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The Canny–Emiris Conjecture for the Sparse Resultant

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Abstract

We present a product formula for the initial parts of the sparse resultant associated with an arbitrary family of supports, generalizing a previous result by Sturmfels. This allows to compute the homogeneities and degrees of this sparse resultant, and its evaluation at systems of Laurent polynomials with smaller supports. We obtain an analogous product formula for some of the initial parts of the principal minors of the Sylvester-type square matrix associated with a mixed subdivision of a polytope. Applying these results, we prove that under suitable hypothesis, the sparse resultant can be computed as the quotient of the determinant of such a square matrix by one of its principal minors. This generalizes the classical Macaulay formula for the homogeneous resultant and confirms a conjecture of Canny and Emiris.

Keywords Sparse resultant · Initial part · Mixed subdivision · Macaulay formula

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1 Introduction

In [27], Macaulay introduced the notion of homogeneous resultant, extending the Sylvester resultant to systems of homogeneous polynomials in several variables with given degrees. In the same paper, he also presented an intriguing family of formulae, each of them allowing to compute it as the quotient of the determinant of a Sylvester-type square matrix by one of its principal minors.

The sparse resultant is a generalization of the homogeneous resultant to systems of multivariate Laurent polynomials with prefixed monomials. It is a basic tool of elimination theory and polynomial equation solving, and it is also connected to combinatorics, toric geometry, and hypergeometric functions, see for instance [8,16,18,33]. As a consequence, there has been a lot of interest in efficient methods for computing it, see also [5–7,10,13,22] and the references therein.

In [4,5], Canny and Emiris introduced a family of Sylvester-type square matrices whose determinants are nonzero multiples of the sparse resultant, and showed that the sparse resultant can be expressed as the gcd of several of these determinants. Besides, for each of these matrices they identified a certain principal submatrix and, following Macaulay, conjectured that the quotient of their determinants coincides with the sparse resultant, at least in some cases. Their construction relies heavily on the combinatorics of the polytopes defined as the convex hull of the exponents of the given monomials, and of a chosen family of affine functions on them. Shortly afterward, Sturmfels extended the method by allowing the use of convex piecewise affine functions on these polytopes [33].

Using this circle of ideas, the first author found a recursive procedure to build Sylvester-type square matrices with a distinguished principal submatrix, and obtained another family of formulae for the sparse resultant extending those of Macaulay for the homogeneous resultant [7]. Some connections between the D'Andrea construction and that of Canny and Emiris were explored by Emiris and Konaxis for families of monomials whose associated polytopes are scaled copies of a fixed one [11]. There are also some determinantal formulae for sparse resultants, but their applicability is limited to a short list of special cases [1,3,9,12,19,25,26,34,35].

The main result of this paper is a proof of a generalized version of the Canny–Emiris conjecture, with precise conditions for its validity. Our approach is based on a systematic study of the Canny–Emiris matrices and their interplay with mixed subdivisions of polytopes. In particular, we compute the orders and initial parts of its principal minors and establish the compatibility of this construction with the restriction of the defining data. We also prove a product formula for the initial parts of the sparse resultant, generalizing a previous one by Sturmfels [33].

Classically, sparse resultants and Canny–Emiris matrices were studied in the situation where the family of exponents of the given monomials is essential in the sense of Sturmfels, that is, when the sparse resultant does depend on all the sets of variables and, in addition, the affine span of these exponents coincides with the ambient lattice, see [33, §1] or Remark 3.5 for details. Whereas this is, without any doubt, the main case of interest, a crucial part of our analysis consists in extending and studying these notions in full generality. Having constructions and properties that behave uniformly allows us to descend to the simple cases where the result can be proved directly.

We also show that the Macaulay formula for the homogeneous resultant corresponding to the critical degree appears as a particular case of our result, thus obtaining an independent proof for it.

We next explain these results with more detail. Let $M \simeq \mathbb{Z}^n$ be a lattice of rank n. Set $\mathbb{T}_M = \mathrm{Hom}(M, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^n$ for the associated torus and, for $a \in M$, denote by $\chi^a : \mathbb{T}_M \to \mathbb{C}^\times$ the corresponding character. For $i = 0, \dots, n$ let $\mathcal{A}_i \subset M$ be a nonempty finite subset, $\boldsymbol{u}_i = \{u_{i,a}\}_{a \in \mathcal{A}_i}$ a set of $\#\mathcal{A}_i$ variables and

$$F_i = \sum_{a \in \mathcal{A}_i} u_{i,a}\, \chi^a \in \mathbb{Z}[\boldsymbol{u}_i][M]$$

the general Laurent polynomial with support equal to the subset $\mathcal{A}_i$, where $\mathbb{Z}[\boldsymbol{u}_i][M] \simeq \mathbb{Z}[\boldsymbol{u}_i][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ denotes the group $\mathbb{Z}[\boldsymbol{u}_i]$-algebra of M.

Let $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\boldsymbol{u}] = \mathbb{Z}[\boldsymbol{u}_0, \dots, \boldsymbol{u}_n]$ be the sparse resultant and the sparse eliminant associated with the family of supports $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ in the sense of [8,16]. The sparse resultant is the resultant of the multiprojective toric variety with torus $\mathbb{T}_M$ associated with $\boldsymbol{\mathcal{A}}$ in the sense of Rémond's multiprojective elimination theory, whereas the sparse eliminant corresponds to what is classically referred to as the sparse resultant, as is done in [6,18,33] for instance. Both are well defined up to the sign, the sparse resultant is a power of the sparse eliminant, and they coincide when the family of supports $\boldsymbol{\mathcal{A}}$ is essential and its affine span coincides with M, see [8] or Sect. 3 for precisions.

For each i denote by Δ_i the convex hull of $\mathcal{A}_i$ in the vector space $M_\mathbb{R} = M \otimes \mathbb{R}$ and set $\Delta = \sum_{i=0}^n \Delta_i$ for the Minkowski sum of these lattice polytopes. For a vector $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \dots, \boldsymbol{\omega}_n) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}} = \prod_{i=0}^n \mathbb{R}^{\mathcal{A}_i}$ set

$$\vartheta_{\boldsymbol{\omega}_i} : \Delta_i \longrightarrow \mathbb{R}, \ i = 0, \dots, n, \quad \text{and} \quad \Theta_{\boldsymbol{\omega}} : \Delta \longrightarrow \mathbb{R} \tag{1.1}$$

for the convex piecewise affine functions parametrizing the lower envelope of the convex hull of the lifted supports $\widehat{\mathcal{A}_i} = \{(a, \omega_{i,a})\}_{a \in \mathcal{A}_i} \subset M \times \mathbb{R}$, $i = 0, \dots, n$, and of their sum $\sum_{i=0}^n \widehat{\mathcal{A}_i} \subset M \times \mathbb{R}$, respectively. These functions define a mixed subdivision $S(\Theta_{\boldsymbol{\omega}})$ of Δ, and for each n-cell D of $S(\Theta_{\boldsymbol{\omega}})$ they also determine a decomposition

$$D = \sum_{i=0}^n D_i$$

where each D_i is a cell of the subdivision $S(\vartheta_{\boldsymbol{\omega}_i})$ of Δ_i, called the i-th component of D. We can then consider the restriction

$$\boldsymbol{\mathcal{A}}_D = (\mathcal{A}_0 \cap D_0, \dots, \mathcal{A}_n \cap D_n)$$

of the given family of supports to these components.

Our first main result, contained in Theorem 3.12, is the following factorization for the initial part of the sparse resultant with respect to $\boldsymbol{\omega}$, defined as the sum of the monomial terms whose exponents have minimal weight with respect to this vector. It generalizes a previous one by Sturmfels for the case when $\boldsymbol{\mathcal{A}}$ is essential [33, Theorem 4.1].

Theorem 1.1 *Let $\boldsymbol{\omega} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$. Then,*

$$\operatorname{init}_{\boldsymbol{\omega}}(\operatorname{Res}_{\boldsymbol{\mathcal{A}}}) = \pm \prod_{D} \operatorname{Res}_{\boldsymbol{\mathcal{A}}_D},$$

the product being over the n-cells of $S(\Theta_{\boldsymbol{\omega}})$.

A result by Philippon and the third author for the Chow weights of a multiprojective toric variety [31, Proposition 4.6] implies that the order of the sparse resultant with respect to $\boldsymbol{\omega}$ can be expressed as the mixed integral of the $\vartheta_{\boldsymbol{\omega}_i}$'s (Theorem 3.12). Applying this together with Theorem 1.1, we derive product formulae for the evaluation of $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}$ by setting some of the coefficients of the input Laurent polynomials to zero (Theorem 3.19 and Proposition 3.22), correcting and generalizing a previous one by Minimair [29], see Remark 3.23. These factorizations might be interesting from the computational point of view, since they allow to extract the sparse resultant associated with a family of supports contained in those of $\boldsymbol{\mathcal{A}}$ as a factor of such an evaluation (Remark 3.21).

Apart from being homogeneous with respect to the sets of variables $\boldsymbol{u}_i$, the sparse resultant is also homogeneous with respect to a weighted grading on $\mathbb{C}[\boldsymbol{u}]$ associated with the action of $\mathbb{T}_M$ by pullbacks on the system of Laurent polynomials $\boldsymbol{F} = (F_0, \dots, F_n)$. As another application of the Philippon–Sombra formula, we compute its degree with respect to this grading, extending a result by Gelfand, Kapranov, and Zelevinsky [18, Chapter 9, Proposition 1.3] and by Sturmfels [33, §6] (Theorem 3.16).

To state our second main result, let

$$\rho_i \colon \Delta_i \longrightarrow \mathbb{R},\ i = 0, \dots, n, \quad \text{and} \quad \rho \colon \Delta \longrightarrow \mathbb{R} \tag{1.2}$$

be the family of convex piecewise affine functions and its inf-convolution defined by a vector of $\mathbb{R}^{\boldsymbol{\mathcal{A}}}$ as in (1.1). Set $\boldsymbol{\rho} = (\rho_0, \dots, \rho_n)$ and suppose that the mixed subdivision $S(\rho)$ is tight (Definition 2.3).

Following Canny and Emiris [4,5] and Sturmfels [33], this data together with a generic translation vector $\delta \in M_{\mathbb{R}}$ determines linear subspaces of $\mathbb{C}(\boldsymbol{u})[M]^{n+1}$ and of $\mathbb{C}(\boldsymbol{u})[M]$, both of them generated by monomials indexed by the lattice points in the translated polytope $\Delta + \delta$, and such that the expression

$$(G_0, \dots, G_n) \longmapsto \sum_{i=0}^{n} G_i\, F_i$$

defines a linear map between them, see Sect. 4.1 for details. The matrix of this linear map is denoted by $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$, and we denote by $\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ the principal submatrix corresponding to the lattice points in $\Delta + \delta$ contained in the translated nonmixed n-cells of $S(\rho)$ (Definition 4.5).

There is a nice interplay between these square matrices and the mixed subdivisions of Δ that are coarser than $S(\rho)$. Let

$$\phi_i : \Delta_i \longrightarrow \mathbb{R},\ i = 0, \dots, n, \quad \text{and} \quad \phi : \Delta \longrightarrow \mathbb{R}$$

be another family of convex piecewise affine functions and its respective inf-convolution, and suppose that $S(\phi)$ is coarser than $S(\rho)$, a condition that is denoted by $S(\phi) \preceq S(\rho)$. For an n-cell D of $S(\phi)$ denote by $\boldsymbol{\rho}_D = (\rho_0|_{D_0}, \dots, \rho_n|_{D_n})$ the restriction to its components of this family of functions.

Theorem 1.2 *For $\boldsymbol{\omega} = (\phi_i(a))_{i,a} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ we have that*

$$\text{init}_{\boldsymbol{\omega}}(\det(\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})) = \prod_D \det(\mathcal{H}_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}),$$

the product being over the n-cells of $S(\phi)$.

More generally, a similar factorization holds for all the principal minors of the Canny–Emiris matrix and in particular, for the determinant of $\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ (Theorem 4.10). Hence, for the vector defined by the ϕ_i's, the initial part of each of these minors factorizes in the same way as the corresponding initial part of the sparse resultant. In contrast with the situation for the sparse resultant, we do not know if this factorization holds for every $\boldsymbol{\omega} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ and as a matter of fact, it would be most interesting to extend it to a larger class of vectors.

Another important property is that the Canny–Emiris matrices associated with the restricted data $\boldsymbol{\mathcal{A}}_D$ and $\boldsymbol{\rho}_D$ can be retrieved as the evaluation of a principal submatrix of $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ by setting some of its coefficients to zero, and that this construction is compatible with refinements of mixed subdivisions (Propositions 4.8 and 4.9). We also determine the homogeneities and degrees of $\det(\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})$ (Proposition 4.6) and show that, under a mild hypothesis, this determinant is a nonzero multiple of the sparse resultant (Proposition 4.16). As a side question, such a hypothesis does not seem necessary, and it would be interesting to get rid of it (Remark 4.20).

The Canny–Emiris conjecture [5, Conjecture 13.1] states that, if the family of supports $\boldsymbol{\mathcal{A}}$ is essential and its affine span coincides with M, then there is a family $\boldsymbol{\rho}$ of affine functions on the Δ_i's and a translation vector $\delta \in M_{\mathbb{R}}$ such that

$$\text{Elim}_{\boldsymbol{\mathcal{A}}} = \pm \frac{\det(\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})}{\det(\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})}. \tag{1.3}$$

As noted in [5, §13], this identity does not hold unconditionally since there are examples of families of convex piecewise affine functions whose associated Canny–Emiris matrix and distinguished principal submatrix do not verify it (Example 5.16).

In [7], the first author presented a recursive procedure, using several mixed subdivisions on polytopes of every possible dimension up to n, for constructing a square matrix with a distinguished principal submatrix such that the quotient of the determinants of these matrices coincides with $\text{Elim}_{\boldsymbol{\mathcal{A}}}$. In [11], Emiris and Konaxis showed that in the generalized unmixed case, the D'Andrea formula can be produced by a single mixed subdivision of Δ, at the price of adding many more points to the supports.

Our third main result gives a positive answer to a generalized version of the Canny–Emiris conjecture. To bypass the recursive steps of the previous approaches, we consider chains of mixed subdivisions of Δ

$$S(\theta_0) \preceq \cdots \preceq S(\theta_n)$$

with $S(\theta_n) \preceq S(\rho)$. The tight mixed subdivision $S(\rho)$ is said to be admissible if there is such a chain which is incremental in the sense of Definition 2.4 and satisfies the conditions in Definition 4.22.

Not every tight mixed subdivision of Δ is admissible (Example 5.16). However, for the family of supports $\boldsymbol{\mathcal{A}}$ one can always find convex piecewise affine functions $\boldsymbol{\rho} = (\rho_0, \dots, \rho_n)$ whose associated mixed subdivision $S(\rho)$ is admissible. For instance, this can be realized by considering convex piecewise affine functions as in (1.1) associated with a generic vector $\boldsymbol{\nu} = (\boldsymbol{\nu}_0, \dots, \boldsymbol{\nu}_n) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ such that $\boldsymbol{\nu}_0 \gg \cdots \gg \boldsymbol{\nu}_n = \mathbf{0}$. Moreover, this vector can be chosen so that the ρ_i's are affine (Example 2.12 and Corollary 4.25).

Theorem 1.3 *If $S(\rho)$ is admissible, then*

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm \frac{\det(\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})}{\det(\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})}.$$

In the setting of the Canny–Emiris conjecture (1.3), the sparse eliminant coincides with the sparse resultant. Hence, this statement follows from Theorem 1.3 taking a family of affine functions whose associated mixed subdivision is admissible.

The statement of Theorem 1.3 is contained in Theorem 4.27 and its proof uses a descent argument similar to that of Macaulay in [27] and the first author in [7], but its implementation is different. In contrast to these references, our approach works directly with the Canny–Emiris matrices associated with restrictions of the given data, without any need of extending the Canny–Emiris construction to a larger one. On the other hand, it is interesting to note that such an enlargement is possible, in analogy with the situation in [7,27]: the Canny–Emiris construction can be enlarged by replacing the translation vector δ by a convex piecewise affine function on a polytope, and Theorem 1.3 extends to this more general situation (Remark 4.28).

This result calls in for several research questions. To begin with, it would be interesting to extend the class of mixed subdivisions to which the quotient formula for the sparse resultant holds. Indeed, such an extension might be possible by enlarging the range of validity of Theorem 1.2. In the mean time, for computational purposes it would be interesting to have a fast way of checking if a given tight mixed subdivision of Δ is admissible. In the same line, it would be interesting to determine the probability that a given tight mixed subdivision is admissible, with respect to a suitable probability distribution.

As an application, we show that the Macaulay formula for the homogeneous resultant corresponding to the critical degree is a particular case of Theorem 1.3, thus providing an independent proof for it (Corollary 5.13). This is done by considering a specific admissible mixed subdivision of scalar multiples of the standard simplex such that its Canny–Emiris matrix and distinguished principal submatrix coincide with those in that formula (Proposition 5.9).

The paper is organized as follows. In Sect. 2 we explain the necessary notions and results from polyhedral geometry, including convex piecewise affine functions on polyhedra and their associated mixed subdivisions, and mixed volumes and integrals. In Sect. 3, we recall the basic definitions and properties of sparse resultants and study some further aspects, including their orders and initial parts, their homogeneities and corresponding degrees, and their behavior under the evaluation at systems of Laurent polynomials with smaller supports. In Sect. 4, we study Canny–Emiris matrices: their behavior under restriction of the data, the orders, initial parts, homogeneities and degrees of their principal minors, some divisibility properties of their determinants, and we give the proof of the Canny–Emiris conjecture. In Sect. 5, we study the Macaulay formula for the homogeneous resultant in the framework of the Canny–Emiris construction, and give some additional examples and observations.

2 Polyhedral Geometry

2.1 Convex Piecewise Affine Functions and Mixed Subdivisions

In this section, we study the mixed subdivisions of convex polyhedra produced by families of convex piecewise affine functions. We also introduce some notions that will play a key role in our analysis of Canny–Emiris matrices, and establish their feasibility for a given family of supports. Some of the techniques we use are similar to those in [20,21]. The necessary background on polyhedral geometry can be found in [17, Part 1].

Let $M \simeq \mathbb{Z}^n$ be a lattice of rank $n \in \mathbb{N}$ and $N = M^\vee = \mathrm{Hom}(M, \mathbb{Z}) \simeq \mathbb{Z}^n$ its dual lattice. Set $M_\mathbb{R} = M \otimes \mathbb{R} \simeq \mathbb{R}^n$ and $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$ for the associated n-dimensional vector spaces, and denote by $\langle v, x\rangle$ the pairing between $v \in N_\mathbb{R}$ and $x \in M_\mathbb{R}$.

A *convex polyhedron* of $M_\mathbb{R}$ is a subset of this vector space given as the intersection of a finite family of closed half-spaces. For a convex polyhedron Δ of $M_\mathbb{R}$ we denote by $\mathrm{ri}(\Delta)$ its *relative interior*, that is, the interior of this convex polyhedron relative to the minimal affine subspace containing it. Its *support function* is the function $h_\Delta \colon N_\mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$h_\Delta(v) = \inf\{\langle v, x\rangle \mid x \in \Delta\}. \tag{2.1}$$

The assignment $\Delta \mapsto h_\Delta$ is additive with respect to the Minkowski sum of convex polyhedra and the pointwise sum of functions.

For a vector $v \in N_\mathbb{R}$, the *face of* Δ *in the direction of* v is defined as

$$\Delta^v = \{x \in \Delta \mid \langle v, x\rangle = h_\Delta(v)\}. \tag{2.2}$$

Let $\rho \colon \Delta \to \mathbb{R}$ be a convex piecewise affine function. Its *graph* and its *epigraph* are the subsets of $M_{\mathbb{R}} \times \mathbb{R}$, respectively, defined as

$$\mathrm{gr}(\rho) = \{(x, \rho(x)) \mid x \in \Delta\} \quad \text{and} \quad \mathrm{epi}(\rho) = \{(x, z) \mid x \in \Delta, z \geq \rho(x)\}.$$

The epigraph is a convex polyhedron, whose faces of the form $\mathrm{epi}(\rho)^{(v,1)}$, $v \in N_{\mathbb{R}}$, are contained in the graph, and are called the *faces* of $\mathrm{gr}(\rho)$.

The *subdivision of* Δ *induced by* ρ, denoted by $S(\rho)$, is the polyhedral subdivision of Δ given by the image of the faces of the graph of ρ with respect to the projection $\pi \colon M_{\mathbb{R}} \times \mathbb{R} \to M_{\mathbb{R}}$. Its elements are called the *cells* of this subdivision. For $j \geq -1$, we denote by $S(\rho)^j$ the set of cells of $S(\rho)$ of dimension j, or j-*cells*. Their union gives the j-*skeleton* of $S(\rho)$, denoted by $|S(\rho)^j|$. For a vector $v \in N_{\mathbb{R}}$, the corresponding cell of $S(\rho)$ is denoted by

$$\Gamma(\rho, v) = \pi\big(\mathrm{epi}(\rho)^{(v,1)}\big). \tag{2.3}$$

For $x \in \Delta$, we have that

$$\rho(x) \geq \langle -v, x \rangle + h_{\mathrm{epi}(\rho)}(v, 1), \tag{2.4}$$

and the equality holds if and only if $x \in \Gamma(\rho, v)$.

Let $\rho \colon \Delta \to \mathbb{R}$ and $\rho' \colon \Delta' \to \mathbb{R}$ be convex piecewise affine functions on convex polyhedra. Their *inf-convolution*, denoted by $\rho \boxplus \rho'$, is the convex piecewise affine function on the Minkowski sum $\Delta + \Delta'$ defined by

$$(\rho \boxplus \rho')(x) = \inf\{\rho(y) + \rho'(y') \mid y \in \Delta, y' \in \Delta' \text{ and } x = y + y'\}. \tag{2.5}$$

Alternatively, it can be defined as the function parametrizing the lower envelope of $\mathrm{epi}(\rho) + \mathrm{epi}(\rho')$, that is,

$$(\rho \boxplus \rho')(x) = \inf\{z \in \mathbb{R} \mid (x, z) \in \mathrm{epi}(\rho) + \mathrm{epi}(\rho')\}.$$

The Minkowski sum $\mathrm{epi}(\rho) + \mathrm{epi}(\rho')$ is a convex polyhedron, and so $\rho \boxplus \rho'$ is a convex piecewise affine function on $\Delta + \Delta'$ and for every point x in this set, the infimum in (2.5) is attained.

Now for $s \in \mathbb{N}$ let $\rho_i \colon \Delta_i \to \mathbb{R}$, $i = 0, \ldots, s$, be a family of $s+1$ convex piecewise affine functions on convex polyhedra and set $\rho = \boxplus_{i=0}^{s} \rho_i$ for their inf-convolution, which is a convex piecewise affine function on the Minkowski sum $\Delta = \sum_{i=0}^{s} \Delta_i$. The subdivision $S(\rho)$ of Δ is called a *mixed subdivision* of Δ.

For $i = 0, \ldots, s$ we, respectively, denote by

$$\Delta_i^{\mathrm{c}} = \sum_{j \neq i} \Delta_j \quad \text{and} \quad \rho_i^{\mathrm{c}} = \boxplus_{j \neq i} \rho_j \tag{2.6}$$

the convex polyhedron and the convex piecewise affine function, respectively, defined by the i-th complementary Minkowski sum and by the i-th complementary inf-convolution. We have that $\Delta_i^{\mathrm{c}} + \Delta_i = \Delta$ and $\rho_i^{\mathrm{c}} \boxplus \rho_i = \rho$.

For $C \in S(\rho)$ consider the subset of $M_{\mathbb{R}}^{s+1}$ defined as

$$\Pi_C = \Big\{(x_0, \dots, x_s) \in \prod_{i=0}^{s} \Delta_i \;\Big|\; \sum_{i=0}^{s} x_i \in C \text{ and } \rho\Big(\sum_{i=0}^{s} x_i\Big) = \sum_{i=0}^{s} \rho_i(x_i)\Big\}.$$

For $i = 0, \dots, s$ let $\pi_i : M_{\mathbb{R}}^{s+1} \to M_{\mathbb{R}}$ denote the projection onto the i-th factor. The *i-th component* of C is the nonempty subset of Δ_i defined as

$$C_i = \pi_i(\Pi_C). \tag{2.7}$$

The next two results give the basic properties of the components of the cells of a mixed subdivision.

Proposition 2.1 *Let $C \in S(\rho)$. Then,*

(1) for $v \in N_{\mathbb{R}}$ such that $C = \Gamma(\rho, v)$ we have that $C_i = \Gamma(\rho_i, v) \in S(\rho_i)$ for all i,

(2) $C = \sum_{i=0}^{s} C_i$,

(3) for $x \in C$ and $x_i \in \Delta_i$, $i = 0, \dots, s$, such that $x = \sum_{i=0}^{s} x_i$ we have that $\rho(x) = \sum_{i=0}^{s} \rho_i(x_i)$ if and only if $x_i \in C_i$ for all i.

Proof Let $v \in N_{\mathbb{R}}$ and set for short $\kappa = h_{\mathrm{epi}(\rho)}(v, 1)$ and $\kappa_i = h_{\mathrm{epi}(\rho_i)}(v, 1)$ for each i. We have that $\mathrm{epi}(\rho) = \sum_{i=0}^{s} \mathrm{epi}(\rho_i)$ and so, by the additivity of the support function,

$$\kappa = \sum_{i=0}^{s} \kappa_i.$$

Let $i \in \{0, \dots, s\}$ and $x_i \in C_i$. Choose $x_j \in C_j$, $j \neq i$, such that $(x_0, \dots, x_s) \in \Pi_C$ and set $x = \sum_{j=0}^{s} x_j$, so that $x \in C$ and $\rho(x) = \sum_{j=0}^{s} \rho_j(x_j)$. Hence, $(x, \rho(x)) \in \mathrm{epi}(\rho)^{(v,1)}$ and $(x_j, \rho_j(x_j)) \in \mathrm{epi}(\rho_j)$ for all j and so

$$\kappa = \langle (v, 1), (x, \rho(x)) \rangle = \sum_{j=0}^{s} \langle (v, 1), (x_j, \rho_j(x_j)) \rangle \geq \sum_{j=0}^{s} \kappa_j = \kappa.$$

Thus, $\langle (v, 1), (x_i, \rho_i(x_i)) \rangle = \kappa_i$ or equivalently $(x_i, \rho_i(x_i)) \in \mathrm{epi}(\rho_i)^{(v,1)}$, which implies that $x_i \in \Gamma(\rho_i, v)$.

Conversely, let $x_i \in \Gamma(\rho_i, v)$. Choose $x_j \in \Gamma(\rho_j, v)$, $j \neq i$, and set $x = \sum_{j=0}^{s} x_j$ and $t = \sum_{j=0}^{s} \rho_j(x_j)$. We have that $t \geq \rho(x)$ and so $(x, t) \in \mathrm{epi}(\rho)$. Moreover

$$\langle (v, 1), (x, t) \rangle = \sum_{j=0}^{s} \langle (v, 1), (x_j, \rho_j(x_j)) \rangle = \sum_{j=0}^{s} \kappa_j = \kappa.$$

Hence, $(x, t) \in \mathrm{epi}(\rho)^{(v,1)}$ and so $x \in C$ and $t = \rho(x)$. In particular, $x_i \in C_i$ and we conclude that $C_i = \Gamma(\rho_i, v)$, proving (1).

Now let $x_i \in C_i$, $i = 0, \dots, s$, and set $x = \sum_{i=0}^{s} x_i$. By (1), for any $v \in N_{\mathbb{R}}$ such that $C = \Gamma(\rho, v)$ we have that $C_i = \Gamma(\rho_i, v)$, and the last part of the proof of this statement shows that $x \in C$ and $\rho(x) = \sum_{i=0}^{s} \rho_i(x_i)$. This proves both that $\sum_{i=0}^{s} C_i \subset C$ and the "if" part in (3).

Conversely, for each $x \in C$ the infimum in (2.5) is attained, and so there are $x_i \in C_i$, $i = 0, \dots, s$, with $x = \sum_{i=0}^{s} x_i$. Hence, $C = \sum_{i=0}^{s} C_i$ as stated in (2), whereas the "only if" part in (3) is immediate from the definition of the components in (2.7). □

Proposition 2.2 *Let $C, C' \in S(\rho)$ and $i \in \{0, \dots, s\}$ such that their respective i-th components have both dimension n and coincide. Then, $C = C'$.*

Proof Since both C_i and C'_i have dimension n and coincide, there is a unique $v \in N_{\mathbb{R}}$ with $C_i = C'_i = \Gamma(\rho_i, v)$. Proposition 2.1(1) then implies that $C = \Gamma(\rho, v) = C'$. □

Definition 2.3 A mixed subdivision $S(\rho)$ on Δ is *tight* if for every n-cell C of $S(\rho)$,

$$\sum_{i=0}^{s} \dim(C_i) = n.$$

If this condition holds, when $s = n-1$ an n-cell of $S(\rho)$ is *mixed* if all its components are segments and when $s = n$, for $k = 0, \dots, n$ an n-cell of $S(\rho)$ is *k-mixed* if its i-th component is a segment for all $i \neq k$ (and so its k-th component is a point).

