Potential Analysis
https://doi.org/10.1007/s11118-021-09903-6

From A_1 to A_∞: New Mixed Inequalities for Certain Maximal Operators

Fabio Berra[1]

Received: 8 July 2020 / Accepted: 21 January 2021 / Published online: 19 March 2021

Abstract

In this article we prove mixed inequalities for maximal operators associated to Young functions, which are an improvement of a conjecture established in Berra (Proc. Am. Math. Soc. **147**(10), 4259–4273, 2019). Concretely, given $r \geq 1$, $u \in A_1$, $v^r \in A_\infty$ and a Young function Φ with certain properties, we have that inequality

$$uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{v(x)} > t\right\}\right) \leq C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) u(x)v^r(x)\,dx$$

holds for every positive t. As an application, we furthermore exhibe and prove mixed inequalities for the generalized fractional maximal operator $M_{\gamma,\Phi}$, where $0 < \gamma < n$ and Φ is a Young function of $L\log L$ type.

Keywords Young functions · Maximal operators · Muckenhoupt weights · Fractional operators

Mathematics Subject Classification (2010) 26A33 · 42B25

1 Introduction

One of the most classical and extensively studied problems in Harmonic Analysis is the characterization of all the functions w for which the Hardy-Littlewood maximal operator is bounded in $L^p(w)$, for $1 < p < \infty$. This problem was first solved by B. Muckenhoupt in [17], where the author proved that the inequality

$$\int_{\mathbb{R}} (Mf(x))^p w(x)\,dx \leq C\int_{\mathbb{R}} |f(x)|^p w(x)\,dx \tag{1.1}$$

holds for $1 < p < \infty$ if and only if $w \in A_p$. Later on, this result was extended to higher dimensions and even to spaces of homogeneous type.

The author was supported by CONICET and UNL

✉ Fabio Berra
fberra@santafe-conicet.gov.ar

1 CONICET and Departamento de Matemática (FIQ-UNL), Santa Fe, Argentina

It is well known that, for the limiting case $p = 1$, the inequality above is not true. Instead, the estimate

$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| w(x)\, dx$$

holds if and only if $w \in A_1$.

In [21] Sawyer proved that if u, v are A_1 weights, then the estimate

$$uv\left(\left\{x \in \mathbb{R} : \frac{M(fv)(x)}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)| u(x) v(x)\, dx \tag{1.2}$$

holds for every positive t. From now on, we will refer to this type of inequality as mixed because of the interaction of two different weights in it. This estimate can be seen as the weak $(1, 1)$ type of the operator $Sf = M(fv)/v$, with respect to the measure $d\mu(x) = u(x)v(x)\, dx$. One of the motivations to study this kind of estimate was the fact that inequality (1.2) combined with the Jones' factorization theorem and the Marcinkiewicz interpolation theorem allows to give, in a very easy way, a proof of Eq. 1.1 when we assume $w \in A_p$.

The proof of Eq. 1.2 is, however, a bit tricky. Since S can be seen as the product of two functions, this produces a perturbation of the level sets of M by the weight v. So it is not clear that classical covering lemmas or decomposition techniques work in this case. To overcome this difficulty, the author uses a decomposition of level sets into an adequate class of intervals with certain properties, called "principal intervals", an idea that had already been used to prove some weak estimates previously in [18].

It was also conjectured in [21] that an analogous estimate to Eq. 1.2 should still hold for the Hilbert transform. That conjecture was settled twenty years later by Cruz-Uribe, Martell and Pérez in [8]. In this paper mixed weak inequalities were given, generalizing Eq. 1.2 to $\mathbb{R}^n$, not only for M but also for Calderón-Zygmund operators (CZO) and proving the conjecture made by Sawyer. The authors considered two different types of hypotheses on the weights u and v: $u, v \in A_1$ and $u \in A_1$ and $v \in A_\infty(u)$. For the first condition, the proof follows similar lines as in [21]. On the other hand, the second condition is more suitable, since it implies that the product uv belongs to A_∞ and therefore is a doubling measure. This allows to apply classical techniques, like Calderón-Zygmund decomposition to achieve the estimate.

It is also convenient to use sparse domination techniques [7] and it also has been explored to provide quantitative estimates [19] and results in the multilinear setting [16].

Another conjecture arose from [8]: the authors claimed that the mixed estimate

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{M(fv)(x)}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| u(x) v(x)\, dx$$

should still hold under the weaker assumption $u \in A_1$ and $v \in A_\infty$. It is easy to note that both conditions on the weights above imply it. This conjecture was recently proved in [15], where the authors apply the "principal cubes" decomposition with adequate modifications to avoid the use of an A_1 condition on the weight v.

Mixed weak estimates have also been explored for a more general class of maximal functions, such as the operator M_Φ associated to the Young function Φ and defined by

$$M_\Phi f(x) = \sup_{Q \ni x} \|f\|_{\Phi, Q},$$

where the supremum is taken over averages of Luxemburg type (see Section 2 for details). For instance, in [4] it was proved that if $\Phi(t) = t^r(1+\log^+ t)^\delta$, with $r \geq 1$ and $\delta \geq 0$, $u \geq 0$, $v = |x|^\beta$ with $\beta < -n$ and $w = 1/\Phi(1/v)$ then the estimate

$$uw\left(\left\{x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{v(x)} > t\right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|v(x)}{t}\right) Mu(x)\, dx$$

holds for every positive t.

The same estimate is true if we consider this family of Young functions and two weights u and v such that $u, v^r \in A_1$. This result is contained in [2], and generalizes the corresponding estimate given in [8] for the case $\Phi(t) = t$. However, this inequality turns out to be non-homogeneous in the weight v. This problem was overcome in [3], where the authors proved that under the same condition on the weights and even for a more general class of Young functions Φ the inequality

$$uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{v(x)} > t\right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) u(x)v^r(x)\, dx \tag{1.3}$$

holds for every positive t. In the same way that inequality (1.2) allows to obtain a different proof of the boundedness of M, the estimate above can be used to give an alternative proof of the boundedness of M_Φ, when Φ is a Young function of $L\log L$ type.

By virtue of the extension proved in [15] and of the discussed results above, a natural interrogant arises: does inequality (1.3) hold in the more general context $u \in A_1$ and $v^r \in A_\infty$? This fact is an improvement of a conjecture established in [2], which remained open until now.

In this paper we answer the question positively. We will be dealing with a family of Young functions with certain properties, as follows. Given $r \geq 1$ we say that a Young function belongs to the family $\mathfrak{F}_r$ if Φ is submultiplicative, has lower type r and satisfies the condition

$$\frac{\Phi(t)}{t^r} \leq C_0(\log t)^\delta, \qquad \text{for } t \geq t^*$$

for some constants $C_0 > 0$, $\delta \geq 0$ and $t^* \geq 1$. Concretely, our main result is the following.

Theorem 1 *Let $r \geq 1$ and $\Phi \in \mathfrak{F}_r$. If $u \in A_1$ and $v^r \in A_\infty$ then there exists a positive constant C such that the inequality*

$$uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{v(x)} > t\right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) u(x)v^r(x)\, dx,$$

holds for every positive t.

The key for the proof of this inequality is to combine some ideas that appear in [15] with a subtle Hölder inequality that allows to split the expression $\Phi(|f|v)$ into $\Phi(|f|)v^r$. The proof also follows the "principal cubes" decomposition introduced by Sawyer in [21].

Since $M_\Phi v \gtrsim v$ we can obtain, as a consequence of the theorem above, the following result.

Corollary 2 *Under the assumptions in Theorem 1 we have that*

$$uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{M_\Phi v(x)} > t\right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) u(x)v^r(x)\, dx.$$

This estimate will be useful to derive, as an application, mixed inequalities for certain fractional maximal operators. Notice that when $v \in A_1$ the operator S defined by Sawyer in [21] is equivalent to

$$\mathcal{T} f(x) = \frac{M(fv)(x)}{Mv(x)}.$$

For the case of M_Φ, it can be proven that $M_\Phi v \approx v$ when $v^r \in A_1$. Therefore inequality (1.3) can be rewritten as follows

$$uv^r \left(\left\{ x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{M_\Phi v(x)} > t \right\} \right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t} \right) u(x) v^r(x)\, dx. \tag{1.4}$$

It is not difficult to see that the operator Sf defined by Sawyer in [21] is bounded in $L^\infty(uv)$ under the assumptions on these weights. The same statement is true if we consider the operator $S_\Phi = M_\Phi(fv)/v$ that appears in Eq. 1.3 and the space $L^\infty(uv^r)$, but this is not clear when we have $v^r \in A_\infty$. However, if we modify the operator by considering the variant

$$\mathcal{T}_\Phi f(x) = \frac{M_\Phi(fv)(x)}{M_\Phi v(x)}$$

we have an alternative operator that is bounded in $L^\infty(uv^r)$ when we only assume $v^r \in A_\infty$. This new operator seems to be a good extension for the case $v \in A_\infty$ since it is continuous in $L^\infty(uv^r)$, a property that will be useful later.

We consider now the modified operator $\mathcal{T}_\Phi$, which coincides with S_Φ when v^r belongs to A_1. The advantage of dealing with it is that we can give a version of Corollary 2 where we weaken a bit the assumption on Φ. That is, if Ψ is an arbitrary Young function that behaves like one in the family $\mathfrak{F}_r$ but only for large t, we have the following result.

Corollary 3 *Let $r \geq 1$, $\Phi \in \mathfrak{F}_r$, $u \in A_1$ and $v^r \in A_\infty$. Let Ψ be a Young function that verifies $\Psi(t) \approx \Phi(t)$, for every $t \geq t^* \geq 0$. Then, there exist two positive constants C_1 and C_2 such that the inequality*

$$uv^r \left(\left\{ x \in \mathbb{R}^n : \frac{M_\Psi(fv)(x)}{M_\Psi v(x)} > t \right\} \right) \leq C_1 \int_{\mathbb{R}^n} \Psi\left(\frac{C_2|f(x)|}{t} \right) u(x) v^r(x)\, dx$$

holds for every $t > 0$.

Corollary 3 will play a fundamental role in obtaining mixed inequalities for a fractional version of the operators considered above. That is, as an important application of these results we can give mixed estimates for the *generalized fractional maximal operator* $M_{\gamma,\Phi}$, defined by

$$M_{\gamma,\Phi} f(x) = \sup_{Q \ni x} |Q|^{\gamma/n} \|f\|_{\Phi,Q},$$

where $0 < \gamma < n$ and Φ is a Young function.

When we consider the family of functions $\Phi(t) = t^r(1 + \log^+ t)^\delta$, for $r \geq 1$ and $\delta \geq 0$, the operator $M_{\gamma,\Phi}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ if and only if $w^r \in A_{p/r,q/r}$, for $0 < \gamma < n/r$, $r < p < n/\gamma$ and $1/q = 1/p - \gamma/n$. This result was set and proved in [1], and generalizes the strong (p,q) type of M_γ between Lebesgue spaces when we consider $r = 1$ and $\delta = 0$. As it occurs with M_γ, this estimate fails in the limit case $p = r$. In [12] it was proved an endpoint weak type estimate for this operator in the setting of spaces of homogeneous type. The corresponding inequality for the Euclidean case is

$$w\left(\left\{ x \in \mathbb{R}^n : M_{\gamma,\Phi} f(x) > t \right\} \right) \leq C\varphi\left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t} \right) \Psi(Mw(x))\, dx \right), \tag{1.5}$$

where $\Psi(t) = t^{n-r\gamma}(1 + \log^+ t^{-r\gamma/n})^\delta$ and $\varphi(t) = (t(1 + \log^+ t^{r\gamma/n})^\delta)^{n/(n-r\gamma)}$.

