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Towards a definition of the quantum ergodic hierarchy: Ergodicity and mixing

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ABSTRACT

In a previous paper we have given a general framework for addressing the definition of quantum chaos by identifying the conditions that a quantum system must satisfy to lead to non-integrability in its classical limit. In this paper we will generalize those results, with the purpose of defining the two lower levels of the quantum ergodic hierarchy: ergodicity and mixing. We will also argue for the physical relevance of this approach by considering a particular example where our formalism has been successfully applied.

1. Introduction

In spite of the increasing attention that quantum chaos has received in recent times, the usual opinion is that there is some kind of tension between quantum mechanics and chaos. Two general strategies have been used to stress this supposed tension:

1. The first one is a "top-down" strategy, which consists in the quantization of simple classical chaotic models: the conflict arises because the resulting quantum models are usually non-chaotic according to some feature considered as an indicator of chaos. For instance, Ford and his collaborators [1,2] take the notion of complexity as the key concept for defining chaos: they argue that the quantization of a classical chaotic system has null complexity and, therefore, is intrinsically non-chaotic. On this basis, these authors consider to have refuted the correspondence principle and conclude the incompleteness of quantum mechanics as a fundamental theory.
2. The second general strategy consists in seeking the usual indicators of chaos directly in quantum systems and verifying that those indicators are not present in quantum evolutions. The particular arguments differ from each other with respect to the specific feature to be regarded as the relevant indicator of chaotic behavior.
   - When the exponential divergence of trajectories is focused, the usual claim is that quantum mechanics suppresses chaos because it is not possible to define precise trajectories in quantum evolutions as a consequence of the uncertainty principle (see Refs. [3,4]).
   - Since a necessary condition for chaos in classical systems is non-linearity, the fact that quantum evolutions are solutions of the linear Schrödinger equation has led some authors to conclude that quantum systems are necessarily non-chaotic (see Refs. [5,6]). One way out to this conclusion is the attempt to recover quantum chaos by introducing non-linear terms in the Schrödinger equation, for instance, by means of non-linear operators (see Ref. [7]).

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- Another feature that has been used to explain the relative scarcity of quantum chaos is the unitarity of the evolutions described by the Schrödinger equation. On this basis, some authors have seek the way to quantum chaos in non-unitary approaches to quantum mechanics (e.g. the GRW theory [8]). A different path is followed by the authors who search for chaotic behavior in open quantum systems [9–11].

However, these two strategies do not take into account that *the real conflict, which would pose a threat to the correspondence principle, would arise only if the classical limit of quantum systems did not display chaotic behavior*. But the results obtained from the quantization of classical chaotic models are, at least, inconclusive, since at present it is well known that the explanation of the classical limit of quantum mechanics involves certain elements that are much more subtle than the inverse of the traditional quantization. On the other hand, the fact that the usual indicators of chaos are not present in quantum evolutions does not prove yet the absence of chaos in the classical description emerging as the result of the classical limit: only if the transference of those indicators from the quantum level to the classical level is assumed in advance, the scarcity of chaos in quantum systems could be considered a real problem in the light of the ubiquity of classical chaos.

For these reasons, following other authors [12,13] we will study the problem of the emergence of classical chaos (and of other levels of instability in the ergodic hierarchy) from quantum descriptions of physical systems. Therefore, although different characterizations of quantum chaos may be useful for other purposes, in the context of the supposed threat to the correspondence principle we will consider the problem of quantum chaos as a particular aspect of the more general issue of the classical limit of quantum mechanics, that is, how classical behavior can emerge from the quantum realm. In the context of this particular problem, the relevant definition of quantum chaos reads: *a quantum system is chaotic if its classical limit exhibits chaos*.

In the light of this idea, and on the basis of our previous works on decoherence and the classical limit of quantum mechanics [14–24], in paper [25] we have presented a general framework for addressing the definition of quantum chaos by identifying the conditions that a quantum system must satisfy to lead to *non-integrability* in the classical limit. In paper [26] we have discussed the conceptual foundations of our approach in the context of Belot–Earman's program (see Ref. [27]), which proposes four requirements that any definition of quantum chaos should fulfil: (i) that it possess *generality* and *mathematical rigor*, (ii) that it agree with *common intuition*, namely, the natural idea that we have of chaos, (iii) that it be clearly related to the criteria of *classical chaos*, and (iv) that it be *physically relevant*.

In this paper, our aim is to carry our approach a step further by defining the two lowest levels of the ergodic hierarchy for non-integrable quantum systems: quantum ergodicity and quantum mixing. For this purpose, we will have to generalize the results obtained in our previous works on non-integrability. In the light of this general aim, the paper is organized as follows. In Section 2, we will introduce a mathematical background necessary for the development of the following sections. In Section 3, we will generalize the results obtained in Ref. [25] for decoherence in quantum systems with continuous energy spectrum to the case of a discrete+continuous spectrum. Section 4 will be devoted to obtain the classical statistical limit of the systems treated in the previous section through the Wigner transformation. In Section 5 we will apply the Ehrenfest theorem to obtain the classical limit with trajectories in phase space. On the basis of these results, in Section 6 we will define quantum ergodicity and quantum mixing. In Section 7 we will argue for the physical relevance of our approach by discussing the Casati–Prosen model, which can be successfully treated with our formalism. Finally, in the conclusions we will explain in what sense our approach satisfies the four requirements of Belot–Earman's program.

2. Mathematical background

2.1. Weak and Cèsaro limits

Our presentation is based on the algebraic formalism of quantum mechanics [28,29]. Let us consider an algebra $\mathcal{A}$ of operators, whose self-adjoint elements $O = O^{\dagger}$ are the observables belonging to the space $\mathcal{O}$. The states ρ are functionals belonging to the dual space $\mathcal{O}'$, and they satisfy the usual conditions: self-adjointness, positivity and normalization. If $\mathcal{A}$ is a C*-algebra, it can be represented by a Hilbert space (GNS theorem). If $\mathcal{A}$ is a general nuclear algebra, it can be represented by a rigged Hilbert space, as proved by a generalization of the GNS theorem [30,31]. In this case, the van Hove states with singular diagonal can be properly defined (see Ref. [32]; for a rigorous presentation of the formalism, see also Refs. [33,34]).

If we write the action of the functional ρ on the vector O as $(\rho|O)$, then we can say that:

- The evolution $U_t\rho = \rho(t)$ has a *Weak-limit* if, for any $O \in \mathcal{O}$ and any $\rho \in \mathcal{O}'$, there is a unique ρ_*^W such that

$$\lim_{t\to\infty} (\rho(t)|O) = (\rho_*^W|O). \tag{1}$$

We will symbolize this limit as

$$W\text{-}\lim_{t\to\infty} \rho(t) = \rho_*^W. \tag{2}$$

- The evolution $U_t\rho = \rho(t)$ has a *Cèsaro-limit* if, for any $O \in \mathcal{O}$ and any $\rho \in \mathcal{O}'$, there is a unique ρ_*^C such that

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} \mathrm{d}t\ (\rho(t)|O) = (\rho_*^C|O). \tag{3}$$

We will symbolize this limit as

$$C\text{-}\lim_{t\to\infty}\rho(t)=\rho_*^C. \tag{4}$$

- If the evolution $\rho(t)$ has a Weak-limit according to Eq. (1), it has a Cèsaro-limit according to Eq. (3): *Weak-convergence implies Cèsaro-convergence.*

2.2. Riemann–Lebesgue theorem

The idea of destructive interference is embodied in the Riemann–Lebesgue theorem, according to which, if $f(\nu)\in\mathbb{L}^1$, then

$$\lim_{t\to\infty}\int_a^b f(\nu)\mathrm{e}^{-\mathrm{i}\nu t}\mathrm{d}\nu=0. \tag{5}$$

- If we can express the action of the functional $\rho(t)\in\mathcal{O}'$ on the vector $O\in\mathcal{O}$ as

$$(\rho(t)|O)=\int_a^b[A\,\delta(\nu)+f(\nu)]\,\mathrm{e}^{-\mathrm{i}\nu t}\mathrm{d}\nu \tag{6}$$

with $f(\nu)\in\mathbb{L}^1$, then

$$\lim_{t\to\infty}(\rho(t)|O)=\lim_{t\to\infty}\int_a^b[A\,\delta(\nu)+f(\nu)]\,\mathrm{e}^{-\mathrm{i}\nu t}\mathrm{d}\nu=A=(\rho_*^W|O). \tag{7}$$

We will call this result "*Weak Riemann–Lebesgue limit*".

- If we can express the action of the functional $\rho(t)\in\mathcal{O}'$ on the vector $O\in\mathcal{O}$ as

$$(\rho(t)|O)=\sum_j f_j\ \mathrm{e}^{-\mathrm{i}\nu_j t}+\int_a^b[A\,\delta(\nu)+f(\nu)]\,\mathrm{e}^{-\mathrm{i}\nu t}\mathrm{d}\nu \tag{8}$$

with $f(\nu)\in\mathbb{L}^1$, then[1]

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau\mathrm{d}t\,\left[\sum_j f_j\ \mathrm{e}^{-\mathrm{i}\nu_j t}+\int_a^b[A\,\delta(\nu)+f(\nu)]\,\mathrm{e}^{-\mathrm{i}\nu t}\mathrm{d}\nu\right]=A=(\rho_*^C|O). \tag{9}$$

We will call this result "*Cèsaro Riemann–Lebesgue limit*".

2.3. Generalized projection

As it is well known, in order to describe an irreversible process in terms of a unitary evolution it is necessary to break the underlying unitarity by means of some sort of projection, which retains the "relevant" information and discards the "irrelevant" information about the system. In this section we will generalize this idea in two senses.

i.- In its traditional form, the action of a projection is to eliminate some components of the state vector corresponding to the finer description (see Ref. [35]). If this idea is generalized, the action of a functional $\rho\in\mathcal{O}'$ on a vector $O\in\mathcal{O}$ can be conceived as the result of a generalized projection. In fact, we can define a projector Π_W belonging to the space $\mathcal{O}\otimes\mathcal{O}'$ such that

$$\bullet\ \Pi_W\doteq(\bullet|O)\,\rho_o \tag{10}$$

where $\rho_o\in\mathcal{O}'$ satisfies $(\rho_o|O)=1$.[2] Therefore, the action of $\rho\in\mathcal{O}'$ on $O\in\mathcal{O}$ involves a projection leading to a state ρ_P such that

$$(\rho|O)\,\rho_o=\rho\,\Pi_W\doteq\rho_P. \tag{11}$$

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[1] In fact, the integral of Eq. (9) reads

$$\sum_j\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau\mathrm{d}t\ f_j\mathrm{e}^{-\mathrm{i}\nu_j t}+\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau\mathrm{d}t\int_a^b[A\,\delta(\nu)+f(\nu)]\,\mathrm{e}^{-\mathrm{i}\nu t}\mathrm{d}\nu.$$

The second term is equal to A because Weak-convergence implies Cèsaro-convergence. The first term vanishes since

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau\mathrm{d}t f_j\mathrm{e}^{-\mathrm{i}\nu_j t}=\lim_{\tau\to\infty}\frac{i}{\nu_j\tau}\,f_i\,\left(\mathrm{e}^{-\mathrm{i}\nu_j\tau}-1\right)=0.$$

[2] In fact, Π_W is a projector since

$$\bullet\Pi_W^2=(\bullet|O)\,\rho_o\Pi_W=(\bullet|O)\,(\rho_o|O)\rho_o=(\bullet|O)\rho_o=\bullet\Pi_W.$$

If the evolution $\rho(t)$ has a Weak-limit, we can guarantee that (see Eq. (1))

$$\lim_{t\to\infty}\rho_P(t)=\lim_{t\to\infty}\rho(t)\Pi_W=\lim_{t\to\infty}(\rho(t)|O)\rho_o=(\rho_*^W|O)\rho_o=\rho_*^W\,\Pi_W=\rho_{P*}^W \tag{12}$$

where ρ_{P*}^W is the projection of the final functional ρ_*^W.[3]

When ρ denotes a state and O denotes an observable, this result means that the expectation value of the observable O in the state ρ, $\langle O\rangle_\rho=(\rho|O)$, can be conceived as a projected magnitude that provides the partial description of ρ from the perspective given by O.

ii.- A further projection can be defined by introducing a time integration on the state $\rho_P(t)$. Let us define a time-projected state $\rho_\tau(t)$ such that

$$\rho_\tau(t)\doteq\rho_P(t)\,\Pi_\tau \tag{13}$$

where the projector Π_τ is the integral[4]

$$\bullet\,\Pi_\tau\doteq\frac{1}{\tau}\int_0^\tau\bullet\mathrm{d}t. \tag{14}$$

Then,

$$\rho_\tau(t)=\rho_P(t)\,\Pi_\tau=\rho(t)\Pi_W\Pi_\tau=(\rho(t)|O)\,\rho_o\Pi_\tau=\frac{1}{\tau}\int_0^\tau(\rho(t)|O)\,\rho_o\,\mathrm{d}t. \tag{15}$$

If the evolution $\rho(t)$ has a Cèsaro-limit, we can guarantee that (see Eq. (3))

$$\lim_{t\to\infty}\rho_\tau(t)=\lim_{t\to\infty}\rho_P(t)\,\Pi_\tau=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau(\rho(t)|O)\,\rho_o\,\mathrm{d}t=(\rho_*^C|O)\rho_o=\rho_*^C\,\Pi_W=\rho_{P*}^C \tag{16}$$

where, again, ρ_{P*}^C is the projection of the final functional ρ_*^C.

