetime.

The loss of the Lorentz gauge freedom can be avoided from the beginning, by providing the TEGR Lagrangian with a divergence term to guarantee its full local-Lorentz invariance. However, this would lead to a sort of $f(R)$ theory (nonlinear extension of the GR Lagrangian), which would depart from the teleparallel foundations. A different strategy consists of improving the Weitzenböck connection, by replacing it with a more general one that is able to provide a better behavior under local-Lorentz transformations. In this way, we endow the theory with a full Lorentz gauge invariance. For reasons that will be clearer in the next section, this strategy is called *covariantization*.

In recent years we have been witnessing intense discussions about the local-Lorentz invariance in modified teleparallel theories of gravity, such as $f(\mathbb{T})$ [16]. Some authors object the covariantization procedure as inconsistent since the components of the more general curvatureless connection cannot be fixed by varying the covariantized action [17,18]. Therefore the connection must be chosen by resorting to some criterion, which is not very different to directly adopting the Weitzenböck connection. Of course, there might be some preferences based on symmetry considerations, but those would often be not unique, neither available in cases of no symmetry.

Some others view the local-Lorentz-covariant representations as the only consistent approach [19,20], by assuming that the fundamental tetrad variable of teleparallel gravity must be regarded as the frame an observer is free to choose, and with the spin connection somehow related to inertial effects. In such case, the Lorentz gauge freedom is frozen by parallelizing according to some physical criterion. But if one thinks of the tetrad as a geometric object to describe gravity, just a set of orthonormal vectors without caring about observers, then such a "physical" criterion becomes meaningless. In any case, the extra d.o.f. will enter into play and they turn out to be rather unhealthy, at least for the standard $f(\mathbb{T})$ theory [21,22] and NGR [23,24].

All these disagreements seem odd, as there is evidence that the covariant and the (original) pure-tetrad approaches would be equivalent [16,25]. Moreover, the general curvatureless connection introduces trivial primary constraints in the Hamiltonian analysis [26,27] which are evidently first class [28]. For instance, the so-called preferred frame effects or "frame-dependent artifacts" are resolved only very superficially [19,20,29]. In a covariantized model one is allowed to choose absolutely any available tetrad for a given metric, but at the price of taking an appropriate spin connection. On the other hand, two different solutions for the same metric ansatz of the pure-tetrad approach would also give two different solutions in the covariantized version, in disguise of two different possible choices of the spin connection for a given frame [25].

It is not to say that the covariant approach has no physical meaning, since for example, it naturally comes from taking teleparallel gravity as a gauge theory of translations [30]. Of course, this gives good hope for formulating conserved quantities such as energy, for authors who believe that these quantities should be well defined for gravity despite that there is no fundamental symmetry which would make it justified, so that in a really canonical way one gets rather a holographic notion of conserved quantities, necessarily paying attention to boundary terms [31].

Leaving aside the controversies, the covariant methods can be used for more convenience of doing calculations [25]. However, what we will show in this article is that the formulation is precisely the same as simply taking the gauge invariant variables in the covariant version, for any teleparallel model which is globally Lorentz invariant. This is yet another, very simple and direct proof that the two versions have no physical difference.

II. TELEPARALLEL AND MODIFIED TELEPARALLEL GRAVITY

In its standard formulation, teleparallel gravity is built from the Weitzenböck connection, which is the connection that makes the tetrad e^b_ν, and its inverse e^μ_b, parallel transported as $0 = \nabla_\mu e^b_\nu = \partial_\mu e^b_\nu - \Gamma^\alpha_{\mu\nu} e^b_\alpha$. Then, its components are

$$\Gamma^\alpha_{\mu\nu} \equiv e^\alpha_a \partial_\mu e^a_\nu. \tag{1}$$

The Weitzenböck connection is compatible with the metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, or $g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}$ (the basis $\{\mathbf{e}_a\}$ is orthonormal). $\Gamma^\alpha_{\mu\nu}$ results to be a curvatureless connection whose torsion is

$$T^\alpha{}_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = e^\alpha_a (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu). \tag{2}$$

