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REVIEW ARTICLE

Recent Results on Nonlinear Elliptic Free Boundary Problems

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Abstract
In this paper we give an overview of some recent and older results concerning free boundary problems governed by elliptic operators.

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1 Introduction

In this survey paper we consider free boundary problems governed by elliptic equations in which the state variable can assume two phases and the condition across the free boundary is expressed by an energy balance involving the fluxes from both sides (so called Bernoulli type problems). Typical examples come from constraint minimization problems, flame propagation as limits of singular perturbation problems with forcing term (flame propagation), [51, 64], the Stokes and Prandtl–Batchelor models in classical hydrodynamics, [7, 10], free transmission problems, [6].

To Alfio Quarteroni: a great mathematician, a great person, a great friend.

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Among the several concepts of solutions, we use the notion of viscosity solution introduced by Luis Caffarelli in the seminal papers [11–13], which seems to be the most appropriate to study optimal regularity of both solutions and free boundaries in great generality. In those papers, in the homogeneous case, Caffarelli developed a general strategy to attack the existence and the optimal regularity of both the solution and the free boundary based on a powerful monotonicity formula, already proved in [5, 16, 59, 65], Harnack principles and families of continuous perturbations.

A different approach to regularity, relying on Harnack principles and linearization, has been introduced by De Silva in [22], for a one phase model motivated by the classical Stokes problem in hydrodynamics, and subsequently refined in [23–25, 28], to cover a broad spectrum of applications, in particular, for problems with distributed sources.

The main common underlying idea of both techniques to obtain the regularity of the free boundary, frequently written simply f.b. in the sequel, is to set up an iterative improvement of flatness argument in a neighborhood of a point where one of the two phases enjoys a non-degeneracy condition (e.g. a linear growth).

After the quoted seminal papers by Caffarelli, the theory can be developed according to a well established paradigm:

a) Existence and optimal regularity of solutions, e.g. viscosity or variational solutions, or solutions obtained as a limit of singular perturbations.
b) Weak regularity properties of the f.b., such as finite perimeter and density properties for the positivity set.
c) Strong regularity properties of the f.b. For instance Lipschitz or "flat" free boundaries are C^1 or better.
d) Higher regularity: Schauder type estimates and analyticity for both solution and f.b.

Also of great importance, we believe, is to have information on the Hausdorff measure or dimension of the *singular* (*nonflat*) points of the free boundary. The best existing results in this direction are obtained by De Philippis, Spolaor, Velichkov for the classical Bernoulli problem in [31, 33]. Nothing is known in the nonhomogeneous case.

In Sections 2 to 5 of this paper we focus on f.b. problems governed by uniformly elliptic equation, while in Section 6 we describe some recent results on nonlinear operators with non-standard growth, see [38].

2 Free Boundary Problems Governed by Fully Nonlinear Operators

In the last years the theory for two phase problems governed by uniformly elliptic operators has reached a considerable level of completeness. Here we mostly focus on problems governed by fully nonlinear equations with distributed sources. Our class of free boundary problems and their viscosity solutions can be stated as follows.

Let Sym_n denote the space of $n \times n$ symmetric matrices and let $F : \mathrm{Sym}_n \to \mathbb{R}$ with $F(O) = 0$ and such that there exist constants $0 < \lambda \leq \Lambda$ with

$$\lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\| \qquad \text{for every } M, N \in \mathrm{Sym}_n \text{ with } N \geq 0,$$

where $\|M\| = \max_{|x|=1} |Mx|$ denotes the (L^2, L^2)-norm of the matrix M. Observe that if F is an operator in our class then for every $r > 0$

$$F_r(M) = \frac{1}{r} F(rM) \tag{1}$$

is still an operator in the same class.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $f_1, f_2 \in C(\Omega) \cap L^\infty(\Omega)$. We consider the following two-phase inhomogeneous free boundary problem (f.b.p. in the sequel).

$$\begin{cases} F(D^2u^+) = f_1 & \text{in } \Omega^+(u) := \{u > 0\}, \\ F(D^2u^-) = f_2\chi_{\{u<0\}} & \text{in } \Omega^-(u) = \{u \le 0\}^o, \\ u_\nu^+(x) = G(u_\nu^-, x, \nu) & \text{along } \mathcal{F}(u) := \partial\{u > 0\} \cap \Omega. \end{cases} \tag{2}$$

Here $\nu = \nu(x)$ denotes the unit normal to the free boundary $\mathcal{F} = \mathcal{F}(u)$ at the point x, pointing toward $\Omega^+(u)$, while the function $G(\beta, x, \nu)$ is Lipschitz continuous, strictly increasing in β, and

$$\inf_{x\in\Omega, |\nu|=1} G(0, x, \nu) > 0.$$

Moreover, u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively.

For any u continuous in Ω we say that a point $x_0 \in \mathcal{F}(u)$ is *regular from the right* (resp. left) if there exists a ball $B \subset \Omega^+(u)$ (resp. $B \subset \Omega^-(u)$) such that $\overline{B} \cap \mathcal{F}(u) = x_0$. In both cases, we denote with $\nu = \nu(x_0)$ the unit normal to ∂B at x_0, pointing toward $\Omega^+(u)$.

Definition 1 A *viscosity solution* of the free boundary problem (2) is a continuous function u which satisfies the first two equality of (2) in viscosity sense and such that the free boundary condition is satisfied in the following viscosity sense:

(i) (supersolution condition) if $x_0 \in \mathcal{F}(u)$ is regular from the right with touching ball B, then, near x_0,

$$u^+(x) \ge \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad \text{in } B, \text{ in with } \alpha \ge 0$$

and

$$u^-(x) \le \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad \text{in } B^c, \text{ in with } \beta \ge 0,$$

with equality along every non-tangential direction, and $\alpha \le G(\beta, x_0, \nu(x_0))$;

(ii) (subsolution condition) if $x_0 \in \mathcal{F}(u)$ is regular from the left with touching ball B, then, near x_0,

$$u^+(x) \le \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad \text{in } B^c, \text{ in with } \alpha \ge 0$$

and

$$u^-(x) \ge \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad \text{in } B, \text{ in with } \beta \ge 0,$$

with equality along every non-tangential direction, and

$$\alpha \ge G(\beta, x_0, \nu(x_0)).$$

The notion of viscosity solution can also be given in terms of test functions (see [15] for the equivalence). Given $u, \varphi \in C(\Omega)$, we say that φ touches u by below (resp. above) at $x_0 \in \Omega$, if $u(x_0) = \varphi(x_0)$ and

$$u(x) \ge \varphi(x) \quad (\text{resp. } u(x) \le \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

Then $u \in C(\Omega)$ is a viscosity solution to (2) if i) and ii) are replaced by

ii') Let $x_0 \in \mathcal{F}(u)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ ($B = B_\delta(x_0)$, δ small) with $\mathcal{F}(v) \in C^2$. If v touches u by below (resp. above) at x_0, then

$$v_\nu^+(x_0) \le G\left(v_\nu^-, x_0, \nu(x_0)\right) \quad (\text{resp. } \ge).$$

2.1 Existence of Lipschitz Viscosity Solutions and Weak Regularity Properties of the Free Boundary

A solution to (2) that involves partial differential equations that are not in divergence form can be constructed via Perron's method, by taking the infimum over the following class of *admissible supersolutions* $\mathcal{S}$.

Definition 2 A locally Lipschitz continuous function $w \in C(\overline{\Omega})$ is in the class $\mathcal{S}$ if

(a) w is a solution in viscosity sense to

$$\begin{cases} F(D^2w^+) \leq f_1 & \text{in } \Omega^+(w), \\ F(D^2w^-) \geq f_2\chi_{\{w<0\}} & \text{in } \Omega^-(w); \end{cases}$$

(b) if $x_0 \in \mathcal{F}(w)$ is regular from the left, with touching ball B, then

$$w^+(x) \leq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad \text{in } B^c, \text{ with } \alpha \geq 0$$

and

$$w^-(x) \geq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad \text{in } B, \text{ with } \beta \geq 0,$$

with

$$\alpha \leq G(\beta, x_0, \nu(x_0));$$

(c) if $x_0 \in \mathcal{F}(w)$ is not regular from the left then

$$w(x) = o(|x - x_0|).$$

The last ingredient one needs is that of minorant subsolution.

