C. Cimadamore
J. P. Díaz Varela

Monadic MV-algebras are Equivalent to Monadic ℓ-groups with Strong Unit

Abstract. In this paper we extend Mundici's functor Γ to the category of monadic MV-algebras. More precisely, we define monadic ℓ-groups and we establish a natural equivalence between the category of monadic MV-algebras and the category of monadic ℓ-groups with strong unit. Some applications are given thereof.

Keywords: Monadic MV-algebras, monadic ℓ-groups, natural equivalence.

1. Introduction

Monadic MV-algebras (monadic Chang algebras in Rutledge's terminology) were introduced and studied by J. D. Rutledge in [10] as an algebraic model for the (monadic) predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. Monadic MV-algebras have been studied by several authors. In [7], Di Nola and Grigolia study monadic MV-algebras as pairs of MV-algebras one of which is a special case of relatively complete subalgebra. In [1], Belluce, Grigolia and Lettieri obtain a representation theorem for certain classes of monadic MV-algebras and give a characterization of the monadic operators over a finite MV-algebra. Finally in [8], Lattanzi and Petrovich give categorical equivalences between the varieties of monadic $(n + 1)$-valued MV-algebras and the class of monadic Boolean algebras endowed with certain family of their filters.

In [3, p. 75] Chang defined a map from totally ordered abelian groups with order unit into totally ordered MV-algebras. As a natural generalization, Mundici [9] defined a functor Γ between the category of ℓ-groups with strong unit and the category of MV-algebras and he proved that this functor is a natural equivalence. The purpose of this article is to extend Mundici's functor Γ to the category of monadic MV-algebra. In that sense, we introduce the notion of monadic ℓ-groups and we establish a natural equivalence between the category of monadic MV-algebras and the category of monadic ℓ-groups with strong unit.

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This paper is organized as follows. In Section 2 we include the basic definitions and results on MV-algebras and ℓ-groups that we need in the rest of the paper. This material is extracted from the monograph [4] and is included in the paper to make its reading as self-contained as possible. Section 3 is devoted to define the variety of monadic MV-algebras and to prove the main properties of this variety. In Section 4 we define quantifiers on the lattice monoid of good sequences of an MV-algebra $\mathbf{A}$ and prove the properties that allow us to define quantifiers on Chang's ℓ-group $\mathbf{G}_A$. In Section 5 we introduce the notion of monadic ℓ-group and we construct the monadic Chang ℓ-group. The natural equivalence between the categories of monadic MV-algebras and monadic ℓ-groups with strong unit is established in Section 6. Finally we give some applications of this equivalence in Section 7.

2. Basic facts on MV-algebras and ℓ-groups

We include in this section the basic definitions and results on MV-algebras and ℓ-groups that we need in the rest of the paper. We start by recalling the definition of MV-algebras. The reader is referred to [4, 5, 6] and [9].

Definition 2.1. An *MV-algebra* is an algebra $\mathbf{A} = \langle A; \oplus, \neg, 0\rangle$ of type $(2, 1, 0)$ satisfying the following equations:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV2) $x \oplus y = y \oplus x$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus \neg 0 = \neg 0$,

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

On each MV-algebra $\mathbf{A}$ we define the constant 1 and the operation $\odot$ as follows: $1 := \neg 0$, $x \odot y := \neg(\neg x \oplus \neg y)$. We write $x \leq y$ if and only if $\neg x \oplus y = 1$. It follows that $\leq$ is a partial order, called the natural order of $\mathbf{A}$. An MV-algebra whose natural order is total is called an MV-chain. On each MV-algebra the natural order determines a lattice structure. Specifically, $x \vee y = (x \odot \neg y) \oplus y$ and $x \wedge y = x \odot (\neg x \oplus y)$.

An *ideal* of an MV-algebra $\mathbf{A}$ is a subset M of A satisfying the following conditions: $0 \in M$, if $x \in M$, $y \in A$ and $y \leq x$ then $y \in M$, and if $x, y \in M$ then $x \oplus y \in M$.

A *partially ordered abelian group* is an abelian group $\mathbf{G} = \langle G; +, -, 0\rangle$ endowed with an order relation $\leq$ which is invariant under translations, that is, if $a \leq b$ then $a + t \leq b + t$, for every $a, b, t \in G$.

The *positive cone* G^+ of $\mathbf{G}$ is the set of all $x \in G$ such that $0 \leq x$. If the relation $\leq$ is a total order then $G = G^+ \cup -G^+$. If the order on $\mathbf{G}$ determines a lattice structure, $\mathbf{G}$ is called a *lattice-ordered abelian group* or an *ℓ-group*. In every ℓ-group, $t + (a \vee b) = (t + a) \vee (t + b)$, $t + (a \wedge b) = (t + a) \wedge (t + b)$ and $t - (a \vee b) = (t - a) \wedge (t - b)$.

For each $x \in G$ we define the *positive part* of x, the *negative part* of x and the *absolute value* of x respectively by $x^+ = x \vee 0$, $x^- = -x \vee 0$ and $|x| = x^+ + x^- = x \vee -x$.

A *strong (order) unit* u of an ℓ-group $\mathbf{G}$ is an archimedian element of G, that is, an element $0 \leq u$ such that for each $x \in G$ there is an integer $n \geq 0$ such that $|x| \leq nu$. An ℓ-group $\mathbf{G}$ with strong unit u will be denoted by $\langle G, u\rangle$.

We say that $\mathbf{K}$ is an *ℓ-subgroup* of an ℓ-group $\mathbf{G}$ if $\mathbf{K}$ is both a subgroup and a sublattice of $\mathbf{G}$. An ℓ-subgroup $\mathbf{K}$ is *convex* if $h, k \in K$ and $h < g < k$ imply $g \in K$.

An *ℓ-ideal* of an ℓ-group $\mathbf{G}$ is a convex ℓ-subgroup J of $\mathbf{G}$. It is known that J is an ℓ-ideal of an ℓ-group $\mathbf{G}$ if and only if J is a subgroup of $\mathbf{G}$ that satisfies the condition if $x \in J$ and $|y| \leq |x|$ then $y \in J$.

If $\mathbf{G}$ and $\mathbf{H}$ are ℓ-groups, a function $f\colon \mathbf{G} \to \mathbf{H}$ is said to be an *ℓ-homomorphism* if h is both a group homomorphism and a lattice-homomorphism.

If J is an ℓ-ideal of $\mathbf{G}$, then $\mathbf{G}/J$ is an ℓ-group, and hence the natural epimorphism $\pi_J\colon \mathbf{G} \to \mathbf{G}/J$ is an ℓ-epimorphism such that $\operatorname{Ker}(\pi_J) = J$. Moreover, if u is a strong unit of $\mathbf{G}$ then $\pi_J(u)$ is a strong unit of $\mathbf{G}/J$. Conversely, if $f\colon \mathbf{G} \to \mathbf{H}$ is an ℓ-homomorphism then $\operatorname{Ker}(f) = f^{-1}(\{0\})$ is an ℓ-ideal of $\mathbf{G}$ and $\mathbf{G}/\operatorname{Ker}(f)$ is isomorphic to the ℓ-subgroup $f(\mathbf{G})$ of $\mathbf{H}$.

An ℓ-ideal J of $\mathbf{G}$ is called *prime* if J is proper and $\mathbf{G}/J$ is totally ordered. The following is a well-known result of ℓ-groups.

Proposition 2.2. *Every ℓ-group* $\mathbf{G}$ *is a subdirect product of totally ordered ℓ-groups.*

Proof. Let $a \neq 0 \in G$. Consider the family of ℓ-ideals of $\mathbf{G}$ not containing a. If J is maximal with this property then J is prime. So the intersection of all prime ℓ-ideals of $\mathbf{G}$ is $\{0\}$, and consequently $\mathbf{G}$ can be embedded into $\prod \mathbf{G}/J$, with J prime, which is a product of totally ordered ℓ-groups. ■

In what follows we shall present a summary of the equivalence between the category of ℓ-groups with strong unit and the category of MV-algebras.

Let $\mathbf{G}$ be an ℓ-group and $u \in G$, $u > 0$ (not necessarily u being a strong unit of $\mathbf{G}$). Consider $[0, u] = \{x \in G : 0 \leq x \leq u\}$, and for each $x, y \in [0, u]$ define the operations:

$$x \oplus y = u \wedge (x + y), \quad \neg x = u - x, \quad 0 = 0_G.$$

Then $\Gamma(\mathbf{G}, u) = \langle [0, u]; \oplus, \neg, 0 \rangle$ in an MV-algebra.

The function $h\colon \langle G, u \rangle \to \langle H, v \rangle$ is a *unital ℓ-homomorphism* if h is an ℓ-homomorphism and $h(u) = v$.

If $h\colon \langle G, u \rangle \to \langle H, v \rangle$ is a unital ℓ-homomorphism then

$$\Gamma(h)\colon \Gamma(\mathbf{G}, u) \to \Gamma(\mathbf{H}, v)$$

defined by $\Gamma(h) = h\restriction_{[0,u]}$ is an MV-algebra homomorphism. Moreover, Γ is a functor from the category of ℓ-groups with strong unit into the category of MV-algebras.

Our next objective is to define a functor Ξ from the category of MV-algebras into the category of ℓ-groups with strong unit.

A sequence $\bar{a} = (a_1, a_2, \ldots, a_n, \ldots)$ of elements of an MV-algebra $\mathbf{A}$ is said to be *good* if for each $i = 1, 2, \ldots, n, \ldots$, $a_i = a_i \oplus a_{i+1}$, and there is an integer n such that $a_r = 0$ for all $r > n$. We shall often write $\bar{a} = (a_1, a_2, \ldots, a_n, 0, \ldots) = (a_1, a_2, \ldots, a_n)$. In particular $(a, 0, \ldots) = (a)$.

Observe that in an MV-chain $\mathbf{A}$, $a \oplus b = a$ if and only if $a = 1$ or $b = 0$, and as a consequence, the good sequences in an MV-chain are of the form $(1^p, a)$ for some integer $p \geq 0$ and $a \in A$, where 1^p denotes a p-tuple of p consecutive 1's.

Let M_A be the set of all good sequences of an MV-algebra $\mathbf{A}$. Then $\mathbf{M}_A = \langle M_A; +, (0), \leq \rangle$ is a lattice abelian monoid, where the *sum* $\bar{c} = \bar{a} + \bar{b}$ of two good sequences $\bar{a}$ and $\bar{b}$ is defined by

$$c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus (a_{i-2} \odot b_2) \oplus \ldots \oplus (a_1 \odot b_{i-1}) \oplus b_i,$$

for $i = 1, 2, \ldots, n$, and where $\bar{b} \leq \bar{a}$ if and only if $b_i \leq a_i$ for all $i = 1, \ldots, n$ if and only if there exists a sequence $\bar{c}$ such that $\bar{b} + \bar{c} = \bar{a}$. This $\bar{c}$, denoted $\bar{a} - \bar{b}$, is given by $\bar{c} = (a_1, \ldots, a_n) + (\neg b_n, \ldots, \neg b_1)$ omitting the first n terms. In addition, the order defined in M_A is invariant under translations. If $\bar{a}$ and $\bar{b}$ are two good sequences then $\bar{a} \vee \bar{b} = (a_1 \vee b_1, \ldots, a_n \vee b_n)$ is the join of $\bar{a}$ and $\bar{b}$, and $\bar{a} \wedge \bar{b} = (a_1 \wedge b_1, \ldots, a_n \wedge b_n)$ is the meet of $\bar{a}$ and $\bar{b}$.