In [6] we study mixed inequalities for this operator when a power of the weight v belongs to A_1. These inequalities arose from the fact that those mixed inequalities allow to obtain an alternative proof of the continuity properties of $M_{\gamma,\Phi}$ discussed above (see Section 4.3 in [6] for further details). The proof relies on a pointwise estimate that relates the operators $M_{\gamma,\Phi}$ and M_Φ (see Proposition 10 in Section 3) and it is a generalization of a Hedberg type inequality used in [5] to give mixed estimates for the fractional maximal operator M_γ, when $0 < \gamma < n$.

The following two theorems contain mixed inequalities for $M_{\gamma,\Phi}$ for the cases $r < p < n/\gamma$ and the limiting case $p = r$, respectively. In the particular case in which the power of v belongs to A_1, both results were established and proved in [6].

Theorem 4 *Let $\Phi(t) = t^r(1 + \log^+ t)^\delta$, with $r \geq 1$ and $\delta \geq 0$. Let $0 < \gamma < n/r$, $r < p < n/\gamma$ and $1/q = 1/p - \gamma/n$. If $u \in A_1$ and $v^{q(1/p+1/r')} \in A_\infty$, then we have that*

$$uv^{q(1/p+1/r')}\left(\left\{x \in \mathbb{R}^n : \frac{M_{\gamma,\Phi}(fv)(x)}{M_\eta v(x)} > t\right\}\right)^{1/q} \leq C\left[\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{t}\right)^p u^{p/q}(x)(v(x))^{1+p/r'}\,dx\right]^{1/p},$$

where $\eta(t) = t^{q/p+q/r'}(1 + \log^+ t)^{n\delta/(n-r\gamma)}$.

Theorem 5 *Let $\Phi(t) = t^r(1 + \log^+ t)^\delta$, with $r \geq 1$ and $\delta \geq 0$. Let $0 < \gamma < n/r$ and $1/q = 1/r - \gamma/n$. If $u \in A_1$ and $v^q \in A_\infty$, then there exists a positive constant C such that*

$$uv^q\left(\left\{x \in \mathbb{R}^n : \frac{M_{\gamma,\Phi}(fv)(x)}{v(x)} > t\right\}\right) \leq \varphi\left(\int_{\mathbb{R}^n}\Phi_\gamma\left(\frac{|f(x)|}{t}\right)\Psi\left(u^{1/q}(x)v(x)\right)dx\right),$$

where $\varphi(t) = [t(1 + \log^+ t)^\delta]^{q/r}$, $\Psi(t) = t^r(1 + \log^+(t^{1-q/r}))^{n\delta/(n-r\gamma)}$ and $\Phi_\gamma(t) = \Phi(t)(1 + \log^+ t)^{\delta r\gamma/(n-r\gamma)}$.

Since $M_\eta v \gtrsim v$ when $\eta(t) = t^q(1 + \log^+ t)^{n\delta/(n-r\gamma)}$, as an immediate consequence of Theorem 5 we have that

$$uv^q\left(\left\{x \in \mathbb{R}^n : \frac{M_{\gamma,\Phi}(fv)(x)}{M_\eta v(x)} > t\right\}\right) \leq \varphi\left(\int_{\mathbb{R}^n}\Phi_\gamma\left(\frac{|f(x)|}{t}\right)\Psi\left(u^{1/q}(x)v(x)\right)dx\right).$$

Remark 1 These two last theorems are very important since they give good extensions to many results concerning to mixed inequalities.

- In [5] we use mixed inequalities for M proved in [8] to give the corresponding estimates for the fractional maximal operator M_γ. A stronger version can be obtained if we use in the proof the mixed estimate for M given in [15]. However, this can only be done for the limiting case $p = 1$ and $q = n/(n - \gamma)$, since the proof for the remaining cases $1 < p < n/\gamma$ depends heavily on an auxiliary lemma which uses A_1 condition of $v^{q/p}$. We can use Theorem 4 to overcome this problem. Indeed, if we set $r = 1$ and $\delta = 0$ we have $\Phi(t) = t$. Therefore $M_{\gamma,\Phi} = M_\gamma$, and we obtain the corresponding extension of Theorem 1 in [5] for the case $v^{q/p} \in A_\infty$, for every $1 < p < n/\gamma$.
- If we assume $v^{q(1/p+1/r')} \in A_1$ in Theorem 4, then $M_\eta v \approx v$. In this case we precisely obtain Theorem 4.9 in [6]. If we furthermore set $r = 1$ and $\delta = 0$, then $M_\eta v = M_{q/p}v \approx v$. This recovers Theorem 1 in [5], for the case $v^{q/p} \in A_1$.

- Theorem 5 extends the limiting case $p = 1$ and $q = n/(n-\gamma)$ for M_γ in [5] when we set $\Phi(t) = t$.
- If we assume $v^q \in A_1$ in Theorem 5, we recover Theorem 4.11 in [6].
- When we take $v = 1$ in Theorem 5 we obtain an estimate similar to Eq. 1.5.

The remainder of this paper is organized as follows: in Section 2 we give the required preliminaries and basic definitions. Section 3 contains some auxiliary results that will be useful in the main proofs. In Section 4 we prove both Theorem 1 and Corollary 3. Finally, we prove Theorem 4 and Theorem 5 in Section 5, as an application of the main result.

2 Preliminaries and Definitions

We shall say that $A \lesssim B$ if there exists a positive constant C such that $A \le CB$. The constant C may change on each occurrence. If we have $A \lesssim B$ and $B \lesssim A$, this will be denoted as $A \approx B$.

Given a function φ, we will say that $f \in L^\varphi_{loc}$ if $\varphi(|f|)$ is locally integrable. In the case $\varphi(t) = t$, the corresponding space is the usual L^1_{loc}.

By a *weight* w we understand a function that is locally integrable, positive and finite in almost every x. Given $1 < p < \infty$, the A_p-Muckenhoupt class is defined to be the set of weights w that verify

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} \le C,$$

for some positive constant C and for every cube $Q \subseteq \mathbb{R}^n$. We shall consider cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. In the limiting case $p = 1$, we say that $w \in A_1$ if there exists a positive constant C such that for every cube Q

$$\frac{1}{|Q|}\int_Q w \le C \inf_Q w,$$

where $\inf_Q$ denotes the essential infimum of w in Q.

The smallest constants C for which the corresponding inequalities above hold are denoted by $[w]_{A_p}$, $1 \le p < \infty$ and called the characteristic A_p constants of w.

Finally, the A_∞ class is defined as the collection of all the A_p classes, that is, $A_\infty = \bigcup_{p\ge 1} A_p$. It is well known that the A_p classes are increasing on p, that is, if $p \le q$ then $A_p \subseteq A_q$. For further details and other properties of weights see [10] or [13].

There are many conditions that characterize A_∞ (see [11] for a detailed account). In this paper we will use the following one: $w \in A_\infty$ if there exist positive constants C and ε such that, for every cube $Q \subseteq \mathbb{R}^n$ and every measurable set $E \subseteq Q$ we have

$$\frac{w(E)}{w(Q)} \le C\left(\frac{|E|}{|Q|}\right)^\varepsilon,$$

where $w(E) = \int_E w$.

Every Muckenhoupt weight satisfies a *reverse Hölder condition*. That is, if $w \in A_p$ for some $1 \le p < \infty$, then there exist positive constants C and $s > 1$ that depend only on the dimension n, p and $[w]_{A_p}$, such that

$$\left(\frac{1}{|Q|}\int_Q w^s(x)\,dx\right)^{1/s} \le \frac{C}{|Q|}\int_Q w(x)\,dx$$

for every cube Q. We write $w \in \mathrm{RH}_s$ to indicate that the inequality above holds, and we denote by $[w]_{\mathrm{RH}_s}$ the smallest constant C associated to this condition. It is easy to see that $\mathrm{RH}_s \subseteq \mathrm{RH}_q$, for every $1 < q < s$.

Given a locally integrable function f, the *Hardy-Littlewood maximal operator* is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

We say that $\varphi : [0, \infty) \to [0, \infty]$ is a *Young function* if it is convex, increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ when $t \to \infty$. Given a Young function φ, the maximal operator M_φ is defined, for $f \in L^{\varphi}_{loc}$, by

$$M_\varphi f(x) = \sup_{Q \ni x} \|f\|_{\varphi, Q}\,,$$

where $\|f\|_{\varphi,Q}$ denotes the *Luxemburg type average* of the function f in the cube Q, defined by

$$\|f\|_{\varphi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \varphi\left(\frac{|f(y)|}{\lambda}\right) dy \le 1 \right\}.$$

Given a weight w, we can also consider the *weighted Luxemburg type average* $\|f\|_{\varphi,Q,w}$ to be defined as

$$\|f\|_{\varphi,Q,w} = \inf \left\{ \lambda > 0 : \frac{1}{w(Q)} \int_Q \varphi\left(\frac{|f(y)|}{\lambda}\right) w(y)\, dy \le 1 \right\}.$$

It is easy to check that from the definition above we have

$$\frac{1}{w(Q)} \int_Q \varphi\left(\frac{|f(y)|}{\|f\|_{\varphi,Q,w}}\right) w(y)\, dy \le 1.$$

When $w \in A_\infty$, the measure given by $d\mu(x) = w(x)\, dx$ is doubling. Thus, by following the same arguments as in the result of Krasnosel'skiĭ and Rutickiĭ ([14], see also [20]) we can get that

$$\|f\|_{\varphi,Q,w} \approx \inf_{\tau > 0} \left\{ \tau + \frac{\tau}{w(Q)} \int_Q \varphi\left(\frac{|f(x)|}{\tau}\right) w(x)\, dx \right\}. \tag{2.1}$$

A Young function φ is *submultiplicative* if there exists a positive constant C such that

$$\varphi(st) \le C\varphi(s)\varphi(t)$$

for every $s, t \ge 0$. We say φ has *lower type* p, $0 < p < \infty$ if there exists a positive constant C_p such that

$$\varphi(st) \le C_p s^p \varphi(t),$$

for every $0 < s \le 1$ and $t > 0$. Also, φ has *upper type* q, $0 < q < \infty$ if there exists a positive constant C_q such that

$$\varphi(st) \le C_q s^q \varphi(t),$$

for every $s \ge 1$ and $t > 0$. As an immediate consequence of these definitions we have that, if φ has lower type p then φ has lower type $\tilde{p}$, for every $0 < \tilde{p} < p$. Also, if φ has upper type q, then it has upper type $\tilde{q}$, for every $\tilde{q} > q$.

Given a function $\varphi : [0, \infty) \to [0, \infty]$ we define the *generalized inverse* of φ as

$$\varphi^{-1}(t) = \inf\{s \ge 0 : \Phi(s) \ge t\},$$

with the convention that $\inf \emptyset = \infty$.