Definition 3 A locally Lipschitz continuous function $\underline{u} \in C(\overline{\Omega})$ is a *strict minorant* if

(a) $\underline{u}$ is a viscosity solution to

$$\begin{cases} F(D^2\underline{u}^+) \geq f_1 & \text{in } \Omega^+(\underline{u}), \\ F(D^2\underline{u}^-) \leq f_2\chi_{\{\underline{u}<0\}} & \text{in } \Omega^-(\underline{u}); \end{cases}$$

(b) every $x_0 \in \mathcal{F}(\underline{u})$ is regular from the right, with touching ball B, and near x_0

$$\underline{u}^+(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \qquad \text{in } B, \text{ with } \alpha > 0,$$

and

$$\underline{u}^-(x) \leq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \qquad \text{in } B^c, \text{ with } \beta \geq 0,$$

with

$$\alpha > G(\beta, x_0, \nu(x_0)).$$

The following result holds [63].

Theorem 4 *Let F be concave, homogeneous of degree one, and g be a continuous function on $\partial\Omega$. Assume that*

(a) *there exists a strict minorant $\underline{u}$ with $\underline{u} = g$ on $\partial\Omega$ and*
(b) *the set $\{w \in \mathcal{S} : w \geq \underline{u},\ w = g \text{ on } \partial\Omega\}$ is not empty.*

Let

$$u = \inf\{w : w \in \mathcal{S}, w \geq \underline{u}\}.$$

Then $u \in C(\overline{\Omega})$, $u = g$ on $\partial\Omega$ and it is a (minimal) viscosity solution of (2). Moreover u is locally Lipschitz in Ω with non degenerate positive part:

$$u^+(x) \geq \alpha \, dist(x, \mathcal{F}(u)) \qquad (\alpha > 0).$$

Once existence of a solution is established, we turn to the analysis of the weak regularity properties of the free boundary.

The free boundary $\mathcal{F}(u)$ has finite $(n-1)$-dimensional Hausdorff measure. More precisely, there exists a universal constant $r_0 > 0$ such that for every $r < r_0$, for every $x_0 \in \mathcal{F}(u)$,

$$\mathcal{H}^{n-1}(\mathcal{F}(u) \cap B_r(x_0)) \leq c r^{n-1}.$$

Moreover, the reduced boundary $\mathcal{F}^*(u)$ of $\Omega^+(u)$ has positive density in $\mathcal{H}^{n-1}$-measure at any point of $F(u)$, i.e. for $r < r_0$, r_0 universal

$$\mathcal{H}^{n-1}(\mathcal{F}^*(u) \cap B_r(x)) \geq c r^{n-1},$$

for every $x \in \mathcal{F}(u)$. In particular

$$\mathcal{H}^{n-1}(\mathcal{F}(u) \setminus \mathcal{F}^*(u)) = 0.$$

Using the strong regularity results below we deduce the following result.

Corollary 5 *$\mathcal{F}(u)$ is a $C^{1,\gamma}$ surface in a neighborhood of $\mathcal{H}^{n-1}$ a.e. point $x_0 \in \mathcal{F}(u)$.*

Existence of a continuous viscosity solution through a Perron method has been established for linear operators in divergence form in [13] (homogeneous case) and in [25] (inhomogeneous case), and for a class of concave operators in [68]. The presence of both a right hand side and the nonlinearity of the governing equation presents several delicate points which require new arguments, significantly when no sign condition is posed on the right hand sides f_1 and f_2.

When $F = F(D^2u, Du)$, with F non concave in the Hessian matrix and even for the linear case $F = \mathrm{Tr}(A(x)D^2u) + b(x) \cdot \nabla u$, both existence and weak regularity remain open questions.

We conclude this section by mentioning a different approach to prove existence for a specific class of free boundary problems in divergence form that can be found for example in [64].

3 Lipschitz Continuity and Global Solutions

The Lipschitz continuity of solutions to (2) is a crucial ingredient in the study of the regularity of the free boundary $\mathcal{F}(u)$. Indeed it provides compactness to carry on a blow-up analysis around a point $x_0 \in \mathcal{F}(u)$, reducing the problem to the classification of global Lipschitz solution. For instance, if $F(Du) = -\Delta u$, $f_1 = f_2 = 0$ it is possible to classify global solutions U as either purely *two-plane* functions (ν a unit direction),

$$U(x) = \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^-, \qquad \alpha = G(\beta), \beta > 0$$

or *one-phase solutions*, in case we have

$$U^- \equiv 0.$$

In particular, if u is a solution with $0 \in \mathcal{F}(u)$, then via a blow-up analysis and flatness results (see below) either $u^-(x) = o(|x|)$ or $\mathcal{F}(u)$ is $C^{1,\gamma}$ in a neighborhood of 0 that is, the only types of singular points are the ones that occur in the one-phase setting when the negative phase is identically zero.

On points of the reduced boundary of the minimal Perron solutions, one can conclude that in the case when the blow up is a one-phase solution, then it is in fact a one-plane solution $\alpha\langle x - x_0, \nu\rangle$, hence $C^{1,\gamma}$ regularity of the reduced boundary follows from flatness results below.

For viscosity solutions in general, when the governing equation is a fully non linear operator more robust arguments are required.

3.1 Lipschitz Continuity

In [29] De Silva and Savin proved the following result under the assumptions that G behaves like t for t large.

Theorem 6 *Let u be a viscosity solution to (2) $f_1 = f_2 = 0$, and assume that $G \in C^2([0,\infty))$ and*

$$G'(t) \to 1, \quad G''(t) = O(1/t) \quad \text{as } t \to \infty. \tag{3}$$

Then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\left(1 + \|u\|_{L^\infty(B_1)}\right)$$

with $C = C(n, \lambda, \Lambda, G)$ universal.

The heuristic behind the proof is that "big gradients" force the free boundary condition to become a no-jump condition for ∇u and then interior $C^{1,\alpha}$ estimates for fully nonlinear equations provide gradient estimates.

The dependence on G in the constant above is determined by the rate of convergence in the limit (3). In particular (3) can be relaxed to $G' \in [1-\delta, 1+\delta]$ for large values of t. If F is homogeneous of degree 1, then it suffices to require $G(t)/t \to c_0$ as $t \to \infty$ for some constant c_0.

The method of the proof still works in the presence of a non-zero right hand side.

In dimension $n = 2$ these results can be improved significantly.

3.2 Classification of Global Solutions

In [30] De Silva and Savin proved the following Liouville theorem for global Lipschitz solutions.

Theorem 7 *Let $f_1 = f_2 = 0$ and u be a globally Lipschitz viscosity solution to (2) in $\mathbb{R}^n$. Assume that*

F is concave (or convex) and homogeneous of degree 1.

Then either u is a two plane-solution

$$u = \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- \qquad \text{with } \alpha, \beta > 0, \alpha = G(\beta),$$

or

$$u^- \equiv 0,$$

which means that u solves the one-phase problem for F.

As in the case of the Laplacian, the main consequence of Theorem 7 is that it reduces the question of the regularity of the free boundary for the two-phase problem to the classification of global blow-up solutions to the one-phase problem. In particular, if u is a solution to (2) with $0 \in \mathcal{F}(u)$, then either $u^-(x) = o(|x|)$ or $\mathcal{F}(u)$ is $C^{1,\gamma}$ in a neighborhood of 0.

Theorem 7 can be extended to more general operators $F(D^2u, \nabla u, u)$ if appropriate assumptions are imposed on F. For example this result holds when the problem is governed by quasilinear equations of the type

$$\sum_{i,j=1}^{n} a_{ij}\left(\frac{\nabla u}{|\nabla u|}\right) u_{ij} = 0,$$

with uniformly elliptic coefficients $a_{ij} \in C^1(S^{n-1})$. In [29] Lipschitz continuity of solutions to such a problem is also established.

The proof requires somewhat involved and technical arguments. One of the main steps consists in obtaining a weak Evans–Krylov type estimate for a nonlinear transmission problem. One major difficulty is that $\mathcal{F}(u)$ is not known to be better than $C^{1,\alpha}$ even in the perturbative setting.

