On the relation of free bodies, inertial sets and arbitrariness

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Abstract

We present a fully relational definition of inertial systems based in the No Arbitrariness Principle, that eliminates the need for absolute inertial frames of reference or distinguished reference systems as the "fixed stars" in order to formulate Newtonian mechanics. The historical roots of this approach to mechanics are discussed as well. The work is based in part in the constructivist perspective of space advanced by Piaget. We argue that inertial systems admit approximations and that what is of practical use are precisely such approximations. We finally discuss a slightly larger class of systems that we call "relatively inertial" which are the fundamental systems in a relational view of mechanics.
Keywords: Relationism; Constructivism; Newtonian Mechanics; inertial frames[1]

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1 Introduction

In the wording of Newton, "absolute time", "absolute space" and "absolute motion" [Newton, 1687, Escholium to Definition VIII] are the ingredients of concern in the formulation of the laws of mechanics. These ideas led to a core concept of Newtonian mechanics, namely that of *inertial frame*, i.e., the frame(s) on which free bodies move with constant velocity (or where changes in the velocity of a body correspond to forces/interactions). However, Newton's concept of absolute space rests on multiple grounds, some of them fundamental for his theory of mechanics, others more adapted to his philosophical taste while not mandatory for the laws of motion.

Newton's philosophical position was challenged by Leibniz who questioned the "absolute" character of space on philosophical and theological grounds, proposing that space is not absolute and prior to the objects we observe, but it is indeed given by the relations among objects, a standpoint that came to be called *relationism*. Other relationist positions were later held by C. Neumann, Streintz, Mach, Thomson and Langer [DiSalle, 1990], although arriving to different conclusions. Neumann and Streintz assumed the existence of a privileged frame (the 'Alpha Body' in Neumann and 'any free body' in Streinz, [DiSalle, 1990]) while Mach questioned absolute space as a metaphysical construct. Thomson and Langer, on the other hand, attempted to construct inertial frames based on the motion of free bodies.

The notion of absolute space in Newton presents several aspects (for a recent discussion based in the work by H. Stein, see [DiSalle, 2020]) but not all of them are *operational*, i.e., have consequences in the formulation and explanatory power of the theory. For Newton:

Definition 1.1. Absolute space

> Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is vulgarly taken for immovable space;[...]

and later, further discussing about the properties of space he writes:

Definition 1.2. True motion

> The causes by which true and relative motions are distinguished, one from the other, are the forces impressed upon bodies to generate motion. True motion is neither generated nor altered, but by some force

> impressed upon the body moved; but relative motion may be generated or altered without any force impressed upon the body. [Newton, 1687]

It is true motion what is operational in Mechanics and furthermore, there is no deductive reasoning that brings us from absolute space into true motion and much less, there is a form to logically infer the first notion from the acceptance of the second. In Newton's belief absolute space is where absolute (or true) motion takes place ("Absolute motion is the translation of a body from one absolute place into another;[...]" [Newton, 1687]).

The relationist position sustains that there are only spatial relations between bodies, and consequently, space is not more than a construction that allows to present such relations, easily connecting with our intuitions. Piaget [Piaget, 1999, Ch. 1] observed that basic notions of space emerge during the first month of childhood simultaneously with the notion of object and the discovery of "permanence" (and then change) as well as groups of operations. The relationist position was presented by Leibniz in his exchange with Clarke, who defended Newton's view. The centre of the exchange between Clarke and Leibniz is precisely what goes from Definition 1.1 to Definition 1.2 and the necessity, as well as correctness, of the first one. No dispute emerges regarding the second notion. Leibniz initiates the discussion by objecting

> Newton says that space is an organ— · like a sense-organ · — by which God senses things. But if God needs an organ to sense things by, it follows that they don't depend entirely on him and weren't produced by him.[Leibniz, 2007, first letter]

a theological argument indeed. By the fifth exchange, the objection had evolved into an epistemic one:

> "I answer that indeed motion doesn't depend on being · observed, but it does depend on being · observable. When there is no *observable* change there is no motion—indeed there is no change of any kind. The contrary opinion is based on the assumption of *real absolute space*, and I have conclusive refuted that through the principle of the need for a sufficient reason." [Leibniz, 2007, fifth letter]

Leibniz' positioning is epistemic, and concerns a sort of "hygiene of reason" that can later be found in Peirce:

> A hypothesis is something which looks as if it might be true and were true, and which is capable of verification or refutation by comparison

with facts. The best hypothesis, in the sense of the one most recommending itself to the inquirer, is the one which can be the most readily refuted if it is false. [Peirce, 1994, CP 1.120]

and later

Long before I first classed abduction as an inference it was recognised by logicians that the operation of adopting an explanatory hypothesis –which is just what abduction is– was subject to certain conditions. Namely, the hypothesis cannot be admitted, even as a hypothesis, unless it be supposed that it would account for the facts or some of them. [Peirce, 1994, CP 5.189]

The first notion of absolute space (1.1) does not admit experimental refutations since nothing is logically deduced from it. The totality of Newton's arguments such as the famous "bucket" [2][Barbour, 1982] rests upon the insight on absolute space given by "true motion", Definition 1.2.

Thomson went one step further in the discussion arguing that the second form can be deduced from spatial relations between objects and he named "inertial frames" the systems providing the references that fulfil the requirement. In this form, the "sensorium of God" [Leibniz, 2007] is deemed not only epistemologically improper but unnecessary as well.

A better known line of argumentation was entertained by Mach:

Newton's experiment with the rotating vessel of water simply informs us, that the relative rotation of the water with respect to the sides of the vessel produces *no* noticeable centrifugal forces, but that such forces *are* produced by its relative rotation with respect to the mass of the earth and the other celestial bodies. [Mach, 1919, p. 232]

Later he returns to the point with his famous view with regard to the fixed stars, while confronting Streintz about Newton's distinction between absolute and relative rotation (that Streintz accepts):

For me, only relative motions exist [...], and I can see, in this regard, no distinction between rotation and translation. When a body moves relatively to the fixed stars, centrifugal forces are produced; when it moves relatively to some different body, and not relatively to the fixed stars, no centrifugal forces are produced. I have no objection to calling the first rotation "absolute" rotation, if it be remembered that nothing is meant by such a designation except *relative rotation with respect to the fixed stars*. [Mach, 1919, p. 543]

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[2]Search for "vessel" in [Newton, 1687]

It has been already indicated that Mach in this respect misses the point [Borzeszkowski and Wahsner, 1995]. Let us explain the matter with a simple example that may be reproduced at home. Consider first the Olympic sport of "hammer throw". To make the throw the athlete makes the hammer rotate around a vertical axis that goes approximately from his feet to a point in the line from his body to the hammer. The hammer can be said to rotate around the vertical axis and the athlete rotates around the same axis. To play the sport strong muscles are needed to pull from the hammer producing the internal force needed to rotate. Newton would call this absolute rotation because force and motion are in correspondence. Let the hammer rest on the floor and spin around your feet. Observe how all objects around appear to rotate (in a geometrical sense) relative to you. Despite some of the objects being far more massive than the hammer, they rotate almost effortless. Newton calls this apparent or relative rotation as it lacks correspondence between force and geometrical description. Since reality cannot be reduced to geometrical appearance we have to distinguish both situations. As indicated in [Borzeszkowski and Wahsner, 1995], Mach's difficulties are rooted in his philosophical standing as empiricist.[3] It is interesting to recall that Newton had rejected the idea of the absolute space being in correspondence with fixed stars.

> ... For it may be that there is no body really at rest, to which the places and motions of other may be referred.

> But we may distinguish rest and motion, absolute and relative, one from the other by their properties, causes and effects. It is a property of rest, that bodies really at rest do rest in respect to one another. And therefore, as it is possible that in remote regions of the fixed stars, or perhaps far beyond them, there may be some body absolutely at rest; but impossible to know, from the position of bodies to one another in our regions whether any of these do keep the same position to that remote body; it follows that absolute rest cannot be determined from the position of bodies in our regions. [Newton, 1687]

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[3]The idea that natural science rests on experience is not controversial. What is controversial is the form in which the observed (or sensed) becomes an idea, the process of ideation. Most empiricists consider the observed as the fact and the fact as the real, thus effectively suppressing the subject, they become realists (objective) by the unconscious action of ignoring the process of production of ideas. For other philosophers such as Peirce [CP 5.145 1994], there is an ideated reality and an observable reality; they bear the relation that exists between universal and particular. Experience is experience of the particular (uniquely situated in time and space an in several other forms) and cannot be used in any other case. This particular experience fosters and tests explanatory ideas, conjectures; by the processes of abduction experiences nourish the real (ideated) which is always transitory, a temporary belief. By the process of interpretation and contrast (induction in the words of Peirce) we control the validity of our beliefs. For Peirce then, the process of abduction is and inexcusable part of science.McAuliffe [2015]