The torsion scalar

$$\mathbb{T} = \frac{1}{4} T_{\alpha\beta\mu} T^{\alpha\beta\mu} + \frac{1}{2} T_{\alpha\beta\mu} T^{\beta\alpha\mu} - T_\mu T^\mu, \tag{3}$$

where $T_\mu = T^\alpha{}_{\alpha\mu}$ is the torsion vector, is directly related to the Levi-Civita curvature scalar $\overset{\circ}{R}$ of the metric $g_{\mu\nu}$ (that is the GR Lagrangian),

$$e\overset{\circ}{R} = -e\mathbb{T} + \partial_\mu(2eT^\mu), \tag{4}$$

where $e = \det e^a_\mu = (-\det g_{\mu\nu})^{1/2}$. On this basis, the TEGR Lagrangian density is taken equal to $\pm e\mathbb{T}$, with the sign depending on the chosen signature. Thus we obtain a TEGR Lagrangian quadratic in first derivatives of the tetrad. The second derivatives contained in the GR Lagrangian have been cornered in the boundary term of Eq. (4), which is not relevant for the dynamics.

In Eq. (4), the lhs depends just on the metric, which is invariant under local-Lorentz transformations of the tetrad,

$$e^a_\mu \to L^a_c(x)e^c_\mu \Rightarrow g_{\mu\nu} \to g_{\mu\nu}, \tag{5}$$

where L^a_c is a matrix belonging to the Lorentz group (so it is $\eta_{ab}L^a_cL^b_d = \eta_{cd}$). We remark that we are not talking about the behavior under diffeomorphisms, since each term in the rhs of Eq. (4) is separately a scalar density. Indeed the expression (1) transforms as an affine connection under coordinate changes, which implies that $T^\alpha{}_{\mu\nu}$ in Eq. (2) really transforms as the components of a tensor. We are focusing our analysis on the behavior of the rhs of Eq. (4) under local changes of the orientation of the tetrad. Of course, if the lhs of Eq. (4) possesses local-Lorentz invariance, then the rhs will possess it too. However, separately each term in the rhs is only *globally* Lorentz invariant. This is because the components of the torsion in Eq. (2) are made of the components $\partial_\mu e^a_\nu - \partial_\nu e^a_\mu$ of the exterior derivative of the tetrad $d\mathbf{e}^a$, which changes as

$$d\mathbf{e}^a \to L^a_c\, d\mathbf{e}^c + d(L^a_c) \wedge \mathbf{e}^c. \tag{6}$$

Only if the Lorentz transformation is global, then the 1-forms $d(L^a_c)$ will be zero; thus $d\mathbf{e}^a$ will behave as a 2-form valued in the tangent space. In a more prosaic language, only for global transformations, the label "a" in $d\mathbf{e}^a$ will behave as a "contravariant" index under Lorentz transformations of the tetrad, in the sense that $d\mathbf{e}^a$ will transform like $\mathbf{e}^a$.

The violation of the local-Lorentz invariance does not show up in the equations of motion of TEGR since its Lagrangian density differs only by a boundary term $\mathbb{B} = \partial_\mu(2eT^\mu)$ from the GR one. This makes TEGR *pseudoinvariant* under local-Lorentz transformations of the tetrad [32]. Pseudoinvariance is a rather low price to be paid for substituting the GR Lagrangian, which contains second derivatives of the metric, with the simpler TEGR Lagrangian that is built from first derivatives of the tetrad. However, there are many generalizations, such as NGR, $f(\mathbb{T})$ gravity, or even higher derivative ones, which do violate the local-Lorentz invariance at the level of equations of motion. The most popular higher derivative models are $f(\mathbb{T}, \mathbb{B})$ ones, even though they do not go much beyond the more usual modified gravities and $f(\mathbb{T})$ gravity. Due to the basic relation (4), they can obviously be represented as simply $f(\overset{\circ}{R}, \mathbb{T})$, apparently inheriting all the potential problems that $f(\mathbb{T})$ could have [21,22].