The idea of proof of Theorem 7 is to show a "reversed" improvement of flatness for the solution u, which means that if u is sufficiently close to a two plane solution at a small scale then it remains close to the same two-plane solution at all larger scales.

3.3 Different Operators

When the two phases are governed by two different operators, few results are known. In [6], Amaral and Teixeira consider the case of divergence form operators, with $f_1, f_2 \in L^p$, $p > n/2$ and A_j merely measurable. They prove that local minimizers of the associated energy functional have a universal Hölder modulus of continuity.

In [14], Caffarelli, De Silva and Savin obtained Lipschitz continuity and classification of global Lipschitz solutions, that is Theorems 6 and 7, for a two-phase problem driven by two different operators with measurable coefficients under very general free boundary conditions $u_\nu^+ = G(u_\nu^-, \nu, x)$, in dimension $n = 2$. We remark that Theorem 6 cannot hold in this generality in higher dimensions. Indeed, say for $n = 3$, it is not difficult to construct two homogeneous functions of degree less than one, that solve two different uniformly elliptic equations in complementary domains in $\mathbb{R}^3$ and satisfy the free boundary condition for a specific G.

4 $C^{1,\alpha}$ Regularity of the Free Boundary

4.1 The Homogeneous Case

The regularity theory for the Laplace operator in the homogeneous case has been developed by Caffarelli in the two seminal papers [11, 12].

In particular the "*Lipschitz implies* $C^{1,\gamma}$" part is contained in [11] while the *flat implies Lipschitz* part is shown in [12]. The flatness condition in [12] is stated in terms of "ε-monotonicity" along a cone of directions $\Gamma(\theta_0, e)$ of axis e and large opening θ_0. Precisely, a function u is said to be ε-monotone ($\varepsilon > 0$ small) along the direction τ in the cone $\Gamma(\theta_0, e)$ if for every $\varepsilon' \geq \varepsilon$,

$$u(x + \varepsilon'\tau) \leq u(x).$$

Geometrically, the ε-monotonicity of u can be interpreted as ε-closeness of $\mathcal{F}(u)$ to the graph of a Lipschitz function.

In those papers Caffarelli set up a general strategy to attack the regularity of the free boundary. Let us briefly describe the central idea of the proof in [11].

Starting from a Lipschitz graph, one shows that in a neighborhood of $\mathcal{F}(u)$ the level sets of u are still Lipschitz graphs, locally in the same direction. Then one improves the Lipschitz constant (i.e. the flatness) of the level sets of u away from the free boundary. Here Harnack inequality applied to directional derivatives of u plays a major role. Then the task is to carry this interior gain up to the free boundary. To this aim, Caffarelli introduces a powerful method of continuity based on the construction of a continuous family of deformations, constructed as the supremum of a harmonic function over balls of variable radius (*supconvolutions*). Finally, by rescaling and iterating the last two steps, one obtains a geometric decay of the Lipschitz constant, which amounts to the $C^{1,\gamma}$ regularity of $\mathcal{F}(u)$.

After 10 years Feldman [41] considered different anisotropic operators with constant coefficients and extends to this case the results in [11].

P. Y. Wang managed to extend the results in [11, 12] to a class of concave fully non linear operators (see [66, 67]). One year later Feldman in [42] considered a class of non concave fully non linear operators of the type $F(D^2u, Du)$. He showed that Lipschitz free boundaries are $C^{1,\alpha}$ thus extending to this case the results in [11] see also [57].

The first papers dealing with variable coefficient operators are by Cerutti, Ferrari, Salsa [17] and by Ferrari [37] and Argiolas, Ferrari [9]. They considered respectively, linear elliptic operators in non-divergence form and a rather general class of fully nonlinear operators $F(D^2u, Du, x)$, with Hölder continuity in x, including Bellman's operators. One of the main difficulty in extending the theory to variable coefficients operator is the fact that directional derivatives do not satisfy any reasonable elliptic equation.

A refinement of the techniques in [17] leads to the following results (see [40]), where the drift coefficient is merely bounded measurable, with two different operators

$$F_i(D^2u, Du) = \mathrm{Tr}\left(A_i(x)D^2u\right) + b_i(x)\cdot\nabla u, \quad i = 1, 2$$

governing the two phases.

Theorem 8 *Let u be a weak solution of our free boundary problem in $\Omega = B_1 = B_1(0)$, the unit ball centered at the origin. Suppose $0 \in \mathcal{F}(u)$ and that*

i) $A_i \in C^{0,a}(B_1)$, $0 < a \leq 1$, $b_i \in L^\infty(B_1)$.
ii) $0 < \alpha_1 \leq \frac{u^+(x)}{\mathrm{dist}(x, F(u))} \leq \alpha_2$.
iii) $G(0) > 0$.

There exist $0 < \overline{\theta} < \pi/2$ and $\overline{\varepsilon} > 0$ such that, if for $0 < \varepsilon < \overline{\varepsilon}$, $\mathcal{F}(u)$ is contained in an ε-neighborhood of a graph of a Lipschitz function $x_n = g(x')$ with

$$\mathrm{Lip}(g) \leq \tan\left(\frac{\pi}{2} - \overline{\theta}\right)$$

then $\mathcal{F}(u)$ is a $C^{1,\gamma}$-graph in $B_{1/2}$.

The same conclusion holds if $B_1^+(u) = \{(x', x_n) : x_n > f(x')\} \cap B_1$ where f is a Lipschitz continuous function.

Condition ii) expresses a linear behavior of u^+ at the free boundary while being trapped in a neighborhood of two Lipschitz graph with small Lipschitz constant is another way to express a flatness condition. Thus, *flatness* plus *linear behavior* of the positive part imply smoothness.

This theorem has been extended to the same class of fully nonlinear operators considered in [37] by Argiolas and Ferrari in [9]. Note that, in principle, all these results *do not need the a priory assumption of Lipschitz continuity of the solution*, that comes as a consequence of the regularity of the free boundary.

4.2 The Nonhomogeneous Case

In this section we describe the strategy to investigate free boundary problems with right hand side. Based on a Harnack type theorem and linearization, this technique avoids the use of supconvolutions, that in presence of distributed sources produces several complicacies. The method can be very well adapted to nonhomogeneous two-phase problems to prove that flat (see below) or Lipschitz free boundaries of (2) are $C^{1,\gamma}$. We assume that $f_1 = f_2 = f$.

We have (we will always assume that $0 \in \mathcal{F}(u)$):

Theorem 9 *Let u be Lipschitz viscosity solution to* (2) *in B_1. Assume that f is bounded and continuous in $B_1^+(u) \cup B_1^-(u)$. There exists a universal constant $\bar{\delta} > 0$ such that, if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\}, \qquad (\delta\text{-}flatness) \tag{4}$$

with $0 \leq \delta \leq \bar{\delta}$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Condition (4) expresses that the zero set of u^+ is trapped between two parallel hyperplanes at δ-distance from each other for a small δ, which is a kind of flatness (δ-flatness). While this looks like a somewhat strong assumption, it is indeed a natural one since it is satisfied for example by rescaling a solution around a point of the free boundary where there is a normal in some weak sense (*regular points*), for instance in the measure theoretical one. Thus, for the minimal Perron solution H^{n-1}−a.e. points on $\mathcal{F}(u)$ are of this kind. Moreover, starting from a Lipschitz free boundary, H^{n-1}−a.e. points on $\mathcal{F}(u)$ are regular, by Rademacher Theorem.

When F is (positively) homogeneous of degree one (or when $F_r(M) = \frac{1}{r}F(rM)$ has a limit $F^*(M)$, as $r \to 0$, which is always homogeneous of degree one), we also have:

Theorem 10 (Lipschitz implies $C^{1,\gamma}$) *Let u be a Lipschitz viscosity solution to* (2) *in B_1. Assume that f is bounded and continuous in $B_1^+(u) \cup B_1^-(u)$. If $\mathcal{F}(u)$ is a Lipschitz graph in a neighborhood of* 0*, then $\mathcal{F}(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of* 0.