It is important to understand at this point that Newton's and Mach's views have a point in common: they want to resolve the problem of inertia by introducing a unique, universal, conjectured reference system be it explicitly metaphysical or not. In contrast, Thomson will depart from this view by considering that references are to be found within what is experimentally at disposition which is indeed the actual form in which we use Newton's laws to explain observed phenomena. Think for example of a table-top experiment demonstrating 2d-collisions. The following demonstration was presented by Prof. Miguel Ángel Virasoro [4] to his students: the table was made of well polished marble and was well levelled; the carts consisted of a cylindrical base made of steel with a recipient above it where frozen CO_2 was placed. The CO_2 gas (produced by sublimation) flew out of the recipient between the base and the marble, lifting the cart just above the table reducing friction substantially. The reference system provided by the table may be (approximately) inertial or not. In addition to this real demonstration we have to imagine the possibility of giving the table a rotatory motion. Being almost decoupled from the carts, the rotation of the table does not change the interactions between carts. Yet, it changes the description of the trajectories. We are in the presence of apparent rotation. To confuse matters more, the room where the table is could be rotating as well, so that the inertial system would be given by a table rotating with respect to the room. But yet, in any case, in the idealised friction-less situation, there will always exist an inertial system which is not deductible from appearances.[5]

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[4]https://en.wikipedia.org/wiki/Miguel_Ángel_Virasoro_(physicist)

[5]The position of Mach as relationist cannot be distinguished from that of Neumann, the Alpha Body of Neumann being the distant stars of Mach. This is most evident when Mach discusses inertial mass and he states

> *Definition of equal masses*: All those bodies are bodies of equal mass, which, mutually acting on each other, produce in each other equal and opposite accelerations.[Mach, 1919, p.218]

Unless we define a reference system, there is no such thing as an acceleration for *each* body. Considering two bodies, only the relative distance makes sense, and how the relative acceleration is distributed among the bodies is not known. By adequately choosing the acceleration of the frame of reference we can make the allocation the way it pleases us. Thus, the definition does not make sense unless we specify the reference system as the "distant stars" or we explicit are speaking of "true motion" in Newton's sense. Inertial systems are discussed in Mach in the appendix, p.545, while commenting work of Lange that appeared after the first edition of Mach's book. Thus, despite his claims regarding being a relationist, essential parts of Mach's thoughts are based in a secular version of Newton's absolute space. It is interesting to realise that the context of the discussion are the various attempts to reformulate Newton's mechanics such as Hertz and Walley [1899], Streintz [1883] and Lange [1886] (see [Pfister, 2014] as well) trying in one way or another to dispose of the metaphysical absolute space.

Thomson opens his paper with:

> There is no distinction known to men among states of existence of
> a body which can give reason for any state being regarded as a state
> of absolute rest in space, and any other being regarded as a state of
> uniform motion... the only motion of a point that men can know of
> or can deal with is motion relative to one, two, three bodies or more
> other points. [Thomson, 1884, p. 568-569]

For Thomson, and later for Lange, it is clear that the notion of space must be sought in the relation between bodies ("...qualities or distinctions of motions of one or more bodies can be ascertained through knowable relations between these motions and the motions of one or more other bodies..." [Thomson, 1884, p. 572]). The reference of motion corresponds to the concept of "free bodies", he states:

> The only motion of a point that men can know of or can deal with
> is motion relative to one, two, three, or more other points [...] Any
> arrangement whatever of points, lines, or planes, changeless in mu-
> tual configuration, will, for present purposes, be named as a reference
> frame, or briefly as a *frame*.[Thomson, 1884, p.569-560]

In the bi-dimensional space provided by the marble-table of the suggested demon-stration, the bodies move freely as long as they do not collide with each other (or with the contention rails that prevent them from falling). The motion of such bod-ies, relative to each other will give the references for any other motion. It is then wise to adjust the table so that the free bodies move in straight lines at constant speed with respect to it. In such a form, the table has acquired the same property than free bodies and we can use it as a reference frame in Thomson's view.