The set of mixed subdivisions of Δ is partially ordered by refinements: for another mixed subdivision $S(\rho')$ of Δ given by a family of convex piecewise affine functions $\rho'_i \colon \Delta_i \to \mathbb{R}$, $i = 0, \dots, s$, we say that $S(\rho)$ is a *refinement* of $S(\rho')$, denoted by

$$S(\rho) \succeq S(\rho') \quad \text{or} \quad S(\rho') \preceq S(\rho),$$

if for all $C \in S(\rho)$ there is $D \in S(\rho')$ such that $C \subset D$ and that $C_i \subset D_i$ for all i.

Definition 2.4 An *incremental* chain of mixed subdivisions of Δ is a chain $S(\theta_0) \preceq \cdots \preceq S(\theta_s)$ where, for $k = 0, \dots, s$, the mixed subdivision $S(\theta_k)$ is induced by the inf-convolution $\theta_k \colon \Delta \to \mathbb{R}$ of a family of convex piecewise affine functions $\theta_{k,i} \colon \Delta_i \to \mathbb{R}$, $i = 0, \dots, s$, such that $\theta_{k,i} = 0|_{\Delta_i}$ for $i \geq k$.

This incremental chain is *tight* if, for each k, the mixed subdivision of Δ (considered as the sum of the $k+1$ polytopes $\Delta_0, \dots, \Delta_{k-1}, \sum_{i=k}^{s} \Delta_i$) induced by the convex piecewise affine functions

$$\theta_{k,i} \colon \Delta_i \to \mathbb{R},\ i = 0, \dots, k-1, \quad \text{and} \quad \boxplus_{i=k}^{s} \theta_{k,i} = 0\Big|_{\sum_{i=k}^{s} \Delta_i},$$

is tight in the sense of Definition 2.3.

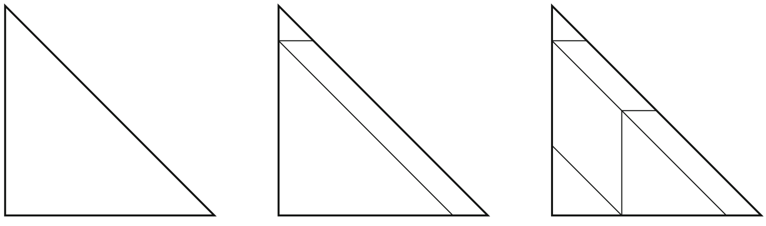

Fig. 1 A tight incremental chain

Remark 2.5 The notion of incremental chain of mixed subdivisions of Δ might be easily extended to chains of arbitrary length. We have chosen to restrict it to chains of length equal to the number of polyhedra Δ_i because it is the only case of interest for the proof of Theorem 4.27.

Remark 2.6 If $S(\theta_0) \preceq \cdots \preceq S(\theta_s)$ is a tight incremental chain, then $S(\theta_s)$ is a tight mixed subdivision.

Example 2.7 Let $n = 2$ and $M = \mathbb{Z}^2$. Set $d_0 = 1$, $d_1 = 3$ and $d_2 = 2$ and for $i = 0, 1, 2$ consider the triangle $\Delta_i = \{(x_1, x_2) \in (\mathbb{R}_{\geq 0})^2 \mid x_1 + x_2 \leq d_i\}$. Consider also the affine functions $\rho_i : \Delta_i \to \mathbb{R}$, $i = 0, 1$, defined by

$$\rho_0(x_1, x_2) = 3\,x_1 + 6\,x_2 \quad \text{and} \quad \rho_1(x_1, x_2) = 2\,x_1 + x_2.$$

For $k, i = 0, 1, 2$ set $\theta_{k,i} = \rho_i$ if $i < k$ and $\theta_{k,i} = 0|_{\Delta_i}$ if $i \geq k$, and then for $k = 0, 1, 2$ set $\theta_k = \theta_{k,0} \boxplus \theta_{k,1} \boxplus \theta_{k,2}$. Hence, $S(\theta_0) \preceq S(\theta_1) \preceq S(\theta_2)$ is a tight incremental chain of mixed subdivisions of the triangle $\Delta = \{(x_1, x_2) \in (\mathbb{R}_{\geq 0})^2 \mid x_1 + x_2 \leq 6\}$ (Fig. 1).

Convex piecewise affine functions on lattice polytopes might be constructed by means of finite sets of lattice points and lifting vectors, as we next describe. For $i = 0, \ldots, s$ let $\mathcal{A}_i \subset M$ be a nonempty finite subset and $\boldsymbol{\nu}_i \in \mathbb{R}^{\mathcal{A}_i}$ a vector. Set $\Delta_i = \operatorname{conv}(\mathcal{A}_i)$ for the lattice polytope of $M_\mathbb{R}$ given by the convex hull of $\mathcal{A}_i$ and

$$\vartheta_{\boldsymbol{\nu}_i} : \Delta_i \longrightarrow \mathbb{R} \tag{2.8}$$

for the convex piecewise affine function parametrizing the lower envelope of the lifted polytope $\operatorname{conv}(\{(a, \nu_{i,a})\}_{a \in \mathcal{A}_i}) \subset M_\mathbb{R} \times \mathbb{R}$. Set also $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \ldots, \mathcal{A}_s)$, $\boldsymbol{\nu} = (\boldsymbol{\nu}_0, \ldots, \boldsymbol{\nu}_s) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}} = \prod_{i=0}^{s} \mathbb{R}^{\mathcal{A}_i}$ and

$$\Theta_{\boldsymbol{\nu}} = \boxplus_{i=0}^{s} \vartheta_{\boldsymbol{\nu}_i}. \tag{2.9}$$

This latter is a convex piecewise affine function on the Minkowski sum $\Delta = \sum_{i=0}^{s} \Delta_i$, and $S(\Theta_{\boldsymbol{\nu}})$ is a mixed subdivision of this polytope.

The next result shows that a generic choice of lifting vectors produces a mixed subdivision that is tight. Furthermore, this choice can be made among the lifting vectors whose associated functions are affine.

Consider the linear map $T_{\mathcal{A}}\colon N^{s+1} \to \mathbb{R}^{\mathcal{A}}$ defined by

$$T_{\mathcal{A}}(v_0, \dots, v_s) = (\langle v_i, a\rangle)_{i\in\{0,\dots,s\}, a\in\mathcal{A}_i}. \tag{2.10}$$

The convex piecewise affine functions associated with the vectors in its image are affine.

The next result is similar to those in [20, page 1546] and [21, Lemma 2.1].

Proposition 2.8 *There is a finite union of hyperplanes $W \subset \mathbb{R}^{\mathcal{A}}$ not containing $T_{\mathcal{A}}(N^{s+1})$ such that for all $\boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}}\setminus W$ the mixed subdivision $S(\Theta_{\boldsymbol{\nu}})$ of Δ is tight.*

To prove it, we need the following auxiliary result. For $i = 0, \dots, s$ let e_i be the $(i+1)$-th vector in the standard basis of $\mathbb{R}^{s+1}$.

Lemma 2.9 *For $\boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}}$ let $\widehat{L} \subset M_{\mathbb{R}} \times \mathbb{R}^{s+1} \times \mathbb{R}$ denote the linear span of the vectors $\{(a, e_i, \nu_{i,a})\}_{i\in\{0,\dots,s\}, a\in\mathcal{A}_i}$. Set also $\widehat{\mathcal{A}}_i = \{(a, \nu_{i,a})\}_{a\in\mathcal{A}_i} \subset M_{\mathbb{R}} \times \mathbb{R}$, $i = 0, \dots, s$. Then,*

$$\dim(\widehat{L}) = \dim\Big(\sum_{i=0}^{s} \operatorname{conv}(\widehat{\mathcal{A}}_i)\Big) + s + 1.$$

Proof For each i let $\widehat{L}_i \subset M_{\mathbb{R}} \times \mathbb{R}$ denote the affine span of the nonempty finite subset $\widehat{\mathcal{A}}_i$. Making linear combinations between the generators of the linear subspace $\widehat{L}$, we easily deduce that $\dim(\widehat{L}) = \dim(\sum_{i=0}^{s} \widehat{L}_i) + s + 1$. The statement then follows from the fact that $\dim(\sum_{i=0}^{s} \widehat{L}_i) = \dim(\sum_{i=0}^{s} \operatorname{conv}(\widehat{\mathcal{A}}_i))$. □

Proof of Proposition 2.8 For $i = 0, \dots, s$ let $\boldsymbol{u}_i$ be a set of $\#\mathcal{A}_i$ variables and let $\boldsymbol{u} = (\boldsymbol{u}_0, \dots, \boldsymbol{u}_s)$. Fix an isomorphism $M_{\mathbb{R}} \simeq \mathbb{R}^n$. Then, for each family $\boldsymbol{\mathcal{D}} = (\mathcal{D}_0, \dots, \mathcal{D}_s)$ of nonempty subsets $\mathcal{D}_i \subset \mathcal{A}_i$ satisfying the conditions

(1) $\sum_{i=0}^{s} \#\mathcal{D}_i = n + s + 2$,
(2) $\dim(\operatorname{conv}(\mathcal{D}_i)) = \#\mathcal{D}_i - 1$, $i = 0, \dots, s$,
(3) $\dim(\sum_{i=0}^{s} \operatorname{conv}(\mathcal{D}_i)) = n$

set $\mathcal{G}_{\boldsymbol{\mathcal{D}}} \in \mathbb{R}[\boldsymbol{u}]^{\boldsymbol{\mathcal{D}}\times(n+s+2)}$ for the square matrix made of the row vectors

$$(a, e_i, u_{i,a}) \in \mathbb{R}[\boldsymbol{u}]^{n+s+2},\ i = 0, \dots, s \quad\text{and}\quad a \in \mathcal{D}_i.$$

Set also $G_{\boldsymbol{\mathcal{D}}} = \det(\mathcal{G}_{\boldsymbol{\mathcal{D}}}) \in \mathbb{R}[\boldsymbol{u}]$, which is a linear form.

The conditions on $\boldsymbol{\mathcal{D}}$ imply that for each i there is $\boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}}$ such that, setting

$$\widehat{\mathcal{D}}_i = \{(a, \nu_{i,a})\}_{a\in\mathcal{D}_i} \subset M_{\mathbb{R}} \times \mathbb{R},\ i = 0, \dots, s, \tag{2.11}$$

we have that $\dim(\sum_{i=0}^{s} \operatorname{conv}(\widehat{\mathcal{D}}_i)) = n + 1$. Moreover, this condition can be fulfilled with a vector $\boldsymbol{\nu} \in T_{\mathcal{A}}(N^{s+1})$. Indeed, the condition (2) implies that the $\mathcal{D}_i$'s are simplexes which by the condition (1), have dimensions that sum up $n + 1$. Assuming without loss of generality that each $\mathcal{D}_i$ contains 0 as one its points, the remaining points in the union of these sets determine $n+1$ directions which by condition (3) span $M_{\mathbb{R}}$. Then, a possible choice for $\boldsymbol{\nu}$ satisfying (2.11) consists in setting $\nu_{i,a}=0$ for each i and all $a \in \mathcal{D}_i$ except one of them, for which this coordinate is set to 1.

Lemma 2.9 then implies $G_{\boldsymbol{\mathcal{D}}}(\boldsymbol{\nu}) \neq 0$ and in particular, the zero set of $G_{\boldsymbol{\mathcal{D}}}$ is a hyperplane not containing the linear subspace $T_{\boldsymbol{\mathcal{A}}}(N^{s+1})$. The set W is then defined as the union of all these hyperplanes.

Now let $\boldsymbol{\nu} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ and suppose that $S(\Theta_{\boldsymbol{\nu}})$ is not tight. Let C be an n-cell of this mixed subdivision such that $\sum_{i=0}^{s} \dim(C_i) > n$. Then, we can choose nonempty finite subsets $\mathcal{D}_i \subset \mathcal{A}_i \cap C_i$, $i=0,\dots,s$, satisfying the conditions (1), (2) and (3). Such a choice may be accomplished by picking simplexes defined by points in the finite subsets $\mathcal{A}_i \cap C_i$, $i=0,\dots,s$, whose dimensions sum up $n+1$.

Let P be the face of the graph of $\Theta_{\boldsymbol{\nu}}$ corresponding to C, and for each i let P_i be the face of the graph of $\vartheta_{\boldsymbol{\nu}_i}$ corresponding to the component C_i. For each i the lifted set $\widehat{\mathcal{D}}_i$ as in (2.11) is contained in P_i and so

$$\sum_{i=0}^{s} \operatorname{conv}(\widehat{\mathcal{D}}_i) \subset \sum_{i=0}^{s} P_i = P.$$

Hence, $\dim(\sum_{i=0}^{s} \operatorname{conv}(\widehat{\mathcal{D}}_i)) \leq \dim(P) = \dim(C) = n$. Lemma 2.9 then implies that $G_{\boldsymbol{\mathcal{D}}}(\boldsymbol{\nu}) = 0$ and so $\boldsymbol{\nu} \in W$, concluding the proof. □

The next corollary shows that we might fix one of the lifting vectors to zero and still get a mixed subdivision that is tight. Set for short

$$\boldsymbol{\mathcal{A}}' = (\mathcal{A}_0, \dots, \mathcal{A}_{s-1})$$

and let $T_{\boldsymbol{\mathcal{A}}'} \colon N^s \to \mathbb{R}^{\boldsymbol{\mathcal{A}}'}$ be the corresponding linear map as in (2.10).

Corollary 2.10 *There is a finite union of hyperplanes $W' \subset \mathbb{R}^{\boldsymbol{\mathcal{A}}'}$ not containing $T_{\boldsymbol{\mathcal{A}}'}(N^s)$ such that for all $\boldsymbol{\nu}' \in \mathbb{R}^{\boldsymbol{\mathcal{A}}'} \setminus W'$ the mixed subdivision $S(\Theta_{\boldsymbol{\nu}})$ of Δ associated with the vector $\boldsymbol{\nu} = (\boldsymbol{\nu}', \mathbf{0}) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ is tight.*

Proof With notation as in Proposition 2.8, choose a vector $(w_0, \dots, w_s) \in N^{s+1}$ whose image with respect to the linear map $T_{\boldsymbol{\mathcal{A}}}$ does not lie in W. Let $\boldsymbol{\zeta}_i = (\langle w_s, a\rangle)_{a \in \mathcal{A}_i} \in \mathbb{R}^{\mathcal{A}_i}$, $i=0,\dots,s$, and

$$W' = \{(\boldsymbol{\nu}'_0, \dots, \boldsymbol{\nu}'_{s-1}) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}'} \mid (\boldsymbol{\nu}'_0 + \boldsymbol{\zeta}_0, \dots, \boldsymbol{\nu}'_{s-1} + \boldsymbol{\zeta}_{s-1}, \boldsymbol{\zeta}_s) \in W\},$$

which is a finite union of hyperplanes of $\mathbb{R}^{\boldsymbol{\mathcal{A}}'}$. We have that

$$(T_{\boldsymbol{\mathcal{A}}'}(w_0 - w_s, \dots, w_{s-1} - w_s), \mathbf{0}) + (\boldsymbol{\zeta}_0, \dots, \boldsymbol{\zeta}_{s-1}, \boldsymbol{\zeta}_s) = T_{\boldsymbol{\mathcal{A}}}(w_0, \dots, w_{s-1}, w_s) \notin W.$$

Hence, $T_{\boldsymbol{\mathcal{A}}'}(w_0 - w_s, \dots, w_{s-1} - w_s) \notin W'$, and so $W' \not\supset T_{\boldsymbol{\mathcal{A}}'}(N^{s-1})$.

By Proposition 2.8, for $\boldsymbol{\nu}' = (\boldsymbol{\nu}'_0, \dots, \boldsymbol{\nu}'_{s-1}) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}'} \setminus W'$ the mixed subdivision associated with $(\boldsymbol{\nu}'_0 + \boldsymbol{\zeta}_0, \dots, \boldsymbol{\nu}'_{s-1} + \boldsymbol{\zeta}_{s-1}, \boldsymbol{\zeta}_s) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ is tight. Hence, this is also the case for the mixed subdivision associated with the vector $(\boldsymbol{\nu}', \mathbf{0}) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$, since the corresponding functions differ by a globally defined linear one. □

The next result shows that small perturbations of a given family of lifting vectors produce finer mixed subdivisions.

Proposition 2.11 *Let $\boldsymbol{\nu} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$. There is a neighborhood U of $\boldsymbol{\nu}$ such that for all $\widetilde{\boldsymbol{\nu}} \in U$ we have that $S(\Theta_{\widetilde{\boldsymbol{\nu}}}) \succeq S(\Theta_{\boldsymbol{\nu}})$.*

Proof For each n-cell C of $S(\Theta_{\boldsymbol{\nu}})$ denote by v_C the unique vector in $N_{\mathbb{R}}$ such that $C = \Gamma(\Theta_{\boldsymbol{\nu}}, v_C)$. By Proposition 2.1(1) and the inequality in (2.4), for each i there is $\kappa_{C,i} \in \mathbb{R}$ such that, for $x \in \Delta_i$,

$$\vartheta_{\boldsymbol{\nu}_i}(x) \geq \langle -v_C, x\rangle + \kappa_{C,i}$$

with equality if and only if $x \in C_i$. Hence, there is $c > 0$ such that for all $a \in \mathcal{A}_i \setminus C_i$,

$$\vartheta_{\boldsymbol{\nu}_i}(a) \geq \langle -v_C, a\rangle + \kappa_{C,i} + c. \tag{2.12}$$

Let $\varepsilon > 0$ and $\widetilde{\boldsymbol{\nu}} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ with $\|\widetilde{\boldsymbol{\nu}} - \boldsymbol{\nu}\|_\infty < \varepsilon$, where $\|\cdot\|_\infty$ denotes the ℓ^∞-norm of $\mathbb{R}^{\boldsymbol{\mathcal{A}}}$. Then, for all i and $x \in \Delta_i$ we have that

$$|\vartheta_{\widetilde{\boldsymbol{\nu}}_i}(x) - \vartheta_{\boldsymbol{\nu}_i}(x)| < \varepsilon. \tag{2.13}$$

Fix a norm $\|\cdot\|$ on $M_{\mathbb{R}}$ and let $\|\cdot\|^\vee$ be the corresponding operator norm on $N_{\mathbb{R}}$, so that for $v \in N_{\mathbb{R}}$ and $x \in M_{\mathbb{R}}$ we have that

$$|\langle v, x\rangle| \leq \|v\|^\vee \, \|x\|. \tag{2.14}$$

Let $\widetilde{C}$ be an n-cell of $S(\Theta_{\widetilde{\boldsymbol{\nu}}})$ and, similarly as before, denote by $v_{\widetilde{C}} \in N_{\mathbb{R}}$ and $\kappa_{\widetilde{C},i} \in \mathbb{R}$, $i = 0, \dots, s$, the corresponding vector and constants. Then, for each n-cell C of $S(\Theta_{\boldsymbol{\nu}})$ with $\dim(\widetilde{C} \cap C) = n$ there is $K > 0$ such that

$$\|v_{\widetilde{C}} - v_C\|^\vee, |\kappa_{\widetilde{C},i} - \kappa_{C,i}| < K\,\varepsilon. \tag{2.15}$$

Since the number of possible pairs $(\widetilde{C}, C)$ for varying $\widetilde{\boldsymbol{\nu}} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ is finite, the constant $K > 0$ can be taken independently of the choice of these n-cells.

From the inequalities in (2.12) and (2.13), we deduce that for all $a \in \mathcal{A}_i \setminus C_i$,

$$\vartheta_{\widetilde{\boldsymbol{\nu}}_i}(a) > \vartheta_{\boldsymbol{\nu}_i}(a) - \varepsilon \geq \langle -v_C, a\rangle + \kappa_{C,i} + c - \varepsilon,$$

and from the inequalities in (2.14) and (2.15),

$$\langle -v_C, a\rangle + \kappa_{C,i} > \langle -v_{\widetilde{C}}, a\rangle + \kappa_{\widetilde{C},i} - K\varepsilon\Big(\sup_{x\in\Delta_i} \|x\| + 1\Big).$$

Then, for $\varepsilon > 0$ sufficiently small we have that $\vartheta_{\widetilde{\boldsymbol{\nu}}_i}(a) > \langle -v_{\widetilde{C}}, a\rangle + \kappa_{\widetilde{C},i}$ for all $a \in \mathcal{A}_i \setminus C_i$, which implies that $\widetilde{C}_i \subset C_i$ for all i. In turn, by Proposition 2.1(2) this implies that $\widetilde{C} \subset C$.

Since this holds for every n-cell of $S(\Theta_{\widetilde{\boldsymbol{\nu}}})$, we deduce that this mixed subdivision refines $S(\Theta_{\boldsymbol{\nu}})$. The statement follows by taking U as the ball of $\mathbb{R}^{\boldsymbol{\mathcal{A}}}$ centered at $\boldsymbol{\nu}$ of radius ε with respect to the ℓ^∞-norm. □

As an application of these results, we can exhibit an explicit family of tight incremental chains of mixed subdivisions of the polytope Δ associated with the data $\boldsymbol{\mathcal{A}}$.

Example 2.12 Set $\mathcal{A}_{>i} = \sum_{j>i} \mathcal{A}_j \subset M$, $i = 0, \dots, s-1$, and for each i denote by W_i' the finite union of hyperplanes given by Corollary 2.10 applied to the family $(\mathcal{A}_0, \dots, \mathcal{A}_i, \mathcal{A}_{>i})$ of $i+2$ nonempty subsets of M.

For $i = 0, \dots, s-1$ choose iteratively $\boldsymbol{\nu}_i \in \mathbb{R}^{\mathcal{A}_i}$ such that $(\boldsymbol{\nu}_0, \dots, \boldsymbol{\nu}_i) \notin W_i'$ and $(\boldsymbol{\nu}_0, \dots, \boldsymbol{\nu}_{i-1}, \boldsymbol{\nu}_i, \mathbf{0}, \dots, \mathbf{0}) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ lies in the neighborhood of $(\boldsymbol{\nu}_0, \dots, \boldsymbol{\nu}_{i-1}, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$ given by Proposition 2.11. Then, for $k = 0, \dots, s$ consider the family of convex piecewise affine functions $\theta_{k,i} \colon \Delta_i \to \mathbb{R}$, $i = 0, \dots, s$, defined as $\theta_{k,i} = \vartheta_{\boldsymbol{\nu}_i}$ if $i < k$ and as $\theta_{k,i} = 0|_{\Delta_i}$ if $i \geq k$. Their inf-convolution

$$\theta_k = \boxplus_{i=0}^{s} \theta_{k,i},$$

is a convex piecewise affine function on Δ. By Corollary 2.10 and Proposition 2.11,

$$S(\theta_0) \preceq \cdots \preceq S(\theta_s)$$

is a tight incremental chain of mixed subdivisions of Δ. Moreover, Corollary 2.10 allows to choose the vector $(\boldsymbol{\nu}_0, \dots, \boldsymbol{\nu}_{s-1}) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}'}$ in the image of the linear map $T_{\boldsymbol{\mathcal{A}}'}$, so that the convex piecewise affine functions $\theta_{k,i}$ are indeed affine.

2.2 Mixed Volumes and Mixed Integrals

The mixed volume of n convex bodies of $M_{\mathbb{R}}$ is a polarization of the notion of volume of a single one. Here, we recall its definition and basic properties, referring to [17, Chapter IV] for the corresponding proofs. We restrict the presentation to polytopes, which are the only convex bodies appearing in this paper.

We denote by vol_M the Haar measure on the vector space $M_{\mathbb{R}}$ that is normalized so that the lattice M has covolume 1.

Definition 2.13 The *mixed volume* of a family of polytopes $\Delta_i \subset M_{\mathbb{R}}$, $i = 1, \dots, n$, is defined as

$$\mathrm{MV}_M(\Delta_1, \dots, \Delta_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \mathrm{vol}_M(\Delta_{i_1} + \cdots + \Delta_{i_j}).$$

For $n = 0$, we agree that $\mathrm{MV}_M = 1$.

For a single polytope Δ, we have that $\mathrm{MV}_M(\Delta, \dots, \Delta) = n! \operatorname{vol}_M(\Delta)$. The mixed volume is symmetric and linear in each variable Δ_i with respect to the Minkowski sum, invariant with respect to linear maps that preserve the measure vol_M, and monotone with respect to the inclusion of polytopes. We have that $\mathrm{MV}_M(\Delta_1, \dots, \Delta_n) \geq 0$, and the equality holds if and only if there is a subset $I \subset \{1, \dots, n\}$ such that $\dim(\sum_{i\in I} \Delta_i) < \#I$. If the Δ_i's are lattice polytopes, then $\mathrm{MV}_M(\Delta_1, \dots, \Delta_n) \in \mathbb{N}$.

Given a family of convex piecewise affine functions $\rho_i : \Delta_i \to \mathbb{R}$, $i = 1, \dots, n$, with inf-convolution $\rho = \boxplus_{i=1}^n \rho_i$ and such that the mixed subdivision $S(\rho)$ is tight (Definition 2.3), the mixed volume of the Δ_i's can be computed as the sum of the volumes of the mixed n-cells [20, Theorem 2.4]:

$$\mathrm{MV}_M(\Delta_1, \dots, \Delta_n) = \sum_{C \text{ mixed}} \operatorname{vol}_M(C). \tag{2.16}$$

Analogously, the mixed integral of a family of $n+1$ concave functions on convex bodies is a polarization of the notion of integral of a single one. It was introduced in [31, §8], and is equivalent to the shadow mixed volume defined in [15, §1]. Here, we recall its definition and properties, translating them to the convex setting and restricting to piecewise affine functions on polytopes. We refer to [31, §4.3] and [32, §8] for the corresponding proofs and more information about this notion.

Definition 2.14 The *mixed integral* of a family of convex piecewise affine functions $\rho_i : \Delta_i \to \mathbb{R}$, $i = 0, \dots, n$, is defined as

$$\mathrm{MI}_M(\rho_0, \dots, \rho_n) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{0 \leq i_0 < \dots < i_j \leq n} \int_{\Delta_{i_0} + \dots + \Delta_{i_j}} \rho_{i_0} \boxplus \dots \boxplus \rho_{i_j} \, \mathrm{dvol}_M .$$

For a convex piecewise affine function on a polytope $\rho : \Delta \to \mathbb{R}$ we have that $\mathrm{MI}_M(\rho, \dots, \rho) = (n+1)! \int_\Delta \rho \, \mathrm{dvol}_M$. The mixed integral is symmetric and additive in each variable ρ_i with respect to the inf-convolution, and monotone.

It is possible to express mixed integrals in terms of mixed volumes. For $i = 0, \dots, n$ choose $\kappa_i \in \mathbb{R}_{\geq 0}$ with $\kappa_i \geq \rho_i(x)$ for all $x \in \Delta_i$ and consider the polytope

$$\Delta_{i,\rho_i,\kappa_i} = \operatorname{conv}(\operatorname{gr}(\rho_i), \Delta_i \times \{\kappa_i\}) \subset M_{\mathbb{R}} \times \mathbb{R}.$$

Then, by [31, Proposition 4.5(d)],

$$\begin{aligned} \mathrm{MI}_M(\rho_0, \dots, \rho_n) = & - \mathrm{MV}_{M\times\mathbb{Z}}(\Delta_{0,\rho_0,\kappa_0}, \dots, \Delta_{n,\rho_n,\kappa_n}) \\ & + \sum_{i=0}^{n} \kappa_i \, \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n). \end{aligned} \tag{2.17}$$

For each i, the convex piecewise affine function $\rho_i : \Delta_i \to \mathbb{R}$ is *lattice* if there are $\mathcal{A}_i \subset M$ and $\boldsymbol{\nu}_i \in \mathbb{Z}^{\mathcal{A}_i}$ such that $\Delta_i = \operatorname{conv}(\mathcal{A}_i)$ and $\rho_i = \vartheta_{\boldsymbol{\nu}_i}$ as in (2.8).

Proposition 2.15 *For* $i = 0, \dots, n$ *let* $\rho_i : \Delta_i \to \mathbb{R}$ *be a lattice convex piecewise affine function. Then,* $\mathrm{MI}_M(\rho_0, \dots, \rho_n) \in \mathbb{Z}$.

Proof This follows directly from (2.17) and the analogous property for the mixed volume. □

3 Sparse Resultants

3.1 Definitions and Basic Properties

In this section, we recall the basic notations, definitions and properties of sparse eliminants and resultants from [8].

We keep the notation of the previous sections. In particular $M \simeq \mathbb{Z}^n$ is a lattice of rank $n \geq 0$ and $N = M^\vee \simeq \mathbb{Z}^n$ its dual lattice. Let

$$\mathbb{T}_M = \mathrm{Hom}(M, \mathbb{C}^\times) = N \otimes_{\mathbb{Z}} \mathbb{C}^\times \simeq (\mathbb{C}^\times)^n$$

be the torus over $\mathbb{C}$ associated with M. Then, $M = \mathrm{Hom}(\mathbb{T}_M, \mathbb{C}^\times)$, and for $a \in M$ we denote by $\chi^a : \mathbb{T}_M \to \mathbb{C}^\times$ the corresponding character of $\mathbb{T}_M$.