Theorem 10 follows from Theorem 9 via a blow-up argument and a Liouville type result for global viscosity solutions to a two-phase free boundary problem, for the details see the original paper [24].

As already pointed out the proof of Theorem 9 follows the strategy developed in [22]. The main difficulty in the analysis in this two-phase problem comes from the case when u^- is *degenerate*, that is very close to zero without being identically zero. In this case the flatness assumption does not guarantee closeness of u to an "optimal" (two-plane or one-plane) configuration. Thus one needs to work only with the positive phase u^+ to balance the situation in which u^+ highly predominates over u^- and the case in which u^- is not too small with respect to u^+.

In particular, the proof is based on a recursive improvement of flatness, obtained via a compactness argument, provided by a geometric type Harnack inequality, which linearizes the problem into a limiting one. The limiting problem turns out to be a transmission problem in the nondegenerate case and a Neumann problem in the other case. The information to set up the iteration towards regularity is precisely stored in the analysis of this problem.

Let us heuristically show how the free boundary condition

$$|\nabla u^+| = G(|\nabla u^-|)$$

linearizes in the *nondegenerate* case. Let $U_\beta(t) = \alpha t^+ - \beta t^-$, $\alpha = G(\beta)$, it is possible to show that for any $\varepsilon > 0$, if $\delta > 0$, depending on ε is small enough, the condition (4) implies the following one

$$U_\beta(x_n - \varepsilon) \le u(x) \le U_\beta(x_n + \varepsilon) \quad \text{in } B_1,$$

with $0 < \beta \le L$, and $L = \mathrm{Lip}(u)$. After rescaling we may assume that

$$|f| \le \varepsilon^2 \min\{\alpha, \beta\}.$$

This suggests the renormalization

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha\varepsilon}, & x \in B_1^+(u) \cup \mathcal{F}(u), \\ \dfrac{u(x) - \beta x_n}{\beta\varepsilon}, & x \in B_1^-(u) \end{cases}$$

or

$$u(x) = \begin{cases} \alpha x_n + \varepsilon\alpha\tilde{u}_\varepsilon(x), & x \in B_1^+(u) \cup \mathcal{F}(u), \\ \beta x_n + \varepsilon\beta\tilde{u}_\varepsilon(x), & x \in B_1^-(u). \end{cases} \tag{5}$$

Recalling (1), we have

$$F_{\alpha\varepsilon}(D^2\tilde{u}_\varepsilon) = \frac{f}{\alpha\varepsilon} \sim \varepsilon \quad \text{in } B_1^+(u)$$

and

$$F_{\beta\varepsilon}(D^2\tilde{u}_\varepsilon) = \frac{f}{\beta\varepsilon} \sim \varepsilon \quad \text{in } B_1^-(u).$$

On $\mathcal{F}(u)$,

$$|\nabla u^+| = \alpha|e_n + \varepsilon\nabla\tilde{u}_\varepsilon(x)| \sim \alpha\left(1 + \varepsilon(\tilde{u}_\varepsilon)_{x_n} + \varepsilon^2|\nabla\tilde{u}_\varepsilon|^2\right)$$

and

$$\begin{aligned} G(|\nabla u^-|) &= G\left(|\beta e_n + \varepsilon\beta\nabla\tilde{u}_\varepsilon|\right) \sim G\left(\beta\left(1 + \varepsilon(\tilde{u}_\varepsilon)_{x_n} + \varepsilon^2|\nabla\tilde{u}_\varepsilon|^2\right)\right) \\ &\sim G(\beta) + \varepsilon G'(\beta)\left(\beta(\tilde{u}_\varepsilon)_{x_n} + \varepsilon\beta|\nabla\tilde{u}_\varepsilon|^2\right) + \varepsilon^2. \end{aligned}$$

Letting $\varepsilon \to 0$, we get formally for "the limit" $\tilde{u}$ the following problem:

$$F^\pm(D^2\tilde{u}) = 0 \quad \text{in } B^\pm_{1/2} \cap \{x_n \ne 0\} \tag{6}$$

and the transmission condition (*linearization of the free boundary condition*)

$$\alpha\left(\tilde{u}_{x_n}\right)^+ - \beta G'(\beta)\left(\tilde{u}_{x_n}\right)^- = 0 \quad \text{on } B_{1/2} \cap \{x_n = 0\},$$

where $F^+(M)$, $F^-(M)$ are limits (of sequences) of operators of the form

$$F_{\alpha\varepsilon}(M) \quad \text{and} \quad F_{\beta\varepsilon}(M),$$

while $(\tilde{u}_{x_n})^+$ and $(\tilde{u}_{x_n})^-$ denote the e_n-derivatives of $\tilde{u}$ restricted to $\{x_n > 0\}$ and $\{x_n < 0\}$, respectively.

Thus, at least formally, we have found an asymptotic problem for the limits of the renormalizations $\tilde{u}_\varepsilon$. The crucial information we were mentioning before is contained in the following regularity result that we state for the fully nonlinear case (see [27] when distributed sources are present). Consider the transmission problem, ($\tilde{a} \neq 0$)

$$\begin{cases} F^{+}(D^2\tilde{u}) = 0 & \text{in } B_1^{+} \cap \{x_n \neq 0\}, \\ F^{-}(D^2\tilde{u}) = 0 & \text{in } B_1^{-} \cap \{x_n \neq 0\}, \\ \tilde{a}(\tilde{u}_{x_n})^{+} - \tilde{b}(\tilde{u}_{x_n})^{-} = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{7}$$

Theorem 11 *Let $\tilde{u}$ be a viscosity solution to (7) in B_1 such that $\|\tilde{u}\|_\infty \leq 1$. Then $\tilde{u} \in C^{1,\gamma}(\bar{B}_1^{\pm})$ and in particular, there exists a universal constant $\bar{C}$ such that*

$$|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'}\tilde{u}(0) \cdot x' + \tilde{p}x_n^{+} - \tilde{q}x_n^{-})| \leq \bar{C}r^{1+\gamma} \quad \text{in } B_r \tag{8}$$

for all $r \leq 1/2$, with $\tilde{a}\tilde{p} - \tilde{b}\tilde{q} = 0$.

Transferring the estimate (8) to $\tilde{u}_\varepsilon$ and then reading it in terms of flatness for u through formulas (5), one deduce that if $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r,

$$U_{\beta'}\left(x \cdot \nu_1 - r\frac{\varepsilon}{2}\right) \leq u(x) \leq U_{\beta'}\left(x \cdot \nu_1 + r\frac{\varepsilon}{2}\right) \quad \text{in } B_r, \tag{9}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde{C}\varepsilon$, and $|\beta - \beta'| \leq \tilde{C}\beta\varepsilon$ for a universal constant $\tilde{C}$.

Rescaling and iterating (9) one deduces uniform pointwise $C^{1,\gamma}$ estimates in neighborhood of the origin.

5 Higher Regularity

In view of the results in Section 5, under suitable flatness assumptions, the free boundary $\mathcal{F}(u)$ is locally $C^{1,\gamma}$ and the same conclusion holds if $\mathcal{F}(u)$ is a graph of a Lipschitz function. Therefore u is a classical solution, i.e. the free boundary condition is satisfied in a pointwise sense. Concerning higher regularity, the result one aims to prove should be the following one.

"**Theorem**". Assume some reasonable starting regularity on $\mathcal{F}(u)$ and let k be a nonnegative integer. Suppose that $f_i \in C^{k,\gamma}(B_1)$, $i = 1, 2$, and G is C^{2+k}. Then $\mathcal{F}(u) \cap B_{1/2}$ is C^{k+2,γ^*}. Moreover, if f_i, $i = 1, 2$, are C^∞ or real analytic in B_1, then $\mathcal{F}(u) \cap B_{1/2}$ is C^∞ or real analytic, respectively.

The state of the art is the following. In the seminal paper [46], the authors used a zero order hodograph transformation and a suitable reflection map, to locally reduce a two-phase problem to an elliptic, coercive nonlinear system of equations. The existing literature on the regularity of solutions to nonlinear systems developed in [3, 61] can be applied as long as the solution u is $C^{2,\alpha}$ for some $\alpha > 0$ up to the free boundary (from either side).