Now recall the construction of Chang's ℓ-group $\mathbf{G}_A$. Consider the equivalence relation on $M_A \times M_A$ defined by

$$(\bar{a}, \bar{b}) \equiv (\bar{c}, \bar{d}) \text{ if and only if } \bar{a} + \bar{d} = \bar{b} + \bar{c}.$$

The equivalence class of the pair $(\bar{a}, \bar{b})$ will be denoted by $[\bar{a}, \bar{b}]$. Let $\mathbf{G}_A = \langle G_A; +, -, [(0),(0)]\rangle$ where G_A is the set of equivalence classes, $[\bar{a}, \bar{b}]+[\bar{c}, \bar{d}] = [\bar{a}+\bar{c}, \bar{b}+\bar{d}]$ and $-[\bar{a}, \bar{b}] = [\bar{b}, \bar{a}]$. It follows that $\mathbf{G}_A$ is an abelian group where $[(0),(0)]$ is the zero for the operation $+$. We say that $[\bar{c}, \bar{d}]$ dominates $[\bar{a}, \bar{b}]$, and we write $[\bar{a}, \bar{b}] \preccurlyeq [\bar{c}, \bar{d}]$, if and only if $[\bar{c}, \bar{d}] - [\bar{a}, \bar{b}] = [\bar{e}, (0)]$, for some $\bar{e} \in M_A$. Note that $[\bar{a}, \bar{b}] \preccurlyeq [\bar{c}, \bar{d}]$ if and only if $\bar{a} + \bar{d} \leq \bar{c} + \bar{b}$. The relation $\preccurlyeq$ is invariant under translations. In fact, $\mathbf{G}_A$ is an ℓ-group where for each pair $[\bar{a}, \bar{b}], [\bar{c}, \bar{d}] \in G_A$,

$$[\bar{a}, \bar{b}]\vee[\bar{c}, \bar{d}] = [(\bar{a}+\bar{d})\vee(\bar{c}+\bar{b}), \bar{b}+\bar{d}], \text{ and } [\bar{a}, \bar{b}]\wedge[\bar{c}, \bar{d}] = [(\bar{a}+\bar{d})\wedge(\bar{c}+\bar{b}), \bar{b}+\bar{d}].$$

It is not difficult to see that $[(1),(0)]$, which will be denoted by u_A, is a strong unit of $\mathbf{G}_A$.

The function $\mathbf{M}_A \to \mathbf{G}_A^+$ defined by $\bar{a} \mapsto [\bar{a}, (0)]$ is a monoid and lattice isomorphism.

THEOREM 2.3. [9] *The correspondence* $\alpha_A \colon \mathbf{A} \to \Gamma(\mathbf{G}_A, u_A)$ *defined by* $\alpha_A(a) = [(a),(0)]$ *is an MV-algebra isomorphism.*

The functor Ξ is defined from the category of MV-algebras into the category of ℓ-groups. Given an MV-algebra $\mathbf{A}$, $\Xi(\mathbf{A}) = \mathbf{G}_A$ is an ℓ-group. Let $h \colon \mathbf{A} \to \mathbf{B}$ be an MV-homomorphism and let

$$h^* \colon \mathbf{M}_A \to \mathbf{M}_B$$

be defined by $h^*((a_1, a_2, \ldots, a_n)) = (h(a_1), h(a_2), \ldots, h(a_n))$. It is easy to see that $h^*(\bar{a}+\bar{b}) = h^*(\bar{a})+h^*(\bar{b})$, $h^*(\bar{a}\vee\bar{b}) = h^*(\bar{a})\vee h^*(\bar{b})$, and $h^*(\bar{a}\wedge\bar{b}) = h^*(\bar{a}) \wedge h^*(\bar{b})$, that is, h^* is a lattice and monoid homomorphism. We now define

$$\Xi(h) \colon \mathbf{G}_A \to \mathbf{G}_B$$

by $\Xi(h)([\bar{a}, \bar{b}]) = [h^*(\bar{a}), h^*(\bar{b})]$. Then $\Xi(h)$ is a unital ℓ-homomorphism. Moreover, Ξ is a functor from the category $\mathcal{MV}$ into the category of ℓ-groups with strong unit.

The following results can be found in [4].

LEMMA 2.4. *Let* $\mathbf{G}$ *be an* ℓ*-group with strong unit* u *and let* $\mathbf{A} = \Gamma(\mathbf{G}, u)$*. For each* $0 \leq a \in G$ *there exists a unique good sequence* $g(a) = (a_1, a_2, \ldots, a_n)$ *of* $\mathbf{A}$ *such that* $a = a_1 + a_2 + \ldots + a_n$*.*

THEOREM 2.5. *The function* $\beta_{\langle G,u\rangle} \colon \mathbf{G} \to \mathbf{G}_{\Gamma(G,u)}$ *defined by* $\beta_{\langle G,u\rangle}(a) = [g(a^+), g(a^-)]$ *is an* ℓ*-isomorphism where* $\beta_{\langle G,u\rangle}(u) = [(u),(0)]$*.*

THEOREM 2.6. *The composite functor* $\Gamma\Xi$ *is naturally equivalent to the identity functor of the category* $\mathcal{MV}$*, and* $\Xi\Gamma$ *is naturally equivalent to the identity functor of the category of* ℓ*-groups with strong unit.*

3. Monadic MV-algebras

In this section we give the basic definitions and investigate some of the properties of monadic MV-algebras. These algebras were introduced by Rutledge [10], under the name of monadic Chang algebras, and were recently developed by Di Nola and Grigolia [7], Belluce, Grigolia and Lettieri [1] and Lattanzi and Petrovich [8].

Definition 3.1. [10] An algebra $\mathbf{A} = \langle A; \oplus, \neg, \exists, 0\rangle$ of type $(2, 1, 1, 0)$ is called a *monadic MV-algebra*, (an MMV-algebra for short), if $\langle A; \oplus, \neg, 0\rangle$ is an MV-algebra and $\exists$ satisfies the following equations:

(MMV1) $x \leq \exists x$,

(MMV2) $\exists(x \vee y) = \exists x \vee \exists y$,

(MMV3) $\exists\neg\exists x = \neg\exists x$,

(MMV4) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,

(MMV5) $\exists(x \odot x) = \exists x \odot \exists x$,

(MMV6) $\exists(x \oplus x) = \exists x \oplus \exists x$.

An MMV-algebra $\mathbf{A} = \langle A; \oplus, \neg, \exists, 0\rangle$ will be denoted simply by $\langle A; \exists\rangle$.

In an MMV-algebra $\mathbf{A}$, we can define $\forall\colon A \to A$ by $\forall a = \neg\exists\neg a$, for every $a \in A$. Clearly we have $\exists a = \neg\forall\neg a$. In Lemma 3.2 we state that $\forall$ satisfies the dual identities of (MMV1)-(MMV6).

Lemma 3.2. *In every MMV-algebra* $\mathbf{A}$ *the following equations are satisfied.*

(MMV7) $\forall x \leq x$,

(MMV8) $\forall(x \wedge y) = \forall x \wedge \forall y$,

(MMV9) $\forall\neg\forall x = \neg\forall x$,

(MMV10) $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$,

(MMV11) $\forall(x \odot x) = \forall x \odot \forall x$,

(MMV12) $\forall(x \oplus x) = \forall x \oplus \forall x$.

A *monadic homomorphism* is a function $h\colon \langle A; \exists\rangle \to \langle B; \exists'\rangle$ such that h is an MV-homomorphism and for each $a \in A$, $h(\exists a) = \exists' h(a)$.

The variety of monadic MV-algebras and the category of MMV-algebras and monadic homomorphisms will be denoted by $\mathcal{MMV}$.

The next lemma collects some basic properties of MMV-algebras.

Lemma 3.3. [10] *Let* $\mathbf{A} \in \mathcal{MMV}$. *For every* $a, b \in A$ *the following properties hold:*

(MMV13) $\forall 0 = 0$, $\exists 1 = 1$,

(MMV14) $\forall 1 = 1$, $\exists 0 = 0$,

(MMV15) $\forall\forall a = \forall a$, $\exists\exists a = \exists a$, $\forall\exists a = \exists a$, $\exists\forall a = \forall a$,

(MMV16) $\forall(\forall a \oplus \forall b) = \forall a \oplus \forall b$, $\exists(\exists a \odot \exists b) = \exists a \odot \exists b$,

(MMV17) $\forall a \leq b$ *if and only if* $\forall a \leq \forall b$, $a \leq \exists b$ *if and only if* $\exists a \leq \exists b$,

(MMV18) *if* $a \leq b$ *then* $\forall a \leq \forall b$ *and* $\exists a \leq \exists b$,

(MMV19) $\forall a \odot \forall b \leq \forall(a \odot b)$, $\exists(a \oplus b) \leq \exists a \oplus \exists b$,

(MMV20) $\forall a \oplus \forall b \leq \forall(a \oplus b)$, $\exists(a \odot b) \leq \exists a \odot \exists b$,

(MMV21) $\forall(\neg a \oplus b) \leq \neg\forall a \oplus \forall b$, $\neg(\exists a) \odot \exists b \leq \exists(\neg a \odot b)$,

(MMV22) $\forall(\forall a \vee \forall b) = \forall a \vee \forall b$,

(MMV23) $\forall a \vee \forall b \leq \forall(a \vee b)$.

In the end of this section, we shall prove that the property $\exists(a \wedge \exists b) = \exists a \wedge \exists b$ also holds for every a, b in an MMV-algebra (Corollary 3.14).

Consider the set $\exists A = \{a \in A \colon a = \exists a\} = \{a \in A \colon a = \forall a\} = \forall A$.

Lemma 3.4. [10] *Let* $\mathbf{A} \in \mathcal{MMV}$. *Then* $\exists\mathbf{A} = \langle \exists A; \oplus, \neg, 0\rangle$ *is an MV-subalgebra of the MV-reduct of* $\mathbf{A}$.

Proof. It is an immediate consequence of (MMV15), (MMV13), (MMV3) and (MMV4). ∎

In every MMV-algebra $\mathbf{A}$, congruences are determined by monadic ideals. An ideal M of an MMV-algebra $\mathbf{A}$ is said to be a *monadic ideal* if M is an ideal of $\mathbf{A}$ and $\exists a \in M$ for every $a \in M$. If M is a monadic ideal of $\mathbf{A}$, then the relation R defined on A by

$$xRy \text{ if and only if } (\neg x \odot y) \oplus (x \odot \neg y) \in M$$

is a congruence. Moreover, there exists an isomorphism between the lattice of monadic ideals and the lattice of congruences of an MMV-algebra. On the other hand, there exists an isomorphism between the lattice of monadic ideals of $\langle A; \exists\rangle$ and the lattice of ideals of $\exists\mathbf{A}$ [10].

Proposition 3.5. [10] *Every MMV-algebra* $\mathbf{A}$ *is a subdirect product of MMV-algebras* $\mathbf{A}_i$, $i \in I$, *such that* $\exists A_i$ *is totally ordered.*

Next we will show that the variety of MMV-algebras is generated by its finite members, and as an important consequence, we will prove that $\mathcal{MMV}$ satisfies the identity $\exists(x \wedge \exists y) \approx \exists x \wedge \exists y$.

For each integer $n \geq 1$, let $\mathcal{K}_n$ be the class of MMV-algebras satisfying the equation

$$(\delta_n)\ \ x^n \approx x^{n+1}.$$

Clearly, if $n \leq l$ then $\mathcal{K}_n \subseteq \mathcal{K}_l$.

Let $\mathbf{S}_n = \langle S_n; \oplus, \neg, 0\rangle$ be the MV-algebra whose universe is the $(n+1)$-element set $S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, $a \oplus b = \min\{1, a+b\}$ and $\neg a = 1-a$, for $a, b \in S_n$. For each integer $k \geq 1$, let us consider the direct product $\langle S_n^k; \oplus, \neg, 0\rangle$, with the MV-operations defined pointwise. Let $\mathbf{S}_n^k = \langle S_n^k; \exists\rangle$ denote the MMV-algebra where $\exists\colon S_n^k \to S_n^k$ is defined by $\exists(\langle a_1, a_2, \ldots, a_k\rangle) = \langle m, m, \ldots, m\rangle$ with $m = a_1 \vee a_2 \vee \ldots \vee a_k = \max\{a_1, a_2, \ldots, a_k\}$.

LEMMA 3.6. *Every subdirectly irreducible algebra of $\mathcal{K}_n$ is simple.*

PROOF. Let $\mathbf{A} \in \mathcal{K}_n$ be subdirectly irreducible. Then $\exists\mathbf{A}$ belongs to the subvariety of MV-algebras that satisfy (δ_n) and it is a subdirectly irreducible MV-chain. Thus $\exists\mathbf{A}$ is isomorphic to $\mathbf{S}_m$, for some $1 \leq m \leq n$ (see [4, Theorem 8.2.2]). In particular, $\exists\mathbf{A}$ is simple. Since there exists an isomorphism between the lattice of monadic ideals of $\langle A; \exists\rangle$ and the lattice of ideals of $\exists\mathbf{A}$, we have that $\mathbf{A}$ is simple. ■

The following lemma characterizes the finite subdirectly irreducible algebras in the subvariety $\mathcal{K}_n$.