The *generalized Hölder inequality* establishes that if φ, ψ and ϕ are Young functions satisfying

$$\psi^{-1}(t)\phi^{-1}(t) \lesssim \varphi^{-1}(t)$$

for every $t \geq t^* > 0$ then there exists a positive constant C such that

$$\|fg\|_{\varphi,Q} \leq C\|f\|_{\psi,Q}\|g\|_{\phi,Q}. \tag{2.2}$$

In this article we shall deal with Young functions of the type $\varphi(t) = t^r(1 + \log^+ t)^\delta$, where $r \geq 1$, $\delta \geq 0$ and $\log^+ t = \max\{0, \log t\}$. It is well known that this class of functions are submultiplicative, have a lower type r and have upper type q, for every $q > r$. Moreover, we have (see, for example, Proposition 1.18 in [6] or p.105 in [9]) that

$$\varphi^{-1}(t) \approx t^{1/r}(1 + \log^+ t)^{-\delta/r}. \tag{2.3}$$

The proof of the main result can be reduced to studying the dyadic version of the operator involved. By a *dyadic grid* $\mathcal{D}$ we understand a collection of cubes of $\mathbb{R}^n$ that satisfies the following properties:

(1) every cube Q in $\mathcal{D}$ has side length 2^k, for some $k \in \mathbb{Z}$;
(2) if $P \cap Q \neq \emptyset$ then $P \subseteq Q$ or $Q \subseteq P$;
(3) $\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\}$ is a partition of $\mathbb{R}^n$ for every $k \in \mathbb{Z}$, where $\ell(Q)$ denotes the side length of Q.

The dyadic maximal operator $M_{\varphi,\mathcal{D}}$ associated to the Young function φ and to the dyadic grid $\mathcal{D}$ is defined in a similar way as above, but the supremum is taken over all cubes in $\mathcal{D}$. It can be shown that

$$M_\Phi f(x) \leq C \sum_{i=1}^{3^n} M_{\Phi,\mathcal{D}^{(i)}} f(x), \tag{2.4}$$

where $\mathcal{D}^{(i)}$ are fixed dyadic grids.

3 Auxiliary Results

The following lemma gives us the decomposition of level sets of dyadic generalized maximal operators into dyadic cubes. A proof of this result can be found in ([2], Lemma 2.1); see also Proposition A.1, p.237, in [9].

Lemma 6 *Given $\lambda > 0$, a bounded function f with compact support, a dyadic grid $\mathcal{D}$ and a Young function φ, there exists a family of maximal cubes $\{Q_j\}$ of $\mathcal{D}$ that satisfies*

$$\{x \in \mathbb{R}^n : M_{\varphi,\mathcal{D}} f(x) > \lambda\} = \bigcup_j Q_j,$$

and $\|f\|_{\varphi,Q_j} > \lambda$ for every j.

The next lemma is purely technical and gives a fundamental fact that will be crucial in the main proof.

Lemma 7 *Let f be the function defined in $[0, \infty)$ by*

$$f(x) = \begin{cases} \left(1 + \frac{1}{x}\right)^{\frac{x}{1+x}} & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then we have that $1 \leq f(x) \leq e^{1/e}$, *for every* $x \geq 0$.

The following proposition establishes that if Ψ and Φ are equivalent Young functions for t large, then they have equivalent Luxemburg norm on every cube Q. As a consequence, we have that $M_\Psi \approx M_\Phi$.

Proposition 8 *Let* Φ *and* Ψ *be Young functions that verify* $\Phi(t) \approx \Psi(t)$ *for every* $t \geq t_0 \geq 0$. *Then* $\|\cdot\|_{\Phi,Q} \approx \|\cdot\|_{\Psi,Q}$, *for every cube* Q.

Proof Fix $f \in L^\Phi_{loc}$. By hypothesis there exist two positive constants C_1 and C_2 such that

$$C_1\Psi(t) \leq \Phi(t) \leq C_2\Psi(t),$$

for $t \geq t_0$. Thus, given $\lambda > 0$ we have that

$$\begin{aligned}\frac{1}{|Q|}\int_Q \Phi\left(\frac{|f|}{\lambda}\right) &= \frac{1}{|Q|}\int_{Q\cap\{|f|\leq t_0\lambda\}} \Phi\left(\frac{|f|}{\lambda}\right) + \frac{1}{|Q|}\int_{Q\cap\{|f|>t_0\lambda\}} \Phi\left(\frac{|f|}{\lambda}\right)\\ &\leq \Phi(t_0) + \frac{C_2}{|Q|}\int_Q \Psi\left(\frac{|f|}{\lambda}\right).\end{aligned}$$

If we set $\lambda = \|f\|_{\Psi,Q}$ then

$$\frac{1}{|Q|}\int_Q \Phi\left(\frac{|f|}{\lambda}\right) \leq \Phi(t_0) + C_2,$$

which implies that $\|f\|_{\Phi,Q} \leq \max\{1, \Phi(t_0) + C_2\}\|f\|_{\Psi,Q}$. By interchanging the roles of Φ and Ψ we can obtain the other inequality. $\square$

The next result gives a version of Jensen inequality for Luxemburg averages.

Lemma 9 *Let* Φ *be a Young function,* $f \in L^\Phi_{loc}$ *and* $r \geq 1$. *Then there exists a positive constant* C *such that for every cube* Q

$$\|f\|^r_{\Phi,Q} \leq C\|f^r\|_{\Phi,Q}.$$

Proof Notice that if $t \geq 1$, then $\Phi(t^{1/r}) \leq \Phi(t)$. Picking $\lambda = \|f^r\|^{1/r}_{\Phi,Q}$ we can estimate

$$\begin{aligned}\frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right) dy &= \frac{1}{|Q|}\int_{Q\cap\{|f|\leq\lambda\}} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy + \frac{1}{|Q|}\int_{Q\cap\{|f|>\lambda\}} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy\\ &\leq \Phi(1) + \frac{1}{|Q|}\int_{Q\cap\{|f|>\lambda\}} \Phi\left(\left(\frac{|f(y)|^r}{\|f^r\|_{\Phi,Q}}\right)^{1/r}\right) dy\\ &\leq \Phi(1) + \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|^r}{\|f^r\|_{\Phi,Q}}\right) dy\\ &\leq \Phi(1) + 1,\end{aligned}$$

and therefore

$$\|f\|_{\Phi,Q} \leq C\|f^r\|^{1/r}_{\Phi,Q}.$$

$\square$

The following proposition provides a pointwise estimate between the operators $M_{\gamma,\Phi}$ and M_ξ, where the functions involved are related in certain way. This result can be seen as a Hedberg type estimate for generalized maximal operators.

Proposition 10 *Let* $0 < \gamma < n$, $1 \le p < n/\gamma$ *and* $1/q = 1/p - \gamma/n$. *Let* Φ, ξ *be Young functions verifying* $t^{\gamma/n}\xi^{-1}(t) \le C\Phi^{-1}(t)$, *for every* $t \ge t_0 \ge 0$. *Then, for every nonnegative functions* w *and* $f \in L^p$ *we have that*

$$M_{\gamma,\Phi}\left(\frac{f}{w}\right)(x) \le CM_{\xi}\left(\frac{f^{p/q}}{w}\right)(x)\left(\int_{\mathbb{R}^n} f^p(y)\,dy\right)^{\gamma/n},$$

for every $x \in \mathbb{R}^n$.

Proof Define $s = 1 + q/p'$ and let $g = f^{p/s}w^{-q/s}$. Then

$$\frac{f}{w} = g^{s/p}w^{q/p-1}.$$

Fix x and a cube Q such that $x \in Q$. By using generalized Hölder inequality (2.2) we obtain

$$\begin{aligned}
|Q|^{\gamma/n}\left\|\frac{f}{w}\right\|_{\Phi,Q} &= |Q|^{\gamma/n}\left\|g^{s/p}w^{q/p-1}\right\|_{\Phi,Q}\\
&= |Q|^{\gamma/n}\left\|g^{1-\gamma/n}g^{s/p+\gamma/n-1}w^{q\gamma/n}\right\|_{\Phi,Q}\\
&\le C|Q|^{\gamma/n}\left\|g^{1-\gamma/n}\right\|_{\xi,Q}\left\|g^{s/p+\gamma/n-1}w^{q\gamma/n}\right\|_{L^{n/\gamma},Q}\\
&= C\left\|f^{p/q}w^{-1}\right\|_{\xi,Q}\left(\int_Q f^p(y)\,dy\right)^{\gamma/n}\\
&\le CM_{\xi}\left(\frac{f^{p/q}}{w}\right)(x)\left(\int_{\mathbb{R}^n} f^p(y)\,dy\right)^{\gamma/n}.
\end{aligned}$$

□

4 Proof of the Main Result

We devote this section to the proof of Theorem 1 and its corollary. We shall present and prove some auxiliary results that will be useful to this purpose. Recall we are dealing with a function $\Phi \in \mathfrak{F}_r$, where $r \ge 1$ is given. By Eq. 2.4, it will be enough to prove that

$$uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_{\Phi,\mathcal{D}}(fv)(x)}{v(x)} > t\right\}\right) \le C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{t}\right)u(x)v^r(x)\,dx,$$

where $\mathcal{D}$ is a given dyadic grid. We can also assume that $t = 1$ and that $g = |f|v$ is a bounded function with compact support. Then, for a fixed number $a > 2^n$, we can write

$$\begin{aligned}
uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_{\Phi,\mathcal{D}}(fv)(x)}{v(x)} > 1\right\}\right) &= \sum_{k\in\mathbb{Z}} uv^r\left(\left\{x : \frac{M_{\Phi,\mathcal{D}}g(x)}{v(x)} > 1, a^k < v \le a^{k+1}\right\}\right)\\
&= \sum_{k\in\mathbb{Z}} uv^r(E_k).
\end{aligned}$$

For every $k \in \mathbb{Z}$ we consider the set

$$\Omega_k = \{x \in \mathbb{R}^n : M_{\Phi,\mathcal{D}}g(x) > a^k\},$$

and by virtue of Lemma 6, there exists a collection of dyadic cubes $\{Q_j^k\}_j$ that satisfies

$$\Omega_k = \bigcup_j Q_j^k,$$

and $\|g\|_{\Phi,Q_j^k} > a^k$ for each j. By maximality, we have

$$a^k < \|g\|_{\Phi,Q_j^k} \leq 2^n a^k, \qquad \text{for every } j. \tag{4.1}$$

We proceed now to split for every $k \in \mathbb{Z}$, as in [15], the obtained cubes in different classes. If $\ell \in \mathbb{N}_0$, we set

$$\Lambda_{\ell,k} = \left\{ Q_j^k : a^{(k+\ell)r} \leq \frac{1}{|Q_j^k|}\int_{Q_j^k} v^r < a^{(k+\ell+1)r} \right\},$$

and also

$$\Lambda_{-1,k} = \left\{ Q_j^k : \frac{1}{|Q_j^k|}\int_{Q_j^k} v^r < a^{kr} \right\}.$$

The next step is to split every cube in the family $\Lambda_{-1,k}$. Fixed $Q_j^k \in \Lambda_{-1,k}$, we perform the Calderón-Zygmund decomposition of the function $v^r\mathcal{X}_{Q_j^k}$ at height a^{kr}. Then we obtain, for each k, a collection of maximal cubes, $\left\{Q_{j,i}^k\right\}_i$, contained in Q_j^k and which satisfy

$$a^{kr} < \frac{1}{|Q_{j,i}^k|}\int_{Q_{j,i}^k} v^r \leq 2^n a^{kr}, \qquad \text{for every } i. \tag{4.2}$$

We now define, for every $\ell \geq 0$ the sets

$$\Gamma_{\ell,k} = \left\{ Q_j^k \in \Lambda_{\ell,k} : \left| Q_j^k \cap \left\{ x : a^k < v \leq a^{k+1} \right\}\right| > 0 \right\},$$

and also

$$\Gamma_{-1,k} = \left\{ Q_{j,i}^k : Q_j^k \in \Lambda_{-1,k} \text{ and } \left| Q_{j,i}^k \cap \left\{ x : a^k < v \leq a^{k+1} \right\}\right| > 0 \right\}.$$

Since $E_k \subseteq \Omega_k$, we can estimate

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} uv^r(E_k) &= \sum_{k\in\mathbb{Z}} uv^r(E_k\cap\Omega_k) \\
&= \sum_{k\in\mathbb{Z}}\sum_j uv^r(E_k\cap Q_j^k) \\
&\leq \sum_{k\in\mathbb{Z}}\sum_{\ell\geq 0}\sum_{Q_j^k\in\Gamma_{\ell,k}} a^{(k+1)r}u(E_k\cap Q_j^k) + \sum_{k\in\mathbb{Z}}\sum_{i:Q_{j,i}^k\in\Gamma_{-1,k}} a^{(k+1)r}u(Q_{j,i}^k).
\end{aligned}$$

If we can prove that given a negative integer N, there exists a positive constant C, independent of N, for which the following estimate

$$\sum_{k\geq N}\sum_{\ell\geq 0}\sum_{Q_j^k\in\Gamma_{\ell,k}} a^{(k+1)r}u(E_k\cap Q_j^k) + \sum_{k\geq N}\sum_{i:Q_{j,i}^k\in\Gamma_{-1,k}} a^{(k+1)r}u(Q_{j,i}^k) \leq C\int_{\mathbb{R}^n}\Phi(|f|)\,uv^r \tag{4.3}$$

holds, then the proof would be completed by letting $N \to -\infty$.