When ρ denotes a state and O denotes an observable, the time integration involved in the Cèsaro-limit can be endowed with a physical meaning. As it is well known, in any realistic measurement the measuring apparatus is a macroscopic device with a certain inertia that delays and averages the result of the measurement. Let us consider, for instance, an evolution $\rho(t)=B(t)+A\sin\omega t$, where the oscillation period $\tau_\omega=\frac{2\pi}{\omega}$ is much smaller than the time variation of $B(t)$. If τ is the characteristic time of the measuring apparatus and $\tau\gg\tau_\omega$, then the result of the measurement can be computed by means of the limit $\tau\to\infty$ as in Eq. (16), where the term $A\sin\omega t$ vanishes. This means that the Cèsaro-limit can be conceived as yielding the result of a measurement performed by a macroscopic apparatus.

2.4. Weyl–Wigner–Moyal mapping

Let $\Gamma=\mathcal{M}_{2(N+1)}\equiv\mathbb{R}^{2(N+1)}$ be the phase space. The functions over Γ will be called $f(\phi)$, where ϕ symbolizes the coordinates of Γ, $\phi=(q^1,\ldots,q^{N+1},p_q^1,\ldots,p_q^{N+1})$. If $\widehat{f},\widehat{g}\in\widehat{\mathcal{A}}$ and $f(\phi),g(\phi)\in\mathcal{A}_q$, where $\widehat{\mathcal{A}}$ is the quantum algebra and $\mathcal{A}_q$ is the "classical-like" algebra, the *Wigner transformation* reads (see Refs. [36–38])

$$symb\widehat{f}\doteq f(\phi)=\int\langle q+\Delta|\widehat{f}\,|q-\Delta\rangle\mathrm{e}^{\mathrm{i}\frac{p\Delta}{\hbar}}\mathrm{d}^{N+1}\Delta. \tag{17}$$

We can also introduce the *star product* (see Ref. [39]),

$$symb(\widehat{f}\,\widehat{g})=symb\widehat{f}*symb\,\widehat{g}=(f*g)(\phi),\qquad(f*g)(\phi)=f(\phi)\exp\left(-\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right)g(\phi) \tag{18}$$

and the *Moyal bracket,* that is, the symbol corresponding to the quantum commutator

$$\{f,g\}_{mb}=\frac{1}{\mathrm{i}\hbar}(f*g-g*f)=symb\left(\frac{1}{\mathrm{i}\hbar}[f,g]\right). \tag{19}$$

It can be proved that (see Ref. [36])

$$(f*g)(\phi)=f(\phi)g(\phi)+0(\hbar),\qquad\{f,g\}_{mb}=\{f,g\}_{pb}+0(\hbar^2). \tag{20}$$

[3] This notion of generalized projection can be applied to the environment-induced decoherence (where the projected state is the result of the tracing over the environmental degrees of freedom, see Refs. [10,11]) as to the self-induced decoherence (where the expectation value approaching a final value can be viewed as a projected magnitude, see Refs. [14,18,20,22]).

[4] The operator Π_τ is a projector since

$$\Pi_\tau^2=\frac{1}{\tau}\int_0^\tau\Pi_\tau\,\mathrm{d}t=\Pi_\tau.$$

To define the inverse $symb^{-1}$, we will use the *symmetrical* or *Weyl* ordering prescription, namely,

$$symb^{-1}[q^i(\phi), p^j(\phi)] \doteq \frac{1}{2}\left(\widehat{q}^i\widehat{p}^j + \widehat{p}^j\widehat{q}^i\right). \quad (21)$$

Therefore, by means of the transformations *symb* and $symb^{-1}$, we have defined an isomorphism between the quantum algebra $\widehat{\mathcal{A}}$ and the algebra $\mathcal{A}_q$,

$$symb^{-1}: \mathcal{A}_q \to \widehat{\mathcal{A}}, \qquad symb: \widehat{\mathcal{A}} \to \mathcal{A}_q. \quad (22)$$

The mapping so defined is the *Weyl–Wigner–Moyal symbol*.[5]

The Wigner transformation for states is

$$\rho(\phi) = symb\,\widehat{\rho} = (2\pi\hbar)^{-(N+1)}\, symb_{(\text{for operators})}\,\widehat{\rho}. \quad (23)$$

As it is well known, an important property of the Wigner transformation is that it yields the correct expectation value of any observable $\widehat{O}$ in a state $\widehat{\rho}$,

$$\langle\widehat{O}\rangle_{\widehat{\rho}} = (\widehat{\rho}|\widehat{O}) = (symb\,\widehat{\rho}\mid symb\,\widehat{O}) = \int \mathrm{d}\phi^{2(N+1)}\rho(\phi)\,O(\phi). \quad (24)$$

This means that the definition of $\widehat{\rho} \in \widehat{\mathcal{A}}'$ as a functional on $\widehat{\mathcal{A}}$ is equivalent to the definition of $symb\,\rho \in \mathcal{A}_q'$ as a functional on $\mathcal{A}_q$.

3. Decoherence in non-integrable systems

In this section we will generalize the results obtained in Ref. [25] for $\widehat{H}$ with continuous spectrum to the case of a discrete+continuous energy spectrum. This task will be essential for explaining quantum ergodicity and quantum mixing in the following sections.

3.1. Local CSCO

This subsection is a short version of the corresponding subsection of paper [25].

a.- In Ref. [25] we have proved that, when the quantum system is endowed with a CSCO of $N+1$ observables containing $\widehat{H}$, that defines an eigenbasis in terms of which the state of the system can be expressed, the corresponding classical system is *integrable*. In fact, if the CSCO is $\{\widehat{H}, \widehat{G}_1, \ldots, \widehat{G}_N\}$, the Moyal brackets of its elements are

$$\{G_I(\phi), G_J(\phi)\}_{mb} = symb\left(\frac{1}{\mathrm{i}\hbar}[\widehat{G}_I, \widehat{G}_J]\right) = 0 \quad (25)$$

where $I, J = 0, 1, \ldots, N$, $\widehat{G}_0 = \widehat{H}$, and $\phi \in \mathcal{M} \equiv \mathbb{R}^{2(N+1)}$. Then, when $\hbar \to 0$, from Eq. (20) we know that

$$\{G_I(\phi), G_J(\phi)\}_{pb} = 0. \quad (26)$$

Thus, since $H(\phi) = G_0(\phi)$, the set $\{G_I(\phi)\}$ is a complete set of $N+1$ constants of motion in involution, *globally* defined all over $\mathcal{M}$; as a consequence, the system is *integrable*.

b.- We have also proved (see Ref. [25]) that, when the CSCO has $A+1 < N+1$ observables, a local CSCO $\{\widehat{H}, \widehat{G}_1, \ldots, \widehat{G}_A, \widehat{O}_{i(A+1)}, \ldots, \widehat{O}_{iN}\}$ can be defined for each domain D_{ϕ_i} around any point $\phi_i \in \Gamma \equiv \mathbb{R}^{2(N+1)}$, where Γ is the Wigner phase space of the system. In this case the system is *non-integrable*.

In order to prove this assertion, we have to recall the *Carathèodory–Jacobi theorem* (see Ref. [40], Theorem 16.29) according to which, when a system with $N+1$ degrees of freedom has $A+1$ global constants of motion in involution $\{G_0(\phi), G_1(\phi), \ldots, G_A(\phi)\}$, then $N-A$ local constants of motion in involution $\{A_{i(A+1)}(\phi), \ldots, A_{iN}(\phi)\}$ can be defined in a maximal domain $\mathcal{D}_{\phi_i}$ around ϕ_i, for any $\phi_i \in \Gamma \equiv \mathbb{R}^{2(N+1)}$.

Let us consider the particular case of a classical system with $N+1$ degrees of freedom, and whose only global constant of motion is the Hamiltonian $H(\phi)$. The Carathèodory–Jacobi theorem tells us that, in this case, the system has N local constants of motion $A_{iI}(\phi)$, with $I = 0, \ldots, N$, in the domain $\mathcal{D}_{\phi_i}$ around ϕ_i, for any $\phi_i \in \Gamma$. If we want to translate these phase space functions into the quantum language, we have to apply the transformation $symb^{-1}$; this can be done in the case of the Hamiltonian, $\widehat{H} = symb^{-1}H(\phi)$, but not in the case of the $A_{iI}(\phi)$ because they are defined in a domain $\mathcal{D}_{\phi_i} \subset \Gamma$ and the Weyl–Wigner–Moyal mapping can be applied only on phase space functions defined on the whole phase space Γ. To face this problem, we can introduce a positive partition of the identity (see Ref. [41]),

$$1 = I(\phi) = \sum_i I_i(\phi) \quad (27)$$

[5] When $\hbar \to 0$, then $\mathcal{A}_q \to \mathcal{A}$, where $\mathcal{A}$ is the classical algebra of observables over phase space.

where each $I_i(\phi)$ is the *characteristic* or *index* function

$$I_i(\phi) = \begin{cases} 1 & \text{if } \phi \in D_{\phi_i} \\ 0 & \text{if } \phi \notin D_{\phi_i} \end{cases} \tag{28}$$

and $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$, $D_{\phi_i} \cap D_{\phi_j} = \emptyset$, $\bigcup_i D_{\phi_i} = \Gamma$. Then we can define the functions $O_{iI}(\phi)$ as[6]

$$O_{iI}(\phi) = A_{iI}(\phi)\, I_i(\phi). \tag{29}$$

Now the $O_{iI}(\phi)$ are defined for all $\phi \in \Gamma$; so, we can obtain the corresponding quantum operators as

$$\widehat{O}_{iI} = symb^{-1} O_{iI}(\phi). \tag{30}$$

Since the original functions $A_{iI}(\phi)$ are local constants of motion in the domain $\mathcal{D}_{\phi_i}$, they make zero the corresponding Poisson brackets in such a domain (see Eq. (26)) and, a fortiori, in $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$. This means that the $O_{iI}(\phi)$ make zero the corresponding Poisson brackets in the whole space space Γ: for $\phi \in D_{\phi_i}$, because $O_{iI}(\phi) = A_{iI}(\phi)$, and trivially for $\phi \notin D_{\phi_i}$. We also know that, in the macroscopic limit $\hbar \to 0$, the Poisson brackets can be identified with the Moyal brackets, that is, the phase space counterpart of the quantum commutator (see Eq. (20)). Therefore, we can guarantee that all the observables of the set $\left\{\widehat{H}, \widehat{O}_{iI}\right\}$ commute to each other:

$$\left[\widehat{H}, \widehat{O}_{iI}\right] = 0 \qquad \left[\widehat{O}_{iI}, \widehat{O}_{iJ}\right] = 0 \tag{31}$$

for $I, J = 1$ to N and for each D_{ϕ_i}. As a consequence, we will say that the set $\left\{\widehat{H}, \widehat{O}_{i1}, \ldots, \widehat{O}_{iN}\right\}$ is the *local CSCO* of $N+1$ observables corresponding to the domain $D_{\phi_i} \subset \Gamma$. If $\widehat{H}$ has a continuous spectrum $0 \le \omega < \infty$, a generic observable $\widehat{O}$ can be decomposed as

$$\widehat{O} = \sum_{im_{iI}\, m'_{iI}} \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega'\ \widetilde{O}_{im_{iI}\, m'_{iI}}(\omega, \omega')\, |\omega, m_{iI}\rangle\langle\omega', m'_{iI}| \tag{32}$$

where the $|\omega, m_{iI}\rangle = |\omega, m_{i1}, \ldots, m_{iN}\rangle$ are the eigenvectors of the local CSCO $\left\{\widehat{H}, \widehat{O}_{iI}\right\}$ corresponding to D_{ϕ_i}. Since it can be proved that (see Ref. [25]), for $i \neq j$,

$$\langle\omega, m_{iI}\, |\omega, m_{jI}\rangle = 0 \tag{33}$$

the decomposition of Eq. (32) is orthonormal, and it generalizes the usual eigen-decomposition of the integrable case to the non-integrable case. Therefore, any $\widehat{O}_{iI}$ corresponding to the domain D_{ϕ_i} commutes with any $\widehat{O}_{jJ}$ corresponding to the domain D_{ϕ_j}, with $i \neq j$,[7]

$$\left[\widehat{O}_{iI}, \widehat{O}_{jJ}\right] = \delta_{ij}\, \delta_{IJ}. \tag{34}$$

3.2. Decoherence in the energy

Let us consider a quantum system endowed with a CSCO consisting only of the Hamiltonian $\widehat{H}$. In order to complete the basis, we can add the observables belonging to the local CSCO as defined in the previous subsection. Thus, we have the set $\left\{\widehat{H}, \widehat{O}_{iI}\right\}$, with $I = 1$ to N and i corresponding to all the domains D_{ϕ_i} obtained from the partition of the phase space.

a.- In paper [25] we have considered the case with continuous spectrum $0 \le \omega < \infty$ for $\widehat{H}$ and discrete spectra $m_{iI} \in \mathbb{N}$ for the $\widehat{O}_{iI}$. Here we need to generalize that case by considering that the energy spectrum has a discrete part $\{w_\alpha\}$ and a continuous part $0 \le w < \infty$. In the eigenbasis of $\widehat{H}$, the elements of any local CSCO can be expressed as (see Eq. (32))

$$\widehat{H} = \sum_{im_{iI}} \left(\sum_\alpha w_\alpha\, |w_\alpha, m_{iI}\rangle\langle w_\alpha, m_{iI}| + \int_0^\infty w\, |w, m_{iI}\rangle\langle w, m_{iI}|\mathrm{d}w \right) \tag{35}$$

$$\widehat{O}_{iJ} = \sum_{im_{iI}} \left(\sum_\alpha m_{iI} |w_\alpha, m_{iI}\rangle\langle w_\alpha, m_{iI}| + \int_0^\infty m_{iI}\, |w, m_{iI}\rangle\langle w, m_{iI}|\mathrm{d}w \right) \tag{36}$$

[6] In paper [25], we have used a "bump" smooth function $B_i(\phi)$ with values belonging to [0, 1] in a boundary zone of the corresponding domain, and we have defined $O_{iI}(\phi) = A_{iI}(\phi)\, B_i(\phi)$. This strategy guarantees smooth connections between the functions $O_{iI}(\phi)$ defined in adjacent domains; in particular, it can be shown that any possible discontinuity in the boundary zones introduces just a $0(\hbar^2)$, which vanishes when $\hbar \to 0$ and, therefore, the Moyal brackets can be replaced with Poisson brackets in such a limit (see Ref. [25]).