LEMMA 3.7. [1] *If $\mathbf{A}$ is a* finite *subdirectly irreducible (simple) MMV-algebra in $\mathcal{K}_n$, then $\mathbf{A} \cong \mathbf{S}_m^k$, $1 \leq m \leq n$, for some $k \in \mathbb{N}$.*

We are going to prove that $\mathcal{K}_n$ is locally finite.

We denote by $\mathbf{S}(X)$ and $\mathbf{S}_{MV}(X)$ the MMV-subalgebra and MV-subalgebra, respectively, generated by a subset X of an algebra $\mathbf{A}$. The proof of the following lemma is immediate.

LEMMA 3.8. *If $\mathbf{A} \in \mathcal{MMV}$ and $G \subseteq A$ is a subset of generators of $\mathbf{A}$, then $\mathbf{A} = \mathbf{S}_{MV}(G \cup \exists A)$.*

LEMMA 3.9. *Every finitely generated simple algebra $\mathbf{A} \in \mathcal{K}_n$ is finite.*

PROOF. Let G be a finite set of generators of $\mathbf{A}$. Since $\mathbf{A} \in \mathcal{K}_n$, it follows that the reduct $\langle \mathbf{A}; \oplus, \neg, 0\rangle$ and $\exists\mathbf{A}$ belong to the subvariety of MV-algebras satisfying (δ_n). Thus $\exists\mathbf{A}$ is isomorphic to $\mathbf{S}_m$, for some $1 \leq m \leq n$. But the subvariety of MV-algebras satisfying (δ_n) is locally finite and $G \cup \exists A$ is a finite set. So $\mathbf{S}_{MV}(G \cup \exists A)$ is finite. By Lemma 3.8, $\mathbf{A} = \mathbf{S}_{MV}(G \cup \exists A)$. Consequently, $\mathbf{A}$ is finite. ■

PROPOSITION 3.10. *$\mathcal{K}_n$ is locally finite.*

PROOF. The free $\mathcal{K}_n$-algebra over a finite set G of free generators, $\mathbf{Free}_{\mathcal{K}_n}(G)$, is isomorphic to a subalgebra of $\prod_{M\in\mathcal{M}} \mathbf{Free}_{\mathcal{K}_n}(G)/M$, where $\mathcal{M}$ is the set of all maximal monadic filters of $\mathbf{Free}_{\mathcal{K}_n}(G)$. By Lemma 3.9, $\mathbf{Free}_{\mathcal{K}_n}(G)/M$ is finite, and it is easy to see that $\mathcal{M}$ is also finite. So $\mathbf{Free}_{\mathcal{K}_n}(G)$ is finite and consequently $\mathcal{K}_n$ is locally finite. ■

COROLLARY 3.11. *$\mathcal{K}_n$ is generated by its finite members. Moreover,*

$$\mathcal{K}_n = \mathcal{V}\left(\left\{\mathbf{S}_m^k : k \in \mathbb{N}, 1 \leq m \leq n\right\}\right).$$

Besides, Rutledge proved that

$$\mathcal{MMV} = \mathcal{V}\left(\bigcup_{n\in\mathbb{N}} \mathcal{K}_n\right),$$

[10, Theorem II.4.1]. So, from this and Corollary 3.11 we have the following result.

COROLLARY 3.12. *The variety $\mathcal{MMV}$ is generated by its finite members.*

Therefore, $\mathcal{MMV}$ is generated by the algebras $\mathbf{S}_n^k$, for non-negative integers n and k.

LEMMA 3.13. *For every $a, b \in S_n^k$, $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.*

PROOF. Let $a = \langle a_1, a_2, \ldots, a_k\rangle, b = \langle b_1, b_2, \ldots, b_k\rangle \in S_n^k$, and $\exists b = \langle m, m, \ldots, m\rangle$, where $m = b_1 \vee b_2 \vee \ldots \vee b_k = \max\{b_1, b_2, \ldots, b_k\}$. Then,

$$a \wedge \exists b = \langle a_1, a_2, \ldots, a_k\rangle \wedge \langle m, m, \ldots, m\rangle = \langle a_1 \wedge m, a_2 \wedge m, \ldots, a_n \wedge m\rangle.$$

Thus

$$\exists(a \wedge \exists b) = \exists(\langle a_1 \wedge m, a_2 \wedge m, \ldots, a_n \wedge m\rangle).$$

Besides

$$(a_1 \wedge m) \vee (a_2 \wedge m) \vee \ldots \vee (a_k \wedge m) = (a_1 \vee a_2 \vee \ldots \vee a_k) \wedge m.$$

and then

$$\exists(a \wedge \exists b) = \langle (a_1 \vee a_2 \vee \ldots \vee a_k) \wedge m, \ldots, (a_1 \vee a_2 \vee \ldots \vee a_k) \wedge m\rangle$$
$$= \langle a_1 \vee a_2 \vee \ldots \vee a_k, \ldots, a_1 \vee a_2 \vee \ldots \vee a_k\rangle \wedge \langle m, m, \ldots, m\rangle = \exists a \wedge \exists b.$$ ■

COROLLARY 3.14. *Let $\mathbf{A} \in \mathcal{MMV}$. For every $a, b \in A$, $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.*

4. The monadic lattice monoid $\mathbf{M}_A$

Let $\mathbf{A}$ be an MMV-algebra and consider the lattice monoid of all good sequences $\mathbf{M}_A$. Since $\exists\mathbf{A}$ is also an MV-algebra, let $\mathbf{M}_{\exists A}$ denote the lattice monoid of good sequences of elements of $\exists A$. It is clear that $\mathbf{M}_{\exists A}$ is a sublattice submonoid of $\mathbf{M}_A$. In this section we define $\exists_M\colon M_A \to M_A$ and $\forall_M\colon M_A \to M_A$ such that $\exists_M(M_A) = M_{\exists A}$.

The following results will be used in order to define the quantifiers on $\mathbf{M}_A$. For completeness, we include the proofs of Lemma 4.1 and Lemma 4.3 since [10] is not easily available. They will be used to prove Lemma 4.5 and Lemma 4.6.

Lemma 4.1. [10] *Let* $\mathbf{A} \in \mathcal{MMV}$. *For each* $a, b \in A$,

$$\forall(a \odot b) = \forall(a \oplus b) \odot \forall(a \odot b).$$

Proof. First note that in any MV-algebra, $a \oplus b = (a \oplus b) \oplus (a \odot b)$. Indeed, it is clear that $a \oplus b \leq (a \oplus b) \oplus (a \odot b)$. In addition,

$$((a \oplus b) \oplus (a \odot b)) \odot \neg(a \oplus b) = (a \odot b) \wedge \neg(a \oplus b) = (a \odot b) \wedge (\neg a \odot \neg b) = 0.$$

This is equivalent to $(a \oplus b) \oplus (a \odot b) \leq a \oplus b$, so $a \oplus b = (a \oplus b) \oplus (a \odot b)$.

Let us see now that $\forall(a \oplus b) = \forall(a \oplus b) \oplus \forall(a \odot b)$. The inequality $\forall(a\oplus b) \leq \forall(a\oplus b)\oplus\forall(a\odot b)$ is immediate. But $\forall(a\oplus b) = \forall((a\oplus b)\oplus(a\odot b)) \geq \forall(a \oplus b) \oplus \forall(a \odot b)$.

Now we prove the statement of the lemma. It is clear that $\forall(a \odot b) \geq \forall(a \oplus b) \odot \forall(a \odot b)$. Besides,

$$\begin{aligned}&\forall(a \odot b) \odot \neg(\forall(a \oplus b) \odot \forall(a \odot b)) = \forall(a \odot b) \odot (\neg\forall(a \oplus b) \oplus \neg\forall(a \odot b)) = \\ &\forall(a \odot b) \wedge \neg\forall(a \oplus b) = \neg\forall(a \oplus b) \odot (\forall(a \oplus b) \oplus \forall(a \odot b)) = \\ &\neg\forall(a \oplus b) \odot \forall(a \oplus b) = 0.\end{aligned}$$

That is, $\forall(a \odot b) \leq \forall(a \oplus b) \odot \forall(a \odot b)$, as required to complete the proof. ∎

Recall that in every MV-algebra we have the following cancellation law.

Lemma 4.2. [2] *Let* $\mathbf{A}$ *be an MV-algebra and* $x, y, z \in A$. *If* $x \odot z = y \odot z$, $\neg x \leq z$ *and* $\neg y \leq z$ *then* $x = y$.

Lemma 4.3. [10] *Let* $\mathbf{A}$ *be an MMV-algebra such that* $\forall A$ *is totally ordered and let* $a, b \in A$. *If* $\forall(a \odot b) > 0$ *then* $a \oplus b = 1$.

PROOF. Let $a, b \in A$ such that $\forall(a \odot b) > 0$. Since $\forall A$ is totally ordered, given the elements $\forall(a \oplus b)$ and $\neg\forall(a \odot b)$, either $\forall(a \oplus b) \leq \neg\forall(a \odot b)$ or $\forall(a \oplus b) \geq \neg\forall(a \odot b)$. Suppose that $\forall(a \oplus b) \leq \neg\forall(a \odot b)$. This is equivalent to $\forall(a \oplus b) \odot \forall(a \odot b) = 0$. So from Lemma 4.1, $\forall(a \odot b) = 0$, a contradiction. Thus $\forall(a \oplus b) \geq \neg\forall(a \odot b)$. Now, from Lemma 4.1, $1 \odot \forall(a \odot b) = \forall(a \oplus b) \odot \forall(a \odot b)$, and since $\forall(a \odot b) > 0 = \neg 1$, and the previous lemma, we have that $\forall(a \oplus b) = 1$ and consequently, $a \oplus b = 1$. ■

COROLLARY 4.4. [10] *Let* $\mathbf{A}$ *be an MMV-algebra such that* $\exists A$ *is totally ordered, and let* $a, b \in A$. *If* $\exists(a \oplus b) < 1$ *then* $a \odot b = 0$.

LEMMA 4.5. *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra. If* $\bar{a} = (a_1, a_2, \ldots, a_n)$ *is a good sequence in* $\mathbf{A}$, *then* $(\exists a_1, \exists a_2, \ldots, \exists a_n)$ *is a good sequence of* $\exists\mathbf{A}$.

PROOF. We have to prove that $\exists a_i \oplus \exists a_{i+1} = \exists a_i$. Since $\exists\mathbf{A}$ is a MV-chain, this is equivalent to prove that either $\exists a_i = 1$ or $\exists a_{i+1} = 0$. Suppose that $\exists a_i < 1$. From $a_i \oplus a_{i+1} = a_i$, it follows that $\exists(a_i \oplus a_{i+1}) < 1$. So $a_i \odot a_{i+1} = 0$ by Corollary 4.4. Since $a_i \oplus a_{i+1} = a_i$ implies that $\neg a_i \oplus \neg a_{i+1} = \neg a_{i+1}$, we also have that $a_i \odot a_{i+1} = \neg(\neg a_i \oplus \neg a_{i+1}) = \neg\neg a_{i+1} = a_{i+1}$. So $a_{i+1} = 0$ and consequently, $\exists a_{i+1} = 0$. ■

LEMMA 4.6. *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra. If* $\bar{a} = (a_1, a_2, \ldots, a_n)$ *is a good sequence in* $\mathbf{A}$, *then* $(\forall a_1, \forall a_2, \ldots, \forall a_n)$ *is a good sequence of* $\forall\mathbf{A}$.

PROOF. The proof is similar to that of the previous lemma, using Lemma 4.3. ■

LEMMA 4.7. *Let* $\mathbf{A} \in \mathcal{MMV}$. *Suppose that* $A \subseteq \prod_{i \in I} A_i$ *is a subdirect product of a family* $\{\mathbf{A}_i\}$ *of subdirectly irreducible MMV-algebras and let* π_i *be the projection map on the ith coordinate of* $\prod_{i \in I} A_i$. *Then the sequence* $\bar{a} = (a_1, a_2, \ldots, a_n)$ *of elements of* A *is a good sequence if and only if for each* $i \in I$, *the sequence*

$$\bar{a}_i = (\pi_i(a_1), \pi_i(a_2), \ldots, \pi_i(a_n))$$

is a good sequence in $\mathbf{A}_i$, *and there exists* $n_0 \geq 0$ *such that* $\pi_i(a_n) = 0$ *whenever* $n > n_0$, *for all* $i \in I$.