For $i = 0, \dots, n$ let $\mathcal{A}_i$ be a nonempty finite subset of M, $\Delta_i = \mathrm{conv}(\mathcal{A}_i)$ the lattice polytope of $M_\mathbb{R}$ given by its convex hull, $\boldsymbol{u}_i = \{u_{i,a}\}_{a \in \mathcal{A}_i}$ a set of $\#\mathcal{A}_i$ variables and

$$F_i = \sum_{a \in \mathcal{A}_i} u_{i,a}\, \chi^a \in \mathbb{Z}[\boldsymbol{u}_i][M]$$

the general Laurent polynomial with support equal to the subset $\mathcal{A}_i$, where $\mathbb{Z}[\boldsymbol{u}_i][M] \simeq \mathbb{Z}[\boldsymbol{u}_i][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ denotes the group $\mathbb{Z}[\boldsymbol{u}_i]$-algebra of M. Set for short

$$\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n), \quad \boldsymbol{\Delta} = (\Delta_0, \dots, \Delta_n), \quad \boldsymbol{u} = (\boldsymbol{u}_0, \dots, \boldsymbol{u}_n) \quad \text{and} \quad \boldsymbol{F} = (F_0, \dots, F_n).$$

The *incidence variety* of $\boldsymbol{F}$ is defined as

$$\Omega_{\boldsymbol{\mathcal{A}}} = Z(\boldsymbol{F}) \subset \mathbb{T}_M \times \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}),$$

that is, the zero set of these Laurent polynomials in that product space. It is an irreducible algebraic subvariety of codimension $n+1$ defined over $\mathbb{Q}$. Denote by $\varpi : \mathbb{T}_M \times \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \to \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$ the projection onto the second factor. The *direct image* of $\Omega_{\boldsymbol{\mathcal{A}}}$ with respect to ϖ is the Weil divisor of $\prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$ defined as

$$\varpi_* \Omega_{\boldsymbol{\mathcal{A}}} = \begin{cases} \deg(\varpi|_{\Omega_{\boldsymbol{\mathcal{A}}}})\, \overline{\varpi(\Omega_{\boldsymbol{\mathcal{A}}})} & \text{if } \overline{\varpi(\Omega_{\boldsymbol{\mathcal{A}}})} \text{ is a hypersurface,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\varpi(\Omega_{\boldsymbol{\mathcal{A}}})}$ is the Zariski closure of the image of the incidence variety with respect to the projection, and $\deg(\varpi|_{\Omega_{\boldsymbol{\mathcal{A}}}})$ is the degree of the restriction of this map to the incidence variety.

Definition 3.1 The *sparse resultant*, denoted by $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$, is defined as any primitive polynomial in $\mathbb{Z}[\boldsymbol{u}]$ giving an equation for $\varpi_*\Omega_{\boldsymbol{\mathcal{A}}}$. The *sparse eliminant*, denoted by $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$, is defined as any irreducible polynomial in $\mathbb{Z}[\boldsymbol{u}]$ giving an equation for $\overline{\varpi(\Omega_{\boldsymbol{\mathcal{A}}})}$, if this a hypersurface, and as 1 otherwise.

Given a ring A and Laurent polynomials $f_i \in A[M]$ with support contained in $\mathcal{A}_i$ for each i, we apply the usual notation

$$\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}(f_0, \dots, f_n) \quad \text{and} \quad \mathrm{Res}_{\boldsymbol{\mathcal{A}}}(f_0, \dots, f_n) \tag{3.1}$$

to denote the evaluation at the coefficients of the f_i's.

Both the sparse resultant and the sparse eliminant are well defined up to the sign, and are both invariant by translations and permutations of the supports [8, Proposition 3.3]. The sparse eliminant does not depend on the lattice M but the sparse resultant does, as the following proposition shows.

Proposition 3.2 *Let $\varphi\colon M \to M'$ be a monomorphism of lattices of rank n. Then,* $\mathrm{Elim}_{\varphi(\boldsymbol{\mathcal{A}})} = \pm\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$ *and* $\mathrm{Res}_{\varphi(\boldsymbol{\mathcal{A}})} = \pm\,\mathrm{Res}_{\boldsymbol{\mathcal{A}}}^{[M':\varphi(M)]}$.

Proof The monomorphism $\varphi\colon M \to M'$ induces a finite map of degree $[M' : \varphi(M)]$

$$\varphi^*\colon \mathbb{T}_{M'} = \mathrm{Hom}(M', \mathbb{C}^\times) \longrightarrow \mathbb{T}_M = \mathrm{Hom}(M, \mathbb{C}^\times).$$

Setting $F_i' = \sum_{a\in\mathcal{A}_i} u_{i,a}\,\chi^{\varphi(a)}$ for the general Laurent polynomial with support $\varphi(\mathcal{A}_i)$ for each i, the system $F_0' = \cdots = F_n' = 0$ has a nontrivial solution in $\mathbb{T}_{M'}$ if and only if $F_0 = \cdots = F_n = 0$ has a nontrivial solution in $\mathbb{T}_M$. Hence, φ^* induces a commutative diagram

$$\begin{array}{ccc} \Omega_{\varphi(\boldsymbol{\mathcal{A}})} & \longrightarrow & \Omega_{\boldsymbol{\mathcal{A}}} \\ \downarrow{\scriptstyle \varpi'} & & \downarrow{\scriptstyle \varpi} \\ \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\varphi(\mathcal{A}_i)}) & = & \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \end{array}$$

which implies the stated equality between the sparse eliminants. From here, we also deduce that $\overline{\varpi'(\Omega_{\varphi(\boldsymbol{\mathcal{A}})})}$ is not a hypersurface if and only if this also holds for $\overline{\varpi(\Omega_{\boldsymbol{\mathcal{A}}})}$, in which case both $\mathrm{Res}_{\varphi(\boldsymbol{\mathcal{A}})}$ and $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ are equal to ± 1, proving the second equality in this case. Otherwise, the multiplicativity of the degree implies that

$$\deg(\varpi'|_{\Omega(\varphi(\boldsymbol{\mathcal{A}}))}) = [M' : \varphi(M)]\,\deg(\varpi|_{\Omega_{\boldsymbol{\mathcal{A}}}})$$

and so $\varpi'_*\Omega_{\varphi(\boldsymbol{\mathcal{A}})} = [M' : \varphi(M)]\,\varpi_*\Omega_{\boldsymbol{\mathcal{A}}}$, which implies the second equality in this other case and completes the proof. □

The sparse resultant is homogeneous in each set of variables $\boldsymbol{u}_i$ of degree [8, Proposition 3.4]:

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n),\ i = 0, \dots, n. \tag{3.2}$$

Let $\mathcal{A} \subset M$ be a nonempty finite subset and $f = \sum_{a \in \mathcal{A}} \alpha_a \chi^a \in \mathbb{C}[M]$ a Laurent polynomial with support contained in $\mathcal{A}$. For $v \in N_{\mathbb{R}}$ we, respectively, set

$$\mathcal{A}^v = \mathcal{A} \cap \operatorname{conv}(\mathcal{A})^v \quad \text{and} \quad \operatorname{init}_v(f) = \sum_{a \in \mathcal{A}^v} \alpha_a \chi^a \tag{3.3}$$

for the *restriction of* $\mathcal{A}$ *to the face* $\operatorname{conv}(\mathcal{A})^v$ as defined in (2.2) and the *initial part of* f *in the direction of* v.

For $v \in N \setminus \{0\}$, the *sparse resultant in the direction of* v, denoted by $\operatorname{Res}_{\mathcal{A}_1^v, \dots, \mathcal{A}_n^v}$, is the sparse resultant associated with the orthogonal lattice $v^\perp \cap M \simeq \mathbb{Z}^{n-1}$ and the supports $\mathcal{A}_i^v$, $i = 1, \dots, n$, modulo suitable translations placing them inside this lattice, see [8, Definition 4.1] for details. By [8, Proposition 3.8], this directional resultant is nontrivial only when v is the inner normal to a face of dimension $n-1$ of the Minkowski sum $\sum_{i=1}^n \Delta_i$. In particular, the number of nontrivial directional sparse resultants of the family of supports $\boldsymbol{\mathcal{A}}$ is finite.

The following result is the Poisson formula for the sparse resultant [8, Theorem 4.2]. For a subset $B \subset M_{\mathbb{R}}$, its *support function* $h_B \colon N_{\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$h_B(v) = \inf\{\langle v, x \rangle \mid x \in B\}.$$

This generalizes the support function of a convex polyhedron in (2.1).

Theorem 3.3 *For* $i = 0, \dots, n$ *let* $f_i \in \mathbb{C}[M]$ *with support contained in* $\mathcal{A}_i$ *and suppose that* $\operatorname{Res}_{\mathcal{A}_1^v, \dots, \mathcal{A}_n^v}(\operatorname{init}_v(f_1), \dots, \operatorname{init}_v(f_n)) \neq 0$ *for all* $v \in N \setminus \{0\}$. *Then,*

$$\operatorname{Res}_{\boldsymbol{\mathcal{A}}}(f_0, f_1, \dots, f_n) = \pm \prod_v \operatorname{Res}_{\mathcal{A}_1^v, \dots, \mathcal{A}_n^v}(\operatorname{init}_v(f_1), \dots, \operatorname{init}_v(f_n))^{-h_{\mathcal{A}_0}(v)} \cdot \prod_p f_0(p)^{m_p},$$

the first product being over the primitive vectors $v \in N$ *and the second over the solutions* $p \in \mathbb{T}_M$ *of the system of equations* $f_1 = \cdots = f_n = 0$*, where* m_p *denotes the intersection multiplicity of this system of equations at the point* p.

For a subset $J \subset \{0, \dots, n\}$ put $\boldsymbol{\mathcal{A}}_J = (\mathcal{A}_i)_{i \in J}$ and $\boldsymbol{u}_J = (\boldsymbol{u}_i)_{i \in J}$.

Definition 3.4 The *fundamental subfamily* of $\boldsymbol{\mathcal{A}}$ is the family of supports $\boldsymbol{\mathcal{A}}_J$ for the minimal subset $J \subset \{0, \dots, n\}$ such that $\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\boldsymbol{u}_J]$ or equivalently, such that $\operatorname{Elim}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\boldsymbol{u}_J]$.

For each i set $L_{\mathcal{A}_i}$ for the sublattice of M generated by the differences of the elements of $\mathcal{A}_i$. For a subset $J \subset \{0, \dots, n\}$ consider the sum $L_{\boldsymbol{\mathcal{A}}_J} = \sum_{i \in J} L_{\mathcal{A}_i}$ and its saturation $L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}} = (L_{\boldsymbol{\mathcal{A}}_J} \otimes_{\mathbb{Z}} \mathbb{R}) \cap M$.

Remark 3.5 By [33, Corollary 1.1 and Lemma 1.2] or [8, Proposition 3.13] when $J \neq \emptyset$ the fundamental subfamily $\boldsymbol{\mathcal{A}}_J$ coincides with the unique essential subfamily of $\boldsymbol{\mathcal{A}}$, that is, the unique subfamily such that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_J}) = \#J - 1$ and $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_{J'}}) \geq \#J'$ for all $J' \subsetneq J$, whereas when $J = \emptyset$, that is when $\operatorname{Res}_{\boldsymbol{\mathcal{A}}} = \pm 1$, we have that $\boldsymbol{\mathcal{A}}_J = \emptyset$ and $\boldsymbol{\mathcal{A}}$ has at least two essential subfamilies.

The sparse eliminant and the sparse resultant are related by

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{d_{\boldsymbol{\mathcal{A}}}} \tag{3.4}$$

with $d_{\boldsymbol{\mathcal{A}}} \in \mathbb{N}_{>0}$.

Proposition 3.6 *Let $\boldsymbol{\mathcal{A}}_J$ be the fundamental subfamily of $\boldsymbol{\mathcal{A}}$ and suppose that $J \neq \emptyset$. Then,* $\mathrm{rank}(L_{\boldsymbol{\mathcal{A}}_J}) = \#J - 1$, *and the exponent in* (3.4) *can be written as*

$$d_{\boldsymbol{\mathcal{A}}} = [L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}} : L_{\boldsymbol{\mathcal{A}}_J}]\ \mathrm{MV}_{M/L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}}(\{\pi(\Delta_i)\}_{i\notin J})$$

where π is the projection $M \to M/L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}$.

Proof The first claim follows from Remark 3.5, whereas the second is [8, Proposition 3.13]. □

Sparse eliminants are particular cases of sparse resultants.

Proposition 3.7 *Let $\boldsymbol{\mathcal{A}}_J$ be the fundamental subfamily of $\boldsymbol{\mathcal{A}}$, suppose that $J \neq \emptyset$ and consider $\boldsymbol{\mathcal{A}}_J$ as a family of $\#J$ nonempty finite subsets of the lattice $L_{\boldsymbol{\mathcal{A}}_J} \simeq \mathbb{Z}^{\#J-1}$. Then,* $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}_J} = \pm\,\mathrm{Res}_{\boldsymbol{\mathcal{A}}_J}$.

Proof The first equality is given by [8, Proposition 3.11], whereas the second follows from the equality in (3.4) and Proposition 3.6. □

We also need the following auxiliary result.

Proposition 3.8 *Let $\widetilde{\boldsymbol{\mathcal{A}}} = (\widetilde{\mathcal{A}}_0, \dots, \widetilde{\mathcal{A}}_n)$ be a further family of supports in M such that $\widetilde{\mathcal{A}}_i \subset \mathcal{A}_i$ for all i. Let $\widetilde{\boldsymbol{\mathcal{A}}}_J$ and $\boldsymbol{\mathcal{A}}_K$ be the respective fundamental subfamilies of supports. Then, $J \subset K$.*

Proof By the degree formula in (3.2), an index $j \in \{0, \dots, n\}$ lies in K if and only if

$$\mathrm{MV}_M(\Delta_0, \dots, \Delta_{j-1}, \Delta_{j+1}, \dots, \Delta_n) > 0.$$

The statement follows then from the monotonicity of the mixed volume with respect to the inclusion of polytopes. □

The notions of sparse eliminant and of sparse resultant include the classical homogeneous resultant introduced by Macaulay [27], as we next explain.

Example 3.9 For $\boldsymbol{d} = (d_0, \dots, d_n) \in (\mathbb{N}_{>0})^{n+1}$ let $\mathrm{Res}_{\boldsymbol{d}}$ be the homogeneous resultant, giving the condition for a system of $n+1$ homogeneous polynomials in $n+1$ variables of degrees $\boldsymbol{d}$ to have a zero in the n-dimensional projective space [6, §3.2]. It coincides, up to the sign, both with the sparse eliminant and the sparse resultant for the lattice $M = \mathbb{Z}^n$ and the family of supports $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ given by

$$\mathcal{A}_i = \{\boldsymbol{a} \in \mathbb{N}^n \mid |\boldsymbol{a}| \leq d_i\},$$

where for a lattice point $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we denote by $|\boldsymbol{a}| = \sum_{i=1}^n a_i$ its *length*. Then, $\Delta_i = \{\boldsymbol{x} \in (\mathbb{R}_{\geq 0})^n \mid |\boldsymbol{x}| \leq d_i\}$ for each i and we can deduce from the degree formula in (3.2) that

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{d}}) = \prod_{j \neq i} d_j, \ i = 0, \dots, n.$$

3.2 Order and Initial Parts

In this section, we study the different orders and initial parts of the sparse resultant.

Definition 3.10 Let $\boldsymbol{\omega} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ and let t be a variable. For $P \in \mathbb{C}[\boldsymbol{u}]\backslash\{0\}$ set

$$P^{\boldsymbol{\omega}} = P((t^{\omega_{i,a}}\, u_{i,a})_{i \in \{0,\dots,n\}, a \in \mathcal{A}_i}) \in \mathbb{C}[\boldsymbol{u}][t^{\mathbb{R}}]\backslash\{0\}. \tag{3.5}$$

The *order* and the *initial part* of P with respect to $\boldsymbol{\omega}$ are the elements $\mathrm{ord}_{\boldsymbol{\omega}}(P) \in \mathbb{R}$ and $\mathrm{init}_{\boldsymbol{\omega}}(P) \in \mathbb{C}[\boldsymbol{u}]\backslash\{0\}$ defined by the equation

$$P^{\boldsymbol{\omega}} = (\mathrm{init}_{\boldsymbol{\omega}}(P) + o(1))\, t^{\mathrm{ord}_{\boldsymbol{\omega}}(P)}, \tag{3.6}$$

where $o(1)$ denotes a sum of terms whose degree in t is strictly positive.

For a nonzero rational function $P \in \mathbb{C}(\boldsymbol{u})^{\times}$ written as $P = P_1/P_2$ with $P_i \in \mathbb{C}[\boldsymbol{u}]\backslash\{0\}$, $i = 1, 2$, the *order* and the *initial part* of P with respect to $\boldsymbol{\omega}$ are defined as

$$\mathrm{ord}_{\boldsymbol{\omega}}(P) = \mathrm{ord}_{\boldsymbol{\omega}}(P_1) - \mathrm{ord}_{\boldsymbol{\omega}}(P_2) \quad \text{and} \quad \mathrm{init}_{\boldsymbol{\omega}}(P) = \frac{\mathrm{init}_{\boldsymbol{\omega}}(P_1)}{\mathrm{init}_{\boldsymbol{\omega}}(P_2)}.$$

These notions do not depend on the choice of P_1 and P_2 and the maps

$$\mathrm{ord}_{\boldsymbol{\omega}} \colon \mathbb{C}(\boldsymbol{u})^{\times} \longrightarrow \mathbb{R} \quad \text{and} \quad \mathrm{init}_{\boldsymbol{\omega}} \colon \mathbb{C}(\boldsymbol{u})^{\times} \longrightarrow \mathbb{C}(\boldsymbol{u})^{\times}$$

are group morphisms. We extend them by setting $\mathrm{ord}_{\boldsymbol{\omega}}(0) = +\infty$ and $\mathrm{init}_{\boldsymbol{\omega}}(0) = 0$. The notion of initial part generalizes the definition in (3.3) for Laurent polynomials.

As pointed out by Sturmfels, the initial part of the sparse resultant in a given direction is closely related to the mixed subdivision of Δ associated with the convex piecewise affine functions defined by that direction [33].

Definition 3.11 For $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \dots, \boldsymbol{\omega}_n) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ let $\vartheta_{\boldsymbol{\omega}_i} \colon \Delta_i \to \mathbb{R}$, $i = 0, \dots, n$, and $\Theta_{\boldsymbol{\omega}} \colon \Delta \to \mathbb{R}$ be the associated convex piecewise affine functions as in (2.8) and (2.9). Let D be an n-cell of the mixed subdivision $S(\Theta_{\boldsymbol{\omega}})$ of Δ and $D_i \in S(\vartheta_{\boldsymbol{\omega}_i})$, $i = 0, \dots, n$, its components as defined in (2.7). The *restriction* of $\boldsymbol{\mathcal{A}}$ to D is the family of nonempty finite subsets of M defined as

$$\boldsymbol{\mathcal{A}}_D = (\mathcal{A}_0 \cap D_0, \dots, \mathcal{A}_n \cap D_n).$$

The next theorem gives formulae for the order and the initial part of the sparse resultant. The first part is a reformulation of a result by Philippon and the third author for the Chow weights of a multiprojective toric variety [31, Proposition 4.6], whereas the second is a generalization of a result by Sturmfels for sparse eliminants in the case when the fundamental subfamily coincides with $\boldsymbol{\mathcal{A}}$ [33, Theorem 4.1].

Theorem 3.12 *Let $\boldsymbol{\omega} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$. Then,*

$$\mathrm{ord}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MI}_M(\vartheta_{\omega_0}, \dots, \vartheta_{\omega_n}) \quad \text{and} \quad \mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \pm \prod_{D \in S(\Theta_{\boldsymbol{\omega}})^n} \mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}.$$

Before proving it, we need to establish some auxiliary results. For $P \in \mathbb{C}[\boldsymbol{u}]$ we denote by $\mathrm{supp}(P)$ its *support*, that is, the finite subset of $\mathbb{N}^{\boldsymbol{\mathcal{A}}}$ of the exponents of the nonzero terms of this polynomial.

Lemma 3.13 *For $\boldsymbol{\omega} = (\omega_0, \dots, \omega_n) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ there is $\widetilde{\boldsymbol{\omega}} = (\widetilde{\omega}_0, \dots, \widetilde{\omega}_n) \in \mathbb{Z}^{\boldsymbol{\mathcal{A}}}$ such that*

(1) $\mathrm{init}_{\widetilde{\boldsymbol{\omega}}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}})$,
(2) $S(\Theta_{\widetilde{\boldsymbol{\omega}}}) = S(\Theta_{\boldsymbol{\omega}})$,
(3) for every n-cell of $S(\Theta_{\widetilde{\boldsymbol{\omega}}})$, its components with respect to the families of convex piecewise affine functions $\vartheta_{\widetilde{\omega}_i}$, $i = 0, \dots, n$, and ϑ_{ω_i}, $i = 0, \dots, n$, coincide.

Proof Set $\mathcal{S} = \mathrm{supp}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) \subset \mathbb{N}^{\boldsymbol{\mathcal{A}}}$ and $\mathcal{S}^{\boldsymbol{\omega}}$ for the subset of $\mathcal{S}$ of lattice points with minimal scalar product with respect to $\boldsymbol{\omega}$. A vector $\widetilde{\boldsymbol{\omega}} = (\widetilde{\omega}_0, \dots, \widetilde{\omega}_n) \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ verifies the condition (1) if and only if

$$\langle \widetilde{\boldsymbol{\omega}}, \boldsymbol{c}' - \boldsymbol{c} \rangle = 0 \text{ for } \boldsymbol{c}, \boldsymbol{c}' \in \mathcal{S}^{\boldsymbol{\omega}} \quad \text{and} \quad \langle \widetilde{\boldsymbol{\omega}}, \boldsymbol{c}' - \boldsymbol{c} \rangle > 0 \text{ for } \boldsymbol{c} \in \mathcal{S}^{\boldsymbol{\omega}} \quad \text{and} \quad \boldsymbol{c}' \in \mathcal{S} \setminus \mathcal{S}^{\boldsymbol{\omega}}. \tag{3.7}$$

With notation as in (2.3), for each $D \in S(\Theta_{\boldsymbol{\omega}})^n$ set $v_D \in N_{\mathbb{R}}$ for the unique vector such that $D = \Gamma(\Theta_{\boldsymbol{\omega}}, v_D)$. For each i let $D_i^0 \subset \mathcal{A}_i$ be the set of vertices of the i-th component of D. Then, $D_i = \Gamma(\vartheta_{\widetilde{\omega}_i}, v_D)$ if and only if

$$\begin{aligned} &\langle v_D, a' - a \rangle + \widetilde{\omega}_{i,a'} - \widetilde{\omega}_{i,a} = 0 \text{ for } a, a' \in D_i^0, \\ &\langle v_D, a' - a \rangle + \widetilde{\omega}_{i,a'} - \widetilde{\omega}_{i,a} \geq 0 \text{ for } a \in D_i^0 \text{ and } a' \in (\mathcal{A}_i \cap D_i) \setminus D_i^0, \\ &\langle v_D, a' - a \rangle + \widetilde{\omega}_{i,a'} - \widetilde{\omega}_{i,a} > 0 \text{ for } a \in D_i^0 \text{ and } a' \in \mathcal{A}_i \setminus D_i. \end{aligned} \tag{3.8}$$

If this condition holds, then $D = \Gamma(\Theta_{\widetilde{\boldsymbol{\omega}}}, v_D)$ by Proposition 2.1(1).

Hence, if $\widetilde{\boldsymbol{\omega}}$ satisfies the condition (3.7) and that in (3.8) for all $D \in S(\Theta_{\boldsymbol{\omega}})^n$, then it also verifies (1), (2) and (3). These conditions amount to the fact that $\widetilde{\boldsymbol{\omega}}$ lies in the relative interior of a polyhedral cone of $\mathbb{R}^{\boldsymbol{\mathcal{A}}}$ defined over $\mathbb{Z}$. This relative interior is nonempty as it contains $\boldsymbol{\omega}$, and so it also contains a vector in $\mathbb{Z}^{\boldsymbol{\mathcal{A}}}$. □

Lemma 3.14 *Let $(v, l) \in N \times \mathbb{Z}$ be a primitive lattice vector with $l > 0$. Let $(v, l)^{\perp}$ be its orthogonal subspace of $M_{\mathbb{R}} \times \mathbb{R}$ and*

$$\varphi \colon (v, l)^{\perp} \cap (M \times \mathbb{Z}) \longrightarrow M$$

the lattice map defined by $(a, q) \mapsto a$*. Then,* $[M : \varphi((v, l)^{\perp} \cap (M \times \mathbb{Z}))] = l$*.*

Proof Set for short $P = (v, l)^{\perp} \cap (M \times \mathbb{Z})$, which is a sublattice of $M \times \mathbb{Z}$ of rank n. The map $\varphi \colon P \to M$ is injective if and only if so is its dual $\varphi^{\vee} \colon M^{\vee} \to P^{\vee}$ and if this is the case, then

$$[M : \varphi(P)] = [P^{\vee} : \varphi^{\vee}(M^{\vee})]. \tag{3.9}$$

We have that $M^{\vee} = N$ and $P^{\vee} \simeq (M \times \mathbb{Z})/\mathbb{Z}(v, l)$. With these identifications, the dual map $\varphi^{\vee} \colon N \to (M \times \mathbb{Z})/\mathbb{Z}(v, l)$ writes down as $\varphi^{\vee}(w) = (w, 0) + \mathbb{Z}(v, l)$. Hence, $\varphi^{\vee}$ is injective because $l > 0$. Moreover, its image is the sublattice $(M \times l\,\mathbb{Z})/\mathbb{Z}(v, l)$ and so

$$[P^{\vee} : \varphi^{\vee}(M^{\vee})] = \#\,(M \times \mathbb{Z}/\mathbb{Z}(v, l))/(M \times l\,\mathbb{Z}/\mathbb{Z}(v, l)) = \#\,\mathbb{Z}/l\mathbb{Z} = l,$$

which together with (3.9) implies the statement. □

Proof of Theorem 3.12 By [32, Proposition 4.5], the degree of a monomial deformation of the sparse resultant can be computed in terms of mixed integrals as

$$\deg_t(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}^{-\boldsymbol{\omega}}) = -\,\mathrm{MI}_M(\vartheta_{\boldsymbol{\omega}_0}, \dots, \vartheta_{\boldsymbol{\omega}_n}).$$

Since $\mathrm{ord}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = -\deg_t(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}^{-\boldsymbol{\omega}})$, this gives the first part of the statement.