In order to prove Eq. 4.3 we need some auxiliary results. The two following lemmas deal with the family of cubes defined above. Both were set and proved in [15]. However, we include the proof of the second one since there are slight changes because we work with Luxemburg averages.

Lemma 11 ([15], Lemma 2.3) *If $\ell \geq 0$ and $Q_j^k \in \Gamma_{\ell,k}$, then there exist two positive constants c_1 and c_2, depending only on u and v^r such that*

$$u(E_k \cap Q_j^k) \leq c_1 e^{-c_2 \ell r} u(Q_j^k). \tag{4.4}$$

Lemma 12 *If Q is a cube in $\Gamma = \cup_{\ell \geq -1} \cup_{k \geq N} \Gamma_{\ell,k}$, then there exists a positive constant C, independent of Q, such that*

$$\left| \bigcup_{Q' \in \Gamma, Q' \subsetneq Q} Q' \right| \leq C|Q|.$$

Proof We shall first prove that if $Q_j^k \subsetneq Q_s^t$ or $Q_j^k \subsetneq Q_{s,m}^t$ or $Q_{j,i}^k \subsetneq Q_s^t$ or $Q_{j,i}^k \subsetneq Q_{s,m}^t$, then $k > t$. By maximality, for the first case we have

$$a^t < \|g\|_{\Phi, Q_s^t} \leq a^k,$$

from where we easily deduce that $k > t$. The second case can be reduced to the first, since $Q_{s,m}^t \subsetneq Q_s^t$. For the third case, notice that $Q_j^k \neq Q_s^t$. Therefore, we must have $Q_s^t \subsetneq Q_j^k$ or $Q_j^k \subsetneq Q_s^t$. If the first condition held, we would have $t > k$. Then

$$\frac{1}{|Q_s^t|} \int_{Q_s^t} v^r \mathcal{X}_{Q_j^k} = \frac{1}{|Q_s^t|} \int_{Q_s^t} v^r \geq a^{tr} > a^{kr},$$

and $Q_{j,i}^k \subsetneq Q_s^t$. This is absurd because $Q_{j,i}^k$ is a maximal cube that verifies

$$\frac{1}{|Q_{j,i}^k|} \int_{Q_{j,i}^k} v^r > a^{kr}.$$

Then we must have $Q_j^k \subsetneq Q_s^t$ and this implies that $k > t$. Finally, the fourth case follows from the third since $Q_{s,m}^t \subsetneq Q_s^t$.

With this fact in mind consider a cube in Γ, say, Q_s^t. We want to estimate

$$\left| \bigcup_{Q \in \Gamma, Q \subsetneq Q_s^t} Q \right|.$$

Note that if $Q \subsetneq Q_s^t$, then the level of Q is greater than t. Therefore

$$\left| \bigcup_{Q \in \Gamma, Q \subsetneq Q_s^t} Q \right| \leq \sum_{k > t} \sum_j |Q_j^k|.$$

Since $a^k < \|g\|_{\Phi, Q_j^k}$ we have that

$$1 < \frac{1}{|Q_j^k|} \int_{Q_j^k} \Phi\left(\frac{g}{a^k}\right), \text{ or equivalently } |Q_j^k| < \int_{Q_j^k} \Phi\left(\frac{g}{a^k}\right).$$

On the other hand, since $\|g\|_{\Phi, Q_s^t} \leq 2^n a^t$ we have

$$\frac{1}{|Q_s^t|} \int_{Q_s^t} \Phi\left(\frac{g}{2^n a^t}\right) \leq 1, \text{ or equivalently } \int_{Q_s^t} \Phi\left(\frac{g}{2^n a^t}\right) \leq |Q_s^t|.$$

By combining these inequalities together with the convexity of Φ we obtain

$$\begin{aligned}\sum_{k>t}\sum_{j}|Q_j^k| &< \sum_{k>t}\sum_{j}\int_{Q_j^k}\Phi\left(\frac{g}{a^k}\right)\\ &\leq \sum_{k>t}\sum_{j}2^n a^{t-k}\int_{Q_j^k}\Phi\left(\frac{g}{2^n a^t}\right)\\ &\leq 2^n\sum_{k>t}a^{t-k}\int_{Q_s^t}\Phi\left(\frac{g}{2^n a^t}\right)\\ &\leq 2^n|Q_s^t|\sum_{k>t}a^{t-k}\\ &= \frac{2^n}{a-1}|Q_s^t|.\end{aligned}$$

□

Proof of Theorem 1 We shall write some parts of the proof as claims, which will be proved at the end, for the sake of clearness. Recall that we have to estimate the two quantities

$$A_N := \sum_{k\geq N}\sum_{\ell\geq 0}\sum_{Q_j^k\in\Gamma_{\ell,k}} a^{(k+1)r}u(E_k\cap Q_j^k)$$

and

$$B_N := \sum_{k\geq N}\sum_{i:Q_{j,i}^k\in\Gamma_{-1,k}} a^{(k+1)r}u(Q_{j,i}^k)$$

by $C\int_{\mathbb{R}^n}\Phi(|f|)\,uv^r$, with C independent of N.

We shall start with the estimate of A_N. Fix $\ell\geq 0$ and let $\Delta_\ell = \cup_{k\geq N}\Gamma_{\ell,k}$. We define a sequence of sets recursively as follows:

$$P_0^\ell = \{Q : Q \text{ is maximal in } \Delta_\ell \text{ in the sense of inclusion}\}$$

and for $m\geq 0$ given we say that $Q_j^k\in P_{m+1}^\ell$ if there exists a cube Q_s^t in P_m^ℓ which verifies

$$\frac{1}{|Q_j^k|}\int_{Q_j^k}u > \frac{2}{|Q_s^t|}\int_{Q_s^t}u \tag{4.5}$$

and it is maximal in this sense, that is,

$$\frac{1}{|Q_{j'}^{k'}|}\int_{Q_j^k}u \leq \frac{2}{|Q_s^t|}\int_{Q_s^t}u \tag{4.6}$$

for every $Q_j^k\subsetneq Q_{j'}^{k'}\subsetneq Q_s^t$.

Let $P^\ell = \cup_{m\geq 0}P_m^\ell$, the set of principal cubes in Δ_ℓ. By applying Lemma 11 and the definition of $\Lambda_{\ell,k}$ we have that

$$\begin{aligned}\sum_{k\geq N}\sum_{\ell\geq 0}\sum_{Q_j^k\in\Gamma_{\ell,k}} a^{(k+1)r}u(E_k\cap Q_j^k) &\leq \sum_{k\geq N}\sum_{\ell\geq 0}\sum_{Q_j^k\in\Gamma_{\ell,k}} c_1 a^{(k+1)r}e^{-c_2\ell r}u(Q_j^k)\\ &\leq \sum_{\ell\geq 0}c_1 e^{-c_2\ell r}a^{r(1-\ell)}\sum_{k\geq N}\sum_{Q_j^k\in\Gamma_{\ell,k}}\frac{v^r(Q_j^k)}{|Q_j^k|}u(Q_j^k).\end{aligned}$$

Let us sort the inner double sum in a more convenient way. We define

$$\mathcal{A}_{(t,s)}^{\ell}=\left\{Q_j^k\in\bigcup_{k\geq N}\Gamma_{\ell,k}:Q_j^k\subseteq Q_s^t \text{ and } Q_s^t \text{ is the smallest cube in } P^{\ell} \text{ that contains it}\right\}.$$

That is, every $Q_j^k\in\mathcal{A}_{(t,s)}^{\ell}$ is not a principal cube, unless $Q_j^k=Q_s^t$. Recall that $v^r\in A_\infty$ implies that there exist two positive constants C and ε verifying

$$\frac{v^r(E)}{v^r(Q)}\leq C\left(\frac{|E|}{|Q|}\right)^{\varepsilon}, \tag{4.7}$$

for every cube Q and every measurable set E of Q.

By using Eq. 4.6 and Lemma 12 we have that

$$\begin{aligned}\sum_{k\geq N}\sum_{Q_j^k\in\Gamma_{\ell,k}}\frac{v^r(Q_j^k)}{|Q_j^k|}u(Q_j^k)&=\sum_{Q_s^t\in P^\ell}\sum_{(k,j):Q_j^k\in\mathcal{A}_{(t,s)}^{\ell}}\frac{u(Q_j^k)}{|Q_j^k|}v^r(Q_j^k)\\&\leq 2\sum_{Q_s^t\in P^\ell}\frac{u(Q_s^t)}{|Q_s^t|}\sum_{(k,j):Q_j^k\in\mathcal{A}_{(t,s)}^{\ell}}v^r(Q_j^k)\\&\leq C\sum_{Q_s^t\in P^\ell}\frac{u(Q_s^t)}{|Q_s^t|}v^r(Q_s^t)\left(\frac{\left|\bigcup_{(k,j):Q_j^k\in\mathcal{A}_{(t,s)}^{\ell}}Q_j^k\right|}{|Q_s^t|}\right)^{\varepsilon}\\&\leq C\sum_{Q_s^t\in P^\ell}\frac{u(Q_s^t)}{|Q_s^t|}v^r(Q_s^t).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k\geq N}\sum_{\ell\geq 0}\sum_{Q_j^k\in\Gamma_{\ell,k}}a^{(k+1)r}u(E_k\cap Q_j^k)&\leq C\sum_{\ell\geq 0}e^{-c_2\ell r}a^{-\ell r}\sum_{Q_s^t\in P^\ell}\frac{v^r(Q_s^t)}{|Q_s^t|}u(Q_s^t)\\&\leq C\sum_{\ell\geq 0}e^{-c_2\ell r}\sum_{Q_s^t\in P^\ell}a^{tr}u(Q_s^t).\end{aligned}$$

Claim 1 Given $\ell\geq 0$ and $Q_j^k\in\bigcup_{k\geq N}\Gamma_{\ell,k}$, there exists a positive constant C, independent of ℓ, such that

$$a^{kr}\leq\frac{C}{|Q_j^k|}\int_{Q_j^k}\Phi\left(|f(x)|\right)v^r(x)\,dx. \tag{4.8}$$

Using this claim, we obtain

$$\begin{aligned}&\sum_{k\geq N}\sum_{\ell\geq 0}\sum_{Q_j^k\in\Gamma_{\ell,k}}a^{(k+1)r}u(E_k\cap Q_j^k)\\&\leq C\sum_{\ell\geq 0}e^{-c_2\ell r}\sum_{Q_s^t\in P^\ell}\frac{u(Q_s^t)}{|Q_s^t|}\int_{Q_s^t}\Phi\left(|f|\right)v^r\end{aligned}$$

$$= C\sum_{\ell\geq 0} e^{-c_2\ell r}\int_{\mathbb{R}^n}\Phi\left(|f(x)|\right)v^r(x)\left(\sum_{Q_s^t\in P^\ell}\frac{u(Q_s^t)}{|Q_s^t|}\mathcal{X}_{Q_s^t}(x)\right)dx$$

$$= C\sum_{\ell\geq 0} e^{-c_2\ell r}\int_{\mathbb{R}^n}\Phi\left(|f(x)|\right)v^r(x)h_1(x)\,dx$$

Claim 2 There exists a positive constant C, independent of ℓ, that satisfies $h_1(x)\leq Cu(x)$.