[7] In this paper we have slightly changed the notation of paper [25], because we consider that the present notation in more explicit than that one.

where m_{iI} is a shorthand for $m_{i1}, \ldots, m_{iN}$, and $\sum_{im_{iI}}$ is a shorthand for $\sum_i \sum_{m_{i1}} \cdots \sum_{m_{iN}}$. In order to simplify the expressions, we will symbolize the energy spectrum with $\omega = (w_\alpha, w)$ and write Eqs. (35) and (36) as

$$\widehat{H} = \sum_{im_{iI}} \left(\sum_\alpha + \int_0^\infty \mathrm{d}w \right) \omega \, |\omega, m_{iI}\rangle \langle \omega, m_{iI}| \tag{37}$$

$$\widehat{O}_{iJ} = \sum_{im_{iI}} \left(\sum_\alpha + \int_0^\infty \mathrm{d}w \right) m_{iI} \, |\omega, m_{iI}\rangle \langle \omega, m_{iI}| \tag{38}$$

where the sum $\sum_\alpha$ corresponds to the discrete spectrum and the integral $\int_0^\infty \mathrm{d}w$ corresponds to the continuous spectrum. With this notation,

$$\widehat{H} \, |\omega, m_{iI}\rangle = \omega \, |\omega, m_{iI}\rangle, \qquad \widehat{O}_{iI} \, |\omega, m_{iI}\rangle = m_{iI} \, |\omega, m_{iI}\rangle \tag{39}$$

where the set of vectors $\{|\omega, m_{iI}\rangle\}$, with $I = 1$ to N and i corresponding to all the D_{ϕ_i}, is an orthonormal basis (see Eq. (33)):

$$\langle w_\alpha, m_{iI} | w_\beta, m'_{iI}\rangle = \delta_{\alpha\beta} \, \delta_{m_{iI} \, m'_{iI}}, \qquad \langle w, m_{iI} | \, w', m'_{iI}\rangle = \delta(w - w') \, \delta_{m_{iI} \, m'_{iI}}, \qquad \langle w_\alpha, m_{iI} | \, w, m'_{iI}\rangle = 0. \tag{40}$$

b.- In the orthonormal basis $\{|\omega, m_{iI}\rangle\}$, a generic observable reads (see Eq. (32))

$$\widehat{O} = \sum_{im_{iI} \, m'_{iI}} \left(\sum_\alpha + \int_0^\infty \mathrm{d}w \right) \left(\sum_\beta + \int_0^\infty \mathrm{d}w' \right) \widetilde{O}_{im_{iI} \, m'_{iI}}(\omega, \omega') \, |\omega, m_{iI}\rangle \langle \omega', m'_{iI}| \tag{41}$$

where $\widetilde{O}_{im_{iI} \, m'_{iI}}(\omega, \omega')$ is a generic *kernel* or *distribution* in ω, ω'. As in paper [25], we will restrict the set of observables by considering only the van Hove observables (see Ref. [32]) such that

$$\widetilde{O}_{im_{iI} \, m'_{iI}}(\omega, \omega') = O_{im_{iI} \, m'_{iI}}(\omega) \, \delta(\omega - \omega') + O_{im_{iI} \, m'_{iI}}(\omega, \omega') \tag{42}$$

where $\delta(\omega - \omega') = \delta_{\alpha\beta}$ for the sums, $\delta(\omega - \omega') = \delta(w - w')$ for the integrals, and the discrete–continuous cross-terms vanish. The first term in the r.h.s. of Eq. (42) is the *singular term* and the second one is the *regular term* since the $O_{im_{iI} \, m'_{iI}}(\omega, \omega')$ are "regular" functions of the variables ω and ω' (where the precise meaning of "regular" will turn out to be clear below). Therefore, the observables of our algebra $\widehat{\mathcal{A}}$ read

$$\widehat{O} = \sum_{im_{iI} \, m'_{iI}} \left(\sum_\alpha + \int_0^\infty \mathrm{d}w \right) O_{im_{iI} \, m'_{iI}}(\omega) \, |\omega, m_{iI}\rangle \langle \omega, m'_{iI}| + \sum_{im_{iI} \, m'_{iI}} \left(\sum_\alpha + \int_0^\infty \mathrm{d}w \right) \left(\sum_{\beta \neq \alpha} + \int_0^\infty \mathrm{d}w' \right) O_{im_{iI} \, m'_{iI}}(\omega, \omega') \, |\omega, m_{iI}\rangle \langle \omega', m'_{iI}|. \tag{43}$$

Since the observables are the self-adjoint operators of the algebra, $\widehat{O}^\dagger = \widehat{O}$, they belong to a space $\widehat{\mathcal{O}} \subset \widehat{\mathcal{A}}$ whose basis $\{|\omega, m_{iI}, m'_{iI}), |\omega, \omega', m_{iI}, m'_{iI})\}$ is defined as

$$|\omega, m_{iI}, m'_{iI}) \doteq |\omega, m_{iI}\rangle \langle \omega, m'_{iI}|, \qquad |\omega, \omega', m_{iI}, m'_{iI}) \doteq |\omega, m_{iI}\rangle \langle \omega', m'_{iI}|. \tag{44}$$

c.- The states belong to a convex set included in the dual of the space $\widehat{\mathcal{O}}$, $\widehat{\rho} \in \widehat{\mathcal{S}} \subset \widehat{\mathcal{O}}'$. The basis of $\widehat{\mathcal{O}}'$ is $\{(\omega, m_{iI}, m'_{iI}|, (\omega, \omega', m_{iI}, m'_{iI}|\}$, whose elements are defined as functionals by the equations

$$\begin{aligned}
&(\omega, m_{iI}, m'_{iI} \mid \eta, n_{iI}, n'_{iI}) \doteq \delta(\omega - \eta) \, \delta_{m_{iI} \, n_{iI}} \, \delta_{m'_{iI} \, n'_{iI}} \\
&(\omega, \omega', m_{iI}, m'_{iI} \mid \eta, \eta', n_{iI}, n'_{iI}) \doteq \delta(\omega - \eta) \, \delta(\omega' - \eta') \, \delta_{m_{iI} \, n_{iI}} \, \delta_{m'_{iI} \, n'_{iI}} \\
&(\omega, m_{iI}, m'_{iI} \mid \eta, \eta', n_{iI}, n'_{iI}) \doteq 0
\end{aligned} \tag{45}$$

and the remaining $(\bullet|\bullet)$ are zero. Then, a generic state reads

$$\widehat{\rho} = \sum_{im_{iI} \, m'_{iI}} \left(\sum_\alpha + \int_0^\infty \mathrm{d}w \right) \overline{\rho_{im_{iI} \, m'_{iI}}(\omega)} (\omega, m_{iI}, m'_{iI}| + \sum_{im_{iI} \, m'_{iI}} \left(\sum_\alpha + \int_0^\infty \mathrm{d}w \right) \left(\sum_{\beta \neq \alpha} + \int_0^\infty \mathrm{d}w' \right) \overline{\rho_{im_{iI} \, m'_{iI}}(\omega, \omega')} (\omega, \omega', m_{iI}, m'_{iI}| \tag{46}$$

where the functions $\overline{\rho_{im_{il}\, m'_{il}}(\omega,\omega')}$ are "regular" functions of the variables ω and ω'. We also require that $\widehat{\rho}^{\dagger}=\widehat{\rho}$, i.e.,

$$\overline{\rho_{im_{il}\, m'_{il}}(\omega,\omega')}=\rho_{im'_{il}\, m_{il}}(\omega',\omega) \tag{47}$$

and that the $\rho_{im_{il}\, m_{il}}(\omega,\omega)\doteq\rho_{im_{il}}(\omega)$ be real and non-negative, satisfying the total probability condition,

$$\rho_{im_{il}}(\omega)\geq 0,\qquad \mathrm{tr}\widehat{\rho}=(\widehat{\rho}|\widehat{I})=\sum_{im_{il}}\left(\sum_{\alpha}+\int_0^{\infty}\mathrm{d}w\right)\rho_{im_{il}}(\omega)=1 \tag{48}$$

where $\widehat{I}=\sum_{im_{il}}\left(\sum_{\alpha}+\int_0^{\infty}\mathrm{d}w\right)|\omega,m_{il}\rangle\langle\omega,m_{il}|$ is the identity operator in $\widehat{\mathcal{O}}$.

d.- On the basis of these characterizations, the expectation value of any observable $\widehat{O}\in\widehat{\mathcal{O}}$ in the state $\widehat{\rho}(t)\in\widehat{\mathcal{S}}$ can be computed as

$$\begin{aligned}\langle\widehat{O}\rangle_{\widehat{\rho}(t)}&=(\widehat{\rho}(t)|\widehat{O})=\sum_{im_{il}\, m'_{il}}\left(\sum_{\alpha}+\int_0^{\infty}\mathrm{d}w\right)\overline{\rho_{im_{il}\, m'_{il}}(\omega)}\,O_{im_{il}\, m'_{il}}(\omega)\\&\quad+\sum_{im_{il}\, m'_{il}}\left(\sum_{\alpha}+\int_0^{\infty}\mathrm{d}w\right)\left(\sum_{\beta\neq\alpha}+\int_0^{\infty}\mathrm{d}w'\right)\overline{\rho_{im_{il}\, m'_{il}}(\omega,\omega')}\;\mathrm{e}^{\mathrm{i}(\omega-\omega')t/\hbar}\,O_{im_{il}\, m'_{il}}(\omega,\omega')\end{aligned} \tag{49}$$

where we can develop the second term as

$$\begin{aligned}&\sum_{im_{il}\, m'_{il}}\sum_{\beta\neq\alpha}\overline{\rho_{im_{il}\, m'_{il}}(w_\alpha,w_\beta)}\;\mathrm{e}^{\mathrm{i}(w_\alpha-w_\beta)t/\hbar}\,O_{im_{il}\, m'_{il}}(w_\alpha,w_\beta)\\&\quad+\int_0^{\infty}\mathrm{d}w\int_0^{\infty}\mathrm{d}w'\,\overline{\rho_{im_{il}\, m'_{il}}(w,w')}\;\mathrm{e}^{\mathrm{i}(w-w')t/\hbar}\,O_{im_{il}\, m'_{il}}(w,w')\\&\quad+\sum_{\alpha}\int_0^{\infty}\mathrm{d}w'\,\overline{\rho_{im_{il}\, m'_{il}}(w_\alpha,w')}\;\mathrm{e}^{\mathrm{i}(w_\alpha-w')t/\hbar}\,O_{im_{il}\, m'_{il}}(w_\alpha,w')\\&\quad+\int_0^{\infty}\mathrm{d}w\sum_{\beta}\overline{\rho_{im_{il}\, m'_{il}}(w,w_\beta)}\;\mathrm{e}^{\mathrm{i}(w-w'_\beta)t/\hbar}\,O_{im_{il}\, m'_{il}}(w,w_\beta).\end{aligned} \tag{50}$$

The requirement of "regularity" for the involved functions means that $\overline{\rho_{im_{il}\, m'_{il}}(w_\alpha,w')}\,O_{im_{il}\, m'_{il}}(w_\alpha,w')\in\mathbb{L}^1$ in variable w', $\overline{\rho_{im_{il}\, m'_{il}}(w,w_\beta)}\,O_{im_{il}\, m'_{il}}(w,w_\beta)\in\mathbb{L}^1$ in variable w, and $\overline{\rho_{im_{il}\, m'_{il}}(w,w')}\,O_{im_{il}\, m'_{il}}(w,w')\in\mathbb{L}^1$ in the variable $\nu=w-w'$. Then, when we take the limit for $t\to\infty$, we can apply the Cèsaro Riemann–Lebesgue limit as introduced in Eq. (9):

$$\lim_{t\to\infty}\frac{1}{\tau}\int_0^{\tau}(\widehat{\rho}(t)|\widehat{O})\,\mathrm{d}t=\lim_{t\to\infty}\frac{1}{\tau}\int_0^{\tau}\langle\widehat{O}\rangle_{\widehat{\rho}(t)}\,\mathrm{d}t=\sum_{im_{il}\, m'_{il}}\left(\sum_{\alpha}+\int_0^{\infty}\mathrm{d}w\right)\overline{\rho_{im_{il}\, m'_{il}}(\omega)}\,O_{im_{il}\, m'_{il}}(\omega)=(\widehat{\rho}_*|\widehat{O}) \tag{51}$$

which, according to Eq. (4) can be expressed as

$$C\text{-}\lim_{t\to\infty}\widehat{\rho}(t)=\sum_{im_{il}\, m'_{il}}\left(\sum_{\alpha}+\int_0^{\infty}\mathrm{d}w\right)\overline{\rho_{im_{il}\, m'_{il}}(\omega)}\,(\omega,m_{il},m'_{il}|=\widehat{\rho}_*. \tag{52}$$

Since only the singular diagonal terms remain in $\widehat{\rho}_*$, we can say that the system *has decohered alla Cèsaro* in the energy.