As noted in the recent work [48], in the case when the governing equation in (2) is linear and in divergence form, the initial assumption to obtain the above theorem is actually $u \in C^{1,\alpha}$. It is not evident that the case of linear nondivergence uniformly elliptic equations can be treated in a similar manner. On the other hand, the case when the leading operator is a fully nonlinear operator definitely requires the solution to have Hölder second derivatives (from both sides).

In [26] we consider the case of nondivergence linear operators with the purpose to develop a general strategy that would apply to a larger class of problem, to include also fully nonlinear operators. The application to the latter case would be rather straightforward once $C^{2,\alpha}$ estimates for the limiting problem (7) were available. However for fully nonlinear operators, this remains an open problem. The main result in [26] is the following.

Theorem 12 *Let F be a linear uniformly elliptic operator in nondivergence form, with $C^{0,\gamma}$ coefficients and let u be a (Lipschitz) viscosity solution to (2) in B_1. Assume that either $\mathcal{F}(u)$ is locally the graph of a Lipschitz continuous function or there exists a universal constant $\eta > 0$ such that,*

$$\{x_n \leq -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\},$$

then $\mathcal{F}(u)$ is C^{2,γ^*} in $B_{1/2}$ for a small γ^* universal, with the C^{2,γ^*} norm bounded by a universal constant.

Hence, the initial proposal, stated in the theoretical expected "Theorem", may be considered achieved by the following corollary contained in [26].

Corollary 13 *Let u be a viscosity solution of (2) when $F := Tr(A(x)D^2u(x))$, and A is a smooth uniformly elliptic matrix of coefficients. Assume that $\mathcal{F}(u)$ is Lipschitz and let k be a nonnegative integer. Suppose that $f_i \in C^{k,\gamma}(B_1)$, $i = 1, 2$, and G is C^{2+k}. Then $\mathcal{F}(u) \cap B_{1/2}$ is C^{k+2,γ^*}. Moreover, if f_i, $i = 1, 2$, are C^∞ or real analytic in B_1, then $\mathcal{F}(u) \cap B_{1/2}$ is C^∞ or real analytic, respectively.*

Other related higher regularity results can be found in [32, 47]. The overall strategy for the proof of Theorem 12 follows the ideas described in the previous section. However, reaching the $C^{2,\gamma}$ regularity requires a much more involved process because of the possible degeneracy of the negative part. Indeed this causes a delicate interplay between the two phases. Ultimately the main source of difficulties is due to the presence of a forcing term of general sign in the negative phase. Indeed, if $f_2 \geq 0$, the Hopf maximum principle would imply nondegeneracy (also) on the negative side, making the two-phases of comparable size and considerably simplifying the final iteration procedure. It is worth noticing, that however even in this easier scenario (and in particular in the homogeneous case), if one wants to attain uniform estimates with universal constants, then one must employ the more involved methods developed in [26] for the degenerate case.

6 Nonlinear Operators with Non-standard Growth

In the second part of this survey we present some recent results concerning a one-phase free boundary problem governed by the $p(x)$-Laplacian. More precisely, let

$$\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

where p is measurable and $1 < p(x) < +\infty$ a.e. in a domain Ω. We recall that if $p : \Omega \to [1, \infty)$ is a measurable bounded function, the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$ is finite. The Luxemburg norm on this space is defined by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space. In addition, setting $p_{\max} = \operatorname{esssup} p(x)$ and $p_{\min} = \operatorname{essinf} p(x)$, the following relation holds between $\varrho_{p(\cdot)}(u)$ and $\|u\|_{L^{p(\cdot)}}$:

$$\min\left\{\left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\min}}, \left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\max}}\right\} \le \|u\|_{L^{p(\cdot)}(\Omega)}$$
$$\le \max\left\{\left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\min}}, \left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\max}}\right\}.$$

Moreover, the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

$W^{1,p(\cdot)}(\Omega)$ denotes the space of measurable functions u such that u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p \in C^1(\Omega)$, $f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \in C^{0,\beta}(\Omega)$, $g \ge 0$. Then our problem reads

$$\begin{cases} \Delta_{p(x)} u = f & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ |\nabla u| = g & \text{on } \mathcal{F}(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{10}$$

In addition we assume that

$$\nabla p \in L^\infty(\Omega)$$

and that

$$1 < p_{\min} \le p(x) \le p_{\max} < \infty.$$

This kind of problems arise naturally from limits of a singular perturbation problem with forcing term as in [52], where the authors analyze solutions to (10), arising in the study of flame propagation with nonlocal and electromagnetic effects. On the other hand, (10) appears by minimizing the energy functional

$$\mathcal{E}(v) = \int_\Omega \left(\frac{|\nabla v|^{p(x)}}{p(x)} + Q^2(x)\chi_{\{v>0\}} + f(x)v\right) dx \tag{11}$$

studied in [54], see [20] as well. We refer also to [55], where (10) appears in the study of an optimal design problem.

In general, partial differential equations with non-standard growth have been receiving a lot of attention and the $p(x)$-Laplacian is a model case in this class. A list of applications of this type of operators includes the modelling of non-Newtonian fluids, for instance, electrorheological [62] or thermorheological fluids [8]. Also non-linear elasticity [70], image reconstruction [1, 18] and the modelling of electric conductors [71], to cite some of them.

We are interested in the regularity of the free boundary for viscosity solutions of (10), following the strategy introduced in [22]. The same technique was applied to the p-Laplace operator ($p(x) \equiv p$ in (10)), with $p \ge 2$, in [56].

Problem (10) was originally studied in the linear homogeneous case in [4], associated to (11). These techniques were generalized to the linear case with $f \not\equiv 0$ in [43, 50] and, in the homogeneous case, to a quasilinear uniformly elliptic situation [5], to the p-Laplacian [21], to an Orlicz setting [58] and to the $p(x)$-Laplacian with $p(x) \ge 2$ [36]. For the case (10) with $1 < p(x) < \infty$ and $f \not\equiv 0$ we refer to [53].

We will present results on problem (10) obtained in [38]. In order to do so, we start with some definitions and preliminaries.

Recall that we have assumed that $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$, with $p(x)$ Lipschitz continuous in Ω, and $f \in L^\infty(\Omega)$. We say that u is a *weak* solution to $\Delta_{p(x)}u = f$ in Ω if $u \in W^{1,p(\cdot)}(\Omega)$ and, for every $\varphi \in C_0^\infty(\Omega)$, there holds that

$$-\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx = \int_\Omega \varphi\, f(x)\, dx. \tag{12}$$

The notion of viscosity solution to (10) is given as follows.

Definition 14 Let u be a continuous nonnegative function in Ω. We say that u is a viscosity solution to (10) in Ω, if the following conditions are satisfied:

1. $\Delta_{p(x)}u = f$ in $\Omega^+(u)$ in the weak sense (i.e., in the sense of (12)).
2. For every $\varphi \in C(\Omega)$, $\varphi \in C^2(\overline{\Omega^+(\varphi)})$. If φ^+ touches u from below (resp. above) at $x_0 \in \mathcal{F}(u)$ and $\nabla\varphi(x_0) \neq 0$, then

$$|\nabla\varphi(x_0)| \leq g(x_0) \quad (\text{resp. } \geq g(x_0)).$$

We point out the difference between the notion of viscosity solution for free boundary problem (10) in Definition 14 and the one for free boundary problem (2) in Definition 1 (Section 2). Namely, equation $\Delta_{p(x)}u = f$ is satisfied here in weak the sense of (12). However, the program followed in [38] required to deal with viscosity solutions of this equation—in the sense of [19]—as well.