For the second part, we reduce without loss of generality to the case when $\boldsymbol{\omega} \in \mathbb{Z}^{\boldsymbol{\mathcal{A}}}$ thanks to Lemma 3.13. Set $\boldsymbol{F}^{\boldsymbol{\omega}} = (F_0^{\boldsymbol{\omega}_0}, \dots, F_n^{\boldsymbol{\omega}_n})$ with

$$F_i^{\boldsymbol{\omega}_i} = F_i((t^{\omega_{i,a}}\, u_{i,a})_{a \in \mathcal{A}_i}) \in \mathbb{C}[\boldsymbol{u}_i][M][t^{\pm 1}],\ i = 0, \dots, n.$$

With notation as in (3.1) and (3.5), we have that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}^{\boldsymbol{\omega}} = \mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{F}^{\boldsymbol{\omega}}). \tag{3.10}$$

Consider the family of $n + 2$ nonempty finite subsets of $M \times \mathbb{Z}$ given by

$$\mathcal{C} = \{(0, 0),\ (0, 1)\} \quad \text{and} \quad \widehat{\mathcal{A}}_i = \{(a, \omega_{i,a})\}_{a \in \mathcal{A}_i},\ i = 0, \dots, n.$$

Let $\boldsymbol{v} = \{v_{(0,0)}, v_{(0,1)}\}$ be a set of variables, so that the general Laurent polynomial with support $\mathcal{C}$ is $v_{(0,0)} + v_{(0,1)}\, z$ and that with support $\widehat{\mathcal{A}}_i$ is $F_i^{\boldsymbol{\omega}}(z)$, the evaluation of $F_i^{\boldsymbol{\omega}}$ at $t = z$ for each i. Set $\widehat{\boldsymbol{\mathcal{A}}} = (\widehat{\mathcal{A}}_0, \dots, \widehat{\mathcal{A}}_n)$. By the "hidden variable" formula in [8, Proposition 4.7], there is $d_{\boldsymbol{\omega}} \in \mathbb{Z}$ such that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{F}^{\boldsymbol{\omega}}) = \pm t^{d_{\boldsymbol{\omega}}}\, \mathrm{Res}_{\mathcal{C}, \widehat{\boldsymbol{\mathcal{A}}}}(z - t, \boldsymbol{F}^{\boldsymbol{\omega}}(z)).$$

Thanks to the formula in (3.6), we have that

$$\mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{Res}_{\mathcal{C}, \widehat{\boldsymbol{\mathcal{A}}}}(z, \boldsymbol{F}^{\boldsymbol{\omega}}(z))$$

provided this latter polynomial is nonzero. To see this, consider a family of Laurent polynomials $f_i \in \mathbb{C}[M]$ with $\mathrm{supp}(f_i) \subset \mathcal{A}_i$, $i = 0, \dots, n$, that is sufficiently generic and set $\boldsymbol{f}^{\boldsymbol{\omega}} = (f_0^{\boldsymbol{\omega}_0}, \dots, f_n^{\boldsymbol{\omega}_n})$. By the invariance of the sparse resultant under translation of the first support and the Poisson formula (Theorem 3.3), with notation as therein we have that

$$\begin{aligned}\mathrm{Res}_{\mathcal{C},\widehat{\boldsymbol{\mathcal{A}}}}(z, \boldsymbol{f}^{\boldsymbol{\omega}}(z)) &= \mathrm{Res}_{\mathcal{C}-(0,1),\widehat{\boldsymbol{\mathcal{A}}}}(1, \boldsymbol{f}^{\boldsymbol{\omega}}(z)) \\ &= \pm \prod_{(v,l)} \mathrm{Res}_{\widehat{\boldsymbol{\mathcal{A}}}^{(v,l)}}(\mathrm{init}_{(v,l)}(\boldsymbol{f}^{\boldsymbol{\omega}}(z)))^{-h_{\mathcal{C}-(0,1)}((v,l))} \\ &= \pm \prod_{(v,l)} \mathrm{Res}_{\widehat{\boldsymbol{\mathcal{A}}}^{(v,l)}}(\mathrm{init}_{(v,l)}(\boldsymbol{f}^{\boldsymbol{\omega}}(z)))^{\max\{0,l\}},\end{aligned} \tag{3.11}$$

the products being over the primitive lattice vectors $(v, l) \in N \times \mathbb{Z}$, and where $\widehat{\boldsymbol{\mathcal{A}}}^{(v,l)}$ denotes the family of supports $(\widehat{\mathcal{A}}_0^{(v,l)}, \dots, \widehat{\mathcal{A}}_n^{(v,l)})$. The last equality follows from the fact that $-h_{\mathcal{C}-(0,1)}((v, l)) = \max\{0, l\}$. Since this holds for every choice of $\boldsymbol{f}$, we have that

$$\mathrm{Res}_{\mathcal{C},\widehat{\boldsymbol{\mathcal{A}}}}(z, \boldsymbol{F}^{\boldsymbol{\omega}}(z)) = \pm \prod_{(v,l)} \mathrm{Res}_{\widehat{\boldsymbol{\mathcal{A}}}^{(v,l)}}^{\max\{0,l\}}$$

because $\mathrm{init}_{(v,l)}(\boldsymbol{F}^{\boldsymbol{\omega}}(z))$ is the general Laurent polynomial with support $\widehat{\boldsymbol{\mathcal{A}}}^{(v,l)}$.

Let $(v, l) \in N \times \mathbb{Z}$ be a primitive vector such that $l > 0$. For the linear map $\varphi\colon (v, l)^{\perp} \cap (M \times \mathbb{Z}) \to M$ induced from the projection onto the first factor we have that $[M : \varphi((v, l)^{\perp} \cap (M \times \mathbb{Z}))] = l$ by Lemma 3.14. For the cell $D = \Gamma\big(\Theta_{\boldsymbol{\omega}}, \frac{1}{l}\, v\big)$, we also have that $\varphi(\widehat{\mathcal{A}}_i^{(v,l)}) = \mathcal{A}_i \cap D_i$ for each i. Proposition 3.2 then implies that

$$\mathrm{Res}_{\widehat{\boldsymbol{\mathcal{A}}}^{(v,l)}}^{\max\{0,l\}} = \pm \mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}\,. \tag{3.12}$$

Since every n-cell of $S(\Theta_{\boldsymbol{\omega}})^n$ appears exactly once in the product (3.11), this second part then follows from (3.10), (3.11) and (3.12). ◻

3.3 Homogeneities and Degrees

The homogeneities of the sparse resultant are of two types: there are $\lambda_i \in \mathbb{Z}$, $i = 0, \dots, n$, and $\mu \in M$ such that for every $\boldsymbol{c} \in \mathbb{N}^{\boldsymbol{\mathcal{A}}}$ in the support of $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ we have that

$$\sum_{a \in \mathcal{A}_i} c_{i,a} = \lambda_i, \ i = 0, \dots, n, \quad \text{and} \quad \sum_{i=0}^{n} \sum_{a \in \mathcal{A}_i} c_{i,a}\, a = \mu,$$

see for instance [18, Chapter 9, Proposition 1.3] or [33, §6]. The first type corresponds to the fact that the sparse resultant is homogeneous in each set of variables $\boldsymbol{u}_i$. As noted in (3.2), its partial degree $\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \lambda_i$ can be computed in terms of mixed volumes.

The second type corresponds to its equivariance with respect to the action of the torus by translations. For $p \in \mathbb{T}_M$ denote by $\tau_p\colon \mathbb{T}_M \to \mathbb{T}_M$ the translation by this point and let $\tau_p^* \boldsymbol{F} = \boldsymbol{F} \circ \tau_p$ be the pullback of the system of general Laurent polynomials $\boldsymbol{F}$ with respect to this map. The fact that the sparse resultant satisfies this type of homogeneity is then equivalent to the validity of identity

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\tau_p^* \boldsymbol{F}) = \chi^{\mu}(p)\, \mathrm{Res}_{\boldsymbol{\mathcal{A}}} \tag{3.13}$$

for all $p \in \mathbb{T}_M$.

Let $\deg_M$ be the grading of the monomials of $\mathbb{C}[\boldsymbol{u}]$ with values in M defined by

$$\deg_M(u_{i,a}) = a \text{ for } i = 0, \dots, n \text{ and } a \in \mathcal{A}_i. \tag{3.14}$$

Then, (3.13) is also equivalent to the fact that the sparse resultant is homogeneous with respect to this grading, of degree μ. As an application of Theorem 3.12, we will reprove this type of homogeneity and compute its degree in terms of mixed integrals.

We first prove an auxiliary lemma. A point $v \in N$ can be seen as a linear function on $M_{\mathbb{R}}$ and, in particular, can be restricted to any subset of this linear space.

Lemma 3.15 *The function $\mu_{\boldsymbol{\Delta}}\colon N \to \mathbb{Z}$ given by*

$$\mu_{\boldsymbol{\Delta}}(v) = \mathrm{MI}_M(v|_{\Delta_0}, \dots, v|_{\Delta_n})$$

is well defined and linear. Therefore $\mu_{\boldsymbol{\Delta}} \in M = N^{\vee}$.

Proof Let $v \in N$. For each subset $I \subset \{0, \dots, n\}$ we have that

$$\boxplus_{i\in I} v|_{\Delta_i} = v|_{\sum_{i\in I} \Delta_i}$$

because v is linear. For $v' \in N$, the definition of the mixed integral then implies that

$$\mathrm{MI}_M((v+v')|_{\Delta_0}, \dots, (v+v')|_{\Delta_n}) = \mathrm{MI}_M(v|_{\Delta_0}, \dots, v|_{\Delta_n}) + \mathrm{MI}_M(v'|_{\Delta_0}, \dots, v'|_{\Delta_n}),$$

which shows that $\mu_{\boldsymbol{\Delta}}$ is linear. Moreover, $v|_{\Delta_i}$ is a lattice convex piecewise affine function and so Proposition 2.15 implies that $\mu_{\boldsymbol{\Delta}}(v) \in \mathbb{Z}$. The last claim follows from the previous ones. □

Theorem 3.16 *The sparse resultant* $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ *is homogeneous with respect to* $\deg_M$ *and*

$$\deg_M(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mu_{\boldsymbol{\Delta}} \in M.$$

Proof Let $v \in N$. For the weight $\boldsymbol{\omega} \in \mathbb{Z}^{\boldsymbol{\mathcal{A}}}$ defined by $\omega_{i,a} = \langle v, a\rangle$ for $i = 0, \dots, n$ and $a \in \mathcal{A}_i$, we have that $\vartheta_{\boldsymbol{\omega}_i} = v|_{\Delta_i}$ for each i and $\Theta_{\boldsymbol{\omega}} = v|_{\Delta}$. Hence, Δ is the unique n-cell of $S(\Theta_{\boldsymbol{\omega}})$, and from Theorem 3.12 we deduce that

$$\mathrm{ord}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MI}_M(v|_{\Delta_0}, \dots, v|_{\Delta_n}) \quad \text{and} \quad \mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{Res}_{\boldsymbol{\mathcal{A}}}.$$

This implies that for all $\boldsymbol{c} \in \operatorname{supp}(\operatorname{Res}_{\boldsymbol{\mathcal{A}}})$ we have that $\langle v, \sum_{i,a} c_{i,a}\, a\rangle = \langle v, \mu_{\boldsymbol{\Delta}}\rangle = \mu_{\boldsymbol{\Delta}}(v)$. Since this holds for all $v \in N$, we deduce the statement. □

Remark 3.17 When $M = \mathbb{Z}^n$ we have that $\mu_{\boldsymbol{\Delta}} = (\mu_{\boldsymbol{\Delta},1}, \dots, \mu_{\boldsymbol{\Delta},n})$ with

$$\mu_{\boldsymbol{\Delta},i} = \operatorname{MI}_M(x_i|_{\Delta_0}, \dots, x_i|_{\Delta_n}),\ i = 1, \dots, n. \tag{3.15}$$

In this case $\mathbb{T}_M = (\mathbb{C}^\times)^n$, and for a point $\boldsymbol{p} = (p_1, \dots, p_n)$ in this torus we have that

$$\operatorname{Res}_{\boldsymbol{\mathcal{A}}}(\tau_{\boldsymbol{p}}^* \boldsymbol{F}) = \operatorname{Res}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{F}(p_1\, x_1, \dots, p_n\, x_n)) = \Big(\prod_{i=1}^n p_i^{\mu_{\boldsymbol{\Delta},i}}\Big) \operatorname{Res}_{\boldsymbol{\mathcal{A}}}.$$

Example 3.18 Let $\operatorname{Res}_{\boldsymbol{d}}$ be the homogeneous resultant corresponding to a sequence of degrees $\boldsymbol{d} = (d_0, \dots, d_n) \in (\mathbb{N}_{>0})^{n+1}$ as in Example 3.9. Since the function $\boldsymbol{x} \mapsto x_i$ is linear, for each i the mixed integral in (3.15) can be computed as

$$\begin{aligned}\mu_{\boldsymbol{\Delta},i} &= \sum_{j=0}^n (-1)^{n-j} \sum_{0 \le k_0 < \dots < k_j \le n} \int_{\Delta_{k_0} + \dots + \Delta_{k_j}} x_i \,\mathrm{d}\boldsymbol{x} \\ &= \sum_{j=0}^n (-1)^{n-j} \sum_{0 \le k_0 < \dots < k_j \le n} \frac{(d_{k_0} + \dots + d_{k_j})^{n+1}}{(n+1)!} = \prod_{l=0}^n d_l,\end{aligned}$$

where the last equality can be proven with elementary algebra as in [17, Theorem 3.7]. This gives the well-known *isobarism* of the homogeneous resultant, a result that goes back to Macaulay [28, page 11].

3.4 Vanishing Coefficients

In this section, we apply Theorem 3.12 to obtain a formula for the evaluation of the sparse resultant by setting some of the coefficients of the system of Laurent polynomials $\boldsymbol{F}$ to zero.

For $i = 0, \dots, n$ let $\widetilde{\mathcal{A}}_i \subset \mathcal{A}_i$ be a nonempty subset, $\widetilde{\Delta}_i \subset M_{\mathbb{R}}$ its convex hull, $\widetilde{\boldsymbol{u}}_i$ the set of variables corresponding to $\widetilde{\mathcal{A}}_i$, and $\widetilde{F}_i$ the general Laurent polynomial with support $\widetilde{\mathcal{A}}_i$, which can be obtained from F_i by setting $u_{i,a} = 0$ for all $a \notin \widetilde{\mathcal{A}}_i$. Set then

$$\widetilde{\boldsymbol{\mathcal{A}}} = (\widetilde{\mathcal{A}}_0, \dots, \widetilde{\mathcal{A}}_n), \quad \widetilde{\boldsymbol{u}} = (\widetilde{\boldsymbol{u}}_0, \dots, \widetilde{\boldsymbol{u}}_n) \quad \text{and} \quad \widetilde{\boldsymbol{F}} = (\widetilde{F}_0, \dots, \widetilde{F}_n).$$

Consider the vector $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \dots, \boldsymbol{\omega}_n) \in \mathbb{Z}^{\boldsymbol{\mathcal{A}}}$ given, for $i = 0, \dots, n$ and $a \in \mathcal{A}_i$, by

$$\omega_{i,a} = \begin{cases} 0 & \text{if } \boldsymbol{a} \in \widetilde{\mathcal{A}}_i, \\ 1 & \text{otherwise,} \end{cases}$$

and let $\vartheta_{\boldsymbol{\omega}_i} \colon \Delta_i \to \mathbb{R}$, $i = 0, \dots, n$, and $\Theta_{\boldsymbol{\omega}} \colon \Delta \to \mathbb{R}$ be the associated convex piecewise affine functions as in (2.8) and (2.9).

Theorem 3.19 *The following conditions are equivalent:*

(1) $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}}) \neq 0$,
(2) $\mathrm{MI}_M(\vartheta_{\boldsymbol{\omega}_0}, \dots, \vartheta_{\boldsymbol{\omega}_n}) = 0$,
(3) for every n-cell D of $S(\Theta_{\boldsymbol{\omega}})$ *we have that* $\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}(\widetilde{\boldsymbol{F}}) \neq 0$.

If any of these conditions holds, then $\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D} \in \mathbb{Z}[\widetilde{\boldsymbol{u}}]$ *for all* $D \in S(\Theta_{\boldsymbol{\omega}})^n$ *and*

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}}) = \pm \prod_{D \in S(\Theta_{\boldsymbol{\omega}})^n} \mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}. \tag{3.16}$$

Proof By Theorem 3.12, following Definition 3.10, we have that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}^{\boldsymbol{\omega}} = \pm \Big(\prod_{D} \mathrm{Res}_{\boldsymbol{\mathcal{A}}_D} + o(1) \Big)\, t^{\mathrm{ord}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}})}, \tag{3.17}$$

the product being over the n-cells D of $S(\Theta_{\boldsymbol{\omega}})$.

Since $\boldsymbol{\omega} \in \mathbb{N}^{\boldsymbol{\mathcal{A}}}$ we have that $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}^{\boldsymbol{\omega}} \in \mathbb{C}[\boldsymbol{u}][t]$ and $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}^{\boldsymbol{\omega}}\big|_{t=0} = \mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}})$. Hence, $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}}) \neq 0$ if and only if $\mathrm{ord}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = 0$, and so the expression in (3.17) gives the equivalence between (1) and (2). If any of these conditions holds, then

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}}) = \mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \pm \prod_{D} \mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}. \tag{3.18}$$

Since the left-hand side of (3.18) lies in $\mathbb{Z}[\widetilde{\boldsymbol{u}}]$ and the right-hand side is a polynomial, the factors of the latter lie in $\mathbb{Z}[\widetilde{\boldsymbol{u}}]$, proving the last part of the statement and implying (3).

Conversely, suppose that the condition (3) holds. Evaluating the expression in (3.17) by setting $u_{i,a} = 0$ for $i = 0, \dots, n$ and $a \in \mathcal{A}_i \setminus \widetilde{\mathcal{A}}_i$ we deduce that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}}) = \pm \Big(\prod_{D} \mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}(\widetilde{\boldsymbol{F}}) + o(1) \Big)\, t^{\mathrm{ord}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}})},$$

which implies (1) and concludes the proof. □

Example 3.20 Let $M = \mathbb{Z}$, $\mathcal{A}_0 = \mathcal{A}_1 = \{0, 1\}$ and set $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \mathcal{A}_1)$. Then,

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \det(u_{i,j})_{i,j \in \{0,1\}} = u_{0,0}\, u_{1,1} - u_{0,1}\, u_{1,0}. \tag{3.19}$$

Set $\widetilde{\mathcal{A}}_i = \{0\}$, $i = 0, 1$, and let $\widetilde{\boldsymbol{F}} = (u_{0,0}, u_{1,0})$ be the corresponding system of Laurent polynomials in $\mathbb{C}[t^{\pm 1}]$. With notation as in Theorem 3.19, in this case we have that $\vartheta_{\boldsymbol{\omega}_i}(x) = x$ for $i = 0, 1$ and $x \in [0, 1]$ and so $\Theta_{\boldsymbol{\omega}}(x) = x$ for $x \in [0, 2]$, as shown in Fig. 2. Hence,

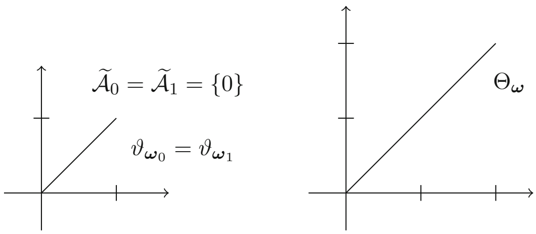

Fig. 2 Convex piecewise affine functions for subsets of the supports

$$\mathrm{MI}_{\mathbb{Z}}(\vartheta_{\omega_0}, \vartheta_{\omega_1}) = \int_0^2 \Theta_{\omega}(x)\,\mathrm{d}x - \int_0^1 \vartheta_{\omega_0}(x)\,\mathrm{d}x - \int_0^1 \vartheta_{\omega_1}(x)\,\mathrm{d}x = 2 - \frac{1}{2} - \frac{1}{2} = 1 \neq 0.$$

This result then tells us that $\mathrm{Res}_{\mathcal{A}}(\widetilde{F}) = 0$, which can also be verified from (3.19).

Set also $\widetilde{\mathcal{A}}_0 = \{0\}$ and $\widetilde{\mathcal{A}}_1 = \{1\}$, and let $\widetilde{F} = (u_{0,0}, u_{1,1}\, t)$ be the corresponding system of Laurent polynomials. Then, $\vartheta_{\omega_0}(x) = x$ and $\vartheta_{\omega_1}(x) = 1 - x$ for $x \in [0, 1]$, and so $\Theta_{\omega}(x) = \max\{1 - x, x - 1\}$ for $x \in [0, 2]$, as shown in Fig. 3. Hence,

$$\mathrm{MI}_{\mathbb{Z}}(\vartheta_{\omega_0}, \vartheta_{\omega_1}) = 1 - \frac{1}{2} - \frac{1}{2} = 0,$$

and so Theorem 3.19 implies that $\mathrm{Res}_{\mathcal{A}}(\widetilde{F}) \neq 0$. The mixed subdivision $S(\Theta_{\omega})$ has the two 1-cells $D = [0, 1]$ and $D' = [1, 2]$, that decompose as $D = 0 + [0, 1]$ and $D' = [0, 1] + 1$. Hence, this result also implies that

$$\mathrm{Res}_{\mathcal{A}}(\widetilde{F}) = \mathrm{Res}_{\mathcal{A}_D} \cdot \mathrm{Res}_{\mathcal{A}_{D'}} = u_{0,0}\, u_{1,1},$$

which can also be verified from (3.19).

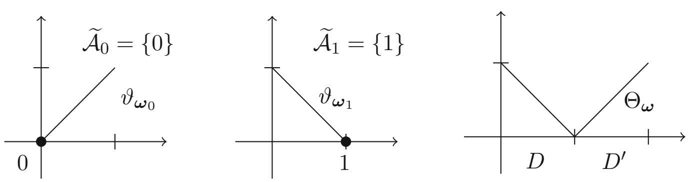

Fig. 3 Convex piecewise affine functions for other subsets

Remark 3.21 The Minkowski sum $\widetilde{\Delta} = \sum_{i=0}^{n} \widetilde{\Delta}_i$ is the cell of the mixed subdivision $S(\Theta_{\omega})$ corresponding to the vector $0 \in N_{\mathbb{R}}$ and its components are the polytopes $\widetilde{\Delta}_i$, $i = 0, \dots, n$. Hence, $\mathcal{A}_{\widetilde{\Delta}} = \widetilde{\mathcal{A}}$. We have that either $\widetilde{\Delta}$ is an n-cell of $S(\Theta_{\omega})$ or

$\operatorname{Res}_{\widetilde{\boldsymbol{\mathcal{A}}}} = \pm 1$. In the presence of any of the equivalent conditions in Theorem 3.19, the factorization in (3.16) holds and it can be alternatively written as

$$\operatorname{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}}) = \pm \operatorname{Res}_{\widetilde{\boldsymbol{\mathcal{A}}}} \cdot \prod_{D \neq \widetilde{\Delta}} \operatorname{Res}_{\boldsymbol{\mathcal{A}}_D} .$$

When $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}$ is known, this factorization can be useful to compute the sparse resultant $\operatorname{Res}_{\widetilde{\boldsymbol{\mathcal{A}}}}$ as a factor of the evaluation $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}(\widetilde{\boldsymbol{F}})$.

The next proposition gives two factorizations for the particular case when $\widetilde{\mathcal{A}}_i = \mathcal{A}_i$ for $i = 1, \dots, n$. The first one follows directly from Theorem 3.19, whereas the second is a consequence of the Poisson formula.

Proposition 3.22 *Let $\widetilde{\mathcal{A}}_0 \subset \mathcal{A}_0$ be a nonempty subset and $\widetilde{F}_0$ the general Laurent polynomial with support $\widetilde{\mathcal{A}}_0$. With notation as in Theorems 3.19 and 3.3, we have that*

$$\operatorname{Res}_{\mathcal{A}_0,\mathcal{A}_1,\dots,\mathcal{A}_n}(\widetilde{F}_0, F_1, \dots, F_n) = \pm \prod_{D \in S(\Theta_{\boldsymbol{\omega}})^n} \operatorname{Res}_{\boldsymbol{\mathcal{A}}_D} = \pm \operatorname{Res}_{\widetilde{\mathcal{A}}_0,\mathcal{A}_1,\dots,\mathcal{A}_n} \cdot \prod_{v} \operatorname{Res}_{\mathcal{A}_1^v,\dots,\mathcal{A}_n^v}^{h_{\widetilde{\mathcal{A}}_0}(v) - h_{\mathcal{A}_0}(v)},$$

the last product being over the primitive vectors $v \in N$.

Proof Let $\widetilde{f}_0 \in \mathbb{C}[M]$ with $\operatorname{supp}(\widetilde{f}_0) \subset \widetilde{\mathcal{A}}_0$ and $f_i \in \mathbb{C}[M]$ with $\operatorname{supp}(f_i) \subset \mathcal{A}_i$, $i = 1, \dots, n$, such that $\operatorname{Res}_{\mathcal{A}_1^v,\dots,\mathcal{A}_n^v}(\operatorname{init}_v(f_1), \dots, \operatorname{init}_v(f_n)) \neq 0$ for all $v \in N \setminus \{0\}$. By Theorem 3.3, we have that

$$\begin{aligned}
&\operatorname{Res}_{\mathcal{A}_0,\mathcal{A}_1,\dots,\mathcal{A}_n}(\widetilde{f}_0, f_1, \dots, f_n) \\
&\quad = \pm \prod_{v} \operatorname{Res}_{\mathcal{A}_1^v,\dots,\mathcal{A}_n^v}(\operatorname{init}_v(f_1), \dots, \operatorname{init}_v(f_n))^{-h_{\mathcal{A}_0}(v)} \cdot \prod_{p} \widetilde{f}_0(p)^{m_p},
\end{aligned}$$

the first product being over the primitive vectors $v \in N$ and the second over the solutions $p \in \mathbb{T}_M$ of $f_1 = \cdots = f_n = 0$, where m_p denotes the corresponding intersection multiplicity, and similarly

$$\begin{aligned}
&\operatorname{Res}_{\widetilde{\mathcal{A}}_0,\mathcal{A}_1,\dots,\mathcal{A}_n}(\widetilde{f}_0, f_1, \dots, f_n) \\
&\quad = \pm \prod_{v} \operatorname{Res}_{\mathcal{A}_1^v,\dots,\mathcal{A}_n^v}(\operatorname{init}_v(f_1), \dots, \operatorname{init}_v(f_n))^{-h_{\widetilde{\mathcal{A}}_0}(v)} \cdot \prod_{p} \widetilde{f}_0(p)^{m_p}.
\end{aligned}$$

Taking the quotient between these two formulae we deduce the second equality in the statement evaluated at $\widetilde{f}_0, f_1, \dots, f_n$. Since these Laurent polynomials are generic, we deduce that this equality holds for the general Laurent polynomials $\widetilde{F}_0, F_1, \dots, F_n$, as stated. This also implies that $\operatorname{Res}_{\mathcal{A}_0,\mathcal{A}_1,\dots,\mathcal{A}_n}(\widetilde{F}_0, F_1, \dots, F_n) \neq 0$, and so the first equality follows from (3.16). □

Remark 3.23 In [29], Minimair also studied the factorization of the evaluation of sparse resultant at systems of Laurent polynomials with smaller supports. Unfortunately, his result is not consistent, since its statement involves the exponent introduced in [29, Remark 3] that, as explained in [8, §5], is not well defined.

4 Canny–Emiris Matrices

4.1 Construction and Basic Properties

In [4,5] Canny and Emiris presented a class of matrices whose determinants are nonzero multiples of the sparse eliminant. These matrices are associated with some data including a family of affine functions on polytopes. Shortly afterward, this construction was extended by Sturmfels to the convex piecewise affine case [33]. Here, we recall it and study its basic properties.

We keep the notations of the previous sections. In particular,

- $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ is a family of $n+1$ supports in the lattice M,
- $\boldsymbol{\Delta} = (\Delta_0, \dots, \Delta_n)$ is the family of $n+1$ polytopes of the vector space $M_{\mathbb{R}}$ given by the convex hull of these supports,
- $\boldsymbol{u} = (\boldsymbol{u}_0, \dots, \boldsymbol{u}_n)$ is the family of $n+1$ sets of variables indexed by the elements of the supports,
- $\boldsymbol{F} = (F_0, \dots, F_n)$ is the associated system of $n+1$ general Laurent polynomials.

For $i = 0, \dots, n$ let $\rho_i : \Delta_i \to \mathbb{R}$ be a convex piecewise affine function on Δ_i defined on $\mathcal{A}_i$, that is, a convex piecewise affine function of the form $\rho_i = \vartheta_{\boldsymbol{\nu}_i}$ with $\boldsymbol{\nu}_i \in \mathbb{R}^{\mathcal{A}_i}$ as in (2.8). Set $\boldsymbol{\rho} = (\rho_0, \dots, \rho_n)$ and consider the Minkowski sum and the inf-convolution, respectively, defined as

$$\Delta = \sum_{i=0}^{n} \Delta_i \quad \text{and} \quad \rho = \boxplus_{i=0}^{n} \rho_i .$$

We assume that the mixed subdivision $S(\rho)$ of Δ is tight (Definition 2.3). Choose also a vector $\delta \in M_{\mathbb{R}}$ such that

$$(|S(\rho)^{n-1}| + \delta) \cap M = \emptyset, \tag{4.1}$$

where $|S(\rho)^{n-1}|$ denotes the $(n-1)$-skeleton of $S(\rho)$.

The *index set* is the finite set of lattice points

$$\mathcal{B} = (\Delta + \delta) \cap M.$$

Each $b \in \mathcal{B}$ lies in a unique *translated* n-cell of $S(\rho)$, that is, a polytope of the form $C + \delta$ with $C \in S(\rho)^n$. Let C_i, $i = 0, \dots, n$, be the components of this cell, as defined in (2.7). Since $S(\rho)$ is tight, there is at least one i such that $\dim(C_i) = 0$, in which case C_i consists of a single lattice point in $\mathcal{A}_i$ because ρ_i is defined on this support. Set then

$$i(b) \in \{0, \dots, n\} \quad \text{and} \quad a(b) \in \mathcal{A}_{i(b)}$$

for the *largest* of those indexes and the unique lattice point in the corresponding component, respectively.

Definition 4.1 The *row content function* associated with $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\rho}$ and δ is the function $\mathrm{rc}\colon \mathcal{B} \to \bigcup_{i=0}^{n}(\{i\} \times \mathcal{A}_i)$ defined by $\mathrm{rc}(b) = (i(b), a(b))$ for $b \in \mathcal{B}$.

Consider the subsets

$$\mathcal{B}_i = \{b \in \mathcal{B} \mid i(b) = i\},\ i = 0, \ldots, n, \tag{4.2}$$

which form a partition of $\mathcal{B}$. Set also $\mathbb{K} = \mathbb{C}(\boldsymbol{u})$ and consider the finite-dimensional linear subspaces of the group algebra $\mathbb{K}[M]$ defined as

$$V_i = \sum_{b \in \mathcal{B}_i} \mathbb{K}\,\chi^{b-a(b)},\ i = 0, \ldots, n, \quad \text{and} \quad V = \sum_{b \in \mathcal{B}} \mathbb{K}\,\chi^{b}. \tag{4.3}$$

Lemma 4.2 *Let $i \in \{0, \ldots, n\}$ and $b \in \mathcal{B}_i$. Then,*

(1) for $b' \in \mathcal{B}_i$ we have that $b' - a(b') = b - a(b)$ if and only if $b' = b$,
(2) $b - a(b) + \mathcal{A}_i \subset \mathcal{B}$.