Remarks. • It is clear that this kind of decoherence involves a *generalized projection*. As we have seen in Section 2.3, expectation values and time integrals are projected magnitudes that restrict the maximum information given by the quantum state. It is precisely this projection what breaks the unitarity of the underlying quantum evolution (for a detailed discussion, see Refs. [20,24]).

- Theoretically, decoherence takes place at $t\to\infty$. Nevertheless, for the continuous spectrum case and for atomic interactions, the *characteristic decoherence time* is $t_D=10^{-15}$ s. For macroscopic systems at room temperature this time is even smaller (e.g. $10^{-37}-10^{-39}$ s, see Ref. [22]). Models with two characteristic times (decoherence and relaxation) can also be considered (see Refs. [22,42]).
- On the basis of the presentation of this subsection, it is also clear that this kind of decoherence strictly obtains when the Hamiltonian has a continuous spectrum. Nevertheless, the process also leads to decoherence in quasi-continuous models, that is, discrete models where (i) the energy spectrum is quasi-continuous, i.e., has a small discrete energy spacing, and (ii) the functions of energy used in the formalism are such that the sums in which they are involved can be approximated by Riemann integrals (see Ref. [42]). This condition is rather weak: the overwhelming majority of the physical models studied in the literature on dynamics, thermodynamics, quantum mechanics and quantum field theory are quasi-continuous, and the well-known strategy for transforming sums in integrals is applied (see Ref. [43]). We will return on this point in Section 7, when the Casati–Prosen model will be discussed.

3.3. *Decoherence in the remaining variables*

Up to this point, $\widehat{\rho}_*$ is diagonal in the variables ω and ω', but in general not in the remaining variables. A further diagonalization of $\widehat{\rho}_*$ in variables m_{il} and m'_{il} can be obtained through a unitary matrix U, which performs the transformation

$$\rho_{im_{il}\,m'_{il}}(\omega) \to \rho_{ip_{il}\,p'_{il}}(\omega)\,\delta_{p_{il}\,p'_{il}} \doteq \rho_{ip_{il}}(\omega)\delta_{p_{il}\,p'_{il}}. \tag{53}$$

Such a transformation defines a new orthonormal basis $\{|\omega, p_{il}\rangle\}$, where p_{il} is a shorthand for $p_{i1}, \ldots, p_{iN}$, and $p_{il} \in \mathbb{N}$. This basis corresponds to a new local CSCO $\{\widehat{H}, \widehat{P}_{il}\}$. Therefore, in each D_{ϕ_i}, the final pointer basis for the observables is $\{|\omega, p_{il}, p'_{il}), |\omega, \omega', p_{il}, p'_{il})\}$, defined as in Eq. (44) but with the indices p instead of m, and the corresponding final pointer basis for the states is $\{(\omega, p_{il}, p'_{il}|, (\omega, \omega', p_{il}, p'_{il}|\}$.

a. When the observables $\widehat{P}_{il}$ have discrete spectra, in the new basis the van Hove observables of our algebra $\widehat{\mathcal{A}}$ will read

$$\widehat{O} = \sum_{ip_{il}} \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right) O_{ip_{il}}(\omega)|\omega, p_{il}) + \sum_{ip_{il}\,p'_{il}} \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right)\left(\sum_{\beta\neq\alpha} + \int_0^{\infty} \mathrm{d}w'\right) O_{ip_{il}\,p'_{il}}(\omega, \omega')\,|\omega, \omega', p_{il}, p'_{il}) = \widehat{O}_S + \widehat{O}_R \tag{54}$$

where $\widehat{O}_S$ is the *singular part* and $\widehat{O}_R$ is the *regular part* of $\widehat{O}$. The states, in turn, will have the following form

$$\widehat{\rho} = \sum_{ip_{il}} \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right) \overline{\rho_{ip_{il}}(\omega)}\,(\omega, p_{il}| + \sum_{ip_{il}\,p'_{il}} \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right)\left(\sum_{\beta\neq\alpha} + \int_0^{\infty} \mathrm{d}w'\right) \overline{\rho_{ip_{il}\,p'_{il}}(\omega, \omega')}(\omega, \omega', p_{il}, p'_{il}| = \widehat{\rho}_S + \widehat{\rho}_R \tag{55}$$

where, again, $\widehat{\rho}_S$ is the *singular part* and $\widehat{\rho}_R$ is the *regular part* of $\widehat{\rho}$. Therefore, when we apply the Cèsaro Riemann–Lebesgue limit of Eq. (52), the regular part vanishes and only the singular part remains:

$$C\text{-}\lim_{t\to\infty} \widehat{\rho}(t) = \sum_{ip_{il}} \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right) \overline{\rho_{ip_{il}}(\omega)}(\omega, p_{il}| = \widehat{\rho}_* = \widehat{\rho}_S. \tag{56}$$

b. In the case of observables $\widehat{P}_{il}$ with continuous spectra, Eq. (54) becomes

$$\widehat{O} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right) O_i(\omega, p_{il})|\omega, p_{il}) + \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_{p'_{il}} \mathrm{d}p_{il}^{\prime N} \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right)\left(\sum_{\beta\neq\alpha} + \int_0^{\infty} \mathrm{d}w'\right) O_i(\omega, \omega', p_{il}, p'_{il})|\omega, \omega', p_{il}, p'_{il}) = \widehat{O}_S + \widehat{O}_R \tag{57}$$

and Eq. (55) becomes

$$\widehat{\rho} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right) \overline{\rho_i(\omega, p_{il})}(\omega, p_{il}| + \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_{p'_{il}} \mathrm{d}p_{il}^{\prime N} \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right)\left(\sum_{\beta\neq\alpha} + \int_0^{\infty} \mathrm{d}w'\right) \overline{\rho_i(\omega, \omega', p_{il}, p'_{il})}(\omega, \omega', p_{il}, p'_{il}| = \widehat{\rho}_S + \widehat{\rho}_R. \tag{58}$$

In this case, through the Cèsaro Riemann–Lebesgue limit we obtain

$$C\text{-}\lim_{t\to\infty} \widehat{\rho}(t) = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \left(\sum_{\alpha} + \int_0^{\infty} \mathrm{d}w\right) \overline{\rho_i(\omega, p_{il})}\,(\omega, p_{il}| = \widehat{\rho}_* = \widehat{\rho}_S. \tag{59}$$

It is worth stressing that the state $\widehat{\rho}(t)$ always evolves unitarily and, as a consequence, its off-diagonal terms never vanish through the evolution. Nevertheless, the Cèsaro-limit means that a projector $\Pi = \Pi_W \Pi_\tau$ can be defined (see Eqs. (10) and (14)), such that the time-projected state $\widehat{\rho}_\tau(t) = \widehat{\rho}(t)\,\Pi_W \Pi_\tau$ has a strong limit (see Eq. (16))

$$\lim_{t\to\infty} \widehat{\rho}_\tau(t) = \lim_{t\to\infty} \widehat{\rho}(t)\,\Pi_W \Pi_\tau = \widehat{\rho}_*. \tag{60}$$

Since $\widehat{\rho}_\tau(t)$ does not denote a quantum state but a projected state, there is no reason that prevents it from evolving non-unitarily: the evolution of $\widehat{\rho}_\tau(t)$ turns out to be analogous to the familiar case of classical unstable systems, where it is completely natural to obtain a non-unitary projected evolution from an underlying unitary dynamics.

In the following sections we will work with the case of observables $\widehat{P_{il}}$ with continuous spectra; the case of $\widehat{P_{il}}$ with discrete spectra is completely analogous.

4. The classical statistical limit

In order to obtain the classical statistical limit, it is necessary to compute the Wigner transformation of observables and states. In paper [25] we have done it in the case of $\widehat{H}$ with continuous spectrum. Here we will generalize that result for the case of a discrete+continuous energy spectrum.

If we develop the expression of the state of Eq. (58), we find that $\widehat{\rho}$ has a discrete part $\widehat{\rho}^{(D)}$, where only the sums in w_α are involved, and a continuous part $\widehat{\rho}^{(C)}$ including the integrals in w:

$$\widehat{\rho} = \widehat{\rho}^{(D)} + \widehat{\rho}^{(C)} \tag{61}$$

where $\widehat{\rho}^{(D)}$ and $\widehat{\rho}^{(C)}$ read

$$\widehat{\rho}^{(D)} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha \overline{\rho_i(w_\alpha, p_{il})}(w_\alpha, p_{il}| + \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_{p'_{il}} \mathrm{d}p_{il}'^N \sum_\alpha \sum_{\beta\neq\alpha} \overline{\rho_i(w_\alpha, w_\beta, p_{il}, p'_{il})}(w_\alpha, w_\beta, p_{il}, p'_{il}| \tag{62}$$

$$\begin{aligned}\widehat{\rho}^{(C)} &= \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_0^\infty \mathrm{d}w\, \overline{\rho_i(w, p_{il})}(w, p_{il}| \\ &\quad + \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_{p'_{il}} \mathrm{d}p_{il}'^N \int_0^\infty \mathrm{d}w \int_0^\infty \mathrm{d}w'\, \overline{\rho_i(w, w', p_{il}, p'_{il})}(w, w', p_{il}, p'_{il}|.\end{aligned} \tag{63}$$

Therefore, Eq. (59) can be written as

$$C\text{-}\lim_{t\to\infty} \widehat{\rho}(t) = C\text{-}\lim_{t\to\infty} \left(\widehat{\rho}(t)^{(D)} + \widehat{\rho}(t)^{(C)}\right) = \widehat{\rho}_* = \widehat{\rho}_S = \widehat{\rho}_S^{(D)} + \widehat{\rho}_S^{(C)} \tag{64}$$

where $\widehat{\rho}_S$ is the singular component of $\widehat{\rho}$, resulting from the Cèsaro Riemann–Lebesgue limit,

$$\widehat{\rho}_S = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \left(\sum_\alpha + \int_0^\infty \mathrm{d}w\right) \overline{\rho_i(\omega, p_{il})}(\omega, p_{il}| \tag{65}$$

and $\widehat{\rho}_S^{(D)}$ and $\widehat{\rho}_S^{(C)}$ are the discrete and the continuous parts of $\widehat{\rho}_S$,

$$\widehat{\rho}_S^{(D)} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha \overline{\rho_i(w_\alpha, p_{il})}(w_\alpha, p_{il}| \tag{66}$$

$$\widehat{\rho}_S^{(C)} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_0^\infty \mathrm{d}w\, \overline{\rho_i(w, p_{il})}(w, p_{il}|. \tag{67}$$

It is easy to see that, in Eq. (64), the limit of $\widehat{\rho}(t)^{(D)}$ is $\widehat{\rho}_S^{(D)}$ and the limit of $\widehat{\rho}(t)^{(C)}$ is $\widehat{\rho}_S^{(C)}$. But whereas in the discrete case the Cèsaro Riemann–Lebesgue limit applies, in the continuous case the regular part vanishes as a consequence of the stronger Weak Riemann–Lebesgue limit (as in the particular case studied in Ref. [25]):

$$C\text{-}\lim_{t\to\infty} \widehat{\rho}(t)^{(D)} = \widehat{\rho}_S^{(D)} \tag{68}$$

$$W\text{-}\lim_{t\to\infty} \widehat{\rho}(t)^{(C)} = \widehat{\rho}_S^{(C)}. \tag{69}$$

Now, the task is to find the classical distribution $\rho_*(\phi) = \rho_S(\phi)$ resulting from applying the Wigner transformation to $\widehat{\rho}_S$ in the limit $\hbar \to 0$,

$$\rho_*(\phi) = \rho_S(\phi) = symb\, \widehat{\rho}_S = symb\, \widehat{\rho}_S^{(D)} + symb\, \widehat{\rho}_S^{(C)} = \rho_S^{(D)}(\phi) + \rho_S^{(C)}(\phi) \tag{70}$$

where (see Eqs. (66) and (67))

$$\rho_S^{(D)}(\phi) = symb\, \widehat{\rho}_S^{(D)} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha \overline{\rho_i(w_\alpha, p_{il})} symb(w_\alpha, p_{il}| \tag{71}$$

$$\rho_S^{(C)}(\phi) = symb \widehat{\rho}_S^{(C)} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_0^\infty \mathrm{d}w\, \overline{\rho_i(w, p_{il})} symb(w, p_{il}|. \tag{72}$$

So, the problem is reduced to compute $symb(w_\alpha, p_{il}|$ and $symb(w, p_{il}\,|$.