The equivalence between weak and viscosity solutions of $\Delta_{p(x)}u = f$ was proved in [44, 45, 60] in the case of the p-Laplacian (i.e., for $p(x) \equiv p$) and in [49] in the case of the homogeneous $p(x)$-Laplacian (i.e., for $f \equiv 0$). For the inhomogeneous $p(x)$-Laplacian the following result was proven in [38]:

Theorem 15 *Assume that $f \in C(\Omega) \cap L^\infty(\Omega)$, $p \in C^1(\Omega)$ with $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ and $\nabla p \in L^\infty(\Omega)$. Let $u \in W^{1,p(\cdot)}(\Omega) \cap C(\Omega)$ be a weak solution to $\Delta_{p(x)}u = f$ in Ω. Then u is a viscosity solution to $\Delta_{p(x)}u = f$ in Ω.*

As a consequence, we have:

Proposition 16 *Let u be a viscosity solution to (10) in Ω. Then the following conditions are satisfied:*

1. *$\Delta_{p(x)}u = f$ in $\Omega^+(u)$ in the viscosity sense, that is:*
   - (ia) *for every $\varphi \in C^2(\Omega^+(u))$ and for every $x_0 \in \Omega^+(u)$, if φ touches u from above at x_0 and $\nabla\varphi(x_0) \neq 0$, then $\Delta_{p(x_0)}\varphi(x_0) \geq f(x_0)$, that is, u is a viscosity subsolution;*
   - (ib) *for every $\varphi \in C^2(\Omega^+(u))$ and for every $x_0 \in \Omega^+(u)$, if φ touches u from below at x_0 and $\nabla\varphi(x_0) \neq 0$, then $\Delta_{p(x_0)}\varphi(x_0) \leq f(x_0)$, that is, u is a viscosity supersolution.*
2. *For every $\varphi \in C(\Omega)$, $\varphi \in C^2(\overline{\Omega^+(\varphi)})$. If φ^+ touches u from below (resp. above) at $x_0 \in \mathcal{F}(u)$ and $\nabla\varphi(x_0) \neq 0$, then*

$$|\nabla\varphi(x_0)| \leq g(x_0) \quad (\textit{resp.} \geq g(x_0)).$$

We now describe the main result in [38]. Namely, flat free boundaries of viscosity solutions to (10) are $C^{1,\alpha}$. In the forthcoming work [39] it is shown that Lipschitz free boundaries of viscosity solutions are $C^{1,\alpha}$. Concerning this last paper, we just say that, remarkably, the regularity step from Lipschitz to $C^{1,\alpha}$ presents new and interesting aspects, requiring deeper techniques.

Precisely, the main result in [38] is the following

Theorem 17 (Flatness implies $C^{1,\alpha}$) *Let u be a viscosity solution to (10) in B_1. Assume that $0 \in \mathcal{F}(u)$, $g(0) = 1$ and $p(0) = p_0$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if the graph of u is $\bar{\varepsilon}$-flat in B_1, in the direction e_n, that is*

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1,$$

and

$$\|\nabla p\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \quad \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \quad [g]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon},$$

then $\mathcal{F}(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

In Theorem 17 the constants $\bar{\varepsilon}$ and α depend only on $p_{\min}$, $p_{\max}$ and n.

We remark that Theorem 17 is crucial in the companion paper [39] to prove that Lipschitz free boundaries of viscosity solutions of (10) are $C^{1,\alpha}$.

The proof of Theorem 17 is based on an improvement of flatness, obtained via a compactness argument (provided by a Harnack type inequality) which linearizes the problem into a limiting one. We point out that the development of new tools was necessary in order to implement this strategy for the inhomogeneous $p(x)$-Laplace operator. In fact, the $p(x)$-Laplacian is a nonlinear operator that appears naturally in divergence form from minimization problems, i.e., in the form $\operatorname{div} A(x, \nabla u) = f(x)$, with

$$\lambda|\eta|^{p(x)-2}|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(x, \eta)\xi_i\xi_j \leq \Lambda|\eta|^{p(x)-2}|\xi|^2, \quad \xi \in \mathbb{R}^n.$$

This operator is singular in the regions where $1 < p(x) < 2$ and degenerate in the ones where $p(x) > 2$.

Some important results for this type of operators are available in the literature only for weak solutions (in the sense of (12)). These results are Harnack inequality, [69], and $C^{1,\alpha}$ estimates, [34, 35]. This is the motivation for the choice of Definition 14. However, the proof of Theorem 17 relies on solutions of inhomogeneous $p(x)$-Laplace equation in the viscosity sense. Then, Theorem 15 becomes essential.

On the other hand, the nondivergence nature of the viscosity solutions requires the construction of suitable barriers. It turns out that barriers of the type $w(x) = c_1|x - x_0|^{-\gamma} - c_2$ are appropriate for the inhomogeneous $p(x)$-Laplace operator, although the proof is delicate, due to the nonlinear singular/degenerate nature of the operator, its x dependence and the presence of the logarithmic term that appears in the nondivergence form of the operator. Precisely, the following key result was proved in [38]:

Lemma 18 *Let $x_0 \in B_1$ and $0 < \bar{r}_1 < \bar{r}_2 \leq 1$. Assume that $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ and $\|\nabla p\|_{L^\infty} \leq \varepsilon^{1+\theta}$, for some $0 < \theta \leq 1$. Let c_0, c_1, c_2 be positive constants and $c_3 \in \mathbb{R}$. There exist positive constants $\gamma \geq 1$, $\bar{c}$, ε_0 and ε_1 such that the functions*

$$\begin{aligned} w(x) &= c_1|x - x_0|^{-\gamma} - c_2, \\ v(x) &= q(x) + \frac{c_0}{2}\varepsilon(w(x) - 1), \quad q(x) = x_n + c_3 \end{aligned}$$

satisfy, for $\bar{r}_1 \le |x - x_0| \le \bar{r}_2$,

$$\Delta_{p(x)} w \ge \bar{c}, \quad \text{for } 0 < \varepsilon \le \varepsilon_0, \tag{13}$$

$$\frac{1}{2} \le |\nabla v| \le 2, \qquad \Delta_{p(x)} v > \varepsilon^2, \quad \text{for } 0 < \varepsilon \le \varepsilon_1. \tag{14}$$

Here $\gamma = \gamma(n, p_{\min}, p_{\max})$, $\bar{c} = \bar{c}(p_{\min}, p_{\max}, c_1)$, $\varepsilon_0 = \varepsilon_0(n, p_{\min}, p_{\max}, \bar{r}_1, c_1)$, $\varepsilon_1 = \varepsilon_1(n, p_{\min}, p_{\max}, \bar{r}_1, c_0, c_1, \theta)$.

Another essential tool leading to the proof of Theorem 17 is given by the following Harnack type inequality, obtained in [38], whose p-Laplace version (i.e., for $p(x) \equiv p$ constant) and with $p \ge 2$, is contained in [56].

Lemma 19 *Assume that* $1 < p_{\min} \le p(x) \le p_{\max} < \infty$ *with* $p(x)$ *Lipschitz continuous in* Ω *and* $\|\nabla p\|_{L^\infty} \le L$, *for some* $L > 0$. *Let* $x_0 \in \Omega$ *and* $0 < R \le 1$ *such that* $\overline{B_{4R}(x_0)} \subset \Omega$. *Let* $v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *be a nonnegative solution to*

$$\operatorname{div}\left(|\nabla v + e|^{p(x)-2}(\nabla v + e)\right) = f \quad \text{in } \Omega,$$

where $f \in L^\infty(\Omega)$ *with* $\|f\|_{L^\infty(\Omega)} \le 1$ *and* $e \in \mathbb{R}^n$ *with* $|e| = 1$. *Then, there exists* C *such that*

$$\sup_{B_R(x_0)} v \le C\left[\inf_{B_R(x_0)} v + R\left(\|f\|_{L^\infty(B_{4R}(x_0))}^{\frac{1}{p_{\max}-1}} + C\right)\right].$$

The constant C *depends only on* n, $p_{\min}$, $p_{\max}$, $\|v\|_{L^\infty(B_{4R}(x_0))}$ *and* L.

Let us also mention that the fact that weak solutions to the inhomogeneous $p(x)$-Laplacian are locally of class $C^{1,\alpha}$ plays a critical role in the results in [38]. We point out that sharp conditions about the regularity of solutions of some elliptic equations with non-standard growth can be found in [2] and [34].