In particular $\dim(V_i) = \#\mathcal{B}_i$ *and for all $G \in V_i$ we have that $G\,F_i \in V$.*

Proof Let $b, b' \in \mathcal{B}_i$ such that $b - a(b) = b' - a(b')$, and denote by C and C' the n-cells of $S(\rho)$ corresponding to these lattice points. With notation as in (2.6), the complementary cells C_i^{c} and $C_i'^{\mathrm{c}}$ have both dimension n and the lattice point $b - a(b) = b' - a(b')$ lies both in $\mathrm{ri}(C_i^{\mathrm{c}}) + \delta$ and in $\mathrm{ri}(C_i'^{\mathrm{c}}) + \delta$, the translates of the relative interiors of these cells. This implies that $C_i^{\mathrm{c}} = C_i'^{\mathrm{c}}$, and so $C = C'$ by Proposition 2.2. We deduce that $\{a(b)\} = C_i = C_i' = \{a(b')\}$ and so $b = b'$, proving (1).

We also have that $b - a(b) \in C_i^{\mathrm{c}} + \delta \subset \Delta_i^{\mathrm{c}} + \delta$ and so

$$b - a(b) + \mathcal{A}_i \subset (\Delta_i^{\mathrm{c}} + \delta + \Delta_i) \cap M = (\Delta + \delta) \cap M = \mathcal{B}$$

as stated in (2). The last two claims follow directly from (1) and (2). □

Consider the linear map $\Phi_{\boldsymbol{\mathcal{A}}}\colon \mathbb{K}[M]^{n+1} \to \mathbb{K}[M]$ defined, for $\boldsymbol{G} = (G_0, \ldots, G_n) \in \mathbb{K}[M]^{n+1}$, by

$$\Phi_{\boldsymbol{\mathcal{A}}}(\boldsymbol{G}) = \sum_{i=0}^{n} G_i\,F_i.$$

By Lemma 4.2(2), if $\boldsymbol{G} \in \bigoplus_{i=0}^{n} V_i$ then $\Phi_{\boldsymbol{\mathcal{A}}}(\boldsymbol{G}) \in V$.

Fixing an order on $\mathcal{B}$, the right decomposition in (4.3) gives a basis of V indexed by this finite subset. This order induces an order on each $\mathcal{B}_i$ through the row content function, and thanks to Lemma 4.2(1) the left decomposition in (4.3) gives a basis for the linear subspace V_i indexed by $\mathcal{B}_i$. The induced basis for the direct sum $\bigoplus_i V_i$ is then indexed by $\mathcal{B}$.

For a subset $\mathcal{C} \subset \mathcal{B}$ with the induced order, we denote by $\mathbb{K}^{\mathcal{C}\times\mathcal{C}}$ the set of matrices with entries in $\mathbb{K}$ and whose rows and columns are indexed by the elements of $\mathcal{C}$.

Definition 4.3 The *Sylvester map* associated with $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\rho}$ and δ is the linear map $\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\delta}\colon \bigoplus_{i=0}^{n} V_i \to V$ given by the restriction of $\Phi_{\boldsymbol{\mathcal{A}}}$ to these linear subspaces. The *Canny–Emiris matrix* associated with $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\rho}$ and δ, denoted by $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\delta} \in \mathbb{K}^{\mathcal{B}\times\mathcal{B}}$, is the matrix of this linear map in terms of row vectors. We set $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\delta} = \det(\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\delta}) \in \mathbb{Z}[\boldsymbol{u}]$ for the corresponding *Canny–Emiris determinant*.

Since the vector δ is fixed throughout our constructions, we omit it from the notation, and so this linear map, matrix and determinant will be, respectively, denoted by

$$\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}, \quad \mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} \quad \text{and} \quad H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}.$$

Remark 4.4 For $\boldsymbol{G} \in \bigoplus_i V_i$ we have that $[\boldsymbol{G}] \cdot \mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = [\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{G})]$, where $[\boldsymbol{G}]$ and $[\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{G})]$ denote the row vectors of $\boldsymbol{G}$ and of $\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{G})$ with respect to the bases of $\bigoplus_i V_i$ and of V given by the decomposition in (4.3). Hence, the row of the Canny–Emiris matrix corresponding to an element $b \in \mathcal{B}$ codifies the coefficients of the Laurent polynomial $\chi^{b-a(b)} F_{i(b)}$. Precisely, the entry corresponding to a pair $b, b' \in \mathcal{B}$ is

$$\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}[b, b'] = \begin{cases} u_{i(b),b'-b+a(b)} & \text{if } b'-b+a(b) \in \mathcal{A}_{i(b)}, \\ 0 & \text{otherwise.} \end{cases}$$

For a subset $\mathcal{C} \subset \mathcal{B}$, we, respectively, denote by

$$\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}} = (\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}[b, b'])_{b,b'\in\mathcal{C}} \in \mathbb{K}^{\mathcal{C}\times\mathcal{C}} \quad \text{and} \quad H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}} = \det(\mathcal{H}_{\boldsymbol{\mathcal{A}},\mathcal{C}}) \in \mathbb{Z}[\boldsymbol{u}]$$

the corresponding principal submatrix and minor of the Canny–Emiris matrix.

Definition 4.5 The *nonmixed index subset*, denoted by $\mathcal{B}^{\circ}$, is the set of elements of $\mathcal{B}$ lying in the translated n-cells of $S(\rho)$ that are not i-mixed for any i (Definition 2.3). We denote by

$$\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{B}^{\circ}} \in \mathbb{K}^{\mathcal{B}^{\circ}\times\mathcal{B}^{\circ}} \quad \text{and} \quad E_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{B}^{\circ}} \in \mathbb{Z}[\boldsymbol{u}]$$

the corresponding principal submatrix and minor of $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$.

We next compute the homogeneities and corresponding degrees of the Canny–Emiris determinants and, more generally, of its principal minors.

Proposition 4.6 *For $\mathcal{C} \subset \mathcal{B}$, the principal minor $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}$ is homogeneous in each set of variables $\boldsymbol{u}_i$ and with respect to the grading* $\deg_M$ *defined in* (3.14). *Moreover,*

$$\deg_{\boldsymbol{u}_i}(H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}) = \#(\mathcal{B}_i \cap \mathcal{C}),\ i = 0, \ldots, n, \quad \text{and} \quad \deg_M(H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}) = \sum_{b\in\mathcal{C}} a(b).$$

Proof Let $i \in \{0, \ldots, n\}$. For $b \in \mathcal{C}$, the entries in the corresponding row of $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}$ are homogeneous in $\boldsymbol{u}_i$ of degree 1 if $i(b) = i$ and of degree 0 otherwise. Expanding $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}$ along rows, we deduce that it is homogeneous in $\boldsymbol{u}_i$ of degree $\#(\mathcal{B}_i \cap \mathcal{C})$.

For the claims concerning $\deg_M$, first extend this grading to $\mathbb{C}[\boldsymbol{u}][M]$ by declaring that $\deg_M(\chi^a) = a$ for $a \in M$. Consider then the matrix $\widetilde{\mathcal{H}} \in \mathbb{C}[\boldsymbol{u}][M]^{\mathcal{C}\times\mathcal{C}}$ obtained from $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}$ multiplying by $\chi^{b-a(b)}$ the row corresponding to a lattice point b, for each $b \in \mathcal{C}$. By Remark 4.4, the entry corresponding to a pair $b, b' \in \mathcal{C}$ is

$$\widetilde{\mathcal{H}}[b, b'] = \begin{cases} \chi^{b-a(b)}\, u_{i(b),b'-b+a(b)} & \text{if } b' - b + a(b) \in \mathcal{A}_{i(b)}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $b' \in \mathcal{C}$, the entries in the corresponding column of $\widetilde{\mathcal{H}}$ are homogeneous with respect to $\deg_M$ of degree b'. Expanding the determinant $\widetilde{H} = \det(\widetilde{\mathcal{H}})$ along columns, we deduce that it is homogeneous with respect to $\deg_M$ of degree

$$\sum_{b' \in \mathcal{C}} b'.$$

These claims then follow from the fact that $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}} = \widetilde{H} \cdot \prod_{b\in\mathcal{C}} \chi^{-b+a(b)}$. □

Remark 4.7 The argument for the homogeneity with respect to $\deg_M$ of the principal minors of a Canny–Emiris matrix is an extension of that of Macaulay for the isobarism of the homogeneous resultant in [28, page 11].

4.2 Restriction of Data and Initial Parts

In this section, we study the interplay between the Canny–Emiris matrix associated with the data $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\rho}$ and δ, and the mixed subdivisions of Δ that are coarser than the tight mixed subdivision $S(\rho)$. We first introduce the notion of restriction of data to an n-cell of a mixed subdivision and study the compatibility of the Canny–Emiris construction with this operation.

Let $\phi_i : \Delta_i \to \mathbb{R}$, $i = 0, \dots, n$, be another family of convex piecewise affine functions, set $\phi = \boxplus_{i=0}^n \phi_i$ for their inf-convolution, and let $S(\phi)$ be the associated mixed subdivision of Δ.

Let D be an n-cell of $S(\phi)$. Similarly as in Definition 3.11, we define the *restriction* of $\boldsymbol{\mathcal{A}}$ and of $\boldsymbol{\rho}$ to D as

$$\boldsymbol{\mathcal{A}}_D = (\mathcal{A}_0 \cap D_0, \dots, \mathcal{A}_n \cap D_n) \quad \text{and} \quad \boldsymbol{\rho}_D = (\rho_0|_{D_0}, \dots, \rho_n|_{D_n}).$$

We suppose that $S(\phi) \preceq S(\rho)$ for the rest of this section. We are not assuming that $S(\phi)$ is tight and in the sequel, the considered row content function is the one induced by the family $\boldsymbol{\rho}$.

Proposition 4.8 *Let D be an n-cell of $S(\phi)$. Then,*

(1) $\rho_i|_{D_i}$ is a convex piecewise affine function on D_i defined on $\mathcal{A}_i \cap D_i$ for each i,
(2) $\boxplus_{i=0}^n \rho_i|_{D_i} = \rho|_D$,
(3) the mixed subdivision $S\big(\boxplus_{i=0}^n \rho_i|_{D_i}\big)$ of D is tight,
(4) the vector $\delta \in M_\mathbb{R}$ is generic with respect to $S\big(\boxplus_{i=0}^n \rho_i|_{D_i}\big)$ in the sense of (4.1).

Proof Clearly, the restriction $\rho_i|_{D_i}$ is a convex piecewise affine function on D_i. Since $S(\phi) \preceq S(\rho)$ we have that D_i is a union of n-cells of $S(\rho_i)$. Hence, $\rho_i|_{D_i}$ is defined on the set of vertices of these n-cells and so on $\mathcal{A}_i \cap D_i$, which proves (1).

For (2), note that for $x \in D$ we have that $(\boxplus_i \rho_i|_{D_i})(x)$ (respectively, $\rho|_D(x)$) is defined as the infimum of the sum

$$\sum_{i=0}^{n} \rho_i(x_i) \tag{4.4}$$

with $x_i \in D_i$ (respectively, $x_i \in \Delta_i$) for all i such that $\sum_{i=0}^{n} x_i = x$. Let $C \in S(\rho)$ such that $x \in C$ and $C \subset D$. By Proposition 2.1(3), the infimum of the sum in (4.4) with $x_i \in \Delta_i$, $i = 0, \dots, n$, such that $\sum_{i=0}^{n} x_i = x$ is attained when $x_i \in C_i$ for all i. Since $S(\phi) \preceq S(\rho)$, we have that $C_i \subset D_i$ and so $x_i \in D_i$ for all i. This implies that $(\boxplus_i \rho_i|_{D_i})(x) = \rho|_D(x)$ and so $\boxplus_{i=0}^{n} \rho_i|_{D_i} = \rho|_D$, as stated.

The statements in (3) and (4) follow directly from that in (2). □

By Proposition 4.8, for $D \in S(\phi)^n$ the data $(\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D, \delta)$ satisfies the hypothesis in Definition 4.3, and so we can consider its corresponding Sylvester map, Canny–Emiris matrix, and determinant. To set up the notation, for $i = 0, \dots, n$ consider the set of variables $\boldsymbol{u}_{D,i} = \{u_{i,a}\}_{a \in \mathcal{A}_i \cap D_i}$ and the general Laurent polynomial with support $\mathcal{A}_i \cap D_i$ defined as

$$F_{D,i} = \sum_{a \in \mathcal{A}_i \cap D_i} u_{i,a}\, \chi^a \in \mathbb{C}[\boldsymbol{u}_{D,i}][M].$$

Let $\boldsymbol{u}_D = (\boldsymbol{u}_{D,0}, \dots, \boldsymbol{u}_{D,n})$ and $\mathbb{K}_D = \mathbb{C}(\boldsymbol{u}_D)$. Set then

$$\mathcal{B}_D = \mathcal{B} \cap (D + \delta) \quad \text{and} \quad \mathcal{B}_{D,i} = \mathcal{B}_i \cap (D + \delta),\ i = 0, \dots, n, \tag{4.5}$$

and consider the linear subspaces of $\mathbb{K}_D[M]$ defined as

$$V_{D,i} = \sum_{b \in \mathcal{B}_{D,i}} \mathbb{K}_D\, \chi^{b-a(b)},\ i = 0, \dots, n, \quad \text{and} \quad V_D = \sum_{b \in \mathcal{B}_D} \mathbb{K}_D\, \chi^b. \tag{4.6}$$

Then, the corresponding Sylvester map $\Phi_{\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D} \colon \bigoplus_{i=0}^{n} V_{D,i} \to V_D$ is defined by

$$\Phi_{\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D}(\boldsymbol{G}) = \sum_{i=0}^{n} G_i\, F_{D,i},$$

the Canny–Emiris matrix $\mathcal{H}_{\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D} \in \mathbb{K}_D^{\mathcal{B}_D \times \mathcal{B}_D}$ is the matrix of this linear map with respect to the bases of $\bigoplus_i V_{D,i}$ and of V_D given by the decomposition in (4.6), and $H_{\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D} \in \mathbb{Z}[\boldsymbol{u}_D]$ is its determinant.

For $\mathcal{C} \subset \mathcal{B}$ let $\mathcal{C}_D = \mathcal{C} \cap (D+\delta)$. This is a subset of $\mathcal{B}_D$, and so we can consider the corresponding principal submatrix and minor of $\mathcal{H}_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}$, respectively, denoted by

$$\mathcal{H}_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D,\mathcal{C}_D} \in \mathbb{K}_D^{\mathcal{C}_D\times\mathcal{C}_D} \quad \text{and} \quad H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D,\mathcal{C}_D} \in \mathbb{Z}[\boldsymbol{u}_D].$$

The next result shows that the Canny–Emiris matrix of $(\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D, \delta)$ coincides with a principal submatrix of the evaluation of the Canny–Emiris matrix of $(\boldsymbol{\mathcal{A}}, \boldsymbol{\rho}, \delta)$ setting to zero the coefficients which are not in $\boldsymbol{\mathcal{A}}_D$.

Proposition 4.9 *The matrix $\mathcal{H}_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D,\mathcal{C}_D}$ is the evaluation of the principal submatrix $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}_D}$ by setting $u_{i,a} = 0$ for $i = 0, \dots, n$ and $a \in \mathcal{A}_i \setminus D_i$.*

Proof By Proposition 4.8, the row content function associated with the restricted data $(\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D, \delta)$ coincides with that of $(\boldsymbol{\mathcal{A}}, \boldsymbol{\rho}, \delta)$ restricted to the index set $\mathcal{B}_D$.

For each i, the general Laurent polynomial $F_{D,i}$ is the evaluation of F_i setting $u_{i,a} = 0$ for $a \in \mathcal{A}_i \setminus D_i$. Hence, the Sylvester map $\Phi_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}$ is the restriction of $\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ to the linear subspace $\bigoplus_{i=0}^n V_{D,i}$ composed with the evaluation that sets $u_{i,a} = 0$ for all i and $a \in \mathcal{A}_i \setminus D_i$.

This implies the statement when $\mathcal{C} = \mathcal{B}$. The case of an arbitrary subset $\mathcal{C} \subset \mathcal{B}$ follows from this one by considering the corresponding principal submatrices. □

Next we turn to the study of the orders and initial parts of the Canny–Emiris determinant and, more generally, of its principal minors.

Theorem 4.10 *Set $\boldsymbol{\omega} = (\phi_i(a))_{i,a} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ and let $\mathcal{C} \subset \mathcal{B}$. Then,*

$$\operatorname{ord}_{\boldsymbol{\omega}}(H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}) = \sum_{b\in\mathcal{C}} \phi_{i(b)}(a(b)) \quad \text{and} \quad \operatorname{init}_{\boldsymbol{\omega}}(H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}) = \prod_{D\in S(\phi)^n} H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D,\mathcal{C}_D}.$$

Before proving the theorem, we will establish some necessary results. The next lemma is a wide generalization of [5, Lemma 4.5] and it plays a key role in the proof of Theorem 4.10.

Lemma 4.11 *Let $b, b' \in \mathcal{B}$ such that $b' \in b - a(b) + \mathcal{A}_{i(b)}$ and set $a' = b' - b + a(b) \in \mathcal{A}_{i(b)}$. Then,*

$$\phi(b'-\delta) \le \phi(b-\delta) - \phi_{i(b)}(a(b)) + \phi_{i(b)}(a') \tag{4.7}$$

and the equality holds if and only if there is $D \in S(\phi)^n$ with $b, b' \in D + \delta$ and $a' \in D_{i(b)}$.

Proof With notation as in (2.6), we have that $b - \delta - a(b) \in \Delta_{i(b)}^{\mathrm{c}}$ and $a' \in \Delta_{i(b)}$. Since $\phi = \phi_{i(b)}^{\mathrm{c}} \boxplus \phi_{i(b)}$, this implies that

$$\phi(b'-\delta) \le \phi_{i(b)}^{\mathrm{c}}(b-\delta-a(b)) + \phi_{i(b)}(a'). \tag{4.8}$$

Let $C \in S(\rho)^n$ such that $b \in C + \delta$ and $D \in S(\phi)^n$ with $C \subset D$. Then, $b - a(b) - \delta \in C_{i(b)}^{\mathrm{c}}$ and $a(b) \in C_{i(b)}$. Since $S(\rho) \succeq S(\phi)$, we have that $C_{i(b)}^{\mathrm{c}} \subset D_{i(b)}^{\mathrm{c}}$ and $C_{i(b)} \subset D_{i(b)}$ and so

$$b - \delta \in D, \quad b - \delta - a(b) \in D_{i(b)}^{\mathrm{c}} \quad \text{and} \quad a(b) \in D_{i(b)}. \tag{4.9}$$

Proposition 2.1(3) then implies that $\phi(b-\delta)=\phi^{\mathrm{c}}_{i(b)}(b-\delta-a(b))+\phi_{i(b)}(a(b))$. The inequality in (4.7) follows from this together with (4.8).

Now if $b'\in D+\delta$ and $a'\in D_{i(b)}$ then Proposition 2.1(3) together with (4.9) implies that the inequality in (4.8) is an equality, and so is (4.7).

Conversely suppose that (4.7) is an equality or equivalently, that this is the case for (4.8). Let $D'\in S(\phi)^n$ such that $b'\in D'+\delta$. Applying again Proposition 2.1(3),

$$b-\delta-a(b)\in D'^{\mathrm{c}}_{i(b)} \quad\text{and}\quad a'\in D'_{i(b)}.$$

Since $S(\rho)$ is tight we have that $\dim(C^{\mathrm{c}}_{i(b)})=n-\dim(C_{i(b)})=n$, and since $b-\delta\in\mathrm{ri}(C)$ we also have that $b-\delta-a(b)\in\mathrm{ri}(C^{\mathrm{c}}_{i(b)})$. Hence, $\mathrm{ri}(C^{\mathrm{c}}_{i(b)})\subset\mathrm{ri}(D'^{\mathrm{c}}_{i(b)})$ and so $b-\delta-a(b)\in\mathrm{ri}(D'^{\mathrm{c}}_{i(b)})$. Using (4.9) we deduce that the n-cells $D'^{\mathrm{c}}_{i(b)}$ and $D^{\mathrm{c}}_{i(b)}$ coincide. Proposition 2.2 then implies that $D'=D$, completing the proof. □

Corollary 4.12 *Let $b,b'\in\mathcal{B}$ such that $b'\in b-a(b)+\mathcal{A}_{i(b)}$ and set $a'=b'-b+a(b)\in\mathcal{A}_{i(b)}$. Then,*

$$\rho(b'-\delta)\le\rho(b-\delta)-\rho_{i(b)}(a(b))+\rho_{i(b)}(a')$$

and the equality holds if and only if $b'=b$.

The next result generalizes [5, Theorem 6.4] which is stated for the case when the ρ_i's are affine, the fundamental subfamily of supports coincides with $\boldsymbol{\mathcal{A}}$ and the lattice $L_{\boldsymbol{\mathcal{A}}}$ coincides with M. The proof follows *mutatis mutandis* the scheme in [33, Theorem 3.1] and [5, Theorem 6.4].

Proposition 4.13 *Let $\boldsymbol{\omega}=(\boldsymbol{\omega}_0,\dots,\boldsymbol{\omega}_n)\in\mathbb{R}^{\boldsymbol{\mathcal{A}}}$ such that $\vartheta_{\boldsymbol{\omega}_i}=\rho_i$ for all i and $\mathcal{C}\subset\mathcal{B}$. Then,*

$$\mathrm{ord}_{\boldsymbol{\omega}}(H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}})=\sum_{b\in\mathcal{C}}\omega_{i(b),a(b)} \quad\text{and}\quad \mathrm{init}_{\boldsymbol{\omega}}(H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}})=\prod_{b\in\mathcal{C}}u_{i(b),a(b)}.$$

In particular $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}\neq 0$.

Proof Set $\mathcal{H}^{\boldsymbol{\omega}}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}=\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}((t^{\omega_{i,a}}u_{i,a})_{i\in\{0,\dots,n\},a\in\mathcal{A}_i})\in\mathbb{K}(t)^{\mathcal{C}\times\mathcal{C}}$ and let $\widetilde{\mathcal{H}}$ be the matrix obtained from it multiplying by $t^{\rho(b-\delta)-\rho_{i(b)}(a(b))}$ the row corresponding to a lattice point b, for each $b\in\mathcal{C}$. The entry corresponding to a pair $b,b'\in\mathcal{C}$ is

$$\widetilde{\mathcal{H}}[b,b']=\begin{cases}t^{\rho(b-\delta)-\rho_{i(b)}(a(b))+\omega_{i(b),a'}}\,u_{i(b),a'} & \text{if } a'\in\mathcal{A}_i,\\ 0 & \text{otherwise,}\end{cases}$$

with $a'=b'-b+a(b)$. For $b'\in\mathcal{C}$ we have that $\rho_{i(b)}(a')\le\omega_{i(b),a'}$ by the definition of this piecewise affine function. Moreover, let $C\in S(\rho)^n$ such that $b\in C+\delta$. Then, $C_{i(b)}=\{a(b)\}$, and since $\rho_{i(b)}=\vartheta_{\boldsymbol{\omega}_{i(b)}}$ this implies that $\rho_{i(b)}(a(b))=\omega_{i(b),a(b)}$. By Corollary 4.12

$$\rho(b'-\delta)\le\rho(b-\delta)-\rho_{i(b)}(a(b))+\omega_{i(b),a'}$$

and the equality holds if and only if $b' = b$. Hence, for $b' \in \mathcal{B}$ the entry in the corresponding column of $\widetilde{\mathcal{H}}$ for $b \in \mathcal{C}$ is of order at least $\rho(b' - \delta)$, and this value is only attained when $b = b'$. We have that $\widetilde{\mathcal{H}}[b, b] = u_{i(b),a(b)}\, t^{\rho(b-\delta)}$ and so

$$H^{\boldsymbol{\omega}}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}} = \det(\mathcal{H}^{\boldsymbol{\omega}}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}) = \det\big(\widetilde{\mathcal{H}}\big) \cdot \prod_{b\in\mathcal{C}} t^{-\rho(b-\delta)+\rho_{i(b)}(a(b))}$$
$$= \Big(\prod_{b\in\mathcal{C}} u_{i(b),a(b)} + o(1)\Big)\, t^{\sum_{b\in\mathcal{C}} \omega_{i(b),a(b)}},$$

proving the statement. □

Proof of Theorem 4.10 This result can be proven similarly as it was done for Proposition 4.13, by considering the matrix $\mathcal{H}^{\boldsymbol{\omega}}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}} = \mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}((t^{\omega_{i,a}}\, u_{i,a})_{i,a}) \in \mathbb{K}(t)^{\mathcal{C}\times\mathcal{C}}$ and the modified matrix $\widetilde{\mathcal{H}}$ obtained multiplying by $t^{\phi(b-\delta)-\phi_{i(b)}(a(b))}$ the row of $\mathcal{H}^{\boldsymbol{\omega}}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}$ corresponding to a lattice point b, for each $b \in \mathcal{C}$.

Let $D \in S(\phi)^n$. By Lemma 4.11, the lowest order in t in the column of $\widetilde{\mathcal{H}}$ corresponding to a lattice point $b' \in \mathcal{C} \cap (D + \delta)$ is $\phi(b' - \delta)$, and it is attained exactly when $b \in D + \delta$ and $a' = b' - b + a(b) \in D_{i(b)}$. Hence, the matrix extracted from $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}}$ by keeping only these entries of minimal order in each column is block diagonal, with blocks corresponding to the n-cells of $S(\phi)$. Moreover, the block corresponding to an n-cell D coincides with $\mathcal{H}_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D,\mathcal{C}_D}$. Hence,

$$H^{\boldsymbol{\omega}}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\mathcal{C}} = \det(\widetilde{\mathcal{H}}) \cdot \prod_{b\in\mathcal{C}} t^{-\phi(b-\delta)+\phi_{i(b)}(a(b))} = \Big(\prod_{D\in S(\phi)^n} H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D,\mathcal{C}_D} + o(1)\Big)\, t^{\sum_{b\in\mathcal{C}} \phi_{i(b)}(a(b))}.$$

By Proposition 4.13, all the $H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D,\mathcal{C}_D}$'s are nonzero, which completes the proof. □

4.3 Divisibility Properties

An important feature of Canny–Emiris determinants is that they provide nonzero multiples of the sparse eliminant. The next proposition generalizes [5, Theorem 6.2] and [33, Theorem 3.1], which are stated for the case when the fundamental subfamily of supports coincides with $\boldsymbol{\mathcal{A}}$.

Proposition 4.14 $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \mid H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ *in* $\mathbb{Z}[\boldsymbol{u}]$.

To prove it, we need the following lemma giving a formula for the right multiplication of a Canny–Emiris matrix by column vectors of a certain type. For a point $p \in \mathbb{T}_M$ consider the vectors

$$\zeta_p \in \mathbb{C}^{\mathcal{B}} \text{ and } \eta_{p,i} \in \mathbb{C}^{\mathcal{B}},\ i = 0, \dots, n, \tag{4.10}$$

respectively, defined for $b \in \mathcal{B}$ by $\zeta_{p,b} = \chi^b(p)$, and by $\eta_{p,i,b} = \chi^{b-a(b)}(p)$ if $b \in \mathcal{B}_i$ and by $\eta_{p,i,b} = 0$ otherwise, for the subset $\mathcal{B}_i$ defined in (4.2).

Lemma 4.15 *For* $p \in \mathbb{T}_M$ *we have that* $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} \cdot \zeta_p^{\mathrm{T}} = \sum_{i=0}^n F_i(p)\, \eta_{p,i}^{\mathrm{T}}$.