PROOF. It is enough to observe that $a_j = a_{j+1} \oplus a_j$ if and only if $\pi_i(a_j) = \pi_i(a_{j+1}) \oplus \pi_i(a_j)$, for every $i \in I$. ■

Corollary 4.8. *Let* $\mathbf{A} \in \mathcal{MMV}$. *If* $\bar{a} = (a_1, a_2, \ldots, a_n) \in M_A$ *then* $(\exists a_1, \exists a_2, \ldots, \exists a_n)$ *and* $(\forall a_1, \forall a_2, \ldots, \forall a_n)$ *are good sequences of* $\exists\mathbf{A}$.

Proof. It is a consequence of Lemma 4.7, Lemma 4.5 and Lemma 4.6. ■

The previous result allows us to define $\exists_M \colon M_A \to M_A$ by $\exists_M(\bar{a}) = (\exists a_1, \exists a_2, \ldots, \exists a_n)$ and $\forall_M \colon M_A \to M_A$ by $\forall_M(\bar{a}) = (\forall a_1, \forall a_2, \ldots, \forall a_n)$.

Note that in the cases in which $\exists\mathbf{A}$ is totally ordered, the good sequences in $M_{\exists A}$ have the form

$$(1^p, \exists a),$$

for some non-negative integer p and some $a \in A$.

The following lemma collects some properties of $\exists_M$ and $\forall_M$.

Lemma 4.9. *Sea* $\mathbf{A} \in \mathcal{MMV}$. *For every* $\bar{a}, \bar{b} \in M_A$ *we have that:*

1. $\bar{a} \leq \exists_M(\bar{a})$, $\forall_M(\bar{a}) \leq \bar{a}$,
2. $\exists_M(\exists_M(\bar{a})) = \exists_M(\bar{a})$, $\forall_M(\forall_M(\bar{a})) = \forall_M(\bar{a})$, $\exists_M(\forall_M(\bar{a})) = \forall_M(\bar{a})$, $\forall_M(\exists_M(\bar{a})) = \exists_M(\bar{a})$,
3. $\exists_M(\bar{a} \vee \bar{b}) = \exists_M\bar{a} \vee \exists_M\bar{b}$, $\forall_M(\bar{a} \wedge \bar{b}) = \forall_M\bar{a} \wedge \forall_M\bar{b}$,
4. *if* $\bar{a} \leq \bar{b}$ *then* $\exists_M(\bar{a}) \leq \exists_M(\bar{b})$,
5. $\exists_M((0)) = (0)$, $\exists_M((1)) = (1)$, $\forall_M((0)) = (0)$, $\forall_M((1)) = (1)$,
6. $\exists_M(\exists_M\bar{a} + \exists_M\bar{b}) = \exists_M\bar{a} + \exists_M\bar{b}$, $\forall_M(\forall_M\bar{a} + \forall_M\bar{b}) = \forall_M\bar{a} + \forall_M\bar{b}$,
7. $\exists_M(\bar{a} \wedge \exists_M\bar{b}) = \exists_M\bar{a} \wedge \exists_M\bar{b}$.

In particular, $\exists_M(\bar{a})$ *is the least element of the set* $[\bar{a}) \cap M_{\exists A}$ *and* $\forall_M(\bar{a})$ *is the greatest element of* $(\bar{a}] \cap M_{\exists A}$.

Proof. Properties 1-6 are immediate from the definition of $\exists_M$ and $\forall_M$, Definition 3.1, Lemma 3.2 and Lemma 3.3. Property 7 is a consequence of Corollary 3.14. ■

Lemma 4.10. *For each good sequence* $\bar{a}$ *of elements of* $\mathbf{A} \in \mathcal{MMV}$, $\exists_M(\bar{a} + \bar{a}) = \exists_M\bar{a} + \exists_M\bar{a}$.

Proof. We may assume that $\mathbf{A}$ is subdirectly irreducible, and then $\exists A$ is totally ordered. If $\bar{a} = (0)$, the property holds. Let $\bar{a} = (a_1, a_2, \ldots, a_n)$ be a good sequence of length n, that is $a_n \neq 0$ and $a_i = 0$ for every $i > n$. Then

$$\exists_M\bar{a} = (1^{n-1}, \exists a_n),$$

where the exponent $n-1$ could be zero. Thus,

$$\exists_M\bar{a}+\exists_M\bar{a}=(1^{n-1},\exists a_n)+(1^{n-1},\exists a_n)=(1^{2(n-1)},\exists a_n\oplus\exists a_n,\exists a_n\odot\exists a_n)$$
$$=(1^{2(n-1)},\exists(a_n\oplus a_n),\exists(a_n\odot a_n)),$$

where if $\exists(a_n\oplus a_n)<1$ then $\exists(a_n\odot a_n)=0$ (Corollary 4.4).

We know that $\bar{a}+\bar{a}=\bar{s}$ with $s_i=a_i\oplus(a_{i-1}\odot a_1)\oplus(a_{i-2}\odot a_2)\oplus\ldots\oplus(a_1\odot a_{i-1})\oplus a_i$, for each $i\in\mathbb{N}$. In particular, and taking into account that $a_n\oplus a_{n-1}=a_{n-1}$ if and only if $\neg a_n\oplus\neg a_{n-1}=\neg a_n$ if and only if $a_n\odot a_{n-1}=a_n$, we have

$$s_{2n-1}=(a_n\odot a_{n-1})\oplus(a_{n-1}\odot a_n)=a_n\oplus a_n,$$

$$s_{2n}=a_n\odot a_n,$$

$$s_{2n+1}=0.$$

Now, $s_{2n-1}\neq 0$, since $a_n\neq 0$. Thus,

$$\exists_M(\bar{a}+\bar{a})=(1^{2n-2},\exists(a_n\oplus a_n),\exists(a_n\odot a_n))=\exists_M\bar{a}+\exists_M\bar{a}. \qquad \blacksquare$$

The following result will play an important role in the definition of the existential quantifier on Chang's ℓ-group.

Lemma 4.11. *Let* $\mathbf{A}\in\mathcal{MMV}$*. If* $\bar{a},\bar{b}$ *are good sequences of* $\mathbf{A}$ *such that* $\bar{a}\wedge\bar{b}=(0)$ *then*

$$\exists_M\bar{a}\wedge\forall_M\bar{b}=(0).$$

Proof. Since $\forall_M\bar{b}\leq\bar{b}$, we have that $\bar{a}\wedge\forall_M\bar{b}\leq\bar{a}\wedge\bar{b}=(0)$. So $\bar{a}\wedge\forall_M\bar{b}=(0)$ and consequently $\exists_M(\bar{a}\wedge\forall_M\bar{b})=(0)$. On the other hand, $\exists_M\forall_M\bar{b}=\forall_M\bar{b}$. Then, by property 7 above

$$\exists_M\bar{a}\wedge\forall_M\bar{b}=\exists_M\bar{a}\wedge\exists_M\forall_M\bar{b}=\exists_M(\bar{a}\wedge\exists_M\forall_M\bar{b})=\exists_M(\bar{a}\wedge\forall_M\bar{b})=(0). \quad \blacksquare$$

5. Monadic ℓ-groups

The objective of this section is to define quantifiers on Chang's ℓ-group $\mathbf{G}_A$ of an MMV-algebra $\mathbf{A}$ in order to transform $\mathbf{G}_A$ into a monadic ℓ-group.

For an MMV-algebra $\mathbf{A}$, consider Chang's ℓ-group $\mathbf{G}_A$. We define $\exists_G\colon\mathbf{G}_A\to\mathbf{G}_A$ by

$$\exists_G([\bar{a},\bar{b}])=[\exists_M((\bar{a}\vee\bar{b})-\bar{b}),\forall_M((\bar{a}\vee\bar{b})-\bar{a})].$$

Let us see that $\exists_G$ is well defined. Suppose that $(\bar{a}, \bar{b}) \equiv (\bar{c}, \bar{d})$. Then $\bar{a}+\bar{d} = \bar{b}+\bar{c}$ and $(\bar{a}+\bar{d})\vee(\bar{b}+\bar{d}) = (\bar{b}+\bar{c})\vee(\bar{b}+\bar{d})$, that is, $(\bar{a}\vee\bar{b})+\bar{d} = (\bar{d}\vee\bar{c})+\bar{b}$. So $(\bar{a}\vee\bar{b})-\bar{b} = (\bar{d}\vee\bar{c})-\bar{d}$. Similarly $(\bar{a}\vee\bar{b})-\bar{a} = (\bar{d}\vee\bar{c})-\bar{c}$. Consequently, $\exists_G([\bar{a},\bar{b}]) = \exists_G([\bar{c},\bar{d}])$.

Lemma 5.1. *Let $\mathbf{A} \in \mathcal{MV}$. If $\bar{a}, \bar{b}$ are good sequences of $\mathbf{A}$ such that $\bar{a}\wedge\bar{b} = (0)$ then $\bar{a}+\bar{b} = \bar{a}\vee\bar{b}$.*

Proof. From the definition of $+$ in $\mathbf{M}_A$ it follows that $\bar{a}+\bar{b} \geq \bar{a}\vee\bar{b}$. Since $\mathbf{M}_A$ is isomorphic to $\mathbf{G}_A^+$ we have that $(\bar{a}+\bar{b})-(\bar{a}\vee\bar{b}) = ((\bar{a}+\bar{b})-\bar{a}) \wedge ((\bar{a}+\bar{b})-\bar{b}) = \bar{a}\wedge\bar{b} = (0)$. ∎

Corollary 5.2. *Let $\mathbf{A}$ be an MMV-algebra and let $\mathbf{G}_A$ be the Chang ℓ-group of $\mathbf{A}$. If $\bar{a}\wedge\bar{b} = (0)$ then $\exists_G([\bar{a},\bar{b}]) = [\exists_M(\bar{a}), \forall_M(\bar{b})]$.*

Observe that $[\bar{a},\bar{b}] = [(\bar{a}\vee\bar{b})-\bar{b}, (\bar{a}\vee\bar{b})-\bar{a}]$ and $((\bar{a}\vee\bar{b})-\bar{b})\wedge((\bar{a}\vee\bar{b})-\bar{a}) = (\bar{a}\vee\bar{b})-(\bar{a}\vee\bar{b}) = (0)$. Consequently, given any class $[\bar{c},\bar{d}] \in \mathbf{G}_A$, we can always choose elements $\bar{a}$ and $\bar{b}$ as representatives of the class $[\bar{c},\bar{d}]$ in such a way that $\bar{a}\wedge\bar{b} = (0)$.

Lemma 5.3. *Let $\mathbf{A}$ be an MMV-algebra and let $\mathbf{G}_A$ be the Chang ℓ-group of $\mathbf{A}$. The mapping $\exists_G\colon \mathbf{G}_A \to \mathbf{G}_A$ defined above satisfies the following properties.*

(1) $[\bar{a},\bar{b}] \preceq \exists_G[\bar{a},\bar{b}]$,

(2) $\exists_G\exists_G[\bar{a},\bar{b}] = \exists_G[\bar{a},\bar{b}]$,

(3) *if* $[\bar{a},\bar{b}] \preceq [\bar{c},\bar{d}]$ *then* $\exists_G[\bar{a},\bar{b}] \preceq \exists_G[\bar{c},\bar{d}]$.

That is, $\exists_G$ is a closure operator.

Proof. Since $\forall_M((\bar{a}\vee\bar{b})-\bar{a}) \leq (\bar{a}\vee\bar{b})-\bar{a}$ and $(\bar{a}\vee\bar{b})-\bar{b} \leq \exists_M((\bar{a}\vee\bar{b})-\bar{b})$, we have $\bar{a}+\forall_M((\bar{a}\vee\bar{b})-\bar{a}) \leq \bar{a}\vee\bar{b} \leq \bar{b}+\exists_M((\bar{a}\vee\bar{b})-\bar{b})$. So $[\bar{a},\bar{b}] \preceq [\exists_M((\bar{a}\vee\bar{b})-\bar{b}), \forall_M((\bar{a}\vee\bar{b})-\bar{a})]$, and we have (1).