With this claim at hand, we can obtain

$$A_N\leq C\int_{\mathbb{R}^n}\Phi\left(|f(x)|\right)u(x)v^r(x)\,dx,$$

where C does not depend on N.

Let us center our attention on the estimate of B_N. Fix $0<\beta<\varepsilon$, where ε is the number appearing in Eq. 4.7. We shall build the set of principal cubes in $\Delta_{-1}=\bigcup_{k\geq N}\Gamma_{-1,k}$. Let

$$P_0^{-1}=\{Q: Q \text{ is a maximal cube in } \Delta_{-1} \text{ in the sense of inclusion}\}$$

and, recursively, we say that $Q_{j,i}^k\in P_{m+1}^{-1}$, $m\geq 0$, if there exists a cube $Q_{s,l}^t\in P_m^{-1}$ such that

$$\frac{1}{|Q_{j,i}^k|}\int_{Q_{j,i}^k}u>\frac{a^{(k-t)\beta r}}{|Q_{s,l}^t|}\int_{Q_{s,l}^t}u \tag{4.9}$$

and it is the biggest subcube of $Q_{s,l}^t$ that verifies this condition, that is

$$\frac{1}{|Q_{j',i'}^{k'}|}\int_{Q_{j',i'}^{k'}}u\leq\frac{a^{(k-t)\beta r}}{|Q_{s,l}^t|}\int_{Q_{s,l}^t}u \tag{4.10}$$

if $Q_{j,i}^k\subsetneq Q_{j',i'}^{k'}\subsetneq Q_{s,l}^t$. Let $P^{-1}=\bigcup_{m\geq 0}P_m^{-1}$, the set of principal cubes in Δ_{-1}. Similarly as before, we define the set

$$\mathcal{A}_{(t,s,l)}^{-1}=\left\{Q_{j,i}^k\in\bigcup_{k\geq N}\Gamma_{-1,k}: Q_{j,i}^k\subseteq Q_{s,l}^t \text{ and } Q_{s,l}^t \text{ is the smallest cube in } P^{-1} \text{ that contains it}\right\}.$$

We can therefore estimate B_N as follows

$$\begin{aligned} B_N &\leq a^r\sum_{k\geq N}\sum_{i:Q_{j,i}^k\in\Gamma_{-1,k}}\frac{v^r(Q_{j,i}^k)}{|Q_{j,i}^k|}u(Q_{j,i}^k)\\ &\leq a^r\sum_{Q_{s,l}^t\in P^{-1}}\sum_{k,j,i:Q_{j,i}^k\in\mathcal{A}_{(t,s,l)}^{-1}}\frac{u(Q_{j,i}^k)}{|Q_{j,i}^k|}v^r(Q_{j,i}^k)\\ &\leq a^r\sum_{Q_{s,l}^t\in P^{-1}}\frac{u(Q_{s,l}^t)}{|Q_{s,l}^t|}\sum_{k\geq t}a^{(k-t)\beta r}\sum_{j,i:Q_{j,i}^k\in\mathcal{A}_{(t,s,l)}^{-1}}v^r(Q_{j,i}^k). \end{aligned}$$

Fixed $k\geq t$, observe that

$$\sum_{j,i:Q_{j,i}^k\in\mathcal{A}_{(t,s,l)}^{-1}}|Q_{j,i}^k|<\sum_{j,i:Q_{j,i}^k\in\mathcal{A}_{(t,s,l)}^{-1}}a^{-kr}v^r(Q_{j,i}^k)\leq a^{-kr}v^r(Q_{s,l}^t)\leq 2^n a^{(t-k)r}|Q_{s,l}^t|.$$

Combining this inequality with the A_∞ condition of v^r we have, for every $k \geq t$, that

$$\sum_{j,i:Q^k_{j,i}\in\mathcal{A}^{-1}_{(t,s,l)}} a^{(k-t)\beta r} v^r(Q^k_{j,i}) \leq C v^r(Q^t_{s,l}) \left(\frac{\sum_{j,i:Q^k_{j,i}\in\mathcal{A}^{-1}_{(t,s,l)}} |Q^k_{j,i}|}{|Q^t_{s,l}|}\right)^{\varepsilon} \leq C a^{(t-k)r\varepsilon}.$$

Thus,

$$\begin{aligned} B_N &\leq C \sum_{Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{|Q^t_{s,l}|} v^r(Q^t_{s,l}) \sum_{k\geq t} a^{(t-k)r(\varepsilon-\beta)} \\ &= C \sum_{Q^t_{s,l}\in P^{-1}} \frac{v^r(Q^t_{s,l})}{|Q^t_{s,l}|} u(Q^t_{s,l}) \\ &\leq C \sum_{Q^t_{s,l}\in P^{-1}} a^{tr} u(Q^t_{s,l}). \end{aligned}$$

Claim 3 If $Q^k_j \in \Lambda_{-1,k}$ then there exists a positive constant C such that

$$a^{kr} \leq \frac{C}{|Q^k_j|} \int_{Q^k_j} \Phi\left(|f(x)|\right) v^r(x)\, dx.$$

We can proceed now as follows

$$\begin{aligned} \sum_{k\geq N} \sum_{i:Q^k_{j,i}\in\Gamma_{-1,k}} a^{(k+1)r} u(Q^k_{j,i}) &\leq C \sum_{Q^t_{s,l}\in P^{-1}} a^{tr} u(Q^t_{s,l}) \\ &\leq C \sum_{Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{|Q^t_s|} \int_{Q^t_s} \Phi\left(|f(x)|\right) v^r(x)\, dx \\ &\leq C \int_{\mathbb{R}^n} \Phi\left(|f(x)|\right) v^r(x) \left[\sum_{Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{|Q^t_s|} \mathcal{X}_{Q^t_s}(x)\right] dx \\ &= C \int_{\mathbb{R}^n} \Phi\left(|f(x)|\right) v^r(x) h_2(x)\, dx. \end{aligned}$$

Claim 4 There exists a positive constant C, independent of N, that verifies $h_2(x) \leq C u(x)$, for almost every x.

This claim allows to obtain the desired estimate for B_N. This completes the proof. □

In order to conclude, we give the proof of the claims.

Proof of Claim 1 Fix $\ell \geq 0$ and a cube $Q^k_j \in \bigcup_{k\geq N} \Gamma_{\ell,k}$. We know that $\|g\|_{\Phi,Q^k_j} > a^k$ or, equivalently, $\left\|\frac{g}{a^k}\right\|_{\Phi,Q^k_j} > 1$. Denote with $A = \{x \in Q^k_j : v(x) \leq t^* a^k\}$ and $B = Q^k_j \setminus A$, where t^* is the number verifying that if $z \geq t^*$, then

$$\frac{\Phi(z)}{z^r} \leq C_0 \left(\log z\right)^{\delta}.$$

Then,

$$1 < \left\|\frac{g}{a^k}\right\|_{\Phi,Q_j^k} \leq \left\|\frac{g}{a^k}\mathcal{X}_A\right\|_{\Phi,Q_j^k} + \left\|\frac{g}{a^k}\mathcal{X}_B\right\|_{\Phi,Q_j^k} = I + II.$$

This inequality implies that either $I > 1/2$ or $II > 1/2$. If the first case holds, since $\Phi \in \mathfrak{F}_r$ we have that

$$\begin{aligned} 1 &< \frac{1}{|Q_j^k|}\int_A \Phi\left(\frac{2|f|v}{a^k}\right) \\ &\leq \frac{\Phi(2t^*)}{|Q_j^k|}\int_A \Phi(|f|)\left(\frac{v}{t^*a^k}\right)^r \\ &\leq \frac{C}{a^{kr}}\frac{1}{|Q_j^k|}\int_{Q_j^k} \Phi(|f|)\, v^r, \end{aligned}$$

and from here we can obtain

$$a^{kr} < \frac{C}{|Q_j^k|}\int_{Q_j^k} \Phi(|f|)\, v^r.$$

On the other hand, if $II > 1/2$ then again

$$\begin{aligned} 1 &< \frac{1}{|Q_j^k|}\int_B \Phi\left(\frac{2|f|v}{a^k}\right) \\ &\leq \Phi(2)C_0\frac{1}{|Q_j^k|}\int_B \Phi(|f|)\frac{v^r}{a^{kr}}\left(\log\left(\frac{v}{a^k}\right)\right)^\delta \mathcal{X}_B, \end{aligned}$$

since $\Phi \in \mathfrak{F}_r$. This implies that

$$a^{kr} \leq \frac{\Phi(2)C_0}{|Q_j^k|}\int_{Q_j^k} \Phi(|f|)\, v^r w_k,$$

where $w_k(x) = \left(\log\left(\frac{v(x)}{a^k}\right)\right)^\delta \mathcal{X}_B(x)$.

Since $v^r \in A_\infty$, there exists $s > 1$ such that $v^r \in \mathrm{RH}_s$. Let $\delta_0 = \max\{\delta/r, 1\}$ and fix $0 < \varepsilon < \varepsilon_0$, where

$$\varepsilon_0 \leq \min\left\{\frac{1}{\delta_0 s' a^{(\ell+1)r/s'}[v^r]_{\mathrm{RH}_s} - 1}, \frac{\log 2}{\log(2\Phi(2)C_0 e^{2/e}a^{\ell r})}\right\}.$$

Now we define $\gamma = 1 + \varepsilon$, therefore $\gamma' = 1 + 1/\varepsilon$. By applying Hölder's inequality with γ and γ' with respect to the measure $d\mu(x) = v^r(x)\,dx$, we get

$$a^{kr} < \Phi(2)C_0\left(\frac{v^r(Q_j^k)}{|Q_j^k|}\right)\left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k}[\Phi(|f|)]^\gamma v^r\right)^{1/\gamma}\left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k} w_k^{\gamma'} v^r\right)^{1/\gamma'}. \tag{4.11}$$

Let us analyze the third factor. From the well-known fact that $\log t \le \xi^{-1} t^{\xi}$ for every $t, \xi > 0$ and applying Hölder's inequality with s and s' we have

$$\begin{aligned}
&\left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k} w_k^{\gamma'} v^r\right)^{1/\gamma'}\\
&= \left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k} \left(\log\left(\frac{v}{a^k}\right)\right)^{\delta\gamma'} v^r\right)^{1/\gamma'}\\
&\le \left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k\cap B} \frac{\delta s'\gamma'}{r}\left(\frac{v}{a^k}\right)^{r/s'} v^r\right)^{1/\gamma'}\\
&\le \left(\delta_0 s'\gamma' \frac{|Q_j^k|}{v^r(Q_j^k)}\right)^{1/\gamma'}\left[\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}\frac{v^r}{a^{kr}}\right)^{1/s'}\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} v^{rs}\right)^{1/s}\right]^{1/\gamma'}\\
&\le \left[\delta_0 s'\gamma' \frac{|Q_j^k|}{v^r(Q_j^k)} a^{(\ell+1)r/s'}[v^r]_{\mathrm{RH}_s}\frac{v^r(Q_j^k)}{|Q_j^k|}\right]^{1/\gamma'}\\
&= \left[\delta_0 s'\gamma' a^{(\ell+1)r/s'}[v^r]_{\mathrm{RH}_s}\right]^{1/\gamma'}\\
&\le (\gamma')^{2/\gamma'}\\
&\le e^{2/e},
\end{aligned}$$

by virtue of the election for ε and Lemma 7.