As it is well known, in its traditional form the Wigner transformation yields the correct expectation value of any observable in a given state when we are dealing with regular functions (see Eq. (24)). In previous papers [23,25] we have extended the Wigner transformation to singular functions in order to apply it to functions as $(w, p_{il}|$. Here we will generalize that result to the discrete+continuous case in two steps: first, we will consider the transformation of observables and, second, we will study the transformation of states. This task will allow us to study the convergence of the classical distribution $\rho(\phi, t)$ in phase space.

4.1. Transformation of observables

As we have seen (see Eq. (57)), our van Hove observables $\widehat{O} \in \widehat{\mathcal{O}}$ have a singular part $\widehat{O}_S$ and a regular part $\widehat{O}_R$. We will direct our attention to the singular operators $\widehat{O}_S$, since the regular operators $\widehat{O}_R$ "disappear" from the expectation values after decoherence, as explained in Section 3:

$$\widehat{O}_S = \sum_i \int_{p_{il}} dp_{il}^N \left(\sum_\alpha + \int_0^\infty dw \right) O_i(\omega, p_{il}) |\omega, p_{il}) = \widehat{O}_S^{(D)} + \widehat{O}_S^{(C)} \tag{73}$$

where $\widehat{O}_S^{(D)}$ and $\widehat{O}_S^{(C)}$ are the discrete and the continuous parts of $\widehat{O}_S$,

$$\widehat{O}_S^{(D)} = \sum_i \int_{p_{il}} dp_{il}^N \sum_\alpha O_i(w_\alpha, p_{il}) |w_\alpha, p_{il}) \tag{74}$$

$$\widehat{O}_S^{(C)} = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty dw \, O_i(w, p_{il}) |w, p_{il}). \tag{75}$$

Then, the Wigner transformation of $\widehat{O}_S$ can be computed as

$$O_S(\phi) = symb\, \widehat{O}_S = symb\, \widehat{O}_S^{(D)} + symb\, \widehat{O}_S^{(C)} = O_S^{(D)}(\phi) + O_S^{(C)}(\phi) \tag{76}$$

where (see Eqs. (74) and (75))

$$O_S^{(D)}(\phi) = symb\, \widehat{O}_S^{(D)} = \sum_i \int_{p_{il}} dp_{il}^N \sum_\alpha O_i(w_\alpha, p_{il}) symb\, |w_\alpha, p_{il}) \tag{77}$$

$$O_S^{(C)}(\phi) = symb\, \widehat{O}_S^{(C)} = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty dw \, O_i(w, p_{il}) symb\, |w, p_{il}). \tag{78}$$

In paper [25] we have proved that, in the case of $\widehat{H}$ with continuous spectrum (that is, when $\widehat{O}_S = \widehat{O}_S^{(C)}$), the function $symb\, |w, p_{il})$ can be computed as

$$symb\, |w, p_{il}) = \delta(H(\phi) - w)\, \delta^N(P_{il}(\phi) - p_{il}). \tag{79}$$

By a completely analogous argument (see Appendix A), it can be proved that, for the discrete part,

$$symb\, |w_\alpha, p_{il}) = \delta_{H(\phi) w_\alpha}\, \delta^N(P_{il}(\phi) - p_{il}). \tag{80}$$

4.2. Transformation of states

As in papers [23,25], in order to compute the *symb* for $(w_\alpha, p_{il}|$ and $(w, p_{il}|$ we will define the Wigner transformation of the singular operator $\widehat{\rho}_S$ on the basis of the only reasonable requirement that such a transformation leads to the correct expectation value of any observable also for singular states; that is (see Eq. (24)),

$$(symb\, \widehat{\rho}_S \mid symb\, \widehat{O}_S) \doteq (\widehat{\rho}_S \mid \widehat{O}_S). \tag{81}$$

In particular,

$$(symb\, \widehat{\rho}_S^{(D)} \mid symb\, \widehat{O}_S^{(D)}) = (\widehat{\rho}_S^{(D)} \mid \widehat{O}_S^{(D)}) \tag{82}$$

$$(symb\, \widehat{\rho}_S^{(C)} \mid symb\, \widehat{O}_S^{(C)}) = (\widehat{\rho}_S^{(C)} \mid \widehat{O}_S^{(C)}). \tag{83}$$

These equations must hold also in the particular case in which $\widehat{O}_S^{(D)} = |w_\beta, p'_{il})$, $\widehat{\rho}_S^{(D)} = (w_\alpha, p_{il}|$ and $\widehat{O}_S^{(C)} = |w', p'_{il})$, $\widehat{\rho}_S^{(C)} = (w, p_{il}|$, for some ϕ_i (see Eq. (33)):

$$(symb(w_\alpha, p_{il}| \mid symb\, |w_\beta, p'_{il})) = (w_\alpha, p_{il} \mid w_\beta, p'_{il}) \tag{84}$$

$$(symb(w, p_{il}| \mid symb\, |w', p'_{il})) = (w, p_{il} \mid w', p'_{il}) \tag{85}$$

and all the remaining cross-terms are zero for any domain D_{ϕ_j}, with $j \neq i$. But from Eqs. (80) and (79) we know how to compute $symb\, |w_\beta, p'_{il})$ and $symb\, |w', p'_{il})$, respectively. Moreover, from the definition of the cobasis (see Eq. (45)) we know that

$$(w_\alpha, p_{il} \mid w_\beta, p'_{il}) = \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il}) \tag{86}$$

$$(w, p_{il} \mid w', p'_{il}) = \delta(w - w')\, \delta^N(p_{il} - p'_{il}). \tag{87}$$

Therefore, from Eqs. (79) and (80),

$$(symb(w_\alpha, p_{il}| \mid \delta_{H(\phi) w_\beta}\, \delta^N(P_{il}(\phi) - p'_{il})) = \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il}) \tag{88}$$

$$(symb(w, p_{il}| \mid \delta(H(\phi) - w')\, \delta^N(P_{il}(\phi) - p'_{il})) = \delta(w - w')\, \delta^N(p_{il} - p'_{il}). \tag{89}$$

From Eq. (89), in paper [25] we have proved that, in the continuous case,

$$symb(w, p_{il}| = \frac{\delta(H(\phi) - w)\, \delta^N(P_{il}(\phi) - p_{il})}{C_i(H, P_{il})} \tag{90}$$

where $C_i(H, P_{il})$ is the configuration volume of the region $\Gamma_{H,P_{il}} \cap D_{\phi_i}$, being $\Gamma_{H,P_{il}} \subset \Gamma$ the hypersurface defined by $H = const.$ and $P_{il} = const$. From Eq. (88), by an analogous argument (see Appendix B) it can be proved that

$$symb(w_\alpha, p_{il}| = \frac{\delta_{H(\phi) w_\alpha}\, \delta^N(P_{il}(\phi) - p_{il})}{C_i(H, P_{il})}. \tag{91}$$

This result shows that the discrete and the continuous parts have an analogous behaviour.

4.3. Convergence in phase space

Finally, we can introduce the results of Eqs. (91) and (90) into Eqs. (71) and (72), respectively, in order to obtain the classical distributions $\rho_S^{(D)}(\phi)$ and $\rho_S^{(C)}(\phi)$:

$$\rho_S^{(D)}(\phi) = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha \frac{\overline{\rho_i(w_\alpha, p_{il})}}{C_i(H, P_{il})}\, \delta_{H(\phi) w_\alpha}\, \delta^N(P_{il}(\phi) - p_{il}) \tag{92}$$

$$\rho_S^{(C)}(\phi) = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_0^\infty \mathrm{d}w\, \frac{\overline{\rho_i(w, p_{il})}}{C_i(H, P_{il})} \delta(H(\phi) - w)\, \delta^N(P_{il}(\phi) - p_{il}). \tag{93}$$

As a consequence, the Wigner transformation of the limits of Eqs. (68) and (69) can be written as

$$C\text{-}\lim_{t\to\infty} \rho(\phi, t)^{(D)} = \rho_S^{(D)}(\phi) = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha \frac{\overline{\rho_i(w_\alpha, p_{il})}}{C_i(H, P_{il})}\, \delta_{H(\phi) w_\alpha}\, \delta^N(P_{il}(\phi) - p_{il}) \tag{94}$$

$$W\text{-}\lim_{t\to\infty} \rho(\phi, t)^{(C)} = \rho_S^{(C)}(\phi) = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_0^\infty \mathrm{d}w\, \frac{\overline{\rho_i(w, p_{il})}}{C_i(H, P_{il})}\, \delta(H(\phi) - w)\, \delta^N(P_{il}(\phi) - p_{il}). \tag{95}$$

5. The classical limit

Up to this point we have obtained the classical distribution $\rho_*(\phi) = \rho_S^{(D)}(\phi) + \rho_S^{(C)}(\phi)$ to which the system converges in phase space. This distribution defines hypersurfaces $H(\phi) = \omega$, $P_{il}(\phi) = p_{il}$ corresponding to the "momentum" variables. But such a distribution does not define the trajectories on those hypersurfaces, that is, it does not fix definite values for the "configuration" variables (the variables canonically conjugated to $H(\phi)$ and $P_{il}(\phi)$). This is reasonable to the extent that definite trajectories would violate the uncertainty principle: we know that, if $\widehat{H}$ and $\widehat{P}_{il}$ have definite values, the values of the observables that non-commute with them will be completely undefined.

Nevertheless, trajectory-like motions can be recovered by means of the Eherenfest theorem, applied to the constants of motion and their conjugated variables. Let us call $\widehat{J}$ the "momentum" variables $\widehat{H}$ and $\widehat{P}_{il}$, and $\widehat{\Theta}$ the corresponding conjugated "configuration" variables, all of them in the domain D_{ϕ_i}. The equations of motion in the Heisenberg picture read

$$\frac{\mathrm{d}\widehat{J}}{\mathrm{d}t} = \frac{i}{\hbar}[\widehat{H}, \widehat{J}] \qquad \frac{\mathrm{d}\widehat{\Theta}}{\mathrm{d}t} = \frac{i}{\hbar}[\widehat{H}, \widehat{\Theta}] \tag{96}$$

where $[\widehat{H}, \widehat{J}] = 0$ and $[\widehat{J}, \widehat{\Theta}] = -i\hbar$. If $\widehat{H} = \widehat{H}(\widehat{J}) = \sum_n A_n \widehat{J}^n$ is a convergent series, we can compute $[\widehat{H}, \widehat{\Theta}]$ and obtain

$$\frac{\mathrm{d}\widehat{J}}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\widehat{\Theta}}{\mathrm{d}t} = \frac{\mathrm{d}\widehat{H}}{\mathrm{d}\widehat{J}} = F(\widehat{J}) \tag{97}$$

which are the quantum version of Hamilton equations. Since $\frac{\mathrm{d}\widehat{H}}{\mathrm{d}\widehat{J}} = F(\widehat{J})$ is constant in time, by integration of these equations we obtain

$$\widehat{J}(t) = \widehat{J}(0) \tag{98}$$

$$\widehat{\Theta}(t) = \widehat{\Theta}(0) + F(\widehat{J})\, t. \tag{99}$$

Now we can consider a wavepacket $\widehat{\rho}$ and compute the mean values in $\widehat{\rho}$ to obtain the Ehrenfest theorem:

$$\frac{\mathrm{d}\langle\widehat{J}\rangle}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\langle\widehat{\Theta}\rangle}{\mathrm{d}t} = \left\langle \frac{\mathrm{d}\widehat{H}}{\mathrm{d}\widehat{J}} \right\rangle = \left\langle F(\widehat{J})\right\rangle. \tag{100}$$

If, following the assumption of the Eherenfest theorem, we can approximate the average of the function $F(\widehat{J})$ with the function of the average of $\widehat{J}$,[8]

$$\left\langle F(\widehat{J})\right\rangle \approx F\left(\langle \widehat{J}\, \rangle\right) \tag{101}$$

then Eq. (100) can be replaced by

$$\frac{\mathrm{d}\langle\widehat{J}\rangle}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\langle\widehat{\Theta}\rangle}{\mathrm{d}t} = \left\langle \frac{\mathrm{d}\widehat{H}}{\mathrm{d}\widehat{J}} \right\rangle \approx F\left(\langle\widehat{J}\rangle\right) \tag{102}$$

and, by integration we obtain

$$\langle\widehat{J}(t)\rangle = \langle\widehat{J}(0)\rangle \tag{103}$$

$$\left\langle\widehat{\Theta}(t)\right\rangle = \left\langle\widehat{\Theta}(0)\right\rangle + F\left(\langle\widehat{J}\rangle\right)\, t. \tag{104}$$

These equations show that the mean values $\langle\widehat{J}\rangle$ and $\langle\widehat{\Theta}\rangle$ also obey the Hamilton equations: the wavepacket $\widehat{\rho}$ will follow a classical trajectory. In fact, since the Wigner transformation preserves the mean values (see Eq. (24)), in the phase space representation Eqs. (103) and (104) become

$$\langle J(\phi, t)\rangle = \langle J(\phi, 0)\rangle \tag{105}$$

$$\langle \Theta(\phi, t)\rangle = \langle \Theta(\phi, 0)\rangle + F\left(\langle J(\phi)\rangle\right)\, t. \tag{106}$$