For (2), let $[\bar{c},\bar{d}] = [\bar{a},\bar{b}]$ such that $\bar{c}\wedge\bar{d} = (0)$. By Corollary 5.2, we have $\exists_G[\bar{a},\bar{b}] = \exists_G[\bar{c},\bar{d}] = [\exists_M\bar{c}, \forall_M\bar{d}]$. By Lemma 4.11 we have $\exists_M\bar{c}\wedge\forall_M\bar{d} = (0)$. Hence, again by Corollary 5.2,

$$\exists_G\exists_G[\bar{c},\bar{d}] = \exists_G[\exists_M\bar{c}, \forall_M\bar{d}] = [\exists_M\exists_M\bar{c}, \forall_M\forall_M\bar{d}] = [\exists_M\bar{c}, \forall_M\bar{d}] = \exists_G[\bar{c},\bar{d}].$$

Let us prove (3). Suppose that $[\bar{a},\bar{b}] \preceq [\bar{c},\bar{d}]$. Then $\bar{a}+\bar{d} \leq \bar{b}+\bar{c}$. So $(\bar{a}+\bar{d})\vee(\bar{a}+\bar{c}) \leq (\bar{b}+\bar{c})\vee(\bar{a}+\bar{c})$, and then $(\bar{c}\vee\bar{d})+\bar{a} \leq (\bar{a}\vee\bar{b})+\bar{c}$, that is, $(\bar{c}\vee\bar{d})-\bar{c} \leq (\bar{a}\vee\bar{b})-\bar{a}$. Hence $\forall_M((\bar{c}\vee\bar{d})-\bar{c}) \leq \forall_M((\bar{a}\vee\bar{b})-\bar{a})$.

On the other hand, from the hypothesis we have $(\bar{a}+\bar{d}) \vee (\bar{b}+\bar{d}) \leq (\bar{b}+\bar{c}) \vee (\bar{b}+\bar{d})$, so $(\bar{a} \vee \bar{b}) + \bar{d} \leq (\bar{c} \vee \bar{d}) + \bar{b}$, that is, $(\bar{a} \vee \bar{b}) - \bar{b} \leq (\bar{c} \vee \bar{d}) - \bar{d}$, and then $\exists_M((\bar{a} \vee \bar{b}) - \bar{b}) \leq \exists_M((\bar{c} \vee \bar{d}) - \bar{d})$. Therefore

$$\exists_M((\bar{a} \vee \bar{b}) - \bar{b}) + \forall_M((\bar{c} \vee \bar{d}) - \bar{c}) \leq \forall_M((\bar{a} \vee \bar{b}) - \bar{a}) + \exists_M((\bar{c} \vee \bar{d}) - \bar{d}),$$

and

$$[\exists_M((\bar{a} \vee \bar{b}) - \bar{b}), \forall_M((\bar{a} \vee \bar{b}) - \bar{a})] \preceq [\exists_M((\bar{c} \vee \bar{d}) - \bar{d}), \forall_M((\bar{c} \vee \bar{d}) - \bar{c})]. \quad \blacksquare$$

Let us consider the ℓ-group $\mathbf{G}_{\exists A}$. It is clear that $\mathbf{G}_{\exists A}$ is a sublattice and a subgroup of $\mathbf{G}_A$. Let $[\bar{c}, \bar{d}] \in G_{\exists A}$ such that $\bar{c}, \bar{d} \in M_{\exists A}$. Hence $\exists_G[\bar{c}, \bar{d}] = [\bar{c}, \bar{d}]$ since $\forall_M((\bar{c} \vee \bar{d}) - \bar{c}) = (\bar{c} \vee \bar{d}) - \bar{c}$ and $\exists_M((\bar{c} \vee \bar{d}) - \bar{d}) = (\bar{c} \vee \bar{d}) - \bar{d}$. Then $\exists_G(G_A) = G_{\exists A}$.

Corollary 5.4. *For each $[\bar{a}, \bar{b}] \in G_A$, $\exists_G[\bar{a}, \bar{b}]$ is the least element of $\big[[\bar{a}, \bar{b}]\big)$ $\cap\, G_{\exists A}$. In addition $\exists_G$ is an additive closure operator, that is, $\exists_G([\bar{a}, \bar{b}] \vee [\bar{c}, \bar{d}]) = \exists_G([\bar{a}, \bar{b}]) \vee \exists_G([\bar{c}, \bar{d}])$.*

Proof. The first statement follows from the fact that $\exists_G$ is a closure operator and $G_{\exists A}$ is the set of closed subsets of $\exists_G$. From this and the property that $\mathbf{G}_{\exists A}$ is a sublattice of $\mathbf{G}_A$ we have that $\exists_G$ is an additive closure operator. $\blacksquare$

Lemma 5.5. *Let $\mathbf{A}$ be an MMV-algebra and let $\mathbf{G}_A$ be the Chang ℓ-group of $\mathbf{A}$. The mapping $\exists_G \colon \mathbf{G}_A \to \mathbf{G}_A$ defined above satisfies the following properties.*

(1) $\exists_G([(0), (0)]) = [(0), (0)]$,

(2) $\exists_G([(1), (0)]) = [(1), (0)]$,

(3) $\exists_G - \exists_G[\bar{a}, \bar{b}] = -\exists_G[\bar{a}, \bar{b}]$,

(4) $\exists_G(\exists_G[\bar{a}, \bar{b}] + \exists_G[\bar{c}, \bar{d}]) = \exists_G[\bar{a}, \bar{b}] + \exists_G[\bar{c}, \bar{d}]$,

(5) *if* $[\bar{a}, \bar{b}] \in G_A^+$ *then* $\exists_G\big([\bar{a}, \bar{b}] + [\bar{a}, \bar{b}]\big) = \exists_G[\bar{a}, \bar{b}] + \exists_G[\bar{a}, \bar{b}]$,

(6) *if* $[\bar{a}, \bar{b}] \in G_A^+$ *then* $\exists_G\big([\bar{a}, \bar{b}] \wedge [(1), (0)]\big) = \exists_G[\bar{a}, \bar{b}] \wedge [(1), (0)]$,

(7) $\exists_G\big([\bar{a}, \bar{b}] \wedge [(0), (0)]\big) = \exists_G[\bar{a}, \bar{b}] \wedge [(0), (0)]$.

Proof. Properties (1) and (2) follow from the definition. From $\exists_G(G_A) = G_{\exists A}$ and the property that $\mathbf{G}_{\exists A}$ is a subgroup of $\mathbf{G}_A$ we have (3) and (4).

To prove (5), suppose that $[\bar{a}, \bar{b}] \in G_A^+$. Then $[\bar{a}, \bar{b}] = [\bar{e}, (0)]$, where $\bar{e} = \bar{a} - \bar{b}$. Thus, $\exists_G\big([\bar{a}, \bar{b}] + [\bar{a}, \bar{b}]\big) = \exists_G([\bar{e}, (0)] + [\bar{e}, \bar{0}]) = \exists_G([\bar{e} + \bar{e}, (0)])$.

Hence, by Lemma 4.10, $[\exists_M(\bar{e}+\bar{e}),(0)] = [\exists_M\bar{e}+\exists_M\bar{e},(0)] = [\exists_M\bar{e},(0)] + [\exists_M\bar{e},(0)] = \exists_G[\bar{e},(0)] + \exists_G[\bar{e},(0)] = \exists_G[\bar{a},\bar{b}] + \exists_G[\bar{a},\bar{b}]$.

For (6), if $[\bar{a},\bar{b}] \in G_A^+$ then $\exists_G\left([\bar{a},\bar{b}] \wedge [(1),(0)]\right) = \exists_G\left([\bar{e},(0)] \wedge [(1),(0)]\right) = [\exists_M(\bar{e}\wedge(1)),(0)] = [\exists_M\bar{e}\wedge(1),(0)] = [\exists_M\bar{e},(0)] \wedge [(1),(0)] = \exists_G[\bar{e},(0)] \wedge [(1),(0)] = \exists_G[\bar{a},\bar{b}] \wedge [(1),(0)]$.

Let us prove (7). Let $[\bar{a},\bar{b}] \in G_A$ and suppose that we have chosen $\bar{a}$ and $\bar{b}$ such that $\bar{a}\wedge\bar{b} = (0)$. Then

$$\exists_G\left([\bar{a},\bar{b}] \wedge [(0),(0)]\right) = \exists_G\left([\bar{a}\wedge\bar{b},\bar{b}]\right) = \exists_G\left([(0),\bar{b}]\right) = [(0),\forall_M\bar{b}].$$

On the other hand, from Corollary 5.2 and Lemma 4.11 we have that

$$\exists_G[\bar{a},\bar{b}] \wedge [(0),(0)] = [\exists_M\bar{a},\forall_M\bar{b}] \wedge [(0),(0)] = [\exists_M\bar{a}\wedge\forall_M\bar{b},\forall_M\bar{b}] = [(0),\forall_M\bar{b}].$$

■

The above induces the notion of a monadic ℓ-group.

Definition 5.6. Let $\langle G;+,-,0,\leq\rangle$ be an ℓ-group and $u>0$ a fixed element of G. We say that $\mathbf{G} = \langle G;+,-,0,\leq,u,\exists\rangle$ is a monadic ℓ-group if $\exists\colon G\to G$ satisfies the following conditions:

(G1) $x \leq \exists x$,

(G2) $\exists(x\vee y) = \exists x \vee \exists y$,

(G3) $\exists 0 = 0$,

(G4) $\exists u = u$,

(G5) $\exists(-\exists x) = -\exists x$,

(G6) $\exists(\exists x + \exists y) = \exists x + \exists y$,

(G7) if $x\in G^+$ then $\exists(x\wedge u) = \exists x \wedge u$,

(G8) if $x\in G^+$ then $\exists(x+x) = \exists x + \exists x$,

(G9) $\exists(x\wedge 0) = \exists x \wedge 0$.

Observe that (G7) and (G8) can be written as equations since for each $x\in G$, $x\vee 0\in G^+$, and if $x\in G^+$ then $x = x\vee 0$. Consequently, the class of all monadic ℓ-groups is a variety.

We shall write $\langle G;u,\exists\rangle$ instead of $\langle G;+,-,0,\leq,u,\exists\rangle$.

Note that in any monadic ℓ-group the following equation holds.

(G10) $\exists\exists x = \exists x$.

Indeed, by (G6) and (G3), $\exists\exists x = \exists(\exists x + 0) = \exists(\exists x + \exists 0) = \exists x + \exists 0 = \exists x$.

Let $\langle G; u, \exists\rangle$ a monadic ℓ-group and $\exists G = \{\exists x : x \in G\}$. From (G10) we have that $x \in \exists G$ if and only if $x = \exists x$. In addition, by (G3), (G6) and (G5), $\exists\mathbf{G}$ is a subgroup of $\mathbf{G}$.

LEMMA 5.7. *The following properties hold for every x, y in a monadic ℓ-group $\langle G; u, \exists\rangle$:*

(G11) *if $x \leq y$ then $\exists x \leq \exists y$,*

(G12) $\exists(x + y) \leq \exists x + \exists y$*,*

(G13) $\exists(x - \exists y) = \exists x - \exists y$*.*

PROOF. If $x \leq y$ then $x \vee y = y$. By (G2) we have $\exists(x \vee y) = \exists x \vee \exists y = \exists y$. That is $\exists x \leq \exists y$, and we have (G11).

For (G12), since $x \leq \exists x$ and $y \leq \exists y$ we have $x + y \leq \exists x + \exists y$. Hence, by (G11) and (G6), $\exists(x + y) \leq \exists(\exists x + \exists y) = \exists x + \exists y$.

Finally, $\exists(x - \exists y) + \exists y = \exists(x + (-\exists y)) + \exists y \leq \exists x + \exists(-\exists y) + \exists y = \exists x - \exists y + \exists y = \exists x$. On the other hand, $\exists x = \exists(x - \exists y + \exists y) \leq \exists(x - \exists y) + \exists\exists y = \exists(x - \exists y) + \exists y$. So, $\exists(x - \exists y) + \exists y = \exists x$, that is, $\exists(x - \exists y) = \exists x - \exists y$. ■

From (G11), (G1), (G10) and (G2), we have that $\exists$ is an additive closure operator on the lattice $\langle G, \leq\rangle$. Furthermore,

(G14) $\exists(\exists x \wedge \exists y) = \exists x \wedge \exists y$,

(G15) $\exists\mathbf{G}$ is a sublattice of $\mathbf{G}$,

(G16) for each $a \in G$, there exists the least element of $[a) \cap \exists G$.