Returning to Eq. 4.11 we have that

$$a^{kr} \le D\left(\frac{v^r(Q_j^k)}{|Q_j^k|}\right)\left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k}[\Phi(|f|)]^{\gamma} v^r\right)^{1/\gamma},$$

where $D = \Phi(2)C_0 e^{2/e}$. By denoting $\Psi(t) = t^{\gamma}$, we have that the second factor is $\|\Phi(f)\|_{\Psi, v^r, Q_j^k}$. Using Eq. 2.1 we have that for every $\tau > 0$

$$\begin{aligned}
a^{kr} &\le D\frac{v^r(Q_j^k)}{|Q_j^k|}\left\{\tau + \frac{\tau^{1-\gamma}}{v^r(Q_j^k)}\int_{Q_j^k}[\Phi(|f|)]^{\gamma} v^r\right\}\\
&\le D\tau a^{(k+\ell)r} + D\frac{\tau^{1-\gamma}}{|Q_j^k|}\int_{Q_j^k}[\Phi(|f|)]^{\gamma} v^r.
\end{aligned}$$

Pick $\tau = 1/(2Da^{\ell r})$ and observe that with this choice

$$D\tau a^{(k+\ell)r} = \frac{a^{kr}}{2}, \quad \text{and} \quad \tau^{1-\gamma} = (2Da^{\ell r})^{\varepsilon} \le 2,$$

by virtue of the definition of ε. Thus,

$$a^{kr} \le \frac{4D}{|Q_j^k|}\int_{Q_j^k}[\Phi(|f|)]^{\gamma} v^r,$$

for every $0 < \varepsilon < \varepsilon_0$, with D independent of ε. The dominate convergence theorem allows to conclude the thesis by letting $\varepsilon \to 0$. □

Proof of Claim 2 Fix $\ell \geq 0$ and $x \in \mathbb{R}^n$ such that $u(x) < \infty$. We shall consider a sequence of nested principal cubes in Δ_ℓ that contain x. Let $Q^{(0)}$ be the maximal cube (in the sense of inclusion) in P^ℓ that contains x. In general, given $Q^{(j)}$ we denote with $Q^{(j+1)}$ the maximal principal cube in $Q^{(j)}$ that contains x. This so-defined sequence has only a finite number of terms. If not, for every j we would have

$$\frac{1}{|Q^{(0)}|}\int_{Q^{(0)}} u \leq \frac{1}{2^j}\frac{1}{|Q^{(j)}|}\int_{Q^{(j)}} u \leq \frac{[u]_{A_1}}{2^j}u(x),$$

or equivalently

$$\frac{2^j}{|Q^{(0)}|}\int_{Q^{(0)}} u \leq [u]_{A_1}u(x),$$

and we would get a contradiction by letting $j \to \infty$. Therefore, x can only belong to a finite number $J = J(x)$ of these cubes. Thus,

$$\begin{aligned}
\sum_{Q_s^t\in P^\ell} \frac{u(Q_s^t)}{|Q_s^t|}\mathcal{X}_{Q_s^t}(x) &\leq \sum_{j=0}^{J}\frac{1}{|Q^{(j)}|}\int_{Q^{(j)}} u \\
&\leq \sum_{j=0}^{J} 2^{j-J}\frac{1}{|Q^{(J)}|}\int_{Q^{(J)}} u \\
&\leq [u]_{A_1}u(x)\sum_{j=0}^{J}2^{j-J} \\
&\leq [u]_{A_1}u(x)2^{-J}(2^{J+1}-1) \\
&\leq 2[u]_{A_1}u(x).
\end{aligned}$$

□

Proof of Claim 3 This proof is similar to the given for Claim 1, with some obvious changes since the average of v^r over Q_j^k is not equivalent to $a^{(\ell+k)r}$. Following the same notation as in Claim 1, since $\left\|\frac{g}{a^k}\right\|_{\Phi,Q_j^k} > 1$, we have that either $I > 1/2$ or $II > 1/2$. If $I > 1/2$, we obtain the thesis exactly in the same way as in this claim. On the other hand, if $II > 1/2$, we have that

$$a^{kr} \leq \frac{C}{|Q_j^k|}\int_{Q_j^k} \Phi\left(|f|\right) v^r w_k,$$

where $w_k = \left(\log\left(\frac{v}{a^k}\right)\right)^\delta \mathcal{X}_B$.

Fix $0 < \varepsilon < \varepsilon_0$, where

$$\varepsilon_0 \leq \frac{1}{[v^r]_{\mathrm{RH}_s}\delta_0 s' - 1}$$

and set $\gamma = 1 + \varepsilon$. We apply Hölder's inequality with γ and γ' with respect to v^r to obtain

$$a^{kr} \leq C\frac{v^r(Q_j^k)}{|Q_j^k|}\left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k}\left[\Phi\left(|f|\right)\right]^\gamma v^r\right)^{1/\gamma}\left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k}\left(\log\left(\frac{v}{a^k}\right)\right)^{\delta\gamma'} v^r\mathcal{X}_B\right)^{1/\gamma'} \tag{4.12}$$

Recall that Q_j^k satisfies $|Q_j^k|^{-1}\int_{Q_j^k} v^r < a^{kr}$. Thus, we can estimate the third factor as follows

$$\begin{aligned}
&\left(\frac{1}{v^r(Q_j^k)}\int_B \left(\log\left(\frac{v}{a^k}\right)\right)^{\delta\gamma'} v^r\right)^{1/\gamma'}\\
&\le \left(\frac{1}{v^r(Q_j^k)}\int_B \frac{\delta s'\gamma'}{r}\left(\frac{v}{a^k}\right)^{r/s'} v^r\right)^{1/\gamma'}\\
&\le \left[\frac{\delta_0 s'\gamma'|Q_j^k|}{v^r(Q_j^k)}\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}\frac{v^r}{a^{kr}}\right)^{1/s'}\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} v^{rs}\right)^{1/s}\right]^{1/\gamma'}\\
&\le \left([v^r]_{\mathrm{RH}_s}\delta_0 s'\gamma'\right)^{1/\gamma'}\\
&\le \left(\gamma'\right)^{2/\gamma'}\\
&\le e^{2/e},
\end{aligned}$$

from the choice for ε and Lemma 7. Returning to Eq. 4.12 we obtain

$$a^{kr} \le D\frac{v^r(Q_j^k)}{|Q_j^k|}\left(\frac{1}{v^r(Q_j^k)}\int_{Q_j^k}[\Phi(|f|)]^\gamma v^r\right)^{1/\gamma},$$

with $D = Ce^{2/e}$. Similarly as we did in the proof of Claim 1 we can conclude that

$$\begin{aligned}
a^{kr} &\le D\frac{v^r(Q_j^k)}{|Q_j^k|}\left\{\tau + \frac{\tau^{1-\gamma}}{v^r(Q_j^k)}\int_{Q_j^k}[\Phi(|f|)]^\gamma v^r\right\}\\
&\le D\tau a^{kr} + D\frac{\tau^{1-\gamma}}{|Q_j^k|}\int_{Q_j^k}[\Phi(|f|)]^\gamma v^r,
\end{aligned}$$

for every $\tau > 0$. Picking $\tau = 1/(2D)$ we get

$$D\tau a^{kr} = \frac{a^{kr}}{2}, \quad \text{y} \quad \tau^{1-\gamma} = (2D)^\varepsilon \le 2D.$$

Therefore,

$$a^{kr} \le \frac{4D^2}{|Q_j^k|}\int_{Q_j^k}[\Phi(|f|)]^\gamma v^r,$$

for every $0 < \varepsilon < \varepsilon_0$. Again, the constant D does not depend on ε. Letting $\varepsilon \to 0$ we obtain the thesis. □

Proof of Claim 4 Let us fix $x \in \mathbb{R}^n$ and assume that $u(x) < \infty$. For every level t, there exists at most one cube Q_s^t such that $x \in Q_s^t$. If this cube does exist, we denoted it by Q^t. Let $G = \{t : x \in Q^t\}$. Since $t \ge N$, G is bounded from below. Then there exists t_0, the minimum of G. We shall build a sequence of elements in G recursively: having chosen t_m, with $m \ge 0$, we pick t_{m+1} as the smallest element in G greater than t_m and that verifies

$$\frac{1}{|Q^{t_{m+1}}|}\int_{Q^{t_{m+1}}} u > \frac{2}{|Q^{t_m}|}\int_{Q^{t_m}} u. \tag{4.13}$$

Observe that if $t \in G$ y $t_m \leq t < t_{m+1}$, then

$$\frac{1}{|Q^t|}\int_{Q^t} u \leq \frac{2}{|Q^{t_m}|}\int_{Q^{t_m}} u. \tag{4.14}$$

This sequence has only a finite number of terms. Indeed, if it was not the case, we would have

$$[u]_{A_1}u(x) \geq \frac{1}{|Q^{t_m}|}\int_{Q^{t_m}} u > \frac{2^m}{|Q^{t_0}|}\int_{Q^{t_0}} u$$

for every $m \geq 0$. By letting $m \to \infty$ we would arrive to a contradiction. Then $\{t_m\} = \{t_m\}_{m=0}^M$. Denoting $\mathcal{F}_m = \{t \in G : t_m \leq t < t_{m+1}\}$, and using Eq. 4.14 we can write

$$h_2(x) = \sum_{Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{|Q^t_s|}\mathcal{X}_{Q^t_s}(x) \leq \sum_{m=0}^M \left(\frac{2}{|Q^{t_m}|}\int_{Q^{t_m}} u\right) \sum_{t\in\mathcal{F}_m}\sum_{s,l:Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{u(Q^t)}.$$

We shall prove that there exists a positive constant C, independent of m, such that

$$\sum_{t\in\mathcal{F}_m}\sum_{s,l:Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{u(Q^t)} \leq C. \tag{4.15}$$

If this inequality holds, we get that

$$\begin{aligned} h_2(x) &\leq 2C\sum_{m=0}^M \frac{1}{|Q^{t_m}|}\int_{Q^{t_m}} u \\ &\leq 2C\sum_{m=0}^M 2^{m-M}\frac{1}{|Q^{t_M}|}\int_{Q^{t_M}} u \\ &\leq 2C[u]_{A_1}u(x)2^{-M}\sum_{m=0}^M 2^m \\ &\leq 4C[u]_{A_1}u(x), \end{aligned}$$

which completes the proof of the claim. To finish, let us prove Eq. 4.15.