But here we are interested in the behavior of the system in the macroscopic limit $\hbar \to 0$. In this limit, and in the case of almost point-like wavepackets, the deviations from the mean values of $J(\phi)$ and $\Theta(\phi)$ are zero:

$$\delta J(\phi) = J(\phi) - \langle J(\phi)\rangle = 0 \tag{107}$$

$$\delta\Theta(\phi) = \Theta(\phi) - \langle\Theta(\phi)\rangle = 0. \tag{108}$$

Therefore,

$$J(\phi, t) = J(\phi, 0) \tag{109}$$

$$\Theta(\phi, t) = \Theta(\phi, 0) + F(J(\phi))\, t. \tag{110}$$

If we now come back to the original variables, calling $\tau(\phi)$, $\theta_{il}(\phi)$ the conjugated variables corresponding to $H(\phi)$, $P_{il}(\phi)$ respectively, these equations become

$$H(\phi) = \omega \qquad P_{il}(\phi) = p_{il} \tag{111}$$

$$\tau(\phi, t) = \tau^0 + \omega t \qquad \theta_{il}(\phi, t) = \theta_{il}^0 + p_{il}\, t. \tag{112}$$

Thus, at any fixed t we can write

$$\sum_i \int_{\tau^0}\int_{\theta_{il}^0} \delta(\tau(\phi, t) - \tau^0 - \omega t)\, \delta(\theta_{il}(\phi, t) - \theta_{il}^0 - p_{il}\, t)\mathrm{d}\tau^0 \mathrm{d}\theta_{il}^0 = 1 \tag{113}$$

and we can include this 1 in the decomposition of Eqs. (94) and (95)

$$\rho_*(\phi) = \rho_S^{(D)}(\phi) + \rho_S^{(C)}(\phi) = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \int_{\tau^0} \mathrm{d}\tau^0 \int_{\theta_{il}^0} \mathrm{d}\theta_{il}^0 \left(\sum_\alpha + \int_0^\infty \mathrm{d}w\right) \overline{\frac{\rho_i(\omega, p_{il})}{C_i(H, P_{il})}}$$
$$\times\, \delta(H(\phi) - \omega)\, \delta^N(P_{il}(\phi) - p_{il})\, \delta(\tau(\phi, t) - \tau^0 - \omega t)\, \delta(\theta_{il}(\phi, t) - \theta_{il}^0 - p_{il}\, t). \tag{114}$$

This means that, in the macroscopic limit, the classical distribution $\rho_*(\phi)$ can be expressed as a sum of classical trajectories weighted by $\overline{\frac{\rho_i(\omega,p_{il})}{C_i(H,P_{il})}}$. This fact is what, in the next section, will allow us to appeal to the definition of the system in terms of a time evolution represented in the phase space by a one-parameter family of invertible transformations with group properties, $T_t : \Gamma \to \Gamma$ $(t \in \mathbb{R}^+)$.

[8] This approximation is exact when the function $F(\widehat{J})$ is a linear function of $\widehat{J}$, as in the case of a free particle or a harmonic oscillator, or in the case of an almost point-like wavepacket.

6. Towards the quantum ergodic hierarchy

Although the existence of chaos in classical mechanics has been rigorously proved only in highly idealized systems, the behavior of many classical systems exhibits features that can be interpreted as symptoms of chaos. In the Introduction we have explained why this fact contrasts with the common opinion that chaos in quantum systems seems to be the exception rather than the rule and that the relative scarcity of quantum chaos poses a severe threat to the correspondence principle. As we have said, we will conceive the problem of quantum chaos as the problem of the emergence of classical chaos from quantum descriptions of physical systems. In this sense, we will say that a quantum system is non-integrable (ergodic, mixing, or K) if its classical limit leads to a non-integrable (ergodic, mixing, or K) classical system.

On the basis of this idea and with the formal tools developed in the previous sections, in this section we will define the two lowest levels of the ergodic hierarchy in quantum mechanics. Our strategy will be to identify the conditions that a quantum system must satisfy to lead to the ergodic properties in the classical limit. In order to formulate the classical ergodic properties, the first step is to introduce some necessary definitions.

Definition 1. A *dynamical system S* is a triple (Γ, T_t, μ) where:

- Γ is a phase space with a σ-algebra $\mathcal{A}$ of μ-measurable sets.
- $T_t : \Gamma \to \Gamma$ $(t \in \mathbb{R}^+)$ is a one-parameter family of invertible transformations with group properties.
- $\mu : \mathcal{A} \to \mathbb{R}^+$ is a measure invariant under the transformation T_t: for all $A \in \mathcal{A}$, $\mu(T_t(A)) = \mu(A)$.

Definition 2. Given a dynamical system S: (Γ, T_t, μ), the *Frobenius–Perron operator* U_t corresponding to T_t is a unitary evolution operator such that, for any integrable function $f : \Gamma \to \mathbb{R}$, $U_t(f(x)) = f(T_t(x))$.

These definitions refer to an abstract dynamical system, whose dynamics is defined on an abstract phase space Γ. In order to apply them to a physical system as treated in the context of classical mechanics, a relevant contextualization is needed. In fact, in a Hamiltonian system represented in a phase space Γ, only a part of the entire Γ is accessible: the accessible space $\Gamma_A \subset \Gamma$ is the hypersurface defined by the global constants of motion of the system. Therefore, it is important to remember that the definitions of all the levels of the ergodic hierarchy on physical systems have to be referred to the corresponding hypersurface Γ_A.

6.1. Quantum non-integrability

In paper [25] and in the previous sections, we have explained how non-integrability may arise from the classical limit of a quantum system. Those explanations were based on the particular case of a CSCO consisting only of the Hamiltonian $\widehat{H}$. In this section we will generalize the results obtained for that particular case.

Given a quantum system which would require $N + 1$ quantum numbers for defining its states, two kinds of observables can be distinguished:

- The $A + 1$ $(0 \le A \le N)$ observables $\widehat{G}_K$, including $\widehat{H}$, that we will call "*global*" because they lead to the global constants of motion in the classical limit described in $\Gamma = \mathbb{R}^{2(N+1)}$. These are the observables belonging to the CSCO $\left\{\widehat{G}_0 = \widehat{H}, \widehat{G}_1, \ldots, \widehat{G}_A\right\}$ of the system.
- The observables $\widehat{L}_{iI}$, with $I = 1$ to $N - A$, that we will call "*local*" because in the classical limit they lead to the local constants of motion in each domain D_{ϕ_i} of the phase space $\Gamma = \mathbb{R}^{2(N+1)}$. These are the observables that constitute, with the $\widehat{G}_K$, the local CSCO $\left\{\widehat{G}_0 = \widehat{H}, \ldots, \widehat{G}_A, \widehat{L}_{i1}, \ldots, \widehat{L}_{i(N-A)}\right\}$ corresponding to the domain D_{ϕ_i}.

On the basis of this characterization, we can distinguish two cases:

- $A = N$. This means that the quantum system has a *total CSCO* $\left\{\widehat{G}_0 = \widehat{H}, \ldots, \widehat{G}_N\right\}$, whose observables turn out to be the $N+1$ global constants of motion in the classical description on $\mathbb{R}^{2(N+1)}$ resulting from the classical limit. As a consequence, such a classical description will be integrable, and we will say that we have an *integrable quantum system*.
- $A < N$. This means that the quantum system has a *partial CSCO* $\left\{\widehat{G}_0 = \widehat{H}, \ldots, \widehat{G}_A\right\}$, whose observables turn out to be the $A + 1$ global constants of motion in the classical description on $\mathbb{R}^{2(N+1)}$ resulting from the classical limit. But since $A+1 < N+1$, such a classical description will be non-integrable, and we will say that we have a *non-integrable quantum system*.

Let us consider each case in more detail.

In the case of *quantum integrability*, if $\left\{\widehat{G}_0 = \widehat{H}, \ldots, \widehat{G}_N\right\}$ is the preferred total CSCO of the system, the classical distribution $\rho_*(\phi) = \rho_S(\phi)$ resulting from the classical limit can be expressed as

$$\rho_*(\phi) = \int_{g_K} \mathrm{d}g_K^{N+1}\, \overline{\rho(g_K)} \prod_K \delta(G_K(\phi) - g_K) \tag{115}$$

where $K = 1$ to N, $G_K(\phi) = symb\,\widehat{G}_K$, and for simplicity we have considered that all the $\widehat{G}_K$ have continuous spectra $0 \le g_K < \infty$ and that the volume $C(G_K)$ is normalized to 1 (see Eqs. (94) and (95)). The $N + 1$ global constants of motion $G_K(\phi) = g_K$ foliate the phase space $\Gamma = \mathbb{R}^{2(N+1)}$ into submanifolds $\mathcal{M}(g_K)$ of dimension N, labeled by the constants $g_K = (g_0 = \omega, g_1, \dots, g_N)$. If the emergent classical system is endowed with action-angle variables, those submanifolds are the tori on which the trajectories are confined. Therefore, the classical system is integrable.

In the case of *quantum non-integrability*, if $\{\widehat{G}_0 = \widehat{H}, \dots, \widehat{G}_A\}$ is the preferred partial CSCO of the system, the classical distribution $\rho_*(\phi)$ reads

$$\rho_*(\phi) = \sum_i \int_{l_{iI}} \mathrm{d}l_{iI}^{(N-A)} \int_{g_K} \mathrm{d}g_K^{A+1}\, \overline{\rho_i(g_K, l_{iI})} \prod_K \delta(G_K(\phi) - g_K) \prod_I \delta(L_{iI}(\phi) - l_{iI}) \tag{116}$$

where $K = 0$ to A, $I = 1$ to $N - A$ and for simplicity we have considered that all the $\widehat{L}_{iI}$ have continuous spectra $0 \le l_{iI} < \infty$ (see Eqs. (94) and (95)). In this case, the emergent classical system has $A + 1$ global constants of motion $G_K(\phi) = g_K$ and $N - A$ local constants of motion $L_{iI}(\phi) = l_{iI}$ in each domain D_{ϕ_i} of the phase space $\Gamma = \mathbb{R}^{2(N+1)}$. As in the previous case, the $A + 1$ global constants of motion foliate the phase space into submanifolds $\mathcal{M}(g_K) = \mathcal{M}(g_0 = \omega, g_1, \dots, g_A)$, but now the submanifolds have dimension $N - A + 1$. Therefore, the classical system is non-integrable.

These considerations were based on the assumption that all the $\widehat{G}_K$ have continuous spectra, but the same conclusions can be drawn when the spectrum ω of $\widehat{H}$ has a discrete part $w_\alpha \in \mathbb{N}$ and a continuous part $w \in \mathbb{R}$: the phase space will also result foliated by the labels $g_K = (g_0 = \omega, g_1, \dots, g_A)$, where $g_0 = \omega$ has the discrete value w_α or the continuous value w. This foliation is particularly relevant because each submanifold $\mathcal{M}(g_K)$, with $K = 0$ to A, will be the accessible hypersurface of Γ to which the ergodic properties will have to be referred to.

Summing up, a quantum system is *quantum non-integrable* if its classical limit leads to a classical non-integrable system, that is, if the system (i) has a well-defined classical limit, and (ii) has a partial CSCO, that is, a CSCO that is not sufficient to define an eigenbasis in terms of which the system's states can be expressed.

6.2. Quantum non-integrable ergodicity

The lowest level of the ergodic hierarchy is ergodicity (see Ref. [44]).[9]

Definition 3. A dynamical system $S : (\Gamma, T_t, \mu)$ is *ergodic* if every invariant $A \in \mathcal{A}$, i.e. $T_t(A) = A$, is a trivial subset of the phase space, i.e. $\mu(A) = 0$ or $\mu(\Gamma - A) = 0$.

Theorem 1. *Given a dynamical system $S : (\Gamma, T_t, \mu)$ and the Frobenius–Perron operator U_t corresponding to T_t, S is* ergodic *iff, for any integrable functions $f : \Gamma \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$,, there is a unique function $f_* : \Gamma \to \mathbb{R}$ such that the evolution $U_t f$ is Cèsaro-convergent to f_*:*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \mathrm{d}t\ (U_t f\,|g) = (f_*|g).$$

When this theorem referred to abstract dynamical systems is to be applied to the quantum case, the involved functions acquire physical meaning and the relevant property has to be restricted to the accessible region of the phase space corresponding to the quantum system. In particular, the functions f and g represent the states ρ and the observables O of the system, respectively, usually belonging to $\mathbb{L}^2(\Gamma, \mu)$, and the accessible region is the region $\Gamma_A \subset \Gamma$ defined by the $A+1$ global constants of motion of the system. Therefore, when Theorem 1 is restated in the language of states and observables used in the previous sections, it plays the role of a definition of ergodicity for quantum systems:

Definition A. Given a dynamical system $S : (\Gamma, T_t, \mu)$ and the Frobenius–Perron operator U_t corresponding to T_t, S is *ergodic* iff, for any $\rho, O \in \mathbb{L}^2(\Gamma, \mu)$, the evolution $\rho(t) = U_t\rho$ is Cèsaro-convergent to ρ_*:

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \mathrm{d}t\ (\rho(t)\,|O) = (\rho_*|O)$$

where ρ_* is the final equilibrium value of the state ρ, and it is *unique* on the hypersurface $\Gamma_A \subset \Gamma$ defined by the global constants of motion of the system.