Finally, the overall strategy of the proof of the Theorem 17 in [38] follows closely the one in [22] and, after a suitable rescaling, it relies on the following

Lemma 20 (Improvement of flatness) *Let* u *satisfy* (10) *in* B_1 *and*

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^2, \quad \|g - 1\|_{L^\infty(B_1)} \le \varepsilon^2, \quad \|\nabla p\|_{L^\infty(B_1)} \le \varepsilon^{1+\theta}, \quad \|p - p_0\|_{L^\infty(B_1)} \le \varepsilon, \tag{15}$$

for $0 < \varepsilon < 1$, *for some constant* $0 < \theta \le 1$. *Suppose that*

$$(x_n - \varepsilon)^+ \le u(x) \le (x_n + \varepsilon)^+ \quad \text{in } B_1, \quad 0 \in \mathcal{F}(u). \tag{16}$$

If $0 < r \le r_0$ *for* r_0 *universal, and* $0 < \varepsilon \le \varepsilon_0$ *for some* ε_0 *depending on* r, *then*

$$(x \cdot \nu - r\varepsilon/2)^+ \le u(x) \le (x \cdot \nu + r\varepsilon/2)^+ \quad \text{in } B_r, \tag{17}$$

with $|\nu| = 1$ *and* $|\nu - e_n| \le \tilde{C}\varepsilon$ *for a universal constant* $\tilde{C}$.

We schematize below the main steps of its proof.

Step 1: Compactness. Fix $r \le r_0$ with r_0 universal (chosen in Step 3). Assume by contradiction that there exists a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (10) in B_1 with right hand side f_k, exponent p_k and free boundary condition g_k satisfying (15) with $\varepsilon = \varepsilon_k$, such that u_k satisfies (16), i.e.,

$$(x_n - \varepsilon_k)^+ \le u_k(x) \le (x_n + \varepsilon_k)^+ \quad \text{for } x \in B_1, 0 \in \mathcal{F}(u_k), \tag{18}$$

but u_k does not satisfy the conclusion (17) of the lemma.

Set

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in \Omega_1(u_k).$$

Then, (18) gives

$$-1 \le \tilde{u}_k(x) \le 1 \quad \text{for } x \in \Omega_1(u_k). \tag{19}$$

With a compactness argument and a sharp application of Ascoli–Arzelà theorem it is possible to prove that there exists a convergent subsequence to a function $\tilde{u}$.

Step 2. Limiting solution. A delicate argument, involving our assumption (15), shows that the function $\tilde{u}$ solves, in the viscosity sense, the following linearized problem

$$\begin{cases} \mathcal{L}_{p_0}\tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{20}$$

Here $1 < p_{\min} \le p_0 \le p_{\max} < \infty$, $\tilde{u}_n$ denotes the derivative in the e_n direction of $\tilde{u}$ and

$$\mathcal{L}_{p_0}u := \Delta u + (p_0 - 2)\partial_{nn}u.$$

We point out that, if $\tilde{u}$ is a continuous function on $B_{1/2} \cap \{x_n \ge 0\}$, we say that $\tilde{u}$ is a viscosity solution to (20), if given a quadratic polynomial $P(x)$ touching $\tilde{u}$ from below (resp. above) at $\bar{x} \in B_{1/2} \cap \{x_n \ge 0\}$,

(i) if $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then $\mathcal{L}_{p_0}P \le 0$ (resp. $\mathcal{L}_{p_0}P \ge 0$), i.e. $\mathcal{L}_{p_0}\tilde{u} = 0$ in the viscosity sense in $B_{1/2} \cap \{x_n > 0\}$;
(ii) if $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$ then $P_n(\bar{x}) \le 0$ (resp. $P_n(\bar{x}) \ge 0$).

Step 3: Improvement of flatness. From the previous step, $\tilde{u}$ solves (20) and from (19),

$$-1 \le \tilde{u}(x) \le 1 \quad \text{in } B_{1/2} \cap \{x_n \ge 0\}. \tag{21}$$

On the other hand it is well known that if $\tilde{u}$ is a viscosity solution to (20) in $B_{1/2} \cap \{x_n \ge 0\}$, then $\tilde{u} \in C^2(B_{1/2} \cap \{x_n \ge 0\})$ and it is a classical solution to (20). Then, using the bound (21) we find that, for the given r,

$$|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0) \cdot x| \le C_0 r^2 \quad \text{in } B_r \cap \{x_n \ge 0\}$$

if $r_0 \le 1/4$, with C_0 universal. Now the iterative argument is the same that has been applied in the linear case, [22, 23].

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References

1. Aboulaich, R., Meskine, D., Souissi, A.: New diffusion models in image processing. Comput. Math. Appl. **56**, 874–882 (2008)

2. Acerbi, E., Mingione, G.: Regularity results for a class of functionals with non-standard growth. Arch. Rational Mech. Anal. **156**, 121–140 (2001)
3. Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Commun. Pure Appl. Math. **12**, 623–727 (1959)
4. Alt, H.W., Caffarelli, L.: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math. **331**, 105–144 (1982)
5. Alt, H.W., Caffarelli, L., Friedman, A.: Variational problems with two phases and their free boundaries. Trans. Amer. Math. Soc. **282**, 431–461 (1984)
6. Amaral, M.D., Teixeira, E.V.: Free transmission problems. Commun. Math. Phys. **337**, 1465–1489 (2015)
7. Amick, C.J., Fraenkel, L.E., Toland, J.F.: On the Stokes conjecture for the wave of extreme form. Acta Math. **148**, 193–214 (1982)
8. Antontsev, S.N., Rodrigues, J.F.: On stationary thermo-rheological viscous flows. Ann. Univ. Ferrara, Sez. VII, Sci. Mat. **52**, 19–36 (2006)
9. Argiolas, R., Ferrari, F.: Flat free boundaries regularity in two-phase problems for a class of fully nonlinear elliptic operators with variable coefficients. Interfaces Free Bound. **11**, 177–199 (2009)
10. Batchelor, G.K.: On steady laminar flow with closed streamlines at large Reynolds number. J. Fluid. Mech. **1**, 177–190 (1956)
11. Caffarelli, L.: A Harnack inequality approach to the regularity of free boundaries, part i: Lipschitz free boundaries are $C^{1,\alpha}$. Rev. Mat. Iberoam. **3**, 139–162 (1987)
12. Caffarelli, L.: A Harnack inequality approach to the regularity of free boundaries, part II: flat free boundaries are Lipschitz. Commun. Pure Appl. Math. **42**, 55–78 (1989)
13. Caffarelli, L.: A Harnack inequality approach to the regularity of free boundaries, part III: existence theory, compactness and dependence on X. Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **15**, 583–602 (1988)
14. Caffarelli, L.A., De Silva, D., Savin, O.: Two-phase anisotropic free boundary problems and applications to the Bellman equation in 2D. Arch. Rational Mech. Anal. **228**, 477–493 (2018)
15. Caffarelli, L., Salsa, S.: A Geometric Approach to Free Boundary Problems. Graduate Studies in Mathematics, vol. 68. American Mathematical Society, Providence (2005)
16. Caffarelli, L., Jerison, D., Kenig, C.: Some new monotonicity theorems with applications to free boundary problems. Ann. Math. **155**, 369–404 (2002)
17. Cerutti, M.C., Ferrari, F., Salsa, S.: Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are $C^{1,\gamma}$. Arch. Rational Mech. Anal. **171**, 329–348 (2004)
18. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**, 1383–1406 (2006)
19. Crandall, M.G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) **27**, 1–67 (1992)
20. Dzhugan, A., Ferrari, F.: Domain variation solutions for degenerate two phase free boundary problems. Math. Eng. **3**, 043 (2021)
21. Danielli, D., Petrosyan, A.: A minimum problem with free boundary for a degenerate quasilinear operator. Calc. Var. Partial Differ. Equ. **23**, 97–124 (2005)
22. De Silva, D.: Free boundary regularity for a problem with right hand side. Interfaces Free Bound. **13**, 223–238 (2011)
23. De Silva, D., Ferrari, F., Salsa, S.: Two-phase problems with distributed sources: regularity of the free boundary. Anal. PDE **7**, 267–310 (2014)
24. De Silva, D., Ferrari, F., Salsa, S.: Free boundary regularity for fully nonlinear non-homogeneous two-phase problems. J. Math. Pures Appl. **103**, 658–694 (2015)
25. De Silva, D., Ferrari, F., Salsa, S.: Perron's solutions for two-phase free boundary problems with distributed sources. Nonlinear Anal. **121**, 382–402 (2015)
26. De Silva, D., Ferrari, F., Salsa, S.: Regularity of higher order in two-phase free boundary problems. Trans. Amer. Math. Soc. **371**, 3691–3720 (2019)
27. De Silva, D., Ferrari, F., Salsa, S.: On the regularity of transmission problems for uniformly elliptic fully nonlinear equations. Two nonlinear days in Urbino 2017. Electron. J. Differ. Eqn. Conf. **25**, 55–63 (2018)
28. De Silva, D., Jerison, D.: A singular energy minimizing free boundary. J. Reine Angew. Math. **635**, 1–22 (2009)
29. De Silva, D., Savin, O.: Lipschitz regularity of solutions to two-phase free boundary problems. Int. Math. Res. Not. IMRN **2019**, 2204–2222 (2019)
30. De Silva, D., Savin, O.: Global solutions to nonlinear two-phase free boundary problems. Commun. Pure Appl. Math. **72**, 2031–2062 (2019)