Fix $0 \leq m \leq M$ and observe that if $t = t_0$, then

$$\sum_{s,l:Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{u(Q^t)} \leq 1.$$

Let $t_m < t < t_{m+1}$. If $Q^{t_m}_{j,i} \cap Q^t_{s,l} \neq \emptyset$, then we must have $Q^t_{s,l} \subsetneq Q^{t_m}_{j,i}$, otherwise we would have $t_m > t$, a contradiction. Let $Q^{t'}_{j',i'}$ the smallest principal cube that contains $Q^{t_m}_{j,i}$ (this cube does exist because we are assuming $t > t_0$). By applying Eq. 4.9 and Eq. 4.10 we conclude that

$$\frac{1}{|Q^t_{s,l}|}\int_{Q^t_{s,l}} u > \frac{a^{(t-t')r\beta}}{|Q^{t'}_{j',i'}|}\int_{Q^{t'}_{j',i'}} u$$

and also

$$\frac{1}{|Q^{t_m}_{j,i}|}\int_{Q^{t_m}_{j,i}} u \leq \frac{a^{(t_m-t')r\beta}}{|Q^{t'}_{j',i'}|}\int_{Q^{t'}_{j',i'}} u.$$

Combining these two estimates with Eq. 4.14 we obtain that, for almost every $y \in Q^t_{s,l}$

$$\begin{aligned} u(y)[u]_{A_1} &\geq \frac{1}{|Q^t_{s,l}|}\int_{Q^t_{s,l}} u > \frac{a^{(t-t_m)r\beta}}{|Q^{t_m}_{j,i}|}\int_{Q^{t_m}_{j,i}} u \\ &\geq a^{(t-t_m)r\beta} \inf_{Q^{t_m}_{j,i}} u \geq a^{(t-t_m)r\beta} \inf_{Q^{t_m}} u \\ &\geq \frac{a^{(t-t_m)r\beta}}{[u]_{A_1}} \frac{1}{|Q^{t_m}|}\int_{Q^{t_m}} u \\ &\geq \frac{a^{(t-t_m)r\beta}}{2[u]_{A_1}} \frac{1}{|Q^{t}|}\int_{Q^{t}} u. \end{aligned}$$

Then

$$u(y) > \frac{a^{(t-t_m)r\beta}}{2[u]^2_{A_1}} \frac{1}{|Q^{t}|}\int_{Q^{t}} u =: \lambda.$$

Since $u \in A_1 \subseteq A_\infty$, there exist two positive constants C and ν for which the inequality

$$\frac{u(E)}{u(Q)} \leq C\left(\frac{|E|}{|Q|}\right)^\nu,$$

holds for every cube Q and every measurable subset E of Q. Therefore,

$$\begin{aligned} \sum_{s,l:Q^t_{s,l}\in P^{-1}} u(Q^t_{s,l}) &\leq \frac{u(\{y \in Q^t : u(y) > \lambda\})}{u(Q^t)} u(Q^t) \\ &\leq C\left(\frac{|\{y \in Q^t : u(y) > \lambda\}|}{|Q^t|}\right)^\nu u(Q^t) \\ &\leq Cu(Q^t)\left(\frac{1}{\lambda|Q^t|}\int_{Q^t} u\right)^\nu \\ &= C\left(2[u]^2_{A_1} a^{(t_m-t)r\beta}\right)^\nu u(Q^t). \end{aligned}$$

If $t = t_m$, we have $Q^t_{s,l} = Q^{t_m}_{j,i}$ and in this case

$$\begin{aligned} u(y)[u]_{A_1} &\geq \frac{1}{|Q^t_{s,l}|}\int_{Q^t_{s,l}} u \geq \inf_{Q^{t_m}_{j,i}} u \\ &\geq \inf_{Q^{t_m}} u \geq \frac{1}{[u]_{A_1}|Q^{t_m}|}\int_{Q^{t_m}} u \\ &\geq \frac{1}{2[u]_{A_1}|Q^t|}\int_{Q^t} u, \end{aligned}$$

which is the corresponding estimate obtained above, with $t = t_m$. Thus,

$$\begin{aligned} \sum_{t\in\mathcal{F}_m}\sum_{s,l:Q^t_{s,l}\in P^{-1}} \frac{u(Q^t_{s,l})}{u(Q^t)} &\leq \sum_{t\geq t_m}\left(2[u]_{A_1} a^{(t_m-t)r\beta}\right)^\nu \\ &\leq C\sum_{t\geq t_m} a^{(t_m-t)r\beta\nu} \\ &= C, \end{aligned}$$

which proves Eq. 4.15. □

In order to prove Corollary 3 we need the following result. A proof can be found in [3].

Lemma 13 *Let μ be a measure, T a sub-additive operator, and φ a Young function. Assume that*

$$\mu(\{x : |Tf(x)| > t\}) \le C \int_{\mathbb{R}^n} \varphi\left(\frac{c|f(x)|}{t}\right) d\mu(x),$$

for some positive constants C and c, and every $t > 0$. Also assume that $\|Tf\|_{L^\infty(\mu)} \le C_0 \|f\|_{L^\infty(\mu)}$. Then

$$\mu\left(\{x : |Tf(x)| > t\}\right) \le C \int_{\{x:|f(x)|>t/(2C_0)\}} \varphi\left(\frac{2c|f(x)|}{t}\right) d\mu(x).$$

Proof of Corollary 3 The equivalence between Φ and Ψ imply that there exist positive constants A and B such that

$$A\Psi(t) \le \Phi(t) \le B\Psi(t),$$

for $t \ge t^*$. Proposition 8 establishes that there exist two positive constants D and E such that

$$DM_\Phi(fv)(x) \le M_\Psi(fv)(x) \le EM_\Phi(fv)(x),$$

for almost every x. By applying Corollary 2 and setting $c_1 = E \max\{\Phi(t^*) + B, 1\}$ we have that

$$\begin{aligned} uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_\Psi(fv)(x)}{M_\Psi v(x)} > t\right\}\right) &\le uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_\Phi(fv)(x)}{M_\Phi v(x)} > \frac{t}{c_1}\right\}\right) \\ &\le C \int_{\mathbb{R}^n} \Phi\left(\frac{c_1|f|}{t}\right) uv^r. \end{aligned}$$

Observe that

$$\|\mathcal{T}_\Psi f\|_{L^\infty} = \left\| \frac{M_\Psi(fv)}{M_\Psi v} \right\|_{L^\infty} \le \|f\|_{L^\infty},$$

which directly implies $\|\mathcal{T}_\Psi f\|_{L^\infty(uv^r)} \le \|f\|_{L^\infty(uv^r)}$ since the measure given by $d\mu(x) = u(x)v^r(x)\,dx$ is absolutely continuous with respect to the Lebesgue measure. We now apply Lemma 13 with $T = \mathcal{T}_\Psi$, $C_0 = 1$, $\varphi = \Phi$ and μ the measure given above to obtain

$$\begin{aligned} uv^r\left(\left\{x \in \mathbb{R}^n : \frac{M_\Psi(fv)(x)}{M_\Psi v(x)} > t\right\}\right) &\le C \int_{\{x:|f(x)|>t/2\}} \Phi\left(\frac{2c_1|f(x)|}{t}\right) u(x)v^r(x)\,dx \\ &\le C\Phi\left(\frac{c_1}{t^*}\right) \int_{\{x:|f(x)|>t/2\}} \Phi\left(\frac{2t^*|f(x)|}{t}\right) u(x)v^r(x)\,dx \\ &\le BC\Phi\left(\frac{c_1}{t^*}\right) \int_{\{x:|f(x)|>t/2\}} \Psi\left(\frac{2t^*|f(x)|}{t}\right) u(x)v^r(x)\,dx \\ &\le C_1 \int_{\mathbb{R}^n} \Psi\left(\frac{C_2|f(x)|}{t}\right) u(x)v^r(x)\,dx. \end{aligned}$$

□

5 Applications: Mixed Inequalities for the Generalized Fractional Maximal Operator

We devote this section to proving Theorem 4 and Theorem 5.

Proof of Theorem 4 Define

$$\sigma=\frac{nr}{n-r\gamma},\quad \nu=\frac{n\delta}{n-r\gamma},\quad \beta=\frac{q}{\sigma}\left(\frac{1}{p}+\frac{1}{r'}\right),$$

and let ξ be the auxiliary function given by

$$\xi(t)=\begin{cases} t^{q/\beta}, & \text{if } 0\le t\le 1,\\ t^{\sigma}(1+\log^+ t)^{\nu}, & \text{if } \quad t>1.\end{cases}$$

By virtue of Eq. 2.3 we have that

$$\xi^{-1}(t)t^{\gamma/n}\approx\frac{t^{1/\sigma+\gamma/n}}{(1+\log^+ t)^{\nu/\sigma}}=\frac{t^{1/r}}{(1+\log^+ t)^{\delta/r}}\approx\Phi^{-1}(t),$$

for every $t\ge 1$. Observe that $\beta>1$: indeed, since $p>r$ we have $q>\sigma$ and thus $q/(\sigma r')>1/r'$. On the other hand, $q/(p\sigma)>1/r$. By combining these two inequalities we have $\beta>1$. Applying Proposition 10 and Lemma 9 with β we can conclude that

$$M_{\gamma,\Phi}\left(\frac{f_0}{w}\right)(x)\le C\left[M_\xi\left(\frac{f_0^{p\beta/q}}{w^\beta}\right)(x)\right]^{1/\beta}\left(\int_{\mathbb{R}^n}f_0^p(y)\,dy\right)^{\gamma/n}. \tag{5.1}$$

Also observe that

$$\left(M_\xi(v^\beta)(x)\right)^{1/\beta}\lesssim M_\eta v(x),\qquad \text{a.e. } x. \tag{5.2}$$

Indeed, it is clear that $\xi(z^\beta)\lesssim\eta(t)$. Given x and a fixed cube Q containing it, we can write

$$\begin{aligned}\frac{1}{|Q|}\int_Q\xi\left(\frac{v^\beta}{\|v\|_{\eta,Q}^\beta}\right)&\lesssim\frac{1}{|Q|}\int_Q\eta\left(\frac{v}{\|v\|_{\eta,Q}}\right)\\ &\le 1,\end{aligned}$$

which directly implies the estimate.

Notice that ξ is equivalent to a Young function in $\mathfrak{F}_\sigma$, for $t\ge 1$. Since $q(1/p+1/r')=\beta\sigma$, if we set $f_0=|f|wv$, then we can use inequalities Eq. 5.1 and Eq. 5.2 and Corollary 3 to estimate

$$\begin{aligned}&uv^{\frac{q}{p}+\frac{q}{r'}}\left(\left\{x:\frac{M_{\gamma,\Phi}(fv)(x)}{M_\eta v(x)}>t\right\}\right)\\ &\lesssim uv^{\beta\sigma}\left(\left\{x:\frac{M_{\gamma,\Phi}(fv)(x)}{\left(M_\xi v^\beta(x)\right)^{1/\beta}}>t\right\}\right)\\ &\le uv^{\beta\sigma}\left(\left\{x:\frac{M_\xi\left(f_0^{p\beta/q}w^{-\beta}\right)(x)}{M_\xi v^\beta(x)}>\frac{t^\beta}{\left(\int|f_0|^p\right)^{\beta\gamma/n}}\right\}\right)\\ &\le C_1\int_{\mathbb{R}^n}\xi\left(C_2\frac{|f|^{p\beta/q}(wv)^{\beta(p/q-1)}}{t^\beta}\left[\int_{\mathbb{R}^n}|f|^p(wv)^p\right]^{\gamma/n\beta}\right)uv^{\sigma\beta}\\ &=C_1\int_{\mathbb{R}^n}\xi(\lambda)uv^{\sigma\beta}\\ &=C_1\left(\int_A\xi(\lambda)uv^{\sigma\beta}+\int_B\xi(\lambda)uv^{\sigma\beta}\right),\end{aligned}$$

where

$$\lambda=C_2\frac{|f|^{p\beta/q}(wv)^{\beta(p/q-1)}}{t^\beta}\left[\int_{\mathbb{R}^n}|f|^p(wv)^p\right]^{\gamma/n\beta},$$

