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REMARKS ON HEYTING ALGEBRAS WITH TENSE OPERATORS

Abstract

The concept of tense operators on Heyting algebras was introduced in [3]. The aim of this paper is to prove, that the set of axioms proposed by I. Chajda in [3, Definition 1], is a dependent axioms system and show that tense operators F and P can not be regarded as existential quantifiers.

1. Introduction

Propositional logics usually does not incorporate the dimension of time. To obtain a tense logic, we enrich a propositional logic by adding new unary operators (or connectives) which are usually denoted by G, H, F and P. We can define F and P by means of G and H as follows: $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x.

Tense operators were first introduced in the classical propositional logic. Tense algebras are algebraic structures corresponding to the propositional tense logic [2]. Recall that an algebra $\langle A, \vee, \wedge, \neg, G, H, 0, 1\rangle$ is a tense algebra if $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and G, H are unary operators on A satisfying the axioms

1. $G(1) = 1$, $H(1) = 1$,
2. $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$,
3. $x \leq GP(x)$, $x \leq HF(x)$, where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

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In the last few years tense operators have been considered by different authors for varied classes of algebras. Some contributions in this area have been the papers by Diaconescu and Georgescu [6], Chiriţă [4, 5], Figallo et al. [7, 9, 8, 10], Chajda [3], and Botur et al. [1]. In particular, in [3], Chajda introduced tense operators on Heyting algebras.

In this short note we prove that the set of axioms proposed by I. Chajda in [3, Definition 1], is a dependent axioms system and show that tense operators F and P can not be regarded as existential quantifiers.

2. Tense operators on Heyting algebras

The concept of tense operators on Heyting algebras was introduced in [3]. We repeat the definition of [3].

DEFINITION 2.1. Let $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ be a Heyting algebra. Denote by $x^* = x \rightarrow 0$ (the so–called pseudocomplement of x). Unary operators G, H on A are called tense operators if the following conditions hold:

(A1) $G(1) = 1$ and $H(1) = 1$,
(A2) $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$ and $H(x \rightarrow y) \leq H(x) \rightarrow H(y)$,
(A3) $G(x) \vee G(y) \leq G(x \vee y)$ and $H(x) \vee H(y) \leq H(x \vee y)$,
(A4) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
(A5) $x \leq GP(x)$ and $x \leq HF(x)$, where $P(x) = H(x^*)^*$ and $F(x) = G(x^*)^*$.

Our aim is to prove that the axioms (A2) and (A3) are redundant. For this we will need the following lemmas.

LEMMA 2.2. *Let G, H be two unary operators on the Heyting algebra $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$, satisfying the axiom* (A4). *Then the following properties hold:*

(a) *$x \leq y$ implies $G(x) \leq G(y)$ and $x \leq y$ implies $H(x) \leq H(y)$,*
(b) *$G(x) \vee G(y) \leq G(x \vee y)$ and $H(x) \vee H(y) \leq H(x \vee y)$.*

PROOF. The assertion (a) follows by (A4) since $x \leq y$ implies $G(x) = G(x \wedge y) = G(x) \wedge G(y)$, thus $G(x) \leq G(y)$, analogously for the operator H. The assertion (b) follows immediately by (a), since G is increasing we have that $G(x) \leq G(x \vee y)$ and $G(y) \leq G(x \vee y)$, thus $G(x) \vee G(y) \leq G(x \vee y)$. Analogously we can reach the second inequality.

LEMMA 2.3. *Let G, H be two unary operations on the Heyting algebra $\langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ such that $G(1) = 1$ and $H(1) = 1$. Then the axiom* (A4) *is equivalent to the axiom* (A2).

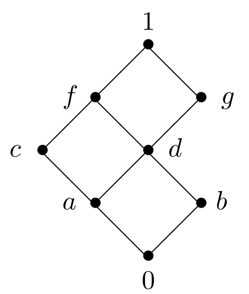
PROOF. We will only prove the equivalence between (A2) and (A4) in the case of G. From (A4) and (a) in Lemma 2.2, we have that $G(x) \wedge G(x \rightarrow y) = G(x \wedge (x \rightarrow y)) = G(x \wedge y) \leq G(y)$. Therefore, $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$. Conversely, let $x, y \in A$ be such that $x \leq y$. Then, $x \rightarrow y = 1$ and so, from (A2) and the hypothesis, we obtain that $1 = G(x \rightarrow y) \leq G(x) \rightarrow G(y)$. Hence, $G(x) \leq G(y)$ from which we get that G is increasing. This last assertion and (A2) we infer that $G(x) \leq G(y \rightarrow (x \wedge y)) \leq G(y) \rightarrow G(x \wedge y)$. Thus, $G(x) \wedge G(y) \leq G(x \wedge y)$. Taking into account that G is increasing we have that $G(x \wedge y) \leq G(x)$ and $G(x \wedge y) \leq G(y)$. Thus, $G(x \wedge y) \leq G(x) \wedge G(y)$. From this statement we conclude that $G(x) \wedge G(y) = G(x \wedge y)$.

Theorem 2.4 follows as an immediate consequence of Lemma 2.2 and 2.3.

THEOREM 2.4. *Axioms* (A2) *and* (A3) *in the definition of tense operators are redundant.*

Chajda in [3, Remark 8], states that F and P can be regarded as existential quantifiers. This statement is not valid as shown in the following example.

EXAMPLE 2.5. *Let us consider the Heyting algebra $A = \{0, a, b, c, d, f, g, 1\}$, which is described as follows:*



x	x^*
0	1
a	b
b	c
d	0
c	b
f	0
g	0
1	0

Define G, H by $G(x) = x = H(x)$, for all $x \in A$. It is easy to see that G and H are tense operators on A. On the other hand,

$$F(a \vee b) = 1 \neq f = F(a) \vee F(b) \text{ and } P(a \vee b) = 1 \neq f = P(a) \vee P(b).$$

Therefore, F and P are not existential quantifiers on A.

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