THEOREM 5.8. *Let* $\mathbf{A}$ *be an MMV-algebra. Then* $\langle G_A; [(1), (0)], \exists_G\rangle$ *is a monadic ℓ-group, which will be called the monadic Chang ℓ-group.*

PROOF. It is a consequence of Lemma 5.3, Corollary 5.4 and Lemma 5.5. ■

THEOREM 5.9. *If* $\langle G; u, \exists\rangle$ *is a monadic ℓ-group then* $\Gamma_\exists(\mathbf{G}, u, \exists) = \langle [0, u]; \oplus, 0, \neg, \exists\rangle$ *is an MMV-algebra, where* $\exists$ *is the restriction to* $[0, u]$ *of the quantifier of* $\mathbf{G}$.

PROOF. By (G11), (G3) and (G4), we have that $\exists\colon [0, u] \to [0, u]$. Let us see that $\exists$ satisfies the identities (MMV1)-(MMV6).

The identity (MMV1) is immediate by (G1).

From (G2) and recalling that the natural order in the MV-algebra $[0, u]$ coincides with the order of the group we have (MMV2).

By (G13) and (G4) we get

$$\exists\neg\exists x = \exists(u - \exists x) = \exists u - \exists x = u - \exists x = \neg\exists x,$$

and we have (MMV3).

By (G6), (G4) and (G14), we have that

$$\begin{aligned}\exists(\exists x \oplus \exists y) &= \exists((\exists x + \exists y) \wedge u) = \exists(\exists(\exists x + \exists y) \wedge \exists u) = \exists(\exists x + \exists y) \wedge \exists u \\ &= (\exists x + \exists y) \wedge u = \exists x \oplus \exists y.\end{aligned}$$

By (G8), (G4), (G13), (G3) and (G2)

$$\begin{aligned}\exists x \odot \exists x &= 0 \vee (\exists x + \exists x - u) = 0 \vee (\exists(x + x) - u) = 0 \vee (\exists(x + x) - \exists u) \\ &= 0 \vee \exists((x + x) - \exists u) = 0 \vee \exists((x + x) - u) = \exists 0 \vee \exists((x + x) - u) \\ &= \exists(0 \vee ((x + x) - u)) = \exists(x \odot x).\end{aligned}$$

Let us prove (MMV6). Note that if $x \in G^+$ then $x + x \in G^+$. So, from (G8) and (G7),

$$\exists x \oplus \exists x = (\exists x + \exists x) \wedge u = \exists(x + x) \wedge u = \exists((x + x) \wedge u) = \exists(x \oplus x). \quad \blacksquare$$

Definition 5.10. We say that $f\colon \langle G; u, \exists\rangle \to \langle H; v, \exists'\rangle$ is a monadic homomorphism of ℓ-groups, or a monadic ℓ-homomorphism, if f is both a group and lattice homomorphism that satisfies $f(u) = v$ and $f(\exists x) = \exists' f(x)$, for each $x \in G$.

If $f\colon \langle G; u, \exists\rangle \to \langle H; v, \exists'\rangle$ is a monadic ℓ-group homomorphism, then

$$\Gamma_\exists(f) = f \restriction [0, u]$$

is an MMV-algebra homomorphism. Indeed, for every $x \in [0, u]$,

$$\Gamma_\exists(f)(\exists x) = f(\exists x) = \exists' f(x) = \exists' \Gamma_\exists(f)(x).$$

Proposition 5.11. $\Gamma_\exists$ *is a functor from the category* $\mathcal{MG}$ *of monadic* ℓ*-groups into the category* $\mathcal{MMV}$ *of MMV-algebras.*

Proof. From Theorem 5.9, $\Gamma_\exists(\mathbf{G}, u)$ is an MMV-algebra, for each monadic ℓ-group $\langle G; u, \exists\rangle$, and by the previous observation, if $f\colon \langle G; u, \exists\rangle \to \langle H; v, \exists'\rangle$ is a homomorphism of monadic ℓ-groups then $\Gamma_\exists(f)$ is a homomorphism of MMV-algebras. Conditions $\Gamma_\exists(id_G) = id_{\Gamma_{\exists(G,u)}}$ and $\Gamma_\exists(g \circ f) = \Gamma_\exists(g) \circ \Gamma_\exists(f)$ are immediate. $\blacksquare$

Now we prove some properties of monadic ℓ-groups.

For a given monadic ℓ-group $\langle G; u, \exists\rangle$ we can define $\forall\colon G \to G$ by

$$\forall x = -\exists -x.$$

Consider the MMV-algebra $\mathbf{A} = \Gamma_\exists(\mathbf{G}, u)$, and define on $[0, u]$ an operator dual to $\exists$ in the following way:

$$\forall_A x = \neg\exists\neg x,$$

where $\neg x = u - x$. That is,

$$\forall_A x = u - \exists(u - x).$$

Let us see that $\forall_A x = \forall x$, for each $x \in [0, u]$, that is $\forall_A$ coincides with the universal quantifier of the monadic ℓ-group $\mathbf{G}$. From (G4), (G5) and (G13) we have that $\exists(-x) + u = \exists(-x) - (-\exists u) = \exists(-x) - \exists(-\exists u) = \exists((-x) - \exists(-\exists u)) = \exists(-x + u)$. Thus, $-u + \exists(-x + u) = \exists(-x)$, or equivalently, $u - \exists(-x + u) = -\exists(-x)$. Hence, $\forall_A x = \forall x$.

Definition 5.12. We say that an ℓ-ideal J of a monadic ℓ-group $\langle G; u, \exists\rangle$ is a *monadic ℓ-ideal* if $\exists x \in J$ whenever $x \in J$.

Observe that if J is a monadic ℓ-ideal of $\mathbf{G}$ and $x \in J$, then $\forall x \in J$. Indeed, if $x \in J$, then $-x \in J$. Thus, $\exists -x \in J$ and $-\exists -x = \forall x \in J$.

Lemma 5.13. *If $f\colon \langle G; u, \exists\rangle \to \langle H; v, \exists'\rangle$ is a monadic ℓ-homomorphism then* $\mathrm{Ker}(f)$ *is a monadic ℓ-ideal.*

Proposition 5.14. *If J is a monadic ℓ-ideal of $\langle G; u, \exists\rangle$ then $\mathbf{G}/J$ is a monadic ℓ-group.*

Proof. We know that if J is an ℓ-ideal of $\langle G; u, \exists\rangle$ then $\mathbf{G}/J$ is an ℓ-group. Define

$$\exists\colon \mathbf{G}/J \to \mathbf{G}/J$$

by $\exists(J + x) = J + \exists x$. Let us see that $\exists$ is well defined. Suppose that $J + x = J + y$. Then there exists $n \in J$ such that $x + n = y$, and thus $\exists y = \exists(x + n)$, and consequently $\exists y \le \exists x + \exists n$. On the other hand, there exists $n' \in J$ such that $y + n' = x$ (note that $n' = -n$). Hence, $\exists x = \exists(y + n') \le \exists y + \exists n'$. So we have

$$-\exists n' \le \exists y - \exists x \le \exists n.$$

Since J is monadic, $\exists n, \exists n' \in J$, and since J is a subgroup, $-\exists n' \in J$. Finally, J is convex, and this implies that $\exists y - \exists x \in J$. Therefore $J + \exists x = J + \exists y$.

It is straightforward to see that $\langle \mathbf{G}/J, \exists\rangle$ is a monadic ℓ-group. ■

PROPOSITION 5.15. *Let* $f\colon \mathbf{G} \to \mathbf{H}$ *be a monadic* ℓ*-homomorphism. Then* $\mathbf{G}/\operatorname{Ker}(f)$ *is isomorphic to the monadic* ℓ*-subgroup* $f(\mathbf{G})$ *of* $\mathbf{H}$.

Let $\mathcal{I}(\exists\mathbf{G})$ denote the lattice of ℓ-ideals of the ℓ-group $\exists\mathbf{G}$ and let $\mathcal{IM}(\mathbf{G})$ denote the lattice of monadic ℓ-ideals of the monadic ℓ-group $\mathbf{G}$.

THEOREM 5.16. *Let* $\langle G; u, \exists\rangle$ *be a monadic* ℓ*-group. Then* $\mathcal{I}(\exists\mathbf{G})$ *and* $\mathcal{IM}(\mathbf{G})$ *are isomorphic.*

PROOF. Let $\rho\colon \mathcal{I}(\exists\mathbf{G}) \to \mathcal{IM}(\mathbf{G})$ be defined by $\rho(F) = \{g \in G : \forall g \in F \text{ and } \exists g \in F\}$, for each $F \in \mathcal{I}(\exists\mathbf{G})$. Let us see that $\rho(F) \in \mathcal{IM}(\mathbf{G})$.

First we prove that $\rho(F)$ is a subgroup of $\mathbf{G}$. Indeed, $0 \in \rho(F)$ since $0 = \exists 0 = \forall 0 \in F$. Let $x \in \rho(F)$. Then $\exists x \in F$ and since F is an ℓ-ideal of $\exists\mathbf{G}$, we have that $-\exists x \in F$. Thus $\forall{-x} \in F$. Similarly, from $\forall x \in F$ we get $-\forall x \in F$, that is, $\exists{-x} \in F$. So $-x \in \rho(F)$. Consider now $x, y \in \rho(F)$ and let us prove that $x+y \in \rho(F)$. Observe that $\forall x+\forall y \leq \forall(x+y) \leq \exists(x+y) \leq \exists x+\exists y$. As F is a subgroup of $\exists\mathbf{G}$, $\forall x+\forall y \in F$ and $\exists x+\exists y \in F$, and since F is convex, we have that $\forall(x+y) \in F$ and $\exists(x+y) \in F$. Consequently, $\rho(F)$ is a subgroup of $\mathbf{G}$.

Let us prove that $\rho(F)$ is a sublattice of $\mathbf{G}$. Let $x, y \in \rho(F)$ and let us see that $x \vee y \in \rho(F)$. Since F is a sublattice of $\exists\mathbf{G}$, then $\exists(x \vee y) = \exists x \vee \exists y \in F$. The convexity of F and the condition $\forall x \vee \forall y \leq \forall(x \vee y) \leq \exists(x \vee y)$ imply $\forall(x \vee y) \in F$. So $x \vee y \in \rho(F)$. In a similar way it can be proved that $x \wedge y \in \rho(F)$.

To prove that $\rho(F)$ is convex, let $x, y \in \rho(F)$ and $g \in G$ such that $x \leq g \leq y$. Then $\exists x \leq \exists g \leq \exists y$ and $\forall x \leq \forall g \leq \forall y$, and since F is convex in $\exists\mathbf{G}$ we have that $\exists g, \forall g \in F$. Consequently $g \in \rho(F)$.

Finally, $\rho(F)$ is closed under $\exists$, being that $\exists\exists x = \exists x$ and $\forall\forall x = \forall x$, and so $\rho(F)$ is monadic.

Now we define for each $J \in \mathcal{IM}(\mathbf{G})$, $\phi(J) = J \cap \exists G$. It is clear that $\phi(J)$ is an ideal of the ℓ-group $\exists\mathbf{G}$.

Let us see now that $F = \phi(\rho(F)) = \rho(F) \cap \exists G$. Clearly, $F \subseteq \rho(F) \cap \exists G$. Let $x \in \rho(F) \cap \exists G$. By definition, $\exists x = x \in F$, and consequently $\rho(F) \cap \exists G \subseteq F$.

Conversely, let $J \in \mathcal{IM}(\mathbf{G})$. Since $\exists J \subseteq J \cap \exists G$ and $\forall J \subseteq J \cap \exists G$, it follows that $J \subseteq \rho(J \cap \exists G)$. If $g \in \rho(J \cap \exists G)$ then $\exists g \in J \cap \exists G$ and $\forall g \in J \cap \exists G$. In particular, $\exists g \in J$ and $\forall g \in J$. So by convexity, $g \in J$. Thus $J \supseteq \rho(J \cap \exists G)$. So $J = \rho(J \cap \exists G)$.