$A = \{x \in \mathbb{R}^n : \lambda(x) \leq 1\}$ y $B = \mathbb{R}^n \backslash A$. By definition of ξ we have that

$$\int_A \xi(\lambda(x)) u(x) [v(x)]^{\sigma\beta}\, dx = \int_A [\lambda(x)]^{q/\beta} u(x) [v(x)]^{\sigma\beta}\, dx.$$

If we set $w = u^{1/q} v^{1/p+1/r'-1}$, then

$$\begin{aligned} \lambda^{q/\beta} u v^{\sigma\beta} &= C_2^{q/\beta} \frac{|f|^p}{t^q} (wv)^{p-q} \left[\int_{\mathbb{R}^n} |f|^p (wv)^p\right]^{q\gamma/n} u v^{\sigma\beta} \\ &= C_2^{q/\beta} \frac{|f|^p}{t^q} \left[\int_{\mathbb{R}^n} |f|^p (wv)^p\right]^{q\gamma/n} u^{p/q} v^{\sigma\beta+(p-q)(1/p+1/r')}. \end{aligned}$$

Observe that

$$\sigma\beta + (p-q)\left(\frac{1}{p} + \frac{1}{r'}\right) = q\left(\frac{1}{p} + \frac{1}{r'}\right) + (p-q)\left(\frac{1}{p} + \frac{1}{r'}\right) = 1 + \frac{p}{r'}.$$

Also, notice that

$$(wv)^p = u^{p/q} v^{1+p/r'-p+p} = u^{p/q} v^{1+p/r'}.$$

Therefore,

$$\begin{aligned} \int_A \xi(\lambda) u v^{\sigma\beta} &\leq \frac{C_2^{q/\beta}}{t^q} \left[\int_{\mathbb{R}^n} |f|^p u^{p/q} v^{1+p/r'}\right]^{q\gamma/n} \left[\int_{\mathbb{R}^n} |f|^p u^{p/q} v^{1+p/r'}\right] \\ &= \frac{C_2^{q/\beta}}{t^q} \left[\int_{\mathbb{R}^n} |f|^p u^{p/q} v^{1+p/r'}\right]^{1+q\gamma/n} \\ &= \frac{C_2^{q/\beta}}{t^q} \left[\int_{\mathbb{R}^n} |f|^p u^{p/q} v^{1+p/r'}\right]^{q/p}. \end{aligned}$$

On the other hand, $\lambda(x) > 1$ over B and since ξ has an upper type q/β, we can estimate the integrand by $\lambda^{q/\beta} u v^{\sigma\beta}$. Then we can conclude the estimate by proceeeding like we did in part A. Thus, we obtain

$$uv^{q(1/p+1/r')}\left(\left\{x \in \mathbb{R}^n : \frac{M_{\gamma,\Phi}(fv)(x)}{M_\eta v(x)} > t\right\}\right)^{1/q} \leq C\left[\int_{\mathbb{R}^n} \left(\frac{|f|}{t}\right)^p u^{p/q} v^{1+p/r'}\right]^{1/p}.$$

□

Proof of Theorem 5 Set $\xi(t) = t^q(1+\log^+ t)^\nu$, where $\nu = \delta q/r$. Thus $t^{\gamma/n}\xi^{-1}(t) \lesssim \Phi^{-1}(t)$. By applying Proposition 10 with $p = r$ we have that

$$M_{\gamma,\Phi}\left(\frac{f_0}{w}\right)(x) \leq C\left[M_\xi\left(\frac{f_0^{r/q}}{w}\right)\right](x)\left(\int_{\mathbb{R}^n} f_0^r(y)\, dy\right)^{\gamma/n}.$$

By setting $f_0 = |f|wv$ we can write

$$\begin{aligned} uv^q\left(\left\{x : \frac{M_{\gamma,\Phi}(fv)(x)}{v(x)} > t\right\}\right) &= uv^q\left(\left\{x : \frac{M_{\gamma,\Phi}(f_0/w)(x)}{v(x)} > t\right\}\right) \\ &\leq uv^q\left(\left\{x : \frac{M_\xi(f_0^{r/q}/w)(x)}{M_\xi v(x)} > \frac{t}{(\int f_0^r)^{\gamma/n}}\right\}\right). \end{aligned}$$

Since $\xi \in \mathfrak{F}_q$, we can use the mixed estimate for M_ξ which leads us to

$$uv^q\left(\left\{x : \frac{M_{\gamma,\Phi}(fv)(x)}{v(x)} > t\right\}\right) \leq C\int_{\mathbb{R}^n} \xi\left(\frac{f_0^{r/q}\left(\int f_0^r\right)^{\gamma/n}}{wvt}\right) uv^q. \tag{5.3}$$

The argument of ξ above can be written as

$$\frac{f_0^{r/q}\left(\int f_0^r\right)^{\gamma/n}}{wvt} = \left(\frac{|f|}{t}\right)^{r/q}(wv)^{r/q-1}\left(\int_{\mathbb{R}^n}\left(\frac{|f|}{t}\right)^r(wv)^r\right)^{\gamma/n}$$
$$= \left[\left(\frac{|f|}{t}\right)(wv)^{1-q/r}\left(\int_{\mathbb{R}^n}\left(\frac{|f|}{t}\right)^r(wv)^r\right)^{\gamma q/(nr)}\right]^{r/q}.$$

Observe that for $0 \le t \le 1$, $\xi(t^{r/q}) = t^r$, and for $t > 1$,

$$\xi(t^{r/q}) = t^r(1+\log t^{r/q})^\nu$$
$$= t^r\left(1+\frac{r}{q}\log t\right)^\nu,$$

which implies $\xi(t^{r/q}) \le \Phi_\gamma(t) = t^r(1+\log^+ t)^\nu$. Then we can estimate as follows

$$\xi\left(\frac{f_0^{r/q}\left(\int_{\mathbb{R}^n} f_0^r\right)^{\gamma/n}}{wvt}\right) \le \Phi_\gamma\left(\left(\frac{|f|}{t}\right)(wv)^{1-q/r}\left(\int_{\mathbb{R}^n}\left(\frac{|f|}{t}\right)^r(wv)^r\right)^{\gamma q/(nr)}\right)$$
$$\le \Phi_\gamma\left(\left[\int_{\mathbb{R}^n}\Phi_\gamma\left(\frac{|f|}{t}\right)(wv)^r\right]^{\gamma q/(nr)}\right)\Phi_\gamma\left(\frac{|f|}{t}(wv)^{1-q/r}\right)$$

Returning to Eq. 5.3 and setting $w = u^{1/q}$, the right hand side is bounded by

$$\Phi_\gamma\left(\left[\int_{\mathbb{R}^n}\Phi_\gamma\left(\frac{|f|}{t}\right)(wv)^r\right]^{\gamma q/(nr)}\right)\int_{\mathbb{R}^n}\Phi_\gamma\left(\frac{|f|}{t}(wv)^{1-q/r}\right)(wv)^q.$$

Notice that $\Phi_\gamma(t^{1-q/r})t^q \le \Psi(t)$. Therefore, the expression above is bounded by

$$\Phi_\gamma\left(\left[\int_{\mathbb{R}^n}\Phi_\gamma\left(\frac{|f|}{t}\right)\Psi(u^{1/q}v)\right]^{\gamma q/(nr)}\right)\int_{\mathbb{R}^n}\Phi_\gamma\left(\frac{|f|}{t}\right)\Psi(u^{1/q}v).$$

To finish, observe that

$$t\Phi_\gamma(t^{\gamma q/(nr)}) \lesssim t^{1+\gamma q/n}(1+\log^+ t)^\nu = t^{q/r}(1+\log^+ t)^{\delta q/r} = \varphi(t).$$ □

References

1. Bernardis, A., Dalmasso, E., Pradolini, G.: Generalized maximal functions and related operators on weighted Musielak-Orlicz spaces. Ann. Acad. Sci. Fenn. Math. **39**(1), 23–50 (2014)
2. Berra, F.: Mixed weak estimates of Sawyer type for generalized maximal operators. Proc. Am. Math. Soc. **147**(10), 4259–4273 (2019)
3. Berra, F., Carena, M., Pradolini, G.: Improvements on Sawyer type estimates for generalized maximal functions, Math. Nachr., in press (2019)
4. Berra, F., Carena, M., Pradolini, G.: Mixed weak estimates of Sawyer type for commutators of generalized singular integrals and related operators. Mich. Math. J. **68**(3), 527–564 (2019)
5. Berra, F., Carena, M., Pradolini, G.: Mixed weak estimates of Sawyer type for fractional integrals and some related operators. J. Math. Anal. Appl. **2**, 1490–1505 (2019)
6. Berra, F.: Desigualdades mixtas para operadores del análisis armónico. Ph.D. thesis, Universidad Nacional del Litoral, 2019. https://hdl.handle.net/11185/5476
7. Caldarelli, M., Rivera-Ríos, I.: A sparse approach to mixed weak type inequalities. Math. Z. **296**(1–2), 787–812 (2020)
8. Cruz-Uribe, D., Martell, J.M., Pérez, C.: Weighted weak-type inequalities and a conjecture of Sawyer. Int. Math. Res Not. **30**, 1849–1871 (2005)

9. Cruz-Uribe, D., Martell, J.M., Pérez, C.: Weights, extrapolation and the theory of Rubio de Francia. Operator Theory: Advances and Applications, 215. Birkhäuser/Springer Basel AG, Basel (2011). xiv+280 pp. ISBN: 978-3-0348-0071-6
10. Duoandikoetxea, J.: Fourier Analysis, Graduate Studies in Mathematics, vol. 29. American Mathematical Society, Providence (2001). Translated and revised from the 1995 Spanish original by David Cruz-Uribe
11. Duoandikoetxea, J., Martín-Reyes, F.J., Ombrosi, S.: On the A_∞ conditions for general bases. Math. Z. **282**(3–4), 955–972 (2016)
12. Gorosito, O., Pradolini, G., Salinas, O.: Weighted weak-type estimates for multilinear commutators of fractional integrals on spaces of homogeneous type, Acta. Math. Sin. (Engl. Ser.) **23**(10), 1813–1826 (2007)
13. Grafakos, L.: Classical and Modern Fourier analysis. Pearson Education, Inc., Upper Saddle River (2004). MR 2449250
14. Krasnoselskiĭ, M.A., Rutickiĭ, J.B.: Convex functions and Orlicz spaces. Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen
15. Li, K., Ombrosi, S., Pérez, C.: Proof of an extension of E. Sawyer's conjecture about weighted mixed weak-type estimates. Math. Ann. **374**(1–2), 907–929 (2019)
16. Li, K., Ombrosi, S., Picardi, M.B.: Weighted mixed weak-type inequalities for multilinear operators. Studia Math. **244**(2), 203–215 (2019)
17. Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. **165**, 207–226 (1972). MR 0293384
18. Muckenhoupt, B., Wheeden, R.L..: Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform. Studia Math. **55**(3), 279–294 (1976)
19. Ombrosi, S., Pérez, C., Recchi, J.: Quantitative weighted mixed weak-type inequalities for classical operators. Indiana Univ. Math. J. **65**(2), 615–640 (2016)
20. Rao, M.M., Ren, Z.D.: Theory of Orlicz spaces, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146. Marcel Dekker, Inc., New York (1991). MR 1113700
21. Sawyer, E.: A weighted weak type inequality for the maximal function. Proc. Am. Math. Soc. **93**(4), 610–614 (1985)