Proof In terms of the dual basis of V, right multiplication of a row vector by ζ_p^{T} corresponds to the linear functional $\mathrm{eval}_p\colon V \to \mathbb{R}$ defined by $G \mapsto G(p)$. In terms of the dual basis of $\bigoplus_j V_j$, right multiplication by $\eta_{p,i}^{\mathrm{T}}$ corresponds to the linear functional $\mathrm{eval}_{p,i}\colon \bigoplus_j V_j \to \mathbb{R}$ defined by $(G_0,\dots,G_n) \mapsto G_i(p)$. With these identifications, for $\boldsymbol{G} \in \bigoplus_j V_j$ we have that

$$[\boldsymbol{G}] \cdot \mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} \cdot \zeta_p^{\mathrm{T}} = \mathrm{eval}_p(\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{G})) = \sum_{i=0}^{n} G_i(p)\, F_i(p)$$
$$= \sum_{i=0}^{n} \mathrm{eval}_{p,i}(\boldsymbol{G})\, F_i(p) = [\boldsymbol{G}] \cdot \Big(\sum_{i=0}^{n} F_i(p)\, \eta_{p,i}^{\mathrm{T}}\Big).$$

The lemma follows from the fact that this equality is valid for every $\boldsymbol{G}$. □

Proof of Proposition 4.14 If $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm 1$ then the statement is trivial. Else, by Definition 3.1 we have that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\boldsymbol{u}]$ is irreducible and the points $\boldsymbol{u} = (\boldsymbol{u}_0,\dots,\boldsymbol{u}_n) \in \mathbb{C}^{\boldsymbol{\mathcal{A}}}$ such that there exists $p \in \mathbb{T}_M(\mathbb{C})$ with $F_0(\boldsymbol{u}_0, p) = \cdots = F_n(\boldsymbol{u}_n, p) = 0$ form a dense subset of the hypersurface $Z(\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}) \subset \prod_i \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$. By Lemma 4.15, for all these points we have that $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{u}) \cdot \zeta_p^{\mathrm{T}} = 0$ and so $\ker(\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{u})) \neq 0$. Hence, $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{u}) = 0$, which implies that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \mid H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ in $\mathbb{Z}[\boldsymbol{u}]$, as stated. □

The next result strengthens Proposition 4.14 by showing that the Canny–Emiris determinant is a multiple of the sparse resultant and not just of the sparse eliminant, under a restrictive hypothesis which nevertheless is sufficiently general for our purposes.

Proposition 4.16 *Let $i \in \{0,\dots,n\}$ such that $\mathcal{B}_i$ is contained in the union of the translated i-mixed n-cells of $S(\rho)$. Then,*

$$\frac{H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}}} \in \mathbb{Q}(\boldsymbol{u}_0,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_n).$$

Moreover, if $\mathcal{B}_i \neq \emptyset$ then $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}/\mathrm{Res}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\boldsymbol{u}_0,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_n]$.

To prove it we need some further lemmas. Set for short

$$m_i = \mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n),\ i = 0,\dots,n. \tag{4.11}$$

Lemma 4.17 *For each i, the function $\mathcal{B}_i \to M$ defined by $b \mapsto b - a(b)$ gives a bijection between*

(1) the set of lattice points of $\mathcal{B}_i$ lying in translated i-mixed n-cells of $S(\rho)$,
(2) the set of lattice points of $\Delta_i^{\mathrm{c}} + \delta$ lying in translated mixed n-cells of $S(\rho_i^{\mathrm{c}})$.

The cardinality of both sets is equal to m_i.

Proof Denote by $\mathcal{C}_i$ and $\mathcal{C}_i'$ the finite subsets of M defined in (1) and in (2), respectively. For $b \in \mathcal{C}_i$ let C be an i-mixed n-cell of $S(\rho)$ with $b \in C + \delta$. Then, C_i^{c} is a mixed

n-cell of $S(\rho_i^c)$ and $b - a(b) \in C_i^c + \delta \subset \Delta_i^c + \delta$, and so $b - a(b) \in \mathcal{C}_i'$. Hence, the assignment $b \mapsto b - a(b)$ defines a function $\mathcal{C}_i \to \mathcal{C}_i'$ which, by Lemma 4.2(1), is injective.

Now for $c \in \mathcal{C}_i'$ let B be a mixed n-cell of $S(\rho_i^c)$ with $c \in B + \delta$. With notation as in (2.3), let $v \in N_\mathbb{R}$ be the unique vector such that $B = \Gamma(\rho_i^c, v)$ and set

$$C = \Gamma(\rho, v) \in S(\rho)^n.$$

By Proposition 2.1(1), C is an i-mixed n-cell of $S(\rho)$ and $C_i^c = B$ and moreover, C_i consists of a single lattice point $a \in \mathcal{A}_i$. Setting $b = c + a \in C + \delta$, we have that $b - a(b) = b - a = c$ and so $\mathcal{C}_i \to \mathcal{C}_i'$ is surjective, proving the first claim.

For the second, note that each mixed n-cell B of $S(\rho_i^c)$ is a lattice parallelepiped, and so the genericity condition in (4.1) implies that $\#(B+\delta) \cap M = \mathrm{vol}_M(B)$. Hence, the cardinality of $\mathcal{C}_i'$ is equal to the sum of the volumes of these n-cells which, by the formula in (2.16), coincides with m_i. □

We also need the next reformulation of a result by Pedersen and Sturmfels [30] and by Emiris and Rege [14] on monomial basis of finite-dimensional algebras.

Lemma 4.18 *For $i = 0, \ldots, n$ let $\mathcal{C}_i$ be the set of lattice points in $\mathcal{B}_i$ lying in translated i-mixed n-cells of $S(\rho)$. Then, there is a proper algebraic subset $Y_i \subset \prod_{j \neq i} \mathbb{C}^{\mathcal{A}_j}$ such that for $(\overline{\boldsymbol{u}}_j)_{j \neq i} \in \prod_{j \neq i} \mathbb{C}^{\mathcal{A}_j} \setminus Y_i$, the zero set*

$$Z_i = Z(\{F_j(\overline{\boldsymbol{u}}_j, \cdot)\}_{j \neq i}) \subset \mathbb{T}_M$$

has cardinality m_i and the matrix $(\chi^{b-a(b)}(p))_{b \in \mathcal{C}_i, p \in Z_i} \in \mathbb{C}^{m_i \times m_i}$ is nonsingular.

Proof Suppose without loss of generality that $i = 0$. By Bernstein's theorem [2] and the Bertini-type theorem in [23, Part I, Theorem 6.3(3)], there is a proper algebraic subset $Y_0 \subset \prod_{j=1}^n \mathbb{C}^{\mathcal{A}_j}$ such that for $(\overline{\boldsymbol{u}}_1, \ldots, \overline{\boldsymbol{u}}_n) \in \prod_{j=1}^n \mathbb{C}^{\mathcal{A}_j} \setminus Y_0$, if we set $f_j = F_j(\overline{\boldsymbol{u}}_j, \cdot) \in \mathbb{C}[M]$, $j = 0, \ldots, n$, and $Z_0 = Z(f_1, \ldots, f_n)$, then the ideal $(f_1, \ldots, f_n) \subset \mathbb{C}[M]$ is radical and Z_0 has cardinality m_0.

By Lemma 4.17, the set $\{b - a(b)\}_{b \in \mathcal{C}_0}$ coincides with the set of lattice points in $\Delta_0^c + \delta = (\sum_{j=1}^n \Delta_j) + \delta$ lying in translated mixed n-cells of $S(\rho_0^c)$. After possibly enlarging Y_0, by [30, Theorem 1.1] or [14, Theorem 4.1] the monomials $\chi^{b-a(b)}$, $b \in \mathcal{C}_0$, form a basis of the quotient algebra $\mathbb{C}[M]/(f_1, \ldots, f_n)$.

Since the ideal $(f_1, \ldots, f_n)$ is radical, the map $\mathbb{C}[M]/(f_1, \ldots, f_n) \to \mathbb{C}^{Z_0}$ defined by $g \mapsto (g(p))_{p \in Z_0}$ is an isomorphism. Hence, the vectors $(\chi^{b-a(b)}(p))_{p \in Z_0} \in \mathbb{C}^{Z_0}$, $b \in \mathcal{C}_0$, are linearly independent, proving the lemma. □

Proof of Proposition 4.16 By its definition in (4.2), the set $\mathcal{B}_i$ contains the set of lattice points in the translated i-mixed n-cells of $S(\rho)$. Thus, the hypothesis in the present statement amounts to the fact that $\mathcal{B}_i$ is equal to this set of lattice points. Lemma 4.17, Proposition 4.6 and the degree formula in (3.2) then imply that

$$\deg_{\boldsymbol{u}_i}(H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}) = \deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = m_i.$$

When $m_i = 0$ the statement is clear. Hence, we suppose that $m_i > 0$. Consider then the 2×2-block decomposition

$$\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \begin{pmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} \end{pmatrix}$$

where the first rows and the first columns correspond to $\mathcal{B}_i$ and the others to $\mathcal{B}_j$ for $j \neq i$. By Proposition 4.13, the matrix $\mathcal{H}_{2,2}$ is nonsingular and so

$$\begin{pmatrix} 1 & -\mathcal{H}_{1,2} \cdot \mathcal{H}_{2,2}^{-1} \\ 0 & 1 \end{pmatrix} \cdot \mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \begin{pmatrix} \mathcal{H}' & 0 \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} \end{pmatrix} \tag{4.12}$$

with $\mathcal{H}' = \mathcal{H}_{1,1} - \mathcal{H}_{1,2} \cdot \mathcal{H}_{2,2}^{-1} \cdot \mathcal{H}_{2,1}$.

With notation as in Lemma 4.18, choose $(\overline{\boldsymbol{u}}_j)_{j \neq i} \in \prod_{j \neq i} \mathbb{C}^{\mathcal{A}_j} \setminus Y_i$. For $\overline{\boldsymbol{u}}_i \in \mathbb{C}^{\mathcal{A}_i}$ set $\overline{\boldsymbol{u}} = (\overline{\boldsymbol{u}}_0, \ldots, \overline{\boldsymbol{u}}_n) \in \mathbb{C}^{\boldsymbol{\mathcal{A}}}$. Set also $f_j = F_j(\overline{\boldsymbol{u}}_j, \cdot) \in \mathbb{C}[M]$ for each j and denote by $Z_i \subset \mathbb{T}_M$ the zero set of the Laurent polynomials f_j, $j \neq i$. With notation as in (4.10), for each $p \in Z_i$ we have that

$$\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\overline{\boldsymbol{u}}) \cdot \zeta_p^{\mathrm{T}} = f_i(p)\, \eta_{p,i}^{\mathrm{T}} \tag{4.13}$$

thanks to Lemma 4.15. Consider the matrices in $\mathbb{C}^{\mathcal{B}_i \times Z_i} \simeq \mathbb{C}^{m_i \times m_i}$ defined as

$$\mathcal{P} = (\chi^b(p))_{b \in \mathcal{B}_i, p \in Z_i} \quad \text{and} \quad \mathcal{Q} = (\chi^{b-a(b)}(p))_{b \in \mathcal{B}_i, p \in Z_i}.$$

From (4.12) and (4.13), we deduce that $\mathcal{H}'(\overline{\boldsymbol{u}}) \cdot \mathcal{P} = \operatorname{diag}((f_i(p))_{p \in Z_i}) \cdot \mathcal{Q}$ and so

$$\begin{aligned} H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\overline{\boldsymbol{u}}) \cdot \det(\mathcal{P}) &= \det(\mathcal{H}_{2,2}(\overline{\boldsymbol{u}})) \cdot \det(\mathcal{H}'(\overline{\boldsymbol{u}})) \cdot \det(\mathcal{P}) \\ &= \det(\mathcal{H}_{2,2}(\overline{\boldsymbol{u}})) \cdot \det(\mathcal{Q}) \cdot \prod_{p \in Z_i} f_i(p). \end{aligned}$$

By Lemma 4.18, the matrix $\mathcal{Q}$ is nonsingular and so $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\overline{\boldsymbol{u}}) = 0$ only if there is $p \in Z_i$ such that $f_i(p) = 0$. Hence, both $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ and $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}$ are polynomials of degree $m_i > 0$ in the set of variables $\boldsymbol{u}_i$ that, for a generic choice of $(\overline{\boldsymbol{u}}_j)_{j \neq i} \in \prod_{j \neq i} \mathbb{C}^{\mathcal{A}_j}$ vanish for $\overline{\boldsymbol{u}}_i \in \mathbb{C}^{\mathcal{A}_i}$ if and only if this also holds for the irreducible polynomial $\operatorname{Elim}_{\boldsymbol{\mathcal{A}}}$ (recall that $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}$ is a power of $\operatorname{Elim}_{\boldsymbol{\mathcal{A}}}$). Thus,

$$H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \gamma \cdot \operatorname{Res}_{\boldsymbol{\mathcal{A}}}$$

with $\gamma \in \mathbb{Q}(\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{i-1}, \boldsymbol{u}_{i+1}, \ldots, \boldsymbol{u}_n)^{\times}$, proving the first claim. The second is a direct consequence of the first together with Gauss' lemma, since $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}$ is a primitive polynomial in $\mathbb{Z}[\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{i-1}, \boldsymbol{u}_{i+1}, \ldots, \boldsymbol{u}_n][\boldsymbol{u}_i]$. □

Corollary 4.19 *We have that $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}/\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Q}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$. Moreover, if $m_0 > 0$ then $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}/\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \in \mathbb{Z}[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n]$.*

Proof For $b \in \mathcal{B}_0$ let C be the n-cell of $S(\rho)$ such that $b \in C + \delta$. Then, $\dim(C_i) > 0$ for all $i > 0$ and so C is 0-mixed. Moreover, if $m_0 > 0$ then $\mathcal{B}_0 \neq \emptyset$ thanks to Lemma 4.17. The corollary follows then from Proposition 4.16. □

Remark 4.20 We conjecture that $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} \mid H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ in the general case. If this is true, then the hypothesis $\mathcal{B}_i \neq \emptyset$ in Proposition 4.16 and that $m_0 > 0$ in Corollary 4.19 would not be necessary.

The next corollary allows to compute the sparse resultant as the greatest common divisor of a family of Canny–Emiris determinants, when the fundamental subfamily of supports coincides with $\boldsymbol{\mathcal{A}}$. It generalizes the method proposed in [5, §7] to the situation when the sublattice $L_{\boldsymbol{\mathcal{A}}}$ is not necessarily equal to M.

Corollary 4.21 *Suppose that the fundamental subfamily of supports coincides with $\boldsymbol{\mathcal{A}}$ and choose a permutation σ_i of the index set $\{0, \dots, n\}$ with $\sigma_i(0) = i$ for each i. Then,*

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{u}) = \pm \gcd(H_{\sigma_0(\boldsymbol{\mathcal{A}}),\sigma_0(\boldsymbol{\rho})}(\sigma_0(\boldsymbol{u})), \dots, H_{\sigma_n(\boldsymbol{\mathcal{A}}),\sigma_n(\boldsymbol{\rho})}(\sigma_n(\boldsymbol{u}))).$$

Proof Corollary 4.19 applied to the data $(\sigma_i(\boldsymbol{\mathcal{A}}), \sigma_i(\boldsymbol{\rho}), \delta)$ implies that $\mathrm{Res}_{\sigma_i(\boldsymbol{\mathcal{A}})}$ divides $H_{\sigma_i(\boldsymbol{\mathcal{A}}),\sigma_i(\boldsymbol{\rho})}$ and that both polynomials have the same degree in the set of variables $\boldsymbol{u}_0$. By the invariance of the sparse resultant under permutations of the supports we deduce that $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{u})$ divides $H_{\sigma_i(\boldsymbol{\mathcal{A}}),\sigma_i(\boldsymbol{\rho})}(\sigma_i(\boldsymbol{u}))$ and that both polynomials have the same degree in the set of variables $\boldsymbol{u}_i$ for each i, which implies the statement. □

4.4 The Macaulay Formula for the Sparse Resultant

In this section, we give the proof of our main result, Theorem 1.3 in the introduction. It is based on the constructions and results from the previous sections, and we keep the notations therein. In particular,

- $\boldsymbol{\rho} = (\rho_0, \dots, \rho_n)$ is a family of $n+1$ convex piecewise affine functions on the polytopes in $\boldsymbol{\Delta} = (\Delta_0, \dots, \Delta_n)$ defined on the supports in $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ such that the mixed subdivision $S(\rho)$ defined by its inf-convolution is tight (Definition 2.3),
- δ is a vector in $M_{\mathbb{R}}$ that is generic with respect to $S(\rho)$ in the sense of (4.1),
- $\mathcal{B}$ is the index set and $\mathcal{B}_i$, $i = 0, \dots, n$, the subsets partitioning it, and $\mathcal{B}^\circ$ is the nonmixed index subset of $\mathcal{B}$,
- $\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$, $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ and $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ are the Sylvester map and the Canny–Emiris matrix and determinant associated with the data $(\boldsymbol{\mathcal{A}}, \boldsymbol{\rho}, \delta)$,
- $\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ and $E_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ are the principal submatrix and minor corresponding to $\mathcal{B}^\circ$,
- $(\boldsymbol{\mathcal{A}}_D, \boldsymbol{\rho}_D)$ is the restriction of $(\boldsymbol{\mathcal{A}}, \boldsymbol{\rho})$ to an n-cell D of a mixed subdivision that is coarser than $S(\rho)$.

To prove Theorem 1.3, we use a descent argument similar to that of Macaulay in [27] and the first author in [7]. The following notion comprises the properties of a tight mixed subdivision that allow us to perform this descent.

Definition 4.22 The tight mixed subdivision $S(\rho)$ is *admissible* if there is an incremental chain of mixed subdivisions of Δ (Definition 2.4)

$$S(\theta_0) \preceq \cdots \preceq S(\theta_n) \tag{4.14}$$

with $S(\theta_n) \preceq S(\rho)$ such that for $k = 0, \ldots, n$, each n-cell D of $S(\theta_k)$ verifies at least one of the conditions:

(1) the fundamental subfamily of $\boldsymbol{\mathcal{A}}_D$ has at most one support,
(2) the subset $\mathcal{B}_{D,k}$ of $\mathcal{B}_k$ defined in (4.5) is contained in the union of the translated k-mixed n-cells of $S(\rho_D)$.

The incremental chain of mixed subdivisions in (4.14) is called *admissible* (for $S(\rho)$).

The next result gives sufficient conditions for a given incremental chain to be admissible that will allow us to recover Macaulay's original formulation with our methods (Proposition 5.8).

Proposition 4.23 *Let $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ be an incremental chain of mixed subdivisions of Δ such that for $k = 0, \ldots, n$, each n-cell D of $S(\theta_k)$ verifies at least one of the conditions:*

(1) there is $J \subset \{0, \ldots, n\}$ such that $\dim(\sum_{j\in J} D_j) < \#J - 1$,
(2) there is $i \in \{0, \ldots, n\}$ such that $\dim(D_i) = 0$,
(3) for all $i < k$ we have that $\dim(\sum_{j\neq i,k} D_j) < n$.

Then, this incremental chain is admissible for any tight mixed subdivision of Δ that refines $S(\theta_n)$.

Proof Let $k = 0, \ldots, n$ and $D \in S(\theta_k)^n$. If this n-cell satisfies the condition (1), then for $i = 0, \ldots, n$ we have that

$$\dim\Big(\sum_{j\in J\setminus\{i\}} D_j\Big) \leq \dim\Big(\sum_{j\in J} D_j\Big) < \#J - 1 \leq \#(J\setminus\{i\}).$$

By the basic properties of the mixed volume, $\mathrm{MV}_M(D_0, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n) = 0$. Hence, $\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D} = \pm 1$ thanks to the degree formula in (3.2) and so the fundamental subfamily of $\boldsymbol{\mathcal{A}}_D$ is empty. Thus, in this case D satisfies the condition (1) in Definition 4.22.

If D satisfies the condition (2), then also $\mathrm{MV}_M(D_0, \ldots, D_{j-1}, D_{j+1}, \ldots, D_n) = 0$ for all $j \neq i$ and so the fundamental subfamily of $\boldsymbol{\mathcal{A}}_D$ is either empty or consists of the single support $\mathcal{A}_i$. In both cases, D also satisfies the condition (1) in Definition 4.22.

Finally suppose that D satisfies the condition (3). For $b \in \mathcal{B}_{D,k}$ let C be the n-cell of $S(\rho_D)$ such that $b \in C + \delta$. We have that $\dim(C_k) = 0$ and that $C_j \subset D_j$ for all j, and so for each $i < k$ we have that

$$\dim(C_i) = n - \sum_{j\neq i,k} \dim(C_j) = n - \dim\Big(\sum_{j\neq i,k} C_j\Big) \geq n - \dim\Big(\sum_{j\neq i,k} D_j\Big) > 0.$$

Since $b \in \mathcal{B}_{D,k}$ we also have that $\dim(C_i) > 0$ for all $i > k$ and so C is k-mixed. Hence, in this case D satisfies the condition (2) in Definition 4.22. We conclude that the incremental chain is admissible for any tight mixed subdivision of Δ refining $S(\theta_n)$. □

As a consequence of this result, we deduce that tight incremental chains of mixed subdivisions are admissible.

Proposition 4.24 *An incremental chain $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ of mixed subdivisions of Δ with $S(\theta_n) \preceq S(\rho)$ that is tight in the sense of Definition 2.4, satisfies the conditions in Proposition 4.23. In particular, it is admissible.*

Proof Since the incremental chain is tight, for $k = 0, \ldots, n$ and $D \in S(\theta_k)^n$ we have that

$$\sum_{j=0}^{k-1} \dim(D_j) + \dim\Big(\sum_{j=k}^{n} D_j\Big) = n. \tag{4.15}$$

If there is $i < k$ such that D_i is a point, then D satisfies the condition (2) in Proposition 4.23. Else $\dim(D_i) > 0$ for all $i < k$ and so the equality in (4.15) implies that

$$\dim\Big(\sum_{j\neq i} D_j\Big) \leq \sum_{i\neq j<k} \dim(D_j) + \dim\Big(\sum_{j=k}^{n} D_j\Big) = n - \dim(D_i) < n,$$

and so D satisfies the condition (3) in Proposition 4.23. This gives the first claim, whereas the second is an application of that proposition. □

Corollary 4.25 *The incremental chains of mixed subdivisions $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ of Δ obtained by setting $s = n$ in Example 2.12, are admissible for $S(\theta_n)$.*

The next result gives the basic particular cases of Theorem 1.3 that can be treated directly. Recall that $m_i = \mathrm{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n)$ for each i, as in (4.11).

Proposition 4.26 *Let $\boldsymbol{\mathcal{A}}_J$ be the fundamental subfamily of $\boldsymbol{\mathcal{A}}$. Then,*

(1) when $\boldsymbol{\mathcal{A}}_J = \emptyset$ we have that $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm 1$ and $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = E_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$,
(2) when $\boldsymbol{\mathcal{A}}_J$ consists of a single support $\mathcal{A}_i$, we have that $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm u_{i,a}^{m_i}$ and $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = u_{i,a}^{m_i}\, E_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ for the unique lattice point $a \in M$ such that $\mathcal{A}_i = \{a\}$.

Proof The first claim in (1) is a direct consequence of the hypothesis that $J = \emptyset$. The same hypothesis together with the degree formula in (3.2) implies that $m_j = 0$ for all j. Lemma 4.17 then implies that $\mathcal{B}^\circ = \mathcal{B}$, which gives the second claim.

For (2), first note that, by the rank condition in Proposition 3.6, if $\#J = 1$ for the fundamental subfamily $\boldsymbol{\mathcal{A}}_J$, its unique element $\mathcal{A}_i$ is a singleton. Then, in this case, $\boldsymbol{u}_i = \{u_{i,a}\}$ and $m_j = 0$ for all $j \neq i$. The first claim follows then from the fact that $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ is a primitive homogeneous polynomial in $\mathbb{Z}[u_{i,a}]$ of degree m_i. The hypothesis that $\mathcal{A}_i = \{a\}$ also implies that $S(\rho)$ has no n-cells that are j-mixed for $j \neq i$. By Lemma 4.17, the subset $\mathcal{C}_i \subset \mathcal{B}_i$ of lattice points lying in the translated n-cells that are i-mixed has cardinality m_i. For each $b \in \mathcal{C}_i$ we have that $\mathrm{rc}(b) = (i, a)$ and so, for $b' \in \mathcal{B}$,

$$\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}[b, b'] = \begin{cases} u_{i,a} & \text{if } b' = b, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{B}^\circ = \mathcal{B} \backslash \mathcal{C}_i$ and $\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ is the principal submatrix of $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ corresponding to this subset, we deduce the second claim. □

Now we are ready for the proof of the main result of this paper, corresponding to Theorem 1.3 in the introduction.

Theorem 4.27 *Suppose that $S(\rho)$ is admissible and let $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ with $S(\theta_n) \preceq S(\rho)$ be an admissible incremental chain for it. Then,*

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm \frac{H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}}{E_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}} \quad \text{and} \quad E_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \prod_{D} E_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D},$$

the product in the second formula being over the n-cells D of $S(\theta_1)$.

Proof Let $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ with $S(\theta_n) \preceq S(\rho)$ be an admissible incremental chain, and for $k = 0, \ldots, n$ let $\theta_{k,j} \colon \Delta_j \to \mathbb{R}$, $j = 0, \ldots, n$, be the family of convex piecewise affine functions corresponding to θ_k. Set also $\theta_{n+1} = \rho$.

We prove by reverse induction on k that for every n-cell D of $S(\theta_k)$ we have that

$$\frac{H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}} = \pm E_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}. \tag{4.16}$$

The first statement in the theorem corresponds to the case when $k = 0$ and $D = \Delta$.

For $k = n + 1$ we note that $S(\theta_{n+1}) = S(\rho)$ is tight. Hence, for $D \in S(\theta_{n+1})$ we have that $\sum_{i=0}^{n} \dim(D_i) = n$ and so $\dim(D_i) = 0$ for at least one i. This implies that the fundamental subfamily of $\boldsymbol{\mathcal{A}}_D$ is either empty or consists of the single support $\mathcal{A}_i \cap D_i$. Hence, $\boldsymbol{\mathcal{A}}_D$ verifies the hypothesis of Proposition 4.26, and so the equality in (4.16) follows from this result.

Hence, suppose that $0 \leq k \leq n$ and let $D \in S(\theta_k)$. In case D satisfies the condition (1) in Definition 4.22, the equality in (4.16) follows similarly from Proposition 4.26. If this does not happen, then D satisfies the condition (2) in this definition, and so the subset $\mathcal{B}_{D,k}$ is contained in the union of the translated k-mixed n-cells of $S(\rho_D)$.

By Proposition 3.8, for all $i < k$ and every n-cell C of $S(\theta_i)$ containing D, the index set of the fundamental subfamily of $\boldsymbol{\mathcal{A}}_C$ contains that of $\boldsymbol{\mathcal{A}}_D$, and so this n-cell cannot satisfy the condition (1) in Definition 4.22. Hence, it satisfies the condition (2) in this definition and so $\mathcal{B}_{C,i}$ is contained in the union of the translated i-mixed cells of $S(\rho_C)$. From here, we deduce that $\mathcal{B}_{D,i}$ is contained in the union of the translated i-mixed cells of $S(\rho_D)$ and as mentioned, the same happens for $i = k$. Together with Proposition 4.16, this implies that

$$E_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D} \in \mathbb{Z}[\boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n] \quad \text{and} \quad \frac{H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}} \in \mathbb{Q}(\boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n).$$

Consider the vector $\boldsymbol{\omega} \in \mathbb{R}^{\boldsymbol{\mathcal{A}}}$ defined by $\omega_{j,a} = \theta_{k+1,j}(a)$ for $j = 0, \dots, n$ and $a \in \mathcal{A}_j$. Since the chain is incremental, we have that $\omega_{j,a} = 0$ for $j \geq k+1$ and $a \in \mathcal{A}_j$ and so

$$E_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D} = \mathrm{init}_{\boldsymbol{\omega}}(E_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}) \quad \text{and} \quad \frac{H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}} = \mathrm{init}_{\boldsymbol{\omega}}\left(\frac{H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D}}\right) = \frac{\mathrm{init}_{\boldsymbol{\omega}}(H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D})}{\mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D})}.$$

Applying Theorem 3.12 and Theorem 4.10 for the subsets $\mathcal{B}_D$ and $\mathcal{B}_D^{\circ}$, we deduce that

$$\mathrm{init}_{\boldsymbol{\omega}}(E_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D}) = \prod_{D'} E_{\boldsymbol{\mathcal{A}}_{D'},\boldsymbol{\rho}_{D'}} \quad \text{and} \quad \frac{\mathrm{init}_{\boldsymbol{\omega}}(H_{\boldsymbol{\mathcal{A}}_D,\boldsymbol{\rho}_D})}{\mathrm{init}_{\boldsymbol{\omega}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}_D})} = \prod_{D'} \frac{H_{\boldsymbol{\mathcal{A}}_{D'},\boldsymbol{\rho}_{D'}}}{\mathrm{Res}_{\boldsymbol{\mathcal{A}}_{D'}}}, \tag{4.17}$$

both products being over the n-cells D' of $S(\Theta_{\boldsymbol{\omega}}) = S(\theta_{k+1})$ that are contained in D. The equality in (4.16) then follows from the inductive hypothesis.

The second statement follows from the first equality in (4.17) applied to the case $k = 0$ and $D = \Delta$. □

Theorem 4.27 generalizes the Canny–Emiris conjecture since, by Proposition 3.7, the sparse eliminant coincides with the sparse resultant of the fundamental subfamily of supports with respect to the minimal lattice containing it.

Remark 4.28 We can extend the Canny–Emiris construction to a larger class of matrices, following an idea of the first author in [7]. The study of its properties can be done similarly as for the original formulation, and so we only indicate the modifications.

Let $\psi : \Lambda \to \mathbb{R}$ be a convex piecewise affine function on a polytope such that the mixed subdivision $S(\rho \boxplus \psi)$ on $\Delta + \Lambda$ is tight and its $(n-1)$-th skeleton does not contain any lattice point. The case treated in this paper corresponds to the situation when $\Lambda = \{\delta\}$ and ψ takes any value at this point.