Finally, it is clear that if $F \subseteq F'$ then $\rho(F) \subseteq \rho(F')$ and if $J \subseteq J'$ then $J \cap \exists G \subseteq J' \cap \exists G$. ■

THEOREM 5.17. *Every monadic ℓ-group $\langle G; u, \exists\rangle$ is a subdirect product of monadic ℓ-groups $\langle G_i; u_i, \exists\rangle$, $i \in I$, such that $\exists G_i$ is totally ordered for each $i \in I$.*

PROOF. Consider the ℓ-group $\exists\mathbf{G}$. Then there exists a family $\mathcal{S}' = \{P_i : i \in I\}$ of prime ℓ-ideals of $\exists\mathbf{G}$ such that $\bigcap \mathcal{S}' = \{0\}$, and such that $\exists G/P_i$ is a chain for all $i \in I$. Let $\mathcal{S} = \{\rho(P_i) : i \in I\}$ and let us see that $\bigcap \mathcal{S} = \{0\}$. Let $a \in G$, $a > 0$. Thus, $\exists a > 0$. Since $\bigcap \mathcal{S}' = \{0\}$, there exists $j \in I$ such that $\exists a \notin P_j$. Hence $a \notin \rho(P_j)$, and then $\bigcap \mathcal{S} = \{0\}$. Let us prove that $\exists\,(G/\rho(P_i))$ is totally ordered. Consider two equivalence classes $\rho(P_i) + \exists x$ and $\rho(P_i) + \exists y$. Since $\exists G/P_i$ is a chain, we have that $P_i + \exists x \leq P_i + \exists y$ or $P_i + \exists y \leq P_i + \exists x$. Suppose without loss of generality that $P_i + \exists x \leq P_i + \exists y$. Then there exists $z \in P_i$ such that $\exists x \leq z + \exists y$. Since $P_i \subseteq \rho(P_i)$ we have that $z \in \rho(P_i)$. Thus, $\rho(P_i) + \exists x \leq \rho(P_i) + \exists y$. Consequently, $\exists\,(G/\rho(P_i))$ is totally ordered for each $i \in I$. ■

6. The functors $\Gamma_\exists$ and $\Xi_\exists$

In the previous section, we have defined a functor $\Gamma_\exists$ from the category of monadic ℓ-groups with strong unit into the category of MMV-algebras. The objective of this section is to invert $\Gamma_\exists$. We will define a functor $\Xi_\exists$ from the category $\mathcal{MMV}$ into the category of monadic ℓ-groups with strong unit and we will prove a natural equivalence between them.

For an MMV-algebra $\mathbf{A}$ we define $\Xi_\exists(\mathbf{A})$ as the monadic Chang ℓ-group of $\mathbf{A}$, that is, $\Xi(\mathbf{A}) = \langle G_A; [(1),(0)], \exists_G\rangle$ (see Theorem 5.8).

Let $h\colon \langle A; \exists\rangle \to \langle B; \exists'\rangle$ be an MMV-algebra homomorphism and consider the monoid and lattice homomorphism

$$h^*\colon \mathbf{M}_A \to \mathbf{M}_B$$

definded by $h^*((a_1, a_2, \ldots, a_n)) = (h(a_1), h(a_2), \ldots, h(a_n))$. Let us see that

$$h^*(\exists_M(a_1, a_2, \ldots, a_n)) = \exists'_M(h^*(a_1, a_2, \ldots, a_n)).$$

Indeed,

$$\begin{aligned} &h^*(\exists_M(a_1, a_2, \ldots, a_n)) = h^*((\exists a_1, \exists a_2, \ldots, \exists a_n)) \\ &= (h(\exists a_1), h(\exists a_2), \ldots, h(\exists a_n)) = (\exists' h a_1, \exists' h a_2, \ldots, \exists' h a_n) \\ &= \exists'_M(h(a_1), h(a_2), \ldots, h(a_n)) = \exists'_M h^*((a_1, a_2, \ldots, a_n)). \end{aligned}$$

Similarly we can prove that for all $\bar{a} \in M_A$

$$h^*(\forall_M \bar{a}) = \forall'_M(h^* \bar{a}).$$

We now define

$$\Xi_\exists(h)\colon \mathbf{G}_A \to \mathbf{G}_B$$

by $\Xi_\exists(h)([\bar{a},\bar{b}]) = [h^*(\bar{a}), h^*(\bar{b})]$. If u_A and u_B are the strong units of $\mathbf{G}_A$ and $\mathbf{G}_B$ respectively then, by definition, $\Xi_\exists(h)(u_A) = u_B$. Moreover, $\Xi_\exists(h)$ is a unital ℓ-group homomorphism [4]. Let us see that $\Xi_\exists(h)$ is a monadic homomorphism, that is, let us prove that

$$\Xi_\exists(h)\exists_G[\bar{a},\bar{b}] = \exists'_{G'}\Xi_\exists(h)[\bar{a},\bar{b}].$$

We have

$$\begin{aligned}
&\Xi_\exists(h)\exists_G[\bar{a},\bar{b}] = \Xi_\exists(h)\left[\exists_M(\bar{a}\vee\bar{b}-\bar{b}), \forall_M(\bar{a}\vee\bar{b}-\bar{a})\right]\\
&= \left[h^*\exists_M(\bar{a}\vee\bar{b}-\bar{b}), h^*\forall_M(\bar{a}\vee\bar{b}-\bar{a})\right]\\
&= \left[\exists'_M h^*(\bar{a}\vee\bar{b}-\bar{b}), \forall'_M h^*(\bar{a}\vee\bar{b}-\bar{a})\right]\\
&= \left[\exists'_M(h^*\bar{a}\vee h^*\bar{b}-h^*\bar{b}), \forall'_M(h^*\bar{a}\vee h^*\bar{b}-h^*\bar{a})\right]\\
&= \exists'_{G'}[h^*\bar{a}, h^*\bar{b}] = \exists'_{G'}\Xi_\exists(h)[\bar{a},\bar{b}].
\end{aligned}$$

Proposition 6.1. $\Xi_\exists$ *is a functor from the category* $\mathcal{MMV}$ *into the category* $\mathcal{MG}$ *of monadic* ℓ*-groups with strong unit.*

Proof. From Theorem 5.8 we have that $\Xi_\exists(\mathbf{A})$ is a monadic ℓ-group, for each $\mathbf{A} \in \mathcal{MMV}$, and from the previous observation if $h\colon \langle A;\exists\rangle \to \langle B;\exists'\rangle$ is an MMV-algebra homomorphism then $\Xi_\exists(h)\colon \mathbf{G}_A \to \mathbf{G}_B$ is a monadic ℓ-homomorphism. The properties about the identity and the composition are consequences of the fact that Ξ is a functor from the category of MV-algebras into the category of ℓ-groups with strong unit. ■

Theorem 6.2. *The composite functor* $\Gamma_\exists\Xi_\exists$ *is naturally equivalent to the identity functor in the category* $\mathcal{MMV}$*. More precisely, the correspondence* $\varphi_A\colon \mathbf{A} \to \Gamma_\exists\Xi_M(\mathbf{A})$ *defined by* $\varphi_A(a) = [(a),(0)]$ *is an MMV-algebra isomorphism, and for each pair of MMV-algebras* $\mathbf{A}$ *and* $\mathbf{B}$*, and each monadic homomorphism* $h\colon \mathbf{A} \to \mathbf{B}$*, the diagram*

$$\begin{array}{ccc}
\mathbf{A} & \xrightarrow{\quad h \quad} & \mathbf{B}\\
\Big\downarrow{\scriptstyle \varphi_A} & & \Big\downarrow{\scriptstyle \varphi_B}\\
\Gamma_\exists\Xi_\exists(\mathbf{A}) & \xrightarrow{\Gamma_\exists\Xi_\exists(h)} & \Gamma_\exists\Xi_\exists(\mathbf{B})
\end{array}$$

commutes.

Proof. Observe that if α_A is the natural isomorphism of the MV-reduct of $\mathbf{A}$ on $\Gamma\Xi(\mathbf{A})$, then $\alpha_A = \varphi_A$. So, in order to prove that φ_A is an MMV-algebra isomorphism it is just enough to check that $\varphi_A(\exists a) = \exists_G \varphi_A(a)$, for all $a \in A$. Indeed,

$$\exists_G \varphi_A(a) = \exists_G[(a),(0)] = [\exists_M((a)), \forall_M((0))] = [(\exists a),(0)] = \varphi_A(\exists a).$$

The commutativity of the diagram is a consequence of Proposition 5.11, Proposition 6.1, the definition of φ and the fact that α is a natural transformation from $\Gamma\Xi$ to the identity. ■

In what follows we are going to prove that $\Xi_\exists\Gamma_\exists$ is naturally equivalent to the identity functor of the category of monadic ℓ-groups with strong unit.

Let $\mathbf{G}$ a monadic ℓ-group with strong unit u and consider $\Gamma_\exists(\mathbf{G}, u)$. From Lemma 2.4 we know that for each $0 \leq a \in G$ there exists a unique good sequence $g(a) = (a_1, \ldots, a_n)$ of elements of $[0, u]$ such that $a = a_1 + a_2 + \ldots + a_n$. This good sequence $g(a)$ is defined in the following way. For each $k \geq 1$ we define a_k inductively by :

$$a_1 = a \wedge u, \quad a_{k+1} = (a - a_1 - \ldots - a_k) \wedge u.$$

In addition, $a - a_1 - \ldots - a_k = (a - ku)^+$ (see [4, Lemma 7.1.3]).

Proposition 6.3. *Let* $\mathbf{G}$ *a monadic* ℓ*-group with strong unit* u *and consider* $\Gamma_\exists(\mathbf{G}, u)$*. For each* $0 \leq a \in G$*, if* $a \in \exists G$ *the good sequence* $g(a) = (a_1, \ldots, a_n)$ *of elements of* $[0, u]$ *satisfies* $\exists a_i = a_i$*. That is, if* $a \in \exists G^+$ *then* $\exists_M(g(a)) = g(\exists a) = g(a)$ *and* $\forall_M(g(a)) = g(\forall a) = g(a)$*.*

Proof. We shall prove that $\exists a_i = a_i$ for all i, by induction.

If $i = 1$ then by (G7), $\exists a_1 = \exists(a \wedge u) = \exists a \wedge u = a \wedge u = a_1$.

Suppose that $\exists a_i = a_i$, for $1 \leq i \leq k$. So,

$$a - a_1 - \ldots - a_k = a - (a_1 + \ldots + a_k) = a - (\exists a_1 + \ldots + \exists a_k) =$$
$$a - \exists(\exists a_1 + \ldots + \exists a_k) = a - \exists(a_1 + \ldots + a_k).$$

Thus, $a - \exists(a_1 + \ldots + a_k) \in G^+$, since $a - a_1 - \ldots - a_k = (a - ku)^+$. By using (G7) and (G13) we get

$$\exists a_{k+1} = \exists((a - \exists(a_1 + \ldots + a_k)) \wedge u) = \exists(a - \exists(a_1 + \ldots + a_k)) \wedge u =$$
$$(\exists a - \exists(a_1 + \ldots + a_k)) \wedge u = (a - a_1 - \ldots - a_k) \wedge u = a_{k+1}.$$ ■

We know that the correspondence $a \mapsto g(a)$ from the positive cone $\mathbf{G}^+$ on the monoid $\mathbf{M}_{\Gamma_\exists(G,u)}$ is a monoid and lattice isomorphism.

Lemma 6.4. *For each* $a \in G^+$, $g(\exists a) = \exists_M g(a)$.

Proof. From Lemma 2.4, for $a \in G^+$ there exists a unique good sequence $(a_1, \ldots, a_n)$ such that $a = a_1 + \ldots + a_n$. By Proposition 6.3, since $\exists a \in G^+$, there exists a unique good sequence $(c_1, \ldots, c_m)$ such that $c_i = \exists c_i$, for each i, and $\exists a = c_1 + \ldots + c_m$.

Since $a \le \exists a$ we have $(a_1, \ldots, a_n) \le (c_1, \ldots, c_m)$, that is, $g(a) \le g(\exists a)$. Then $\exists_M g(a) \le \exists_M g(\exists a) = g(\exists a)$.

On the other hand, $\exists a = \exists(a_1 + \ldots + a_n) \le \exists a_1 + \ldots + \exists a_n$. So, $g(\exists a) \le g(\exists a_1 + \ldots + \exists a_n) = (\exists a_1, \ldots, \exists a_n) = \exists_M g(a)$. ■

The following Lemma can be proved in a similar way.

Lemma 6.5. *For each* $a \in G^+$, $g(\forall a) = \forall_M g(a)$.