As we have seen, when a quantum system with definite classical limit has a partial CSCO of $A + 1 < N + 1$ observables, and the spectrum of its Hamiltonian has a discrete part and a continuous part, the evolution of the classical distribution $\rho(\phi, t)$ is Cèsaro-convergent to a final distribution $\rho_*(\phi)$ (see Eqs. (94) and (95), and remember that Weak-convergence implies Cèsaro-convergence):

$$C\text{-}\lim_{t\to\infty} \rho(\phi, t) = C\text{-}\lim_{t\to\infty} \left(\rho(\phi, t)^{(D)} + \rho(\phi, t)^{(C)}\right) = \rho_S^{(D)}(\phi) + \rho_S^{(C)}(\phi) = \rho_*(\phi) \tag{117}$$

[9] For a rigorous presentation of the properties of ergodicity and mixing from a mathematical viewpoint, see Ref. [45].

where

$$\rho_S^{(D)}(\phi) = \sum_i \int_{l_{il}} \mathrm{d}l_{il}^{(N-A)} \int_{g_K} \mathrm{d}g_K^A \sum_\alpha \overline{\rho_i(w_\alpha, g_K, l_{il})}\ \delta_{H(\phi)w_\alpha} \prod_K \delta(G_K(\phi) - g_K) \prod_I \delta(L_{il}(\phi) - l_{il}) \quad (118)$$

$$\rho_S^{(C)}(\phi) = \sum_i \int_{l_{il}} \mathrm{d}l_{il}^{(N-A)} \int_{g_K} \mathrm{d}g_K^A \int_0^\infty \mathrm{d}w\, \overline{\rho_i(w, g_K, l_{il})}\ \delta(H(\phi) - w) \prod_K \delta(G_K(\phi) - g_K) \prod_I \delta(L_{il}(\phi) - l_{il}). \quad (119)$$

In turn, in Section 3.1 we have proved that the elements of the partial CSCO $\{\widehat{G}_0 = \widehat{H}, \dots, \widehat{G}_A\}$ turn out to be the global constants of the motion of the corresponding classical system; then, the accessible region Γ_A will be the submanifold $\mathcal{M}(g_K)$ of dimension $N - A + 1$. This means that the requirement of the unicity of $\rho_*(\phi)$ on the hypersurface $\Gamma_A = \mathcal{M}(g_K)$ (see Definition A) becomes the requirement of the unicity of $\rho_*(\phi)$ for each set of values of the observables of the preferred CSCO. It is easy to prove that this requirement is satisfied by showing that $\rho_*(\phi)$ depends only on the global variables $G_K(\phi)$ and is independent of the local variables $L_{il}(\phi)$ since they are averaged out. In fact, Eqs. (118) and (119) can be written as

$$\rho_S^{(D)}(\phi) = \int_{g_K} \mathrm{d}g_K^A \sum_\alpha \delta_{H(\phi)w_\alpha} \prod_K \delta(G_K(\phi) - g_K) \sum_i \int_{l_{il}} \mathrm{d}l_{il}^{(N-A)}\, \overline{\rho_i(w_\alpha, g_K, l_{il})} \prod_I \delta(L_{il}(\phi) - l_{il}) \quad (120)$$

$$\rho_S^{(C)}(\phi) = \int_{g_K} \mathrm{d}g_K^A \int_0^\infty \mathrm{d}w\, \delta(H(\phi) - w) \prod_K \delta(G_K(\phi) - g_K) \sum_i \int_{l_{il}} \mathrm{d}l_{il}^{(N-A)}\, \overline{\rho_i(w, g_K, l_{il})} \prod_I \delta(L_{il}(\phi) - l_{il}). \quad (121)$$

If we call

$$\widetilde{\rho}(\omega, g_K) = \sum_i \int_{l_{il}} \mathrm{d}l_{il}^{(N-A)}\, \overline{\rho_i(\omega, g_K, l_{il})} \prod_I \delta(L_{il}(\phi) - l_{il}) \quad (122)$$

where $\omega = w_\alpha$ in the discrete case and $\omega = w$ in the continuous case, then Eqs. (120) and (121) are given by

$$\rho_S^{(D)}(\phi) = \int_{g_K} \mathrm{d}g_K^A \sum_\alpha \widetilde{\rho}(w_\alpha, g_K)\, \delta_{H(\phi)w_\alpha} \prod_K \delta(G_K(\phi) - g_K) \quad (123)$$

$$\rho_S^{(C)}(\phi) = \int_{g_K} \mathrm{d}g_K^A \int_0^\infty \mathrm{d}w\, \widetilde{\rho}(w, g_K)\, \delta(H(\phi) - w) \prod_K \delta(G_K(\phi) - g_K) \quad (124)$$

and we can see that $\rho_S^{(D)}(\phi)$ and $\rho_S^{(C)}(\phi)$ are only dependent of the global variables. In other words, when the observables of the preferred CSCO have the set of values $\{g_0 = \omega = a_0, g_K = a_K\}$, the diagonal coordinates of the initial state $\widehat{\rho}$ read

$$\rho_i(\omega, g_K, l_{il}) = \rho_i'(l_{il})\, \delta(\omega - a_0) \prod_K \delta(g_K - a_K) \quad (125)$$

and, as a consequence,

$$\widetilde{\rho}(\omega, g_K) = \delta(\omega, a_0) \prod_K \delta(g_K - a_K) \quad (126)$$

where $\delta(\omega, a_0)$ stands for $\delta_{w_\alpha a_0}$ in the discrete case and for $\delta(w - a_0)$ in the continuous case. Therefore, in the discrete case $\omega = w_\alpha$, $\rho_*(\phi) = \rho_S^{(D)}(\phi) + \rho_S^{(C)}(\phi)$ will acquire on $\mathcal{M}(a_K)$ the unique value

$$\rho_*(\phi) = \rho_S^{(D)}(\phi) + 0 = \delta_{H(\phi)a_0} \prod_K \delta(G_K(\phi) - a_K) \quad (127)$$

and in the continuous case $\omega = w$,

$$\rho_*(\phi) = 0 + \rho_S^{(C)}(\phi) = \delta(H(\phi) - a_0) \prod_K \delta(G_K(\phi) - a_K). \quad (128)$$

Therefore, in each hypersurface $\mathcal{M}(g_K = a_K)$, the evolution $\rho(\phi, t)$ is Cèsaro-convergent to a unique distribution $\rho_*(\phi)$:

$$C\text{-}\lim_{t\to\infty} \rho(\phi, t) = \rho_*(\phi) = \delta(H(\phi), a_0) \prod_K \delta(G_K(\phi) - a_K). \quad (129)$$

This means that, in spite of the unitary evolution of $\rho(\phi, t)$, a time-projected state $\rho_\tau(\phi, t) = \rho(\phi, t)\, \Pi_W \Pi_\tau$ can be defined (see Eqs. (11) and (13)), such that it evolves non-unitarily with a strong limit

$$\lim_{t\to\infty} \rho_\tau(\phi, t) = \rho_*(\phi) = \delta(H(\phi), a_0) \prod_K \delta(G_K(\phi) - a_K). \quad (130)$$

Summing up, a quantum system is *quantum non-integrable ergodic* if its classical limit leads to a classical ergodic system, that is, if the system (i) is quantum non-integrable, and (ii) the spectrum of its Hamiltonian has a discrete part and a continuous part. Let us note that this condition agrees with the spectral condition for the Liouville operators in classical ergodic systems (see Ref. [46], Proper Value Theorem, p. 34).

6.3. Quantum mixing

The next higher level of the ergodic hierarchy is mixing (see Ref. [44]).

Definition 4. A dynamical system $S : (\Gamma, T_t, \mu)$ is *mixing* if, for any $A, B \in \mathcal{A}$,

$$\lim_{t\to\infty} \mu(A \cap T_t B) = \mu(A)\,\mu(B).$$

Theorem 2. *Given a dynamical system $S : (\Gamma, T_t, \mu)$ and the Frobenius–Perron operator U_t corresponding to T_t, S is* mixing *iff, for any integrable functions $f : \Gamma \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$,, there is a unique function $f_* : \Gamma \to \mathbb{R}$ such that the evolution $U_t f$ is Weak-convergent to f_*:*

$$\lim_{\tau\to\infty} (U_t f\,|g) = (f_*|g).$$

Analogously to the previous case, when Theorem 2 is restated in the language of states and observables, it plays the role of a definition of mixing for quantum systems:

Definition B. Given a dynamical system $S : (\Gamma, T_t, \mu)$ and the Frobenius–Perron operator U_t corresponding to T_t, S is *mixing* iff, for any $\rho, O \in \mathbb{L}^2(\Gamma, \mu)$, the evolution $\rho(t) = U_t\rho$ is Weak-convergent to ρ_*:

$$\lim_{\tau\to\infty} (\rho(t)\,|O) = (\rho_*|O).$$

where ρ_* is the final equilibrium value of the state ρ, and it is unique on the hypersurface $\Gamma_A \subset \Gamma$ defined by the global constants of motion of the system.

As we have seen, when a quantum system with definite classical limit has a partial CSCO of $A + 1 < N + 1$ observables, the evolution of the classical distribution $\rho(\phi, t) = \rho(\phi, t)^{(D)} + \rho(\phi, t)^{(C)}$ is Cèsaro-convergent in its discrete part and Weak-convergent in its continuous part (see Eqs. (94) and (95)),

$$C\text{-}\lim_{t\to\infty} \rho(\phi, t)^{(D)} = \rho_S^{(D)}(\phi) \quad (131)$$

$$W\text{-}\lim_{t\to\infty} \rho(\phi, t)^{(C)} = \rho_S^{(C)}(\phi). \quad (132)$$

In the particular case that the system's Hamiltonian has continuous spectrum, $\rho_S^{(D)}(\phi) = 0$. Then, the classical distribution $\rho(\phi, t) = \rho_S^{(C)}(\phi)$ turns out to be Weak-convergent to $\rho_S(\phi) = \rho_S^{(C)}(\phi)$

$$W\text{-}\lim_{t\to\infty} \rho(\phi, t) = \rho_S(\phi) = \rho_*(\phi) \quad (133)$$

where (se Eq. (119), with $w = \omega$)

$$\rho_*(\phi) = \rho_S(\phi) = \sum_i \int_{l_{il}} \mathrm{d}l_{il}^{(N-A)} \int_{g_K} \mathrm{d}g_K^A \int_0^\infty \mathrm{d}\omega\, \overline{\rho_i(\omega, g_K, l_{il})}\, \delta(H(\phi) - \omega) \times \prod_K \delta(G_K(\phi) - g_K) \prod_I \delta(L_{il}(\phi) - l_{il}). \quad (134)$$

By means of an argument analogous to that of the previous subsection but restricted to the continuous case, it can be proved that $\rho_*(\phi)$ is unique on each hypersurface $\mathcal{M}(g_K = a_K)$, that is, for each set of values $\{g_0 = \omega = a_0, g_K = a_K\}$ of the observables of the preferred CSCO (see Eq. (128)):

$$\rho_*(\phi) = \rho_S(\phi) = \delta(H(\phi) - a_0) \prod_K \delta(G_K(\phi) - a_K). \quad (135)$$

Therefore, in each hypersurface $\mathcal{M}(g_K = a_K)$, the evolution $\rho(\phi, t)$ is Weak-convergent to a unique distribution $\rho_*(\phi)$:

$$W\text{-}\lim_{t\to\infty} \rho(\phi, t) = \rho_*(\phi) = \delta(H(\phi) - a_0) \prod_K \delta(G_K(\phi) - a_K). \quad (136)$$

In this case, a projected state $\rho_P(\phi, t) = \rho(\phi, t)\,\Pi_W$ can be defined (see Eq. (11)). Then, in spite of the unitary evolution of $\rho(\phi, t)$, $\rho_P(\phi, t)$ evolves non-unitarily with a strong limit

$$\lim_{t\to\infty} \rho_P(\phi, t) = \rho_*(\phi) = \delta(H(\phi) - a_0) \prod_K \delta(G_K(\phi) - a_K). \quad (137)$$

It can be expected that $\rho_*(\phi)$ be independent of the local variables (that is, Eqs. (136) and (137) hold) when the local variables have strong fluctuations in the phase space in comparison with the variation of the functions $O(\phi)$. In this situation, the expectation value $\langle O(\phi)\rangle_{\rho_*(\phi)}$ averages out the local variables and mixing is obtained.

Summing up, a quantum system is *quantum mixing* if its classical limit leads to a classical mixing system, that is, if the system (i) is quantum non-integrable, and (ii) its Hamiltonian has continuous spectrum. Let us note that this condition agrees with the spectral condition for the Liouville operators in classical mixing systems (see Ref. [46], Mixing Theorem, p. 39, and Ref. [47], Wiener–Khinchin Theorem, p. 81).

7. Physical relevance

As we have seen in the Introduction, according to Belot–Earman's program, one of the requirements that any definition of quantum chaos should fulfil is physical relevance. In this section we will argue for the physical relevance of our approach by showing that it can be applied to explain the behavior of the Casati–Prosen model [48] in a conceptually precise way.