The index set is defined as $\mathcal{B} = (\Delta + \Lambda) \cap M$. For each $b \in \mathcal{B}$, there is a unique n-cell C of $S(\rho \boxplus \psi)$ containing it, and we denote by $C_i \in S(\rho_i)$, $i = 0, \dots, n$, and $B \in S(\psi)$ its components. The tightness condition implies that there is always an i such that $\dim(C_i) = 0$, in which case C_i consists of a single lattice point of $\mathcal{A}_i$. We then set

$$i(b) \in \{0, \dots, n\} \quad \text{and} \quad a(b) \in \mathcal{A}_{i(b)}$$

for the largest of these i's and the unique lattice point in the corresponding component, respectively. For $b \in \mathcal{B}$ we have that $b - a(b) + \mathcal{A}_{i(b)} \subset \mathcal{B}$ and so we can define a Sylvester map $\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\psi}$ and the corresponding Canny–Emiris matrix $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\psi}$ and determinant $H_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\psi}$ in the same way as it was done before.

For $i \in \{0, \dots, n\}$, we say that an n-cell C of $S(\rho \boxplus \psi)$ is i-mixed if $\dim(C_j) = 1$ for all $j \neq i$. The nonmixed index subset $\mathcal{B}^{\circ}$ is the subset of $\mathcal{B}$ of lattice points lying in the n-cells that are not i-mixed for any i, and we denote by $\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\psi}$ and $E_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho},\psi}$ the corresponding principal submatrix and minor of the Canny–Emiris matrix.

Theorem 4.27 can then be extended to the statement that

$$\mathrm{Res}_{\mathcal{A}} = \pm \frac{H_{\mathcal{A},\rho,\psi}}{E_{\mathcal{A},\rho,\psi}}$$

whenever the mixed subdivision $S(\rho)$ is admissible in the sense of Definition 4.22, together with a factorization for $E_{\mathcal{A},\rho,\psi}$ in terms of the n-cells of the second mixed subdivision in an admissible chain for $S(\rho)$.

Remark 4.29 The extraneous factor $E_{\mathcal{A},\rho}$ does not depend on the choice of the admissible chain $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ but its factorization in Theorem 4.27 in principle depends on the second mixed subdivision $S(\theta_1)$. One can go further and refine this factorization using the subsequent mixed subdivisions in this chain. It would be interesting to exhibit concrete cases where different admissible chains produce different factorizations for this extraneous factor.

In addition, our factorization of $E_{\mathcal{A},\rho}$ consists of a product of extraneous factors of smaller systems in the same dimension, whereas the factorizations presented by Macaulay in [27, page 14] and by the first author in [7, (47)] consist of Canny–Emiris determinants and extraneous factors of systems in lower dimensions. It would be interesting to compare these approaches and put them into a more general framework.

5 Homogeneous Resultants

The goal of this section is to show that Macaulay's classical formula for the homogeneous resultant can be recovered as a particular case of our construction. Macaulay's row content function is given in terms of the exponents of the monomials indexing the matrix, and we will exhibit a family of affine functions on multiples of the standard simplex whose associated mixed subdivision reflects this idea. Unfortunately, this mixed subdivision is not tight in dimension ≥ 3 (Remark 5.6) but we show that any tight refinement of it will do the work (Proposition 5.9). This is done by constructing a chain of mixed subdivisions for this refinement that is admissible (Proposition 5.8).

5.1 The Classical Macaulay Formula

In this section, we describe the Macaulay formula for the homogeneous resultant from [27].

Let $\boldsymbol{d} = (d_0, \ldots, d_n) \in (\mathbb{N}_{>0})^{n+1}$. For $i = 0, \ldots, n$ let $\boldsymbol{u}_i = \{u_{i,\boldsymbol{c}}\}_{|\boldsymbol{c}|=d_i}$ be a set of $\binom{d_i+n}{n}$ variables indexed by the lattice points $\boldsymbol{c} \in \mathbb{N}^{n+1}$ of length $|\boldsymbol{c}| = d_i$, put $\boldsymbol{u} = (\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n)$ and denote by

$$\mathrm{Res}_{\boldsymbol{d}} \in \mathbb{Z}[\boldsymbol{u}]$$

the corresponding homogeneous resultant as in Example 3.9.

Let $\boldsymbol{t} = \{t_0, \ldots, t_n\}$ be a further set of $n+1$ variables. By [6, Chapter 3, Theorem 2.3], $\mathrm{Res}_{\boldsymbol{d}}$ is the unique irreducible polynomial in $\mathbb{Z}[\boldsymbol{u}]$ vanishing when evaluated at the coefficients of a system of $n+1$ homogeneous polynomials in the variables $\boldsymbol{t}$ of degrees $\boldsymbol{d}$ if and only if this system has a zero in the projective space $\mathbb{P}^n_{\mathbb{C}}$, and verifying that $\mathrm{Res}_{\boldsymbol{d}}(t_0^{d_0}, \dots, t_n^{d_n}) = 1$.

Set $\mathbb{K} = \mathbb{C}(\boldsymbol{u})$ and choose an integer $m \ge |\boldsymbol{d}| - n$. For $i = 0, \dots, n$ consider the general homogeneous polynomial in the variables $\boldsymbol{t}$ of degree d_i

$$P_i = \sum_{|\boldsymbol{c}|=d_i} u_{i,\boldsymbol{c}}\, \boldsymbol{t}^{\boldsymbol{c}} \in \mathbb{K}[\boldsymbol{t}]$$

and the linear subspace of the homogeneous part $\mathbb{K}[\boldsymbol{t}]_{m-d_i}$ given by

$$T_i = \{Q_i \in \mathbb{K}[\boldsymbol{t}]_{m-d_i} \mid \deg_{t_j}(Q_i) < d_j \text{ for } j = i+1, \dots, n\}, \tag{5.1}$$

where $\deg_{t_j}(Q_i)$ denotes the degree of Q_i in the variable t_j. Set also $T = \mathbb{K}[\boldsymbol{t}]_m$ and consider the linear map $\Psi_{\boldsymbol{d},m} \colon \bigoplus_{i=0}^n T_i \to T$ defined by

$$\Psi_{\boldsymbol{d},m}(Q_0, \dots, Q_n) = \sum_{i=0}^n Q_i\, P_i.$$

Let $\mathcal{I} = \{\boldsymbol{c} \in \mathbb{N}^{n+1}\}_{|\boldsymbol{c}|=m}$ be the *index set*, and consider also the finite subsets

$$\mathcal{I}_i = \{\boldsymbol{c} = (c_0, \dots, c_n) \in \mathcal{I} \mid c_i \ge d_i \text{ and } c_j < d_j \text{ for } j > i\},\ i = 0, \dots, n,$$

which form a partition of it. Denoting by $\widehat{e_i}$, $i = 0, \dots, n$, the vectors in the standard basis of $\mathbb{R}^{n+1}$, the sets of monomials

$$\{\boldsymbol{t}^{\boldsymbol{c}-d_i\,\widehat{e_i}}\}_{\boldsymbol{c}\in\mathcal{I}_i},\ i = 0, \dots, n, \quad \text{and} \quad \{\boldsymbol{t}^{\boldsymbol{c}}\}_{\boldsymbol{c}\in\mathcal{I}} \tag{5.2}$$

are bases of T_i, $i = 0, \dots, n$, and of T, respectively. Then, we set

$$\mathcal{M}_{\boldsymbol{d},m} \in \mathbb{K}^{\mathcal{I}\times\mathcal{I}}$$

for the matrix of $\Psi_{\boldsymbol{d},m}$ in terms of row vectors and with respect to the monomial bases of $\bigoplus_{i=0}^n T_i$ and of T given by (5.2), both indexed by $\mathcal{I}$.

Set also $\mathcal{I}^{\mathrm{red}} \subset \mathcal{I}$ for the subset of lattice points $\boldsymbol{c} = (c_0, \dots, c_n) \in \mathcal{I}$ with a unique i such that $c_i \ge d_i$. We then denote by $\mathcal{I}^{\circ} = \mathcal{I} \setminus \mathcal{I}^{\mathrm{red}}$ the *nonreduced index subset* and by $\mathcal{N}_{\boldsymbol{d},m}$ the corresponding principal submatrix of $\mathcal{M}_{\boldsymbol{d},m}$, The Macaulay formula for the homogeneous resultant [27] then states that

$$\mathrm{Res}_{\boldsymbol{d}} = \frac{\det(\mathcal{M}_{\boldsymbol{d},m})}{\det(\mathcal{N}_{\boldsymbol{d},m})}, \tag{5.3}$$

see [24, Proposition 3.9.4.4] for a modern treatment.

Example 5.1 Let $n = 2$, $\boldsymbol{d} = (1, 2, 2)$ and $m = |\boldsymbol{d}| - n = 3$. The corresponding general homogeneous polynomials are

$$\begin{aligned}
P_0 &= \alpha_0\, t_0 + \alpha_1\, t_1 + \alpha_2\, t_2,\\
P_1 &= \beta_0\, t_0^2 + \beta_1\, t_0\, t_1 + \beta_2\, t_1^2 + \beta_3\, t_0\, t_2 + \beta_4\, t_1\, t_2 + \beta_5\, t_2^2,\\
P_2 &= \gamma_0\, t_0^2 + \gamma_1\, t_0\, t_1 + \gamma_2\, t_1^2 + \gamma_3\, t_0\, t_2 + \gamma_4\, t_1\, t_2 + \gamma_5\, t_2^2,
\end{aligned}$$

and the index set splits as $\mathcal{I} = \mathcal{I}_0 \sqcup \mathcal{I}_1 \sqcup \mathcal{I}_2$ with

$$\begin{aligned}
\mathcal{I}_0 &= \{(3, 0, 0), (2, 1, 0), (2, 0, 1), (1, 1, 1)\},\\
\mathcal{I}_1 &= \{(1, 2, 0), (0, 3, 0), (0, 2, 1)\},\\
\mathcal{I}_2 &= \{(1, 0, 2), (0, 1, 2), (0, 0, 3)\}.
\end{aligned}$$

The matrix $\mathcal{M}_{\boldsymbol{d},m}$ is constructed by declaring that the row corresponding to each lattice point $\boldsymbol{c} \in \mathcal{I}_i$, $i = 0, 1, 2$, consists of the coefficients of the polynomials $\boldsymbol{t}^{\boldsymbol{c}-d_i\,\widehat{e}_i}\, P_i$ in the monomial basis of the homogeneous part $\mathbb{K}[\boldsymbol{t}]_3$. Hence, this matrix is written as

$$\begin{array}{c} \\ t_0^2\,P_0 \\ t_0\,t_1\,P_0 \\ t_0\,t_2\,P_0 \\ t_1\,t_2\,P_0 \\ t_0\,P_1 \\ t_1\,P_1 \\ t_2\,P_1 \\ t_0\,P_2 \\ t_1\,P_2 \\ t_2\,P_2 \end{array}
\begin{array}{c} \begin{array}{cccccccccc} t_0^3 & t_0^2\,t_1 & t_0^2\,t_2 & t_0\,t_1\,t_2 & t_0\,t_1^2 & t_1^3 & t_1^2\,t_2 & t_0\,t_2^2 & t_1\,t_2^2 & t_2^3 \end{array} \\
\begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_0 & 0 & \alpha_2 & \alpha_1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \alpha_0 & \alpha_1 & 0 & 0 & 0 & \alpha_2 & 0 & 0\\
0 & 0 & 0 & \alpha_0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0\\
\beta_0 & \beta_1 & \beta_3 & \beta_4 & \beta_2 & 0 & 0 & \beta_5 & 0 & 0\\
0 & \beta_0 & 0 & \beta_3 & \beta_1 & \beta_2 & \beta_4 & 0 & \beta_5 & 0\\
0 & 0 & \beta_0 & \beta_1 & 0 & 0 & \beta_2 & \beta_3 & \beta_4 & \beta_5\\
\gamma_0 & \gamma_1 & \gamma_3 & \gamma_4 & \gamma_2 & 0 & 0 & \gamma_5 & 0 & 0\\
0 & \gamma_0 & 0 & \gamma_3 & \gamma_1 & \gamma_2 & \gamma_4 & 0 & \gamma_5 & 0\\
0 & 0 & \gamma_0 & \gamma_1 & 0 & 0 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5
\end{pmatrix} \end{array}. \qquad (5.4)$$

We have that $\mathcal{I} \backslash \mathcal{I}^{\mathrm{red}} = \{(1, 2, 0), (1, 0, 2)\}$ and so

$$\mathcal{N}_{\boldsymbol{d},m} = \begin{pmatrix} \beta_2 & \beta_5 \\ \gamma_2 & \gamma_5 \end{pmatrix}.$$

By the identity in (5.3), $\mathrm{Res}_{\boldsymbol{d}}$ is the quotient of the determinants of these matrices. It is an irreducible trihomogeneous polynomial in $\mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}]$ of tridegree $(4, 2, 2)$ having 234 monomial terms.

5.2 A Mixed Subdivision on a Simplex

In this section, we study a specific mixed subdivision of a scalar multiple of the standard simplex of $\mathbb{R}^n$ that, in the next one, will be applied to the analysis of the classical Macaulay formula for the homogeneous resultant.

For $i = 0, \dots, n$, consider the simplex $\Delta_i = \{\boldsymbol{x} \in (\mathbb{R}_{\geq 0})^n \mid |\boldsymbol{x}| \leq d_i\}$ and the affine function $\varphi_i \colon \Delta_i \to \mathbb{R}$ defined by

$$\varphi_i(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}| & \text{if } i = 0, \\ d_i - x_i & \text{if } i > 0, \end{cases} \tag{5.5}$$

where $|\boldsymbol{x}| = \sum_{i=1}^{n} x_i$ denotes the length of the point $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Set then

$$\Delta = \sum_{i=0}^{n} \Delta_i = \{\boldsymbol{x} \in (\mathbb{R}_{\geq 0})^n \mid |\boldsymbol{x}| \leq |\boldsymbol{d}|\} \text{ and } \varphi = \boxplus_{i=0}^{n} \varphi_i \colon \Delta \longrightarrow \mathbb{R}$$

for the Minkowski sum of these simplexes and the inf-convolution of these affine functions, respectively.

For a subset $I \subset \{0, \dots, n\}$ denote by $I^{\mathrm{c}} = \{0, \dots, n\} \setminus I$ its complement and consider the polytope of Δ defined as

$$C_I = \{\boldsymbol{x} \in \Delta \mid \ell_i(\boldsymbol{x}) \geq 0 \text{ for } i \in I \text{ and } \ell_i(\boldsymbol{x}) \leq 0 \text{ for } i \in I^{\mathrm{c}}\}, \tag{5.6}$$

where $\ell_i \colon \Delta \to \mathbb{R}$ is the affine function given by $\ell_i(\boldsymbol{x}) = \sum_{j=1}^{n} d_j - |\boldsymbol{x}|$ when $i = 0$ and by $\ell_i(\boldsymbol{x}) = x_i - d_i$ when $i > 0$. For a subset $J \subset I^{\mathrm{c}}$ and an index $l \in I$ consider the lattice point defined as

$$v_{J,l} = \Big(\sum_{j \in J} d_j\Big) e_l + \sum_{j \in J^{\mathrm{c}}} d_j \, e_j$$

with e_i equal to the i-th vector in the standard basis of $\mathbb{R}^n$ when $i > 0$ and $e_0 = \mathbf{0} \in \mathbb{R}^n$. For $i = 0, \dots, n$, consider also the face of Δ_i defined as

$$C_{I,i} = \begin{cases} d_i \, e_i & \text{if } i \in I, \\ d_i \operatorname{conv}(e_i, \{e_j\}_{j \in I}) & \text{if } i \in I^{\mathrm{c}}. \end{cases} \tag{5.7}$$

The next result collects the basic information about the mixed subdivision $S(\varphi)$ of Δ that we need for the study of the Macaulay formula for the homogeneous resultant.

Proposition 5.2 *We have that $\varphi = \sum_{i=0}^{n} \max\{0, \ell_i\}$ and the n-cells of $S(\varphi)$ are the polytopes C_I for $I \subset \{0, \dots, n\}$ with $I, I^{\mathrm{c}} \neq \emptyset$. Moreover, for each I we have that*

(1) the vertices of C_I are the lattice points $v_{J,l}$ for $J \subset I^{\mathrm{c}}$ and $l \in I$,
(2) the components of C_I are the polytopes $C_{I,i}$, $i = 0, \dots, n$,
(3) $\sum_{i=0}^{n} \dim(C_{I,i}) = \#I \cdot \#I^{\mathrm{c}}$.

Example 5.3 For $n = 2$, the mixed subdivision $S(\varphi)$ has 6 maximal cells that, with notation as in Fig. 4, decompose as

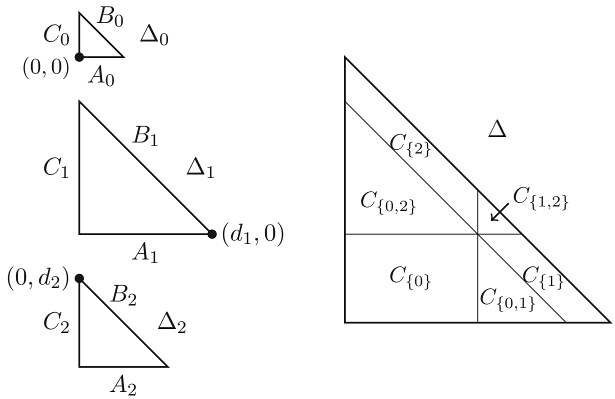

Fig. 4 A mixed subdivision in dimension 2

$$
\begin{aligned}
C_{\{0\}} &= (0,0) + A_1 + C_2, & C_{\{0,1\}} &= (0,0) + (d_1,0) + \Delta_2,\\
C_{\{1\}} &= A_0 + (d_1,0) + B_2, & C_{\{1,2\}} &= \Delta_0 + (d_1,0) + (0,d_2),\\
C_{\{2\}} &= C_0 + B_1 + (0,d_2), & C_{\{0,2\}} &= (0,0) + \Delta_1 + (0,d_2).
\end{aligned}
$$

To prove Proposition 5.2, we lift the previous constructions to $\mathbb{R}^{n+1}$. For $i = 0,\dots,n$ consider the simplex $\widehat{\Delta}_i = \{\boldsymbol{z} = (z_0,\dots,z_n) \in (\mathbb{R}_{\geq 0})^{n+1} \mid |\boldsymbol{z}| = d_i\}$ and the affine function $\widehat{\varphi}_i : \widehat{\Delta}_i \to \mathbb{R}$ defined by $\widehat{\varphi}_i(\boldsymbol{z}) = d_i - z_i$, and set

$$
\widehat{\Delta} = \sum_{i=0}^{n} \widehat{\Delta}_i = \{\boldsymbol{z} \in (\mathbb{R}_{\geq 0})^{n+1} \mid |\boldsymbol{z}| = |\boldsymbol{d}|\} \quad \text{and} \quad \widehat{\varphi} = \boxplus_{i=0}^{n} \widehat{\varphi}_i : \widehat{\Delta} \longrightarrow \mathbb{R}
$$

for the Minkowski sum of these simplexes and for the inf-convolution of these affine functions, respectively.

For a subset $I \subset \{0,\dots,n\}$, consider the polytope of $\widehat{\Delta}$ defined as

$$
\widehat{C}_I = \{\boldsymbol{z} \in \widehat{\Delta} \mid z_i \geq d_i \text{ for } i \in I \text{ and } z_i \leq d_i \text{ for } i \in I^{\mathrm{c}}\},
$$

and for each subset $J \subset I^{\mathrm{c}}$ and each index $l \in I$ consider the lattice point defined as

$$
\widehat{v}_{J,l} = \Big(\sum_{j \in J} d_j\Big)\widehat{e}_l + \sum_{j \in J^{\mathrm{c}}} d_j\, \widehat{e}_j, \tag{5.8}
$$

where $\widehat{e}_i$ denotes the $(i+1)$-th vector in the standard basis of $\mathbb{R}^{n+1}$. For $i = 0,\dots,n$ consider also the face of $\widehat{\Delta}_i$ defined as

$$
\widehat{C}_{I,i} = \begin{cases} d_i\, \widehat{e}_i & \text{if } i \in I,\\ d_i \operatorname{conv}(\widehat{e}_i, \{\widehat{e}_j\}_{j \in I}) & \text{if } i \in I^{\mathrm{c}}. \end{cases} \tag{5.9}
$$

For each i, consider also the convex piecewise affine function $\widehat{\eta}_i : \widehat{\Delta} \to \mathbb{R}$ defined by $\widehat{\eta}_i(z) = \max\{0, z_i - d_i\}$ and set $\widehat{\tau} = \sum_{i=0}^n \widehat{\eta}_i$.

Lemma 5.4 *The n-cells of $S(\widehat{\tau})$ are the polytopes $\widehat{C}_I$ for $I \subset \{0, \dots, n\}$ with $I, I^c \neq \emptyset$. For each I, the vertices of $\widehat{C}_I$ are the lattice points $\widehat{v}_{J,l}$ for $J \subset I^c$ and $l \in I$.*

Proof For each i, the subdivision $S(\widehat{\eta}_i)$ has two n-cells, one defined by $z_i \geq d_i$ and the other by $z_i \leq d_i$. The n-cells of $S(\widehat{\tau})$ are intersections of n-cells of these subdivisions and so they are of the form $\widehat{C}_I$ for $I \subset \{0, \dots, n\}$.

We have that $\sum_{i=0}^n z_i - d_i = 0$ on $\widehat{\Delta}$, and so if either $I = \emptyset$ or $I^c = \emptyset$ then the polytope $\widehat{C}_I$ reduces to the lattice point $\boldsymbol{d}$. Otherwise, take $0 < \varepsilon < 1$ and consider the point $z = (z_0, \dots, z_n)$ defined by $z_j = d_j + \frac{\varepsilon}{\#I}$ if $j \in I$ and as $z_j = d_j - \frac{\varepsilon}{\#I^c}$ if $j \in I^c$. We have that $|z| = |\boldsymbol{d}|$ and that $z_j > 0$ for all j, and so $z \in \mathrm{ri}(\widehat{\Delta})$. Also for each i we have that z lies in the relative interior of the corresponding n-cell of $S(\widehat{\eta}_i)$. Hence, $\widehat{C}_I$ is n-dimensional in this case, proving the first claim.

Now let $I \subset \{0, \dots, n\}$ with $I, I^c \neq \emptyset$. The vertices of $\widehat{C}_I$ are the intersections in this polytope of n of its supporting hyperplanes. These supporting hyperplanes are the zero set of one of the affine functions

$$z_j,\ j \in I^c, \quad \text{and} \quad z_j - d_j,\ j = 0, \dots, n.$$

To compute the vertices, take disjoint subsets $J \subset I^c$ and $K \subset \{0, \dots, n\}$ with $\#J + \#K = n$. The intersection of the corresponding hyperplanes in the affine span of $\widehat{\Delta}$ is

$$\{z \in \mathbb{R}^{n+1} \mid |z| = |\boldsymbol{d}|,\ z_j = 0 \text{ for } j \in J \text{ and } z_j = d_j \text{ for } j \in K\}$$

and it consists of the lattice point

$$\widehat{v}_{J,l} = \Big(|\boldsymbol{d}| - \sum_{j \in K} d_j\Big) \widehat{e}_l + \sum_{j \in K} d_j \widehat{e}_j$$

for the unique index l in the complement of $J \cup K$.

When $J \neq \emptyset$ we have that $\widehat{v}_{J,l} \in \widehat{C}_I$ if and only if $l \in I$. When $J = \emptyset$ we have that $\widehat{v}_{J,l} = \boldsymbol{d}$, which is also realized by taking $l \in I$ and $K = \{0, \dots, n\} \setminus \{l\}$, proving the second claim. □

Recall that for a convex piecewise affine function $\rho : \Delta \to \mathbb{R}$ on a polyhedron Δ of $\mathbb{R}^{n+1}$ and a vector $w \in \mathbb{R}^{n+1}$ we denote by $\Gamma(\rho, w)$ the corresponding cell of the subdivision $S(\rho)$ of Δ as in (2.3).

Lemma 5.5 *Let $I \subset \{0, \dots, n\}$ with $I, I^c \neq \emptyset$ and set $w_I = -\sum_{k \in I} \widehat{e}_k$. Then,*

(1) $\widehat{C}_I = \Gamma(\widehat{\tau}, w_I)$ and for $z \in \widehat{C}_I$ we have that $\widehat{\tau}(z) = \sum_{j \in I} z_j - d_j$,
(2) for $i = 0, \dots, n$ we have that $\widehat{C}_{I,i} = \Gamma(\widehat{\varphi}_i, w_I)$ and for $z \in \widehat{C}_{I,i}$ we have that $\widehat{\varphi}_i(z) = 0$ if $i \in I$ and $\widehat{\varphi}_i(z) = \sum_{j \in I} z_j$ if $i \in I^c$,
(3) $\widehat{C}_I = \sum_{i=0}^n \widehat{C}_{I,i}$,
(4) $\widehat{\varphi} = \widehat{\tau}$.

Proof For $z \in \widehat{\Delta}$ we have that $\widehat{\tau}(z) = \sum_{j=0}^{n} \max\{0, z_j - d_j\}$, which readily implies that

$$\widehat{\tau}(z) \geq \sum_{j \in I} z_j - d_j$$

with equality if and only if $z \in \widehat{C}_I$. By the characterization of the cell $\Gamma(\widehat{\tau}, w_I)$ that follows from (2.4), this proves the statement in (1).

For each i, the vertices of $\widehat{\Delta}_i$ are the lattice points $d_i \widehat{e}_j$, $j = 0, \dots, n$. For each j, we have that $\langle w_I, d_i \widehat{e}_j \rangle = -d_i$ if $j \in I$ and $\langle w_I, d_i \widehat{e}_j \rangle = 0$ if $j \in I^{\mathrm{c}}$, whereas $\widehat{\varphi}_i(d_i \widehat{e}_j) = 0$ if $j = i$ and $\widehat{\varphi}_i(d_i \widehat{e}_j) = d_i$ if $j \neq i$. Hence,

$$\langle w_I, d_i \widehat{e}_j \rangle + \widehat{\varphi}_i(d_i \widehat{e}_j) = \begin{cases} -d_i & \text{if } j \in I \text{ and } j = i, \\ 0 & \text{if } j \in I \text{ and } j \neq i, \text{ or } j \in I^{\mathrm{c}} \text{ and } j = i, \\ d_i & \text{if } j \in I^{\mathrm{c}} \text{ and } j \neq i. \end{cases} \tag{5.10}$$

Then, the definition in (5.9) and the characterization in (2.4) easily imply that $\widehat{C}_{I,i} = \Gamma(\widehat{\varphi}_i, w_I)$ which proves the first part of the statement in (2). Now let $z \in \widehat{C}_{I,i}$. For $i \in I$ the polytope $\widehat{C}_{I,i}$ consists of the single lattice point $d_i \widehat{e}_i$ and so $\widehat{\varphi}_i(z) = \widehat{\varphi}_i(d_i \widehat{e}_i) = 0$. For $i \in I^{\mathrm{c}}$ the vertices of $\widehat{C}_{I,i}$ are the lattice points $d_i \widehat{e}_j$, $j \in \{i\} \cup I$, and so $z = \sum_{j \in \{i\} \cup I} z_j e_j$. Hence, (5.10) gives that $\widehat{\varphi}_i(z) = -\langle w_I, z \rangle = \sum_{j \in I} z_j$, completing the proof of (2).

Let $\widehat{v}_{J,l}$ be the vertex of $\widehat{C}_I$ associated with a subset $J \subset I^{\mathrm{c}}$ and an index $l \in I$ as in (5.8). We have that $d_i \widehat{e}_l \in \widehat{C}_{I,i}$ for all $i \in I^{\mathrm{c}}$ and $d_i \widehat{e}_i \in \widehat{C}_{I,i}$ for all i, and so $\widehat{v}_{J,l} \in \sum_{i=0}^{n} \widehat{C}_{I,i}$. Since this holds for all J and l, Lemma 5.4 implies that

$$\widehat{C}_I \subset \sum_{i=0}^{n} \widehat{C}_{I,i}. \tag{5.11}$$

From (2) and Proposition 2.1(1), we deduce that the $\widehat{C}_{I,i}$'s are the components of the cell $\Gamma(\widehat{\varphi}, w_I)$ of $S(\widehat{\varphi})$ and by Proposition 2.1(2), we have that $\sum_{i=0}^{n} \widehat{C}_{I,i}$ is a cell of this mixed subdivision. On the other hand, the $\widehat{C}_I$'s are polytopes that cover $\widehat{\Delta}$ and so the inclusion in (5.11) is an equality, as stated in (3).