III. LORENTZ COVARIANTIZATION

The components of the torsion in Eq. (2) will become invariant under local-Lorentz transformations of the tetrad if the ordinary exterior derivative $d\mathbf{e}^a$ is *covariantized* by endowing it with a *spin-connection* term:

$$\mathcal{D}\mathbf{e}^a \equiv d\mathbf{e}^a + \omega^a{}_b \wedge \mathbf{e}^b. \tag{7}$$

The 1-forms $\omega^a{}_b$ must accompany the change of the tetrad by transforming as components of a spin connection to absorb the undesirable term in Eq. (6):

$$\mathbf{e}^a \to L^a_c\, \mathbf{e}^c, \quad \omega^a{}_b \to L^a_c\omega^c{}_d(L^{-1})^d_b - (L^{-1})^c_b\, d(L^a_c). \tag{8}$$

Thus it turns out to be $\mathcal{D}\mathbf{e}^a \to L^a_c(x)\,\mathcal{D}\mathbf{e}^c$.

When talking about invariance or covariance, it is very important to precisely specify what kind of transformations are being considered, since the invoked property is not only about the algebraic structure of the involved group but also about how it acts. Let us introduce the following two definitions:

Definition 1: If a teleparallel theory is Lorentz invariant under the simultaneous transformations of the tetrad field and the spin connection of Eq. (8), then it will be called *Lorentz invariant of type I*.

Definition 2: If a teleparallel theory is Lorentz invariant under the transformation (5) of the tetrad field (alone), then it will be called *Lorentz invariant of type II*.

Remark: Naturally, if a theory violates the Lorentz invariance only at the boundary, the word "pseudo" is added to the above definitions.

Any theory with an explicitly introduced spin connection is usually Lorentz invariant of type I, in the full meaning of that, without the prefix pseudo. However, in case of teleparallel models, the type II is at best a "pseudoinvariance" which gets broken by almost every modification away from TEGR. On the other hand, given the fulfillment of type I invariance, the transformations of type II can equivalently be viewed as transformations of the spin connection alone.

In its pure-tetrad formulation, TEGR uses the Weitzenböck spin connection $\omega^a{}_b = 0$. But a teleparallel theory covariantly formulated should be built from the (Lorentz-invariant) torsion $T^\alpha{}_{\mu\nu} = e^\alpha_a(\mathcal{D}\mathbf{e}^a)_{\mu\nu}$.[1] Since

[1] $\mathbf{T}^a = \mathcal{D}\mathbf{e}^a$ is the 1st Cartan's structure equation. It defines the relation between torsion and connection.

teleparallelism uses curvatureless connections, so implying that gravity comes exclusively from the torsion field, the spin connection should be chosen within the family of curvatureless spin connections to which the Weitzenböck connection belongs. The more general connection of this type can be obtained by local-Lorentz transforming the Weitzenböck connection (modulo possible global issues of cohomology type)

$$\omega^a{}_b = -(\Lambda^{-1})^c_b \, d\Lambda^a_c, \tag{9}$$

where the Λ's are matrices belonging to the Lorentz group.[2] We use "Λ" instead of "L" because they are new variables of the theory, characterizing the spin connection, while "L" indicates the local-Lorentz transformations of the variables.

Now the (type I) simultaneous Lorentz transformation (8) can be displayed as

$$e^a_\mu \to L^a_c(x) e^c_\mu, \quad \Lambda^a_b \to L^a_c(x)\Lambda^c_b. \tag{10}$$

As explained above, any (modified) teleparallel model which is globally Lorentz invariant in its pure-tetrad formulation (every model discussed in the literature we know) becomes locally Lorentz invariant upon this covariantization procedure, though with respect to the (type I) simultaneous transformation (8).