The Casati–Prosen model combines a typical case of classical chaos – the Sinai billiard – with a paradigmatic quantum phenomenon – the double slit experiment. The model consists in a triangular *upper billiard* with perfectly reflecting walls and two slices in its base, placed on the top of a box, the *radiating region,* with a photographic film in its base and absorbent walls (see Fig. 1 of Ref. [48]). A quantum state with a Gaussian wavepacket as initial condition "bounces" inside the triangle and produces two centers of radiation in the two slices, which radiate from the billiard to the radiating zone. The question is to explain the results obtained in the photographic film.

Casati and Prosen explain the results in terms of kinematical averages by means of computer simulations. They show that, when the billiard is perfectly triangular and, therefore, integrable (see full lines in Fig. 1 of Ref. [48]), then interference fringes appear in the film (full lines in Fig. 2 of Ref. [48]). But when the billiard is a Sinai billiard (see dotted lines in Fig. 1 of Ref. [48]), the system decoheres and interference vanishes (dotted lines in Fig. 2 of Ref. [48]). According to the authors, the behavior of this model shows that complexity may produce decoherence in a closed system, that is, without the interaction with an environment or with external noise.

We have applied our theoretical framework to the Sinai billiard in Ref. [25] and have used it to explain the Casati–Prosen model in Ref. [49]. In these papers we have represented the walls of the billiard by three potential walls $U_i(x, y)$, with $i = 1$ to 3, which produce the bounces of the wavepacket. The CSCO of the system is given by the Hamiltonian $\widehat{H}$, which, strictly speaking, has a discrete spectrum because the system is finite. Nevertheless, this is the case of a quasi-continuous model in the sense of the third Remark of Section 3.2: the energy spectrum is quasi-continuous, and the functions of energy used in the formalism are such that the sums in which they are involved can be approximated by Riemann integrals (see a full discussion of this point for the Casati–Prosen model in Ref. [49]). Therefore, for times much shorter than the recurrence time, the dynamics of the system can be modeled with a Hamiltonian with continuous spectrum. The other relevant magnitudes are the momenta $\widehat{P}_x$ and $\widehat{P}_y$ which, for the same reason, can be treated as having continuous spectra. In this model, four domains can be distinguished (see Appendix A of Ref. [25]):

- D_0 corresponds to the interior of the triangle, and its local CSCO is $\{\widehat{H}, \widehat{P}_x\}$ (or $\{\widehat{H}, \widehat{P}_y\}$ since $H = \frac{1}{2m}(\widehat{P}_x^2 + \widehat{P}_y^2)$).
- D_1 corresponds to the horizontal wall $U_1(x, y)$, and its local CSCO is $\{\widehat{H}, \widehat{P}_x\}$.
- D_2 corresponds to the vertical wall $U_2(x, y)$, and its local CSCO is $\{\widehat{H}, \widehat{P}_y\}$.
- D_3 corresponds to the third wall $U_3(x, y) = U(ax + by)$, and its local CSCO is $\{\widehat{H}, \widehat{P}_{xy}\}$, where $\widehat{P}_{xy}$ is a linear combination of $\widehat{P}_x$ and $\widehat{P}_y$.

In paper [49] we have showed that, so defined, this model satisfies the conditions required for the Weak-convergence of its state to a final value ρ_*, where ρ_* acquires a unique value for each value of $\widehat{H}$. However, there is a relevant difference between the case of a triangular billiard and the case of a Sinai billiard. In fact, the decoherence time t_D turns out to be proportional to r^2, where r is the radius of the third wall (see Ref. [49]). Therefore,

- When the billiard is perfectly triangular, $r \to \infty$ and, then, $t_D \to \infty$. This means that the initial Gaussian wavepacket remains forever bouncing inside the billiard with no modification of its initial shape and, as a consequence, interference fringes appear on the photographic film.
- When the billiard is a Sinai billiard, $r \neq \infty$ and, then, the system has a finite decoherence time t_D proportional to r^2. In this case, if the distance between the two slits is macroscopic, interference vanishes through the destructive interference embodied in the Riemann–Lebesgue theorem.

In this way, in paper [49] we have explained the computational results obtained by Casati and Prosen by means of our theoretical framework, which is capable of describing systems with partial CSCOs: the non-integrability of the system supplies a necessary condition for the higher levels of instability and complexity. The results of the present paper allow us to carry our conclusions a step further. To the extent that the model is quasi-continuous, that is, it can be modeled with a Hamiltonian with continuous spectrum, the evolution of the quantum state has a Weak-limit that guarantees the Weak-convergence of the classical distribution obtained though the classical limit to a final equilibrium state. Therefore, the resulting description is mixing, and we can say that the model is a *quantum mixing system.*

It is quite clear that this approach can be applied to any model where different domains can be distinguished, each one with its local CSCO. Such a framework supplies a conceptually useful tool for the understanding of the unstable classical behavior emerging from physically relevant quantum systems.

8. Conclusions

If quantum mechanics is a fundamental theory, it should be capable of describing any kind of systems, even those macroscopic systems that can also be adequately described by means of classical theories; in particular, quantum mechanics should be able to account for macroscopic chaotic behavior. In this context it has been argued that there is some kind of tension between quantum mechanics and chaos. In order to face this problem, one has to find out which properties a quantum system must possess to lead to chaotic behavior in the classical limit. When this is the question at issue, the relevant task is not to search for the usual indicators of classical chaos in the quantum domain; what really matters is whether the quantum system possesses the right properties necessary to manifest chaotic behavior in the appropriate classical limit. From this perspective, the claim that quantum chaos puts pressure on the correspondence principle is not legitimate without a clear and precise account of the classical limit of quantum mechanics, since this is the key element for explaining how classical properties can emerge from quantum systems.

On the basis of the general problem of the emergence of the ergodic hierarchy from quantum descriptions, in this paper we have proposed a general theoretical framework to treat quantum systems with unstable behavior in their classical limit. In particular, we have showed that, when the quantum system is endowed with a CSCO whose observables define an eigenbasis for the system's states, the emerging classical description is integrable and, therefore, non-unstable. This result has led us to search for instability in quantum systems with partial CSCOs, that is, CSCOs that are not sufficient to define an eigenbasis of the Hilbert space of the system: in these cases, the classical limit results in a classical non-integrable description characterized by global and local constants of motion. In this situation, the Cèsaro-convergence of the resulting classical distribution (a necessary and sufficient condition for ergodicity) obtains when the quantum Hamiltonian of the system has a discrete+continuous spectrum; in turn, the Weak-convergence of that distribution (a necessary and sufficient condition for mixing) obtains when the quantum Hamiltonian has a continuous spectrum.

We consider that our approach satisfies the four requirements proposed by Belot–Earman's program for any definition of quantum chaos or, in general, of quantum instability:

- It possesses generality in the sense that it supplies general criteria to decide when a quantum system is ergodic or mixing. It possesses mathematical precision (of course, as understood in a physical context) since it is based on a general account of decoherence and of the classical limit of quantum mechanics.
- It agrees with common intuition to the extent that it recovers the characterization of ergodicity and mixing, not only in terms of measurable sets in abstract dynamical systems, but mainly in terms of the different forms of convergence of the system's evolution.
- It is clearly related with the criteria of classical instability because it shows how the features that define classical instability may arise from quantum descriptions without challenging the correspondence principle.
- It is physically relevant since it provides a useful tool for explaining, from a theoretical viewpoint, the results obtained in interesting physical systems by numerical means.

Of course, this work does not exhaust a problem as complex as that of the quantum ergodic properties. Our aim for a future work is to extend the approach presented in this paper in order to complete the quantum ergodic hierarchy with the characterization of the higher levels of instability in the quantum domain.

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Appendix A. Wigner transformation of observables

From Eq. (74) we can write

$$\widehat{O}_S^{(D)} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha O_i(w_\alpha, p_{il})\, |w_\alpha, p_{il}). \tag{A.1}$$

If we Wigner transform this last equation, we obtain

$$symb\, \widehat{O}_S^{(D)} = \sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha O_i(w_\alpha, p_{il}) symb|w_\alpha, p_{il}). \tag{A.2}$$

If we consider, as usual, that $O_i(w_\alpha, p_{il})$ is a polynomial, we know that

$$\widehat{O}_S^{(D)} = \sum_i O_i(\widehat{H}, \widehat{P}_{il}). \tag{A.3}$$

Since $\widehat{H}$ and the $\widehat{P}_{il}$ commute to each other, we can Wigner transform this last equation as

$$symb\,\widehat{O}_S^{(D)} = \sum_i symb\,O_i(\widehat{H}, \widehat{P}_{il}) = \sum_i O_i(H(\phi), P_{il}(\phi)) + 0(\hbar^2). \tag{A.4}$$

where $0(\hbar^2)$ vanishes in the macroscopic limit $\hbar \to 0$. Therefore, in such a limit, Eq. (A.2) reads

$$\sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha O_i(w_\alpha, p_{il})\ \ symb|w_\alpha, p_{il}) = \sum_i O_i(H(\phi), P_{il}(\phi)). \tag{A.5}$$

In the particular case that

$$O_i(w_\alpha, p_{il}) = \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il}), \tag{A.6}$$

Eq. (A.5) becomes

$$\sum_i \int_{p_{il}} \mathrm{d}p_{il}^N \sum_\alpha \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il})\ symb\,|w_\alpha, p_{il}) = \sum_i \delta_{H(\phi) w_\beta}\, \delta^N(P_{il}(\phi) - p'_{il}). \tag{A.7}$$

Therefore, from Eq. (A.7) we can conclude that

$$symb\,|w_\alpha, p_{il}) = \delta_{H(\phi) w_\alpha}\, \delta^N(P_{il}(\phi) - p_{il}) \tag{A.8}$$

as advanced in Eq. (80).

Appendix B. Wigner transformation of states

Form Eq. (88) we know that

$$(symb(w_\alpha, p_{il}|\ |\ \delta_{H(\phi) w_\beta}\, \delta^N(P_{il}(\phi) - p'_{il})) = \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il}) \tag{B.1}$$

which can be computed as

$$\int_{D_{\phi_i}} \mathrm{d}\phi_i^{2(N+1)} symb(w_\alpha, p_{il}|\, \delta_{H(\phi) w_\beta}\, \delta^N(P_{il}(\phi) - p'_{il})) = \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il}). \tag{B.2}$$

Since $\widehat{\rho}_S^{(D)}$ is always time-invariant, the same holds in the particular case that $\widehat{\rho}_S^{(D)} = (w_\alpha, p_{il}|$. As a consequence, $symb(w_\alpha, p_{il}|$ must also be time-invariant, that is, a function of the local constants of motion $\{H(\phi), P_{il}(\phi)\}$ in the corresponding domain D_{ϕ_i}:

$$symb(w_\alpha, p_{il}| = f(H(\phi), P_{il}(\phi)). \tag{B.3}$$

Thus, in each domain D_{ϕ_i} we can define local action-angle variables $(\theta_{il}, J_{il}) = (\theta_{i0}, \theta_{i1}, \dots, \theta_{iN}, J_{i0}, J_{i1}, \dots, J_{iN})$, where $J_{i0} = H(\phi)$ and the $J_{il} = P_{il}(\phi)$ for $I = 1$ to N. If we make the canonical transformation $\phi_i \to (\theta_{il}, J_{il}) = (\theta_{i0}, \theta_{i1}, H, P_{il})$, we obtain

$$\mathrm{d}\phi_i^{2(N+1)} = \mathrm{d}\theta_{il}^{N+1}\, \mathrm{d}H \mathrm{d}P_{il}^N. \tag{B.4}$$

Now we can introduce the change of variables given by Eq. (B.3) and Eq. (B.4) into Eq. (B.2) to obtain

$$\int_{D_{\phi_i}} \mathrm{d}\theta_{il}^{N+1} \mathrm{d}H \mathrm{d}P_{il}^N f(H(\phi), P_{il}(\phi))\, \delta_{H(\phi) w_\beta}\, \delta^N(P_{il}(\phi) - p'_{il}) = \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il}). \tag{B.5}$$

The integration of the l.h.s. of this equation leads us to

$$\int_{D_{\phi_i}} \mathrm{d}H\, \mathrm{d}P_{il}^N\ C_i(H, P_{il})\, f(H(\phi), P_{il}(\phi))\, \delta_{H(\phi) w_\beta}\, \delta^N(P_{il}(\phi) - p'_{il}) = \delta_{w_\alpha w_\beta}\, \delta^N(p_{il} - p'_{il}) \tag{B.6}$$

where we have integrated the angular variables θ_{il}, obtaining the configuration volume $C_i(H, P_{il})$ of the region $\Gamma_{H,P_{il}} \cap D_{\phi_i}$, being $\Gamma_{H,P_{il}} \subset \Gamma$ the hypersurface defined by $H = const.$ and $P_{il} = const.$ Therefore, from Eq. (B.6) we can conclude that

$$C_i(H, P_{il})\, f(H(\phi), P_{il}(\phi)) = \delta_{H(\phi) w_\alpha}\, \delta^N(P_{il}(\phi) - p_{il}). \tag{B.7}$$

If we recall that $fH(\phi), P_{il}(\phi) = symb(w_\alpha, p_{il}|$ (see Eq. (B.3)), we finally obtain

$$symb(w_\alpha, p_{il}| = \frac{\delta_{H(\phi) w_\alpha}\, \delta^N(P_{il}(\phi) - p_{il})}{C_i(H, P_{il})} \tag{B.8}$$

as advanced in Eq. (91).

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