31. De Philippis, G., Spolaor, L., Velichkov, B.: Regularity of the free boundary for the two-phase Bernoulli problem. Invent. Math. **225**, 347–394 (2021)
32. Engelstein, M.: A two-phase free boundary problem for the harmonic measure. Ann. Sci. Éc. Norm. Supér. (4) **49**, 859–905 (2016)
33. Engelstein, M., Spolaor, L., Velichkov, B.: Uniqueness of the blowup at isolated singularities for the Alt–Caffarelli functional. Duke Math. J. **169**, 1541–1601 (2020). arXiv:1801.09276 (2018)
34. Fan, X.: Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form. J. Differ. Equ. **235**, 397–417 (2007)
35. Fan, X., Zhao, D.: A class of De Giorgi type and Hölder continuity. Nonlinear Anal. **36**, 295–318 (1999)
36. Fernández Bonder, J., Martínez, S., Wolanski, N.: A free boundary problem for the $p(x)$-Laplacian. Nonlinear Anal. **72**, 1078–1103 (2010)
37. Ferrari, F.: Two-phase problems for a class of fully nonlinear elliptic operators. Lipschitz free boundaries are $C^{1,\gamma}$. Amer. J. Math. **128**, 541–571 (2006)
38. Ferrari, F., Lederman, C.: Regularity of flat free boundaries for a $p(x)$-Laplacian problem with right hand side. Nonlinear Anal. **212**, 112444 (2021)
39. Ferrari, F., Lederman, C.: Regularity of Lipschitz free boundaries for a $p(x)$-Laplacian problem with right hand side. Preprint
40. Ferrari, F., Salsa, S.: Regularity of the free boundary in two-phase problems for linear elliptic operators. Adv. Math. **214**, 288–322 (2007)
41. Feldman, M.: Regularity for nonisotropic two-phase problems with Lipschitz free boundaries. Differ. Integral Equ. **10**, 1171–1179 (1997)
42. Feldman, M.: Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations. Indiana Univ. Math. J. **50**, 1171–1200 (2001)
43. Gustafsson, B., Shahgholian, H.: Existence and geometric properties of solutions of a free boundary problem in potential theory. J. Reine Angew. Math. **473**, 137–179 (1996)
44. Julin, V., Juutinen, P.: A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation. Commun. Partial Differ. Equ. **37**, 934–946 (2012)
45. Juutinen, P., Lindqvist, P., Manfredi, J.: On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. SIAM J. Math. Anal. **33**, 699–717 (2001)
46. Kinderlehrer, D., Nirenberg, L., Spruck, J.: Regularity in elliptic free boundary problems I. J. Anal. Math. **34**, 86–119 (1978)
47. Koch, H.: Classical solutions to phase transition problems are smooth. Commun. Partial Differ. Equ. **23**, 389–437 (1998)
48. Kriventsov, D., Lin, F.: Regularity for shape optimizers: the nondegenerate case. Commun. Pure Appl. Math. **71**, 1535–1596 (2018)
49. Juutinen, P., Lukkari, T., Parviainen, M.: Equivalence of viscosity and weak solutions for the $p(x)$-Laplacian. Ann. Inst. H. Poincare Anal. Non Lineaire **27**, 1471–1487 (2010)
50. Lederman, C.: A free boundary problem with a volume penalization. Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **23**, 249–300 (1996)
51. Lederman, C., Wolanski, N.: A two phase elliptic singular perturbation problem with a forcing term. J. Math. Pures Appl. **86**, 552–589 (2006)
52. Lederman, C., Wolanski, N.: An inhomogeneous singular perturbation problem for the $p(x)$-Laplacian. Nonlinear Anal. **138**, 300–325 (2016)
53. Lederman, C., Wolanski, N.: Weak solutions and regularity of the interface in an inhomogeneous free boundary problem for the $p(x)$-Laplacian. Interfaces Free Bound. **19**, 201–241 (2017)
54. Lederman, C., Wolanski, N.: Inhomogeneous minimization problems for the $p(x)$-Laplacian. J. Math. Anal. Appl. **475**, 423–463 (2019)
55. Lederman, C., Wolanski, N.: An optimization problem with volume constraint for an inhomogeneous operator with nonstandard growth. Discrete Contin. Dyn. Syst. **41**, 2907–2946 (2021)
56. Leitão, R., Ricarte, G.: Free boundary regularity for a degenerate problem with right hand side. Interfaces Free Bound. **20**, 577–595 (2018)
57. Lu, G., Wang, P.: On the uniqueness of a solution of a two-phase free boundary problem. J. Funct. Anal. **258**, 2817–2833 (2010)
58. Martínez, S., Wolanski, N.: A minimum problem with free boundary in Orlicz spaces. Adv. Math. **218**, 1914–1971 (2008)
59. Matevosyan, N., Petrosyan, A.: Almost monotonicity formulas for elliptic and parabolic operators with variable coefficients. Commun. Pure Appl. Math. **64**, 271–311 (2011)
60. Medina, M., Ochoa, P.: On the viscosity and weak solutions for non-homogeneous p-Laplace equations. Adv. Nonlinear Anal. **8**, 468–481 (2019)

61. Morrey, C.B. Jr.: Multiple Integrals in the Calculus of Variations. Reprint of the 1966 Edition Classics in Mathematics. Springer, Berlin (2008)
62. Růžička, M.: Electrorheological Fluids: Modeling and Mathematical Theory. Springer, Berlin (2000)
63. Salsa, S., Tulone, F., Verzini, G.: Existence of viscosity solutions to two-phase problems for fully nonlinear equations with distributed sources. Math. Eng. **1**, 147–173 (2019)
64. Teixeira, E.V.: A variational treatment for general elliptic equations of the flame propagation type: regularity of the free boundary. Ann. Inst. H. Poincare Anal. Non Lineaire **25**, 633–658 (2008)
65. Teixeira, E.V., Zhang, L.: Monotonicity theorems for Laplace Beltrami operator on Riemannian manifolds. Adv. Math. **226**, 1259–1284 (2011)
66. Wang, P.-Y.: Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order I. Lipschitz free boundaries are $C^{1,\alpha}$. Commun. Pure Appl. Math. **53**, 799–810 (2000)
67. Wang, P.-Y.: Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. II. Flat free boundaries are Lipschitz. Commun. Partial Differ. Equ. **27**, 1497–1514 (2002)
68. Wang, P.Y.: Existence of solutions of two-phase free boundary problems for fully nonlinear elliptic equations of second order. J. Geom. Anal. **13**, 715–738 (2003)
69. Wolanski, N.: Local bounds, Harnack's inequality and Hölder continuity for divergence type elliptic equations with non-standard growth. Rev. Un. Mat. Argent. **56**, 73–105 (2015)
70. Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR. Izv. **29**, 33–66 (1987)
71. Zhikov, V.V.: Solvability of the three-dimensional thermistor problem. Trudy Mat. Inst. Steklova **261**, 101–114 (2008). Proc. Steklov Inst. Math. **261**, 98–111 (2008) (Engl. Ver.)