For many popular models, it has been shown that the equation of motion for the spin connection, which results from varying the action with respect to the Λ's variables, just reproduces the antisymmetric part of the tetrad equations. Why this happens is rather evident [16]: in the Lorentz-covariant action, the antisymmetric variation of the tetrad gets precisely compensated by variation of the spin connection (keeping it inside the flat metric-compatible class).

Therefore, in the covariant version we can always choose the $\omega^a{}_b = 0$ gauge, even right inside the action which otherwise is not always a harmless choice to do, which brings us back to the pure-tetrad formalism.

Moreover, any gauge choice does not influence the value of the torsion tensor, therefore it is not only that this choice does not influence the physical contents of equations of motion, but it does not change the global quantities, like the full value of the action, either. If we found some solution in the covariant version, we can choose a gauge and make it a pure-tetrad solution, with the same metric and the same torsion tensor.

[2]The basis $\{\mathbf{e}_a\}$ will no longer be parallel transported if the Weitzenböck connection is replaced with (9), but the connection will be still metric. Anyway, the "parallelization" can be always retrieved by passing to the Weitzenböck connection through a local-Lorentz transformation.

IV. LORENTZ GAUGE-INVARIANT VARIABLES

Now, when we know that any globally Lorentz-invariant modified teleparallel model can be covariantized by replacing the partial derivatives of the tetrad for the Lorentz-covariant ones, let us make a simple rewriting of the covariant derivative (7) with the connection (9):

$$\begin{aligned}(\mathcal{D}\mathbf{e}^a)_{\mu\nu} &= 2\,\partial_{[\mu} e^a_{\nu]} - 2(\Lambda^{-1})^c_b (\partial_{[\mu}\Lambda^a_c)\, e^b_{\nu]} \\ &= 2\,\partial_{[\mu} e^a_{\nu]} + 2\,\Lambda^a_c (\partial_{[\mu}\,(\Lambda^{-1})^c_b) e^b_{\nu]} \\ &= \Lambda^a_c\, 2\Big[(\Lambda^{-1})^c_b \partial_{[\mu} e^b_{\nu]} + (\partial_{[\mu}(\Lambda^{-1})^c_b) e^b_{\nu]}\Big] \\ &= \Lambda^a_c\, 2\,\partial_{[\mu}\tilde{e}^c_{\nu]},\end{aligned} \tag{11}$$

which means that $\mathcal{D}\mathbf{e}^a = \Lambda^a_c\, d\tilde{\mathbf{e}}^c$, where

$$\tilde{\mathbf{e}}^c \equiv (\Lambda^{-1})^c_b\, \mathbf{e}^b \tag{12}$$

is a Lorentz-invariant quantity which does not change at all under the simultaneous local-Lorentz transformations (10). Thus the Lorentz-covariant torsion tensor is

$$T^\alpha_{\mu\nu} = e^\alpha_a (\mathcal{D}\mathbf{e}^a)_{\mu\nu} = \tilde{e}^\alpha_a (\partial_\mu \tilde{e}^a_\nu - \partial_\nu \tilde{e}^a_\mu), \tag{13}$$

which is the torsion for the Weitzenböck connection associated with $\tilde{\mathbf{e}}^a$. Then, the covariant formulation is dynamically equivalent to a pure-tetrad formulation. The only dynamical object of the covariant formulation is $\tilde{\mathbf{e}}^a$:

$$\mathcal{L}^{\text{covariant}}(e,\Lambda) = \mathcal{L}^{\text{pure tetrad}}(\tilde{e}), \tag{14}$$

and therefore the locally Lorentz-invariant variables (12) simply satisfy the equations of motion of the pure-tetrad version of the model. Thus, conclusions made in the pure-tetrad formulation also apply to the covariant formulation, and it is evident that the spin connection contains purely gauge degrees of freedom.