Now let $z = (z_0, \dots, z_n) \in \widehat{C}_I$ and $z_i \in \widehat{C}_{I,i}$, $i = 0, \dots, n$, such that $z = \sum_{i=0}^{n} z_i$. Necessarily $z_i = d_i \widehat{e}_i$ for $i \in I$ and so

$$\widehat{\tau}(z) = \sum_{j \in I} z_j - d_j = \sum_{j \in I} \Big(\Big(\sum_{i=0}^{n} z_{i,j} \Big) - d_j \Big) = \sum_{i \in I^{\mathrm{c}}, j \in I} z_{i,j} = \sum_{i=0}^{n} \widehat{\varphi}_i(z_i) = \widehat{\varphi}(z),$$

where the first equality follows from (1) and the two last from (2) and Proposition 2.1(3), respectively. Hence, $\widehat{\tau}$ and $\widehat{\varphi}$ coincide on $\widehat{C}_I$ and so on the whole of $\widehat{\Delta}$, proving (4). □

Proof of Proposition 5.2 Consider the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by

$$\pi(z_0, \ldots, z_n) = (z_1, \ldots, z_n).$$

This linear map induces isomorphisms between $\widehat{\Delta}$ and Δ and between $\widehat{\Delta}_i$ and Δ_i for each i, and it also satisfies that $\widehat{\varphi}_i = \varphi_i \circ \pi$ for each i.

For $\boldsymbol{z} \in \widehat{\Delta}$ and $\boldsymbol{x} = \pi(\boldsymbol{z}) \in \Delta$, the condition that $\boldsymbol{z} = \sum_{i=0}^{n} \boldsymbol{z}_i$ with $\boldsymbol{z}_i \in \widehat{\Delta}_i$ translates into $\boldsymbol{x} = \sum_{i=0}^{n} \boldsymbol{x}_i$ with $\boldsymbol{x}_i \in \Delta_i$, and vice versa. Precisely, if the first condition holds then so does the second with $\boldsymbol{x}_i = \pi(\boldsymbol{z}_i)$ and conversely, if the second condition holds then so does the first with $\boldsymbol{z}_i$ defined as the only preimage of $\boldsymbol{x}_i$ in $\widehat{\Delta}_i$.

We deduce from Lemma 5.5(4) that $\varphi \circ \pi = \widehat{\varphi} = \widehat{\tau}$, and the statement then follows from Lemmas 5.4 and 5.5. □

Remark 5.6 Proposition 5.2(3) implies that the mixed subdivision $S(\varphi)$ is tight only when $n \leq 2$.

We next study a specific incremental chain of mixed subdivisions of Δ. For $k = 0, \ldots, n$ consider the affine functions $\theta_{k,i} : \Delta_i \to \mathbb{R}$, $i = 0, \ldots, n$, defined as $\theta_{k,i} = \varphi_i$ if $i < k$ and as $\theta_{k,i} = 0|_{\Delta_i}$ if $i \geq k$, and set $\theta_k = \boxplus_{i=0}^{n} \theta_{k,i}$ for their inf-convolution.

For a subset $I \subset \{0, \ldots, k-1\}$ let $I^{\mathrm{c}} = \{0, \ldots, k-1\} \setminus I$ and consider the polytope of Δ defined as

$$C_{k,I} = \{\boldsymbol{x} \in \Delta \mid \ell_i(\boldsymbol{x}) \geq 0 \text{ for } i \in I \text{ and } \ell_i(\boldsymbol{x}) \leq 0 \text{ for } i \in I^{\mathrm{c}}\}.$$

For $J \subset I^{\mathrm{c}}$ and $l \in I \cup \{k, \ldots, n\}$ consider the lattice point defined as

$$v_{k,J,l} = \Big(\sum_{j \in J \cup \{k,\ldots,n\}} d_j \Big) e_l + \sum_{j \in J^{\mathrm{c}}} d_j \, e_j$$

For $i = 0, \ldots, n$ consider also the face of Δ_i defined as

$$C_{k,I,i} = \begin{cases} d_i \, e_i & \text{if } i \in I, \\ d_i \operatorname{conv}(e_i, \{e_j\}_{j \in I \cup \{k,\ldots,n\}}) & \text{if } i \in I^{\mathrm{c}}, \\ d_i \operatorname{conv}(\{e_j\}_{j \in I \cup \{k,\ldots,n\}}) & \text{if } i \geq k. \end{cases}$$

The next result gives a detailed description of the mixed subdivision $S(\theta_k)$ of Δ.

Proposition 5.7 *We have that $\theta_k = \sum_{i=0}^{k-1} \max\{0, \ell_i\} + \sum_{i=k}^{n} \ell_i$ and the n-cells of the mixed subdivision $S(\theta_k)$ are the polytopes $C_{k,I}$ for $I \subset \{0, \ldots, k-1\}$. Moreover, for each I*

(1) the vertices of $C_{k,I}$ are the lattice points $v_{k,J,l}$ for $J \subset I^{\mathrm{c}}$ and $l \in I \cup \{k, \ldots, n\}$,
(2) the components of $C_{k,I}$ are the polytopes $C_{k,I,i}$, $i = 0, \ldots, n$,
(3) $\sum_{i=0}^{n} \dim(C_{k,I,i}) = \#I^{\mathrm{c}} \cdot (\#I + n - k + 1) + (n - k + 1)(\#I + n - k)$.

Proof Denote with a hat the corresponding objects in $\mathbb{R}^{n+1}$ as it was previously done for the study of φ, and consider also the convex piecewise affine function on Δ defined as

$$\tau_k = \sum_{i=0}^{k-1} \max\{0, \ell_i\} + \sum_{i=k}^{n} \ell_i$$

The proof of these properties is a direct generalization of that for Proposition 5.2 and so we only indicate the main steps:

- show that the n-cells of $S(\widehat{\tau_k})$ are the polytopes $\widehat{C}_{k,I}$ for $I \subset \{0, \ldots, k-1\}$,
- for each I, compute the vertices of $\widehat{C}_{k,I}$ by considering the intersections of the supporting hyperplanes of this polytope,
- compute the face of $\widehat{\Delta}_i$ defined by the slope of $\widehat{\tau_k}$ on $\widehat{C}_{k,I}$,
- show that the Minkowski sum of these faces coincides with $\widehat{C}_{k,I}$,
- show that $\widehat{\theta_k}$ coincides with $\widehat{\tau_k}$ on each $\widehat{C}_{k,I}$, and so on the whole of Δ.

Finally, the obtained results are brought back to $\mathbb{R}^n$ via the projection π, as in the proof of Proposition 5.2. □

Proposition 5.8 *We have that $S(\theta_0) \preceq \cdots \preceq S(\theta_n)$ is an incremental chain of mixed subdivisions of Δ with $S(\theta_n) \preceq S(\varphi)$ that satisfies the conditions in Proposition 4.23. In particular, this incremental chain is admissible for any tight mixed subdivision of Δ that refines $S(\varphi)$.*

Proof For each $I \subset \{0, \ldots, n\}$ with $I, I^{\mathrm{c}} \neq \emptyset$ we have that

$$C_I \subset C_{n, I\cap\{0,\ldots,n-1\}} \text{ and } C_{I,i} \subset C_{n, I\cap\{0,\ldots,n-1\},i}, \ i = 0, \ldots, n.$$

Propositions 5.2 and 5.7 then imply that $S(\varphi) \succeq S(\theta_n)$. Similarly, for $k \in \{1, \ldots, n\}$ and each $I \subset \{0, \ldots, k-1\}$ we have that

$$C_{k,I} \subset C_{k-1, I\cap\{0,\ldots,k-2\}} \text{ and } C_{k,I,i} \subset C_{k-1, I\cap\{0,\ldots,k-2\},i}, \ i = 0, \ldots, n,$$

and Proposition 5.7 implies that $S(\theta_k) \succeq S(\theta_{k-1})$. Hence, $S(\theta_0) \preceq \cdots \preceq S(\theta_n) \preceq S(\varphi)$. Since $\theta_{k,i} = 0|_{\Delta_i}$ for all k and $i \geq k$, the chain

$$S(\theta_0) \preceq \cdots \preceq S(\theta_n)$$

is incremental.

For $k = 0, \ldots, n$, let $I \subset \{0, \ldots, k-1\}$. If $I \neq \emptyset$ then for the n-cell C_I of $S(\theta_k)$, each component $C_{I,i}$ with $i \in I$ consists of a lattice point and so this component verifies the condition (2) in Proposition 4.23. Else $I = \emptyset$ and so $C_{k,I,i} = d_i \operatorname{conv}(e_i, \{e_j\}_{j\in\{k,\ldots,n\}})$ if $i < k$ and $C_{k,I,i} = d_i \operatorname{conv}(\{e_j\}_{j\in\{k,\ldots,n\}})$ if $i \geq k$. For $i = 0, \ldots, k-1$, consider the nonzero vector $w_i \in \mathbb{R}^n$ defined as $w_i = \sum_{j=1}^{n} e_j$ if $i = 0$ and as $w_i = e_i$ if $i > 0$. For all $j \neq i$, we have that $C_{k,I,j}$

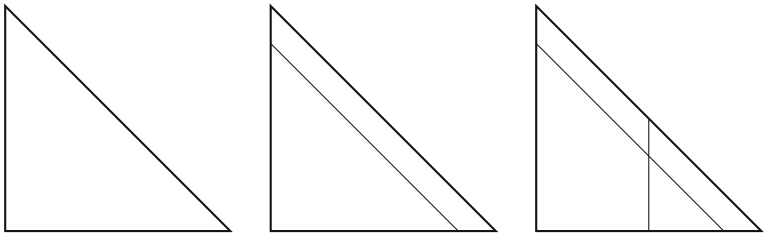

Fig. 5 An admissible incremental chain

lies in a hyperplane that is parallel to $w_i^{\perp}$ and so

$$\dim\Big(\sum_{j\neq i} C_{k,I,j}\Big) < n.$$

Hence, $C_{k,I}$ satisfies the condition (3) in Proposition 4.23. The last statement is a direct consequence of that proposition. □

Figure 5 shows this admissible incremental chain of mixed subdivisions for a case in dimension $n = 2$.

5.3 Polyhedral Interpretation

Let $\boldsymbol{d} = (d_0, \ldots, d_n) \in (\mathbb{N}_{>0})^{n+1}$. In this section, we study the classical Macaulay formula (4.16) for the situation when

$$m = |\boldsymbol{d}| - n,$$

which is the main case of interest. We keep the notation of Sect. 5.1. In particular, for $i = 0, \ldots, n$ we denote by $\boldsymbol{u}_i$ a set of $\binom{d_i+n}{n}$ variables indexed by the lattice points $\boldsymbol{c} \in \mathbb{N}^{n+1}$ of length $|\boldsymbol{c}| = d_i$. Also $\boldsymbol{u} = (\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n)$ and $\mathbb{K} = \mathbb{C}(\boldsymbol{u})$.

In this situation, we, respectively, denote the corresponding index set and nonreduced index subset, linear map, Macaulay matrix and distinguished principal submatrix by

$$\mathcal{I}^{\circ} \subset \mathcal{I} \subset \mathbb{Z}^{n+1}, \quad \Psi_{\boldsymbol{d}} \colon \bigoplus_{i=0}^{n} T_i \to T, \quad \mathcal{M}_{\boldsymbol{d}} \in \mathbb{K}^{\mathcal{I}\times\mathcal{I}} \quad \text{and} \quad \mathcal{N}_{\boldsymbol{d}} \in \mathbb{K}^{\mathcal{I}^{\circ}\times\mathcal{I}^{\circ}},$$

where T_i, $i = 0, \ldots, n$, and T are the finite-dimensional linear subspaces of the polynomial ring $\mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_0, \ldots, t_n]$ in (5.1).

As explained in Example 3.9, the homogeneous resultant $\mathrm{Res}_{\boldsymbol{d}}$ coincides, up to the sign, with the sparse resultant corresponding to the lattice $M = \mathbb{Z}^n$ and the family of supports $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \ldots, \mathcal{A}_n)$ defined by $\mathcal{A}_i = \{\boldsymbol{a} \in \mathbb{N}^n \mid |\boldsymbol{a}| \leq d_i\}$ for each i.

We also use the notation of Sect. 5.2. In particular, for $i = 0, \ldots, n$ we consider the simplex $\Delta_i = \operatorname{conv}(\mathcal{A}_i) = \{\boldsymbol{x} \in (\mathbb{R}_{\geq 0})^n \mid |\boldsymbol{x}| \leq d_i\}$ and the affine function $\varphi_i : \Delta_i \to \mathbb{R}$ defined by $\varphi_i(\boldsymbol{x}) = |\boldsymbol{x}|$ if $i = 0$ and as $\varphi_i(\boldsymbol{x}) = d_i - x_i$ if $i > 0$. Let also $\Delta = \sum_{i=0}^{n} \Delta_i$ be the Minkowski sum of these polytopes and $\varphi = \boxplus_{i=0}^{n} \varphi_i$ the inf-convolution of these affine functions. By Proposition 5.2, we have that

$$\Delta = \{\boldsymbol{x} \in (\mathbb{R}_{\geq 0})^n \mid |\boldsymbol{x}| \leq |\boldsymbol{d}|\} \quad \text{and} \quad \varphi = \sum_{i=0}^{n} \max\{0, \ell_i\},$$

where $\ell_i : \Delta \to \mathbb{R}$ is the affine function defined by $\ell_i(\boldsymbol{x}) = \sum_{j=1}^{n} d_j - |\boldsymbol{x}|$ when $i = 0$ and by $\ell_i(\boldsymbol{x}) = x_i - d_i$ when $i > 0$.

For $i = 0, \ldots, n$ choose a linear function $\mu_i : \mathbb{R}^n \to \mathbb{R}$ and set $\rho_i = \varphi_i + \mu_i$. Set then $\boldsymbol{\rho} = (\rho_0, \ldots, \rho_n)$ and $\rho = \boxplus_{i=0}^{n} \rho_i$, and suppose that the mixed subdivision $S(\rho)$ of Δ is tight and refines $S(\varphi)$. By Propositions 2.8 and 2.11, both conditions are attained when the μ_i's are sufficiently generic and small. Choose also $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ with $\delta_i + 1 > 0$ for all i and $\sum_{i=1}^{n}(\delta_i + 1) < 1$, and satisfying the genericity condition (4.1) with respect to $S(\rho)$.

Consider then the index set and nonmixed index subset, Sylvester map, Canny–Emiris matrix and distinguished principal submatrix corresponding to $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\rho}$ and $\boldsymbol{\delta}$, respectively, denoted by

$$\mathcal{B}^{\circ} \subset \mathcal{B} \subset \mathbb{Z}^n, \quad \Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} : \bigoplus_{i=0}^{n} V_i \to V, \quad \mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} \in \mathbb{K}^{\mathcal{B}\times\mathcal{B}} \quad \text{and} \quad \mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} \in \mathbb{K}^{\mathcal{B}^{\circ}\times\mathcal{B}^{\circ}}$$

where V_i, $i = 0, \ldots, n$, and V are the finite-dimensional linear subspaces of $\mathbb{K}[\mathbb{Z}^n] = \mathbb{K}[\boldsymbol{s}^{\pm 1}] = \mathbb{K}[s_1^{\pm 1}, \ldots, s_n^{\pm 1}]$ defined in (4.3).

The next proposition shows that this Canny–Emiris matrix coincides with that of Macaulay, and that this is also the case for their distinguished principal submatrices.

Proposition 5.9 *The morphism of algebras $\pi_* : \mathbb{K}[\boldsymbol{t}] \to \mathbb{K}[\boldsymbol{s}]$ defined by $\pi_*(t_0) = 1$ and $\pi_*(t_i) = s_i$, $i = 1, \ldots, n$, induces a commutative diagram*

$$\begin{array}{ccc} \bigoplus_{i=0}^{n} T_i & \xrightarrow{\Psi_{\boldsymbol{d}}} & T \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \bigoplus_{i=0}^{n} V_i & \xrightarrow{\Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}} & V \end{array}$$

and bijections between the monomial bases of $\bigoplus_{i=0}^{n} T_i$ and $\bigoplus_{i=0}^{n} V_i$, and between those of T and V. In particular, $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \mathcal{M}_{\boldsymbol{d}}$ and $\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \mathcal{N}_{\boldsymbol{d}}$.

To prove it, we first need to establish some auxiliary lemmas.

Lemma 5.10 *Let C be an n-cell of $S(\rho)$. Let $I \subset \{0, \ldots, n\}$ with $I, I^{\mathrm{c}} \neq \emptyset$ such that $C \subset C_I$ and $i \in \{0, \ldots, n\}$. Then,*

(1) the i-th component C_i is a point if and only if $i \in I$, and if this is the case then $C_i = \{d_i\, e_i\}$,
(2) C is i-mixed if and only if $I = \{i\}$.

Proof Since $S(\rho) \succeq S(\varphi)$, we have that $C_i \subset C_{I,i}$. If $i \in I$ then (5.7) implies that $C_i = \{d_i\, e_i\}$. Conversely, for $i \in I^{\mathrm{c}}$ consider the vector $w_i \in \mathbb{R}^n$ defined as $w_i = \sum_{j=1}^{n} e_j$ if $i = 0$ and as $w_i = e_i$ if $i > 0$. For all $j \neq i$ we have that $C_{I,j}$ lies in a hyperplane that is parallel to $w_i^{\perp}$, and so does C_j. Hence,

$$\dim(C_i) = n - \dim\Big(\sum_{j \neq i} C_j\Big) > 0,$$

proving (1). The statement in (2) is a direct consequence of that in (1): C is i-mixed if and only if $C_{I,i}$ is the unique component of C_I of dimension 0, which is equivalent to the fact that $I = \{i\}$. □

Lemma 5.11 $\mathcal{B} = \{\boldsymbol{b} \in \mathbb{N}^n \mid |\boldsymbol{b}| \leq |\boldsymbol{d}| - n\}$.

Proof Let $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$. Then, $\boldsymbol{b} \in \Delta + \boldsymbol{\delta}$ if and only if $b_i - \delta_i \geq 0$ for all i and $|\boldsymbol{b} - \boldsymbol{\delta}| \leq |\boldsymbol{d}|$. Since $\boldsymbol{b}$ is a lattice point, $\delta_i + 1 > 0$ for all i and $\sum_{i=1}^{n} (\delta_i + 1) < 1$, these conditions are equivalent to $b_i \geq 0$ for all i and $|\boldsymbol{b}| \leq |\boldsymbol{d}| - n$, proving the lemma. □

Lemma 5.12 *Let $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathcal{B}$ and set $b_0 = |\boldsymbol{d}| - n - |\boldsymbol{b}|$. Then,*

(1) for $I \subset \{0, \ldots, n\}$ with $I, I^{\mathrm{c}} \neq \emptyset$ we have that $\boldsymbol{b} \in C_I + \boldsymbol{\delta}$ if and only if $b_i \geq d_i$ for all $i \in I$ and $b_i < d_i$ for all $i \in I^{\mathrm{c}}$,
(2) for each i we have that $\boldsymbol{b} \in \mathcal{B}_i$ if and only if $b_i \geq d_i$ and $b_j < d_j$ for $j > i$, and if this is the case then $a(b) = d_i\, e_i$,
(3) $\boldsymbol{b} \in \mathcal{B} \backslash \mathcal{B}^{\circ}$ if and only if there is a unique i such that $b_i \geq d_i$.

Proof With notation as in (5.6), we have that $\boldsymbol{b} \in C_I + \boldsymbol{\delta}$ if and only if $\ell_i(\boldsymbol{b} - \boldsymbol{\delta}) \geq 0$ for $i \in I$ and $\ell_i(\boldsymbol{b} - \boldsymbol{\delta}) \leq 0$ for $i \in I^{\mathrm{c}}$. Arguing as in the proof of Lemma 5.11, we deduce that these conditions are equivalent to $b_i \geq d_i$ for all $i \in I$ and $b_i < d_i$ for all $i \in I^{\mathrm{c}}$, as stated in (1).

The statements in (2) and (3) follow from that in (1) together with Lemma 5.10. □

Proof of Proposition 5.9 Let $\pi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ be the linear map given by $\pi(c_0, \ldots, c_n) = (c_1, \ldots, c_n)$, so that $\pi_*(\boldsymbol{t}^{\boldsymbol{c}}) = \boldsymbol{s}^{\pi(\boldsymbol{c})} = s_1^{c_1} \ldots s_n^{c_n}$ for all $\boldsymbol{c} = (c_0, \ldots, c_n) \in \mathbb{N}^{n+1}$.

By Lemma 5.11, π induces a bijection between the index sets $\mathcal{I}$ and $\mathcal{B}$, and so the morphism of algebras π_* gives a bijection between the monomial bases of T and of V. Similarly, by Lemma 5.12(2) π also induces a bijection between $\mathcal{I}_i$ and $\mathcal{B}_i$ for each i, and so π_* gives a bijection between the monomial bases of $\bigoplus_{i=0}^{n} T_i$ and $\bigoplus_{i=0}^{n} V_i$, proving the second claim. Moreover, for $\boldsymbol{c} \in \mathcal{I}$ we have that

$$\pi_*(\Psi_{\boldsymbol{d}}(\boldsymbol{t}^{\boldsymbol{c}})) = \pi_*(\boldsymbol{t}^{\boldsymbol{c} - d_i\, \widehat{e}_i}) = \boldsymbol{s}^{\pi(\boldsymbol{c}) - d_i\, e_i} = \Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\boldsymbol{s}^{\pi(\boldsymbol{c})}) = \Phi_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}(\pi_*(\boldsymbol{t}^{\boldsymbol{c}})),$$

which shows the commutativity of the diagram. The last claim is a direct consequence of the two previous. □

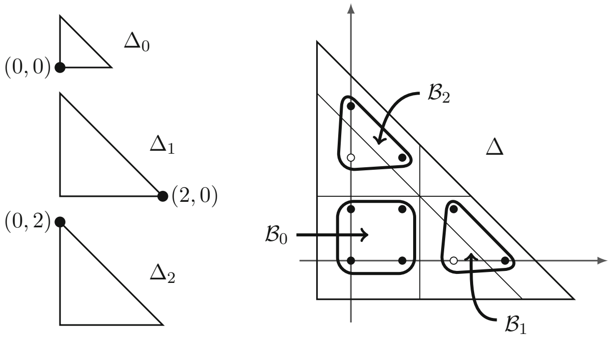

Fig. 6 The index set of a mixed subdivision

Corollary 5.13 $\mathrm{Res}_{\boldsymbol{d}} = \det(\mathcal{M}_{\boldsymbol{d}})/\det(\mathcal{N}_{\boldsymbol{d}})$.

Proof By Proposition 5.8, the mixed subdivision $S(\rho)$ is admissible. Theorem 1.3 and Proposition 5.9 then imply that

$$\mathrm{Res}_{\boldsymbol{d}} = \pm\frac{\det(\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})}{\det(\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}})} = \pm\frac{\det(\mathcal{M}_{\boldsymbol{d}})}{\det(\mathcal{N}_{\boldsymbol{d}})}.$$

The sign can be determined by considering the evaluation of both sides of this equality at the coefficients of systems of polynomials $t_i^{d_i}$, $i = 0, \dots, n$.

Example 5.14 Consider again the case when $n = 2$, $\boldsymbol{d} = (1, 2, 2)$ and $m = 3$. Then,

$$\mathcal{A}_0 = \{(0, 0), (1, 0), (0, 1)\},$$
$$\mathcal{A}_1 = \mathcal{A}_2 = \{(0, 0), (1, 0), (2, 0), (1, 0), (1, 1), (0, 2)\}.$$

By Proposition 5.2(3), the mixed subdivision $S(\varphi)$ is tight and so we can take $\rho_i = \varphi_i$, $i = 0, 1, 2$. We also choose $\boldsymbol{\delta} = (-\frac{2}{3}, -\frac{3}{4}) \in \mathbb{R}^2$.

As shown in Fig. 6, the index set $\mathcal{B}$ splits as $\mathcal{B} = \mathcal{B}_0 \sqcup \mathcal{B}_1 \sqcup \mathcal{B}_2$ with

$$\mathcal{B}_0 = \{(0, 0), (1, 0), (0, 1), (1, 1)\},$$
$$\mathcal{B}_1 = \{(2, 0), (3, 0), (2, 1)\},$$
$$\mathcal{B}_2 = \{(0, 2), (1, 2), (0, 3)\},$$

and the row content function assigns to the elements of $\mathcal{B}_i$, $i = 0, 1, 2$, the vertices $(0, 0) \in \Delta_0$, $(2, 0) \in \Delta_1$, $(0, 2) \in \Delta_2$, respectively. Moreover, the elements of $\mathcal{B}$ lying in the translated nonmixed 2-cells are $(2, 0)$ and $(0, 2)$.

The Canny–Emiris matrix $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ and its principal submatrix $\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$, respectively, coincide with the Macaulay matrices $\mathcal{M}_{\boldsymbol{d}}$ and $\mathcal{N}_{\boldsymbol{d}}$ in Example 5.1, in agreement with Proposition 5.9.

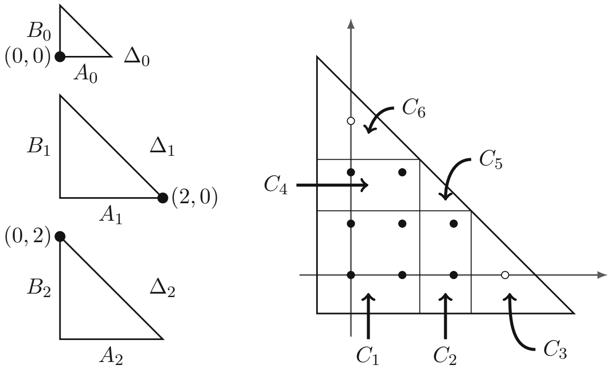

Fig. 7 A nonadmissible mixed subdivision

Remark 5.15 The Macaulay matrix $\mathcal{M}_{\boldsymbol{d}}$ cannot be produced by a mixed subdivision admitting a tight incremental chain (Definition 2.4). This can be shown by inspecting the different mixed subdivisions of Δ that allow such an incremental chain and verifying that none of them coincides with $S(\rho)$. One of these incremental chains for the case when $n = 2$ and $\boldsymbol{d} = (1, 3, 2)$ is shown in Example 2.7.

We next show that when the mixed subdivision $S(\rho)$ is not admissible, the formula in Theorem 1.3 might fail to hold.

Example 5.16 Let notation be as in Example 5.14 and instead of the φ_i's, consider the affine functions $\rho_i : \Delta_i \to \mathbb{R}$, $i = 0, 1, 2$, defined by

$$\rho_0(0,0) = 0,\ \ \rho_0(1,0) = 1,\ \ \rho_0(0,1) = 1,$$
$$\rho_1(0,0) = 0,\ \ \rho_1(2,0) = 0,\ \ \rho_1(0,2) = 3,$$
$$\rho_2(0,0) = 0,\ \ \rho_2(2,0) = 3,\ \ \rho_2(0,2) = 0.$$

Let $\rho : \Delta \to \mathbb{R}$ be their inf-convolution. The mixed subdivision $S(\rho)$ of Δ is tight and has 6 maximal cells as shown in Fig. 7, that decompose as

$$C_1 = (0,0) + A_1 + B_2,\quad C_2 = A_0 + (2,0) + B_2,\qquad C_3 = (1,0) + (2,0) + \Delta_2,$$
$$C_4 = B_0 + A_1 + (0,2),\quad C_5 = \Delta_0 + (2,0) + (0,2),\quad C_6 = (0,1) + \Delta_1 + (0,2).$$

The index set and row function corresponding to the data $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2)$, $\boldsymbol{\rho} = (\rho_0, \rho_1, \rho_2)$ and $\boldsymbol{\delta}$ are equal to those in Example 5.14, and so the Canny–Emiris matrix $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ also coincides with that in (5.4). However, the translated nonmixed cells of $S(\rho)$ differ from those in Example 5.14. Their lattice points are $(3, 0)$ and $(0, 3)$ and the corresponding principal submatrix is

$$\mathcal{E}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}} = \begin{pmatrix} \beta_2 & 0 \\ 0 & \gamma_5 \end{pmatrix}.$$

The determinant of this matrix does not divide that of $\mathcal{H}_{\boldsymbol{\mathcal{A}},\boldsymbol{\rho}}$ and so the formula in Theorem 1.3 does not hold in this case. In particular, $S(\rho)$ is not admissible.

Indeed, this latter observation can be verified directly: let

$$S(\theta_0) \preceq S(\theta_1) \preceq S(\theta_2)$$

be an incremental chain of mixed subdivisions of Δ with $S(\theta_2) \preceq S(\rho)$ and for each $k = 0, 1, 2$ let $\theta_{k,i} \colon \Delta_i \to \mathbb{R}$, $i = 0, 1, 2$, be the corresponding family of convex piecewise affine functions.

If $\theta_{1,0} \colon \Delta_0 \to \mathbb{R}$ is not constant, then $S(\theta_1)$ has a cell that is a translate of the triangle $\Delta_1 + \Delta_2 = \{(x_1, x_2) \in (\mathbb{R}_{\geq 0})^2 \mid x_1 + x_2 \leq 4\}$ which is not compatible with the assumption that $S(\rho)$ is a refinement of $S(\theta_1)$, as it can be verified in Fig. 6. We deduce that $\theta_{1,0} \colon \Delta_0 \to \mathbb{R}$ is constant, but in this case $S(\theta_1)$ is the trivial mixed subdivision of Δ and this incremental chain does not verify the conditions in Definition 4.22 for $k = 1$, and so it is not admissible.

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