Theorem 6.6. *The mapping* $\psi_G \colon \mathbf{G} \to \mathbf{G}_{\Gamma_\exists(G,u)}$ *defined by* $\psi_G(a) = [g(a^+), g(a^-)]$ *is a monadic* ℓ*-isomorphism.*

Proof. If $\beta_{\langle G,u\rangle}$ is the ℓ-isomorphism from the ℓ-group $\mathbf{G}$ on $\mathbf{G}_{\Gamma(G,u)}$, then $\beta_{\langle G,u\rangle} = \psi_G$. Thus ψ_G is an ℓ-isomorphism, where $\psi_G(u) = [(u), (0)]$. Let us see that $\psi_G(\exists a) = \exists_G \psi_G(a)$. Taking into account the definitions of ψ_G and $\exists_G$ we have to prove that

$$\begin{aligned}&[g(\exists a \vee 0), g(-\exists a \vee 0)] = \exists_G[g(a^+), g(a^-)]\\ &= [\exists_M \left(g(a^+) \vee g(a^-) - g(a^-)\right), \forall_M \left(g(a^+) \vee g(a^-) - g(a^+)\right)].\end{aligned}$$

Observe that $a^+ + a^- = a \vee -a \vee 0 = a^+ \vee a^-$, for all $a \in G$. So as g is a $\{+, \vee\}$-preserving operation, we have that $g(a^+) + g(a^-) = g(a^+) \vee g(a^-)$. Thus

$$g(a^+) \vee g(a^-) - g(a^+) = g(a^-) \quad \text{and} \quad g(a^+) \vee g(a^-) - g(a^-) = g(a^+).$$

Hence $\exists_G[g(a^+), g(a^-)] = [\exists_M(g(a^+)), \forall_M(g(a^-))]$.

In addition, $\exists_M(g(a^+)) = g(\exists a \vee 0)$. Indeed, from Lemma 6.4, (G2) and (G3),

$$\exists_M(g(a^+)) = g(\exists(a \vee 0)) = g(\exists a \vee \exists 0) = g(\exists a \vee 0).$$

From the previous lemma and (G9)

$$\forall_M(g(a^-)) = g(\forall(a^-)) = g(\forall(-a \vee 0)) = g(\forall(-a) \vee 0) = g(-\exists a \vee 0).$$

Consequently, ψ_G is a monadic ℓ-isomorphism. ■

THEOREM 6.7. *The composition functor $\Xi_\exists\Gamma_\exists$ is naturally equivalent to the identity functor of the category of monadic ℓ-groups with strong unit. In other words, if $f\colon \langle G; u, \exists\rangle \to \langle H; v, \exists'\rangle$ is a monadic ℓ-homomorphism then the diagram*

$$\begin{array}{ccc}
\langle G; u, \exists\rangle & \xrightarrow{\quad f \quad} & \langle H; v, \exists'\rangle \\
{\scriptstyle \psi_G}\downarrow & & \downarrow{\scriptstyle \psi_H} \\
\Xi_\exists\Gamma_\exists(\mathbf{G}, u) & \xrightarrow{\Xi_\exists\Gamma_\exists(f)} & \Xi_\exists\Gamma_\exists(\mathbf{H}, v)
\end{array}$$

commutes.

PROOF. It is a consequence of Proposition 5.11, Proposition 6.1, Theorem 6.6 and the fact that $\beta = \psi$ is a natural transformation of $\Xi\Gamma$ into the identity. ■

COROLLARY 6.8. *The functor $\Gamma_\exists$ defines a natural equivalence between the category of monadic ℓ-groups with strong unit and the category of MMV-algebras.*

7. Applications

In this section we derive some applications of the equivalence between the category of ℓ-groups with strong unit and the category of MMV-algebras.

The proof of the following lemma is the same as that of [4, Th. 7.2.1].

LEMMA 7.1. *Let $\langle G; u, \exists\rangle$ and $\langle H; v, \exists'\rangle$ be monadic ℓ-groups and let $h\colon \Gamma_\exists(\mathbf{G}, u) \to \Gamma_\exists(\mathbf{H}, v)$ be a homomorphism of MMV-algebras. Then there exists a unique monadic ℓ-homomorphism $f\colon \langle G; u, \exists\rangle \to \langle H; v, \exists'\rangle$ such that $h = \Gamma_\exists(f)$. In addition, f is surjective (injective) whenever h is surjective (injective).*

Recall that if $\mathbf{G}$ is an ℓ-group with strong unit u and $\mathbf{A} = \Gamma(\mathbf{G}, u)$, then the correspondence

$$\sigma\colon \mathcal{I}(\mathbf{A}) \to \mathcal{I}(\mathbf{G})$$

defined by $\sigma(J) = \{x \in G : |x| \wedge u \in J\}$ is an order isomorphism between the set $\mathcal{I}(\mathbf{A})$ of ideals of the MV-algebra $\mathbf{A}$ and the set $\mathcal{I}(\mathbf{G})$ of ℓ-ideals of $\mathbf{G}$. The inverse isomorphism τ is given by $\tau(H) = H \cap [0, u]$ for all $H \in \mathcal{I}(\mathbf{G})$ [4, Theorem 7.2.2]. In the case of monadic ℓ-groups we have the following result.

Theorem 7.2. *Let $\langle G; u, \exists\rangle$ be a monadic ℓ-group with strong unit u, and let $\mathbf{A} = \Gamma_\exists(\mathbf{G}, u)$. Then the correspondence*

$$\sigma\colon J \mapsto \{x \in G : |x| \wedge u \in J\}$$

is an order isomorphism between the set $\mathcal{IM}(\mathbf{A})$ of monadic ideals of the MMV-algebra $\mathbf{A}$, ordered by inclusion, and the ordered set $\mathcal{IM}(\mathbf{G})$ of monadic ℓ-ideals of $\mathbf{G}$. The function $\tau\colon \mathcal{IM}(\mathbf{G}) \to \mathcal{IM}(\mathbf{A})$ defined by $\tau(H) = H \cap [0, u]$ is the inverse isomorphism.

Proof. Let J be a monadic ideal of $\mathbf{A}$. We know that $\sigma(J)$ is an ℓ-ideal of $\mathbf{G}$. Let us see that $\sigma(J)$ is a monadic ℓ-ideal. For this, let $x \in \sigma(J)$. Observe first that $|\exists x| = \exists x^+ + \forall x^-$. Indeed, by (G9),

$$\begin{aligned}|\exists x| &= (\exists x)^+ + (\exists x)^- = \exists x^+ + (-\exists x \vee 0) = \exists x^+ + (\forall{-x} \vee 0)\\ &= \exists x^+ + \forall(-x \vee 0) = \exists x^+ + \forall x^-.\end{aligned}$$

As J is a monadic ideal of $\mathbf{A}$, from $|x| \wedge u \in J$ and (G7), we have that $\exists(|x| \wedge u) = \exists|x| \wedge u \in J$.

On the other hand, $x^+ \leq |x|$, so $\exists x^+ \leq \exists|x|$ and $\exists x^+ \wedge u \leq \exists|x| \wedge u$. Consequently, $\exists x^+ \wedge u \in J$. Similarly we obtain that $\exists x^- \wedge u \in J$. So from $\forall x^- \wedge u \leq \exists x^- \wedge u$, and the fact that J is decreasing we have that $\forall x^- \wedge u \in J$. Since J is an ideal of $\mathbf{A}$, $(\exists x^+ \wedge u) \oplus (\forall x^- \wedge u) \in J$. But

$$\begin{aligned}&(\exists x^+ \wedge u) \oplus (\forall x^- \wedge u) = \big((\exists x^+ \wedge u) + (\forall x^- \wedge u)\big) \wedge u\\ &= \big(((\exists x^+ \wedge u) + \forall x^-) \wedge ((\exists x^+ \wedge u) + u)\big) \wedge u = ((\exists x^+ \wedge u) + \forall x^-) \wedge u\\ &= \big(\exists x^+ + \forall x^-\big) \wedge (u + \forall x^-) \wedge u = (\exists x^+ + \forall x^-) \wedge u \in J.\end{aligned}$$

So $|\exists x| \wedge u \in J$. Hence $\sigma(J)$ is a monadic ℓ-ideal of $\mathbf{G}$. If H is a monadic ℓ-ideal of $\mathbf{G}$, it is clear that $\tau(H) = H \cap [0, u]$ is a monadic ideal of $\mathbf{A}$. ■

From Theorem 7.2 and Theorem 5.16 we have the following.

Corollary 7.3. *Let $\langle G; u, \exists\rangle$ be a monadic ℓ-group with strong unit u, and let $\mathbf{A} = \Gamma_\exists(\mathbf{G}, u)$. Then*

$$\mathcal{IM}(\mathbf{G}) \cong \mathcal{I}(\exists\mathbf{G}) \cong \mathcal{IM}(\mathbf{A}) \cong \mathcal{I}(\exists\mathbf{A}).$$

Corollary 7.4. *Under the isomorphism established in Theorem 7.2 there is a correspondence between the set of maximal monadic ℓ-ideals of $\mathbf{G}$ and the set of maximal monadic ideals of $\Gamma_\exists(\mathbf{G}, u)$, and between the set of prime monadic ℓ-ideals of $\mathbf{G}$ and the set of prime monadic ideals of $\Gamma_\exists(\mathbf{G}, u)$.*

Theorem 7.5. *Let* $\langle G; u, \exists\rangle$ *be a monadic* ℓ*-group with strong unit* u*. Then for each monadic* ℓ*-ideal* J *de* $\mathbf{G}$*,*

$$\Gamma_\exists\,(\mathbf{G}/J, u+J) \cong \Gamma_\exists(\mathbf{G}, u)/(J \cap [0,u]).$$

Proof. Let $\pi\colon \mathbf{G} \to \mathbf{G}/J$ the natural epimorphism. From Theorem 7.2, we know that $J \cap [0,u]$ is a monadic ideal of the MMV-algebra $\Gamma_\exists(\mathbf{G}, u)$. In addition, $\Gamma_\exists(\pi)\colon \Gamma_\exists(\mathbf{G}, u) \to \Gamma_\exists\,(\mathbf{G}/J, u+J)$ is an epimorphism such that $\mathrm{Ker}(\Gamma_\exists(\pi)) = J \cap [0,u]$. Thus $\Gamma_\exists\,(\mathbf{G}/J, u+J) \cong \Gamma_\exists(\mathbf{G}, u)/(J \cap [0,u])$. ■

References

[1] Belluce, L. P., R. Grigolia, and A. Lettieri, 'Representations of Monadic MV-algebras', *Studia Logica*, 81:123–144, 2005.

[2] Chang, C. C., 'Algebraic analysis of many valued logics', *Trans. Amer. Math. Soc.*, 88:467–490, 1958.

[3] Chang, C. C., 'A New Proof of the Completeness of the Łukasiewicz Axioms', *Trans. Amer. Math. Soc.*, 93:74–80, 1959.

[4] Cignoli, R., I. L. D'Ottaviano, and D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Vol. 7 of Trends in Logic - Studia Logica Library, Kluwer Academic Publishers, Dordrecht, 2000.

[5] Cignoli, R., and D. Mundici, 'An Invitation to Chang's MV Algebras', *Advances in Algebra and Model Theory*, Essen, 1994; Dresden, 1995, Algebra Logic Appl., 9, Gordon and Breach, Amsterdam, 171–197, 1997.

[6] Cignoli, R., and D. Mundici, 'An Elementary Presentation of the Equivalence Between MV-algebras and ℓ-groups with Strong Unit', *Studia Logica*, 61:49–64, 1998.

[7] Di Nola, A., and R. Grigolia, 'On Monadic MV-algebras', *Annals of Pure and Applied Logic*, 128:125–139, 2004.

[8] Lattanzi, M., and A. Petrovich, 'A Duality for Monadic $(n+1)$-valued MV-algebras', *Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress*, Univ. Nac. del Sur, Bahía Blanca, 107–117, 2008.

[9] Mundici, D., 'Interpretation of FA C^*-algebras in Łukasiewicz Sentential Calculus', *J. Funct. Anal.* 65(1):15–63, 1986.

[10] Rutledge, J. D., 'A Preliminary Investigation of the Infinitely Many-valued Predicate Calculus', Ph. D. Thesis, Cornell University, 1959.

C. Cimadamore
Departamento de Matemática
Universidad Nacional del Sur
8000 Bahía Blanca, Argentina
crcima@criba.edu.ar

J. P. Díaz Varela
Departamento de Matemática
and Instituto de Matemática de Bahía Blanca
CONICET - Universidad Nacional del Sur
8000 Bahía Blanca, Argentina
usdiavar@criba.edu.ar