The formula (14) is valid for any modified teleparallel model which can be written in terms of the metric (and its Levi-Civita connection) and the spacetime components of the torsion tensor. Indeed, the metric is obviously invariant under any kind of the Lorentz transformations, be it type I or type II, while for the torsion tensor we have

$$T^\alpha_{\mu\nu}(e,\omega(\Lambda)) = T^\alpha_{\mu\nu}(\tilde{e},0).$$

Therefore, it includes $f(\mathbb{T})$ models as well as NGR and its nonlinear generalizations, and even models of $f(\mathbb{T},\mathbb{B})$ and many other types.

What we have shown here is that taking the $\omega^a{}_b = 0$ gauge is equivalent to a simple change of the variables. Very similarly, a Lagrangian of the form

$$\mathcal{L}(\phi,\psi)=\frac{1}{2}\partial_\mu(\phi-\psi)\partial^\mu(\phi-\psi)$$

can be transformed to $\mathcal{L}(\chi,\psi)=\frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi)$ by a simple change of variable $\phi\to\chi=\phi-\psi$ which removes its dependence on ψ. It is nothing but rewriting the Lagrangian in gauge-invariant variables.

A source of concern might be eligibility of restricting to gauge-invariant variables only, inside the action. For example, using the vector potential is important for the action principle of electrodynamics. In fact, its gauge-invariant quantities $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ depend on derivatives of the fundamental variables. The variation of $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with respect to $F_{\mu\nu}$ would give the trivial equation $F_{\mu\nu}=0$ while its variation with respect to A_ν gives the correct result of $\overset{\circ}{\nabla}_\mu F^{\mu\nu}=0$.

The reason is that the condition of δA vanishing at infinity is stronger than the same condition for δF. This is not about a precise type of asymptotic behavior, the same is true for vanishing everywhere outside a big enough ball. For example, vanishing of an otherwise arbitrary function $x(t)$ both in the past and in the future means also that $\dot{x}(t)$ integrates to zero, what gives a nontrivial condition on admissible functions $\dot{x}(t)$ even if the boundary contributions are totally neglected. In other words, the usual variational principle in terms of A_μ imposes extra restrictions on the allowed variations of $F_{\mu\nu}$.

The stronger condition on admissible variations means that the action must be stationary with respect to a smaller class of variations. It makes the corresponding equations admit more solutions. Note also that the vanishing field strength is of course a particular case of the divergenceless one. This is a common effect which plays, for example, an important role in understanding mimetic gravity models [33].

However, this is not the case with the choice of variables presented in this work; covariant and pure-tetrad formulations are equivalent at the level of the action too. This is because the gauge-invariant variables (12) do not depend on any derivatives of the fundamental quantities; we have presented a purely algebraic relation which imposes no restriction on the class of variations of gauge-invariant variables.

There are two different sides of the traditional understanding of the role of spin connection [20]. One assumes a particular spin connection for a given tetrad, which removes inertial effects. This assumption that one could objectively separate gravity from inertia goes against all experimental evidence. And it is related to the picture of (fictional) global translations being gauged, and requires some, necessarily voluntary, choice of the reference tetrad. By now, it is known that the standard recipes [20] of determining the spin connection can give nonunique results [34], and beyond the simplest cases they might even easily get wrong, with equations of motion not being satisfied if not in TEGR [35].

Secondly, there are opinions in the literature concerned about making the action and other global quantities finite, and it has been sometimes claimed that the spin connection plays a regularizing role. However, our change of variables is performed directly at the level of all the fundamental geometrical quantities, and does not involve neglecting any boundary term. It means that the action, or some conserved quantities, can be regularized in the pure-tetrad approach with no less success than in the covariant one. And in fact, if there is a finite-action covariant solution, we can always make a type I (simultaneous) Lorentz transformation which brings this particular solution, preserving all its physical properties, to the zero spin connection case.

To state it once more, there are different types of the local-Lorentz (pseudo-)invariance in the case of TEGR. One way is to write the action fully in terms of the Lorentz-covariant derivatives. Then we have the type I invariance, in practically any model we might think of, in these terms. On the other hand, if we think of the tetrad as four vectors composing the fundamental degrees of freedom, then there is no reason to want this full-fledged local-Lorentz invariance.

However, even the pure-tetrad action of TEGR appears to possess some invariance which goes beyond the explicitly maintained diffeomorphism invariance. This is the type II Lorentz pseudoinvariance. And this is the invariance that gets lost in most generalizations. Even though it is different from what most experts in general metric-affine gravity are used to call local-Lorentz invariance, it is algebraically related with the same group, the Lorentz group. Note also that the influence of all these symmetries of the action can be seen in the Hamiltonian analysis at the level of the primary constraints [28].

Of course, what we get in models of $f(T)$ gravity is nontrivial dependence on which tetrad to choose. But this is not an obstacle, since it grants a new model with more degrees of freedom. These are represented by the tetrad, a fundamental variable which carries more information than what the metric does. The covariantization procedure is equivalent to formally rewriting the very same model in locally Lorentz-invariant terms. This is done by simply sharing the extra modes with the newly introduced variables, the components of the spin connection. However it does not get rid of the very fact that the model does have some new dynamical content which a metric alone cannot provide. Getting something more on top of the metric is the essence of the local-Lorentz symmetry (type II) violation, and the covariant version simply rewrites it in different terms.

V. CONCLUSIONS

We have proven that the covariant and pure-tetrad formulations of teleparallel theories are fully equivalent, as long as matter is not coupled to the *Lorentzian* extra degrees of freedom. Due to the nature of the (curvatureless and metric) Weitzenböck connection (1), its torsion can be covariantized by replacing the tetrad $\mathbf{e}^a$ with the "gauge-invariant" tetrad $\tilde{\mathbf{e}}^a$ [see Eq. (13)], which is equivalent to splitting the original tetrad into two factors [see Eq. (12)]. This splitting introduces six new variables—the Lorentz matrices Λ^a_b—deprived of independent dynamics; they are clearly spurious variables that will have an impact on the constraint algebra. Since the covariantized torsion is the Weitzenböck torsion of the tetrad $\tilde{\mathbf{e}}^a$, then $\tilde{\mathbf{e}}^a$ satisfies the same dynamics as the original variables $\mathbf{e}^a$. Therefore, although the covariant formulation is Lorentz invariant of type I, it is not dynamically different from the pure-tetrad formulation.

When the Lorentz invariance of type II is broken in the pure-tetrad formulation, it gives rise to new dynamics. This dynamics also appears in the covariant formulation, because the Lorentz type II breaking is also present. Certainly, a pure-tetrad Lorentz rotation is no longer an obvious symmetry once nonzero spin connection terms are added. Therefore, what is broken by generalizations of the covariant TEGR would rather be a "type $\tilde{\mathrm{II}}$" invariance, which represents local-Lorentz rotations of our Lorentz-invariant variables (12).

The results here exposed are rooted in the fact that Weitzenböck connection (1) has already the most general form for a curvatureless connection. From this perspective, there is a one-to-one relation between tetrads and curvatureless tetrad-compatible connections (up to global linear transformations and possible topological obstructions). This is analogous to the (metric based) GR formulation, where there exists a one-to-one relation between metrics and torsionless metric-compatible (Levi-Civita) connections. However, since the relation tetrad metric is not one-to-one but is subject to a local-Lorentz invariance, the teleparallel formulations are expected to be endowed with such invariance. This expectation seems to come from considering the metric as more fundamental than the tetrad. However, such invariance is not mandatory for building a dynamical theory of tetrads, since both the gravity Lagrangian and the coupling matter gravity can be written only in terms of tetrads (without prejudice to those matter Lagrangians that in fact exhibit a local-Lorentz invariance).

Our findings are applicable to the entire class of teleparallel theories built from the torsion tensor, which include TEGR, $f(\mathbb{T})$, NGR, $f(T_{ax}, T_{\rm vec}, T_{\rm ten})$, $f(\mathbb{T}, \mathbb{B})$, teleparallel Horndeski gravity, among others.

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