A free body is certainly a concept, an idea, that emerges by the process of idealisation introduced by Galilei [1914, day 4]. We can say that:

Definition 1.3. Approximately free*: A body is said to be approximately free when for a given purpose its interactions with other bodies can be ignored within the established level of tolerance.*

There are several instances in which a body can be considered approximately free because its characteristics (such as charge, mass, volume and others), its rel-ative position to other bodies, and the lapse of time that is relevant for our purpose, allow us to expect no appreciable influence from other bodies on it or, what it is for practical purposes the same, influences are cancelled (as it is usually the case for electrical influences between statistically neutral bodies).

Definition 1.4. Free body*: A free body is the idealised version of an approximately free body.*

Thus, free bodies exist only in our mind and, at the same time, approximately free bodies are not too difficult to find if we exercise some due tolerance. Further, Thomson proposed that inertial reference frames can be introduced by considering the motion of bodies with respect to a reference point and directions derived from the relative motion of free bodies.

In [Solari and Natiello, 2018], notions of space and time were developed elaborating from the concepts of *me* vs. *other* and *permanence* vs. *change*. Starting from the subjective intuitive notion of space we proposed the existence of *inertial frames*, and a relational formulation of Newtonian mechanics implying that absolute space (in the sense objected by Leibniz) was not fundamental to Newton's theory. The development in [Solari and Natiello, 2018] rests on the No Arbitrariness Principle (NAP), namely that *no knowledge of Nature depends on arbitrary decisions.* In other words, in any description of Nature arbitrariness is either absent or "controllable", in the sense that different arbitrary representations are connected by a *group* of transformations.

In the present work we connect, in a mathematical form, the notion of a relational space with (a) the intuitions developed during the early childhood [Piaget, 1999], (b) the notion of intrinsic reference systems as developed by Thomson and (c) the No Arbitrariness Principle. As a result, we show that the class of reference frames that preserve the objective relational dynamics includes non-rotating "accelerated" systems and that only a subclass of them consists of Thomson's inertial frames.Neither the fixed stars or the absolute space of Definition 1.1are required (or relevant) to set the grounds of Newtonian mechanics. The historical approach is then reverted, from "free bodies move in straight uniform motion with respect to an inertial frame" into "inertial frames are those reference systems where free bodies are described as moving in straight uniform motion". Since the idea of "free, non-interacting, body" admits approximations, inertial systems admit approximations as well. Actually, the intuitive use of inertial frames corresponds to this later notion.

In the next Section we start by developing the notion of space from a Piagetian perspective, then connecting it with the Cartesian concept of coordinates. Since Descartes presentation assumes the existence of an external observer, we identify the relational content of the notion of space and finally develop the theory of relatively inertial frames, and its connection to Newtonian Mechanics. We discuss the achievements in the final Section.

2   The relational notion of space

Piaget describes the construction of the notion of space, objects and spatial relations in the child:

> To understand how the budding intelligence constructs the external world, we must first ask whether the child, in its first months of life, conceives and perceives things as we do, as objects that have substance, that are permanent and of constant dimensions. If this is not the case, it is then necessary to explain how the idea of an object (object concept) is built up. The problem is closely connected with that of space. A world without objects would not present the character of spatial homogeneity and of coherence in displacements that marks our universe. Inversely, the absence of " groups" in the changes of position would be equivalent to endless transformations, that is, continuous changes of states in the absence of any permanent object. In this first chapter, then, substance and space should be considered simultaneously, and it is only through abstraction that we shall limit ourselves to object concept.Piaget [1999, p. 3]

> The conclusion to which the analysis of object concept has led us is that in the course of his first twelve to eighteen months the child proceeds from a sort of initial practical solipsism to the construction of a universe which includes himself as an element. At first the object is nothing more, in effect, than the sensory image at the disposal of acts; it merely extends the activity of the subject and, without being conceived as created by the action itself (since the subject knows nothing of himself at this level of his perception of the world), it is only felt and perceived as linked with the most immediate and subjective data of sensorimotor activity. During the first months the object does not, therefore, exist apart from the action, and the action alone confers upon it the quality of constancy. At the other extreme, on the contrary, the object is envisaged as a permanent substance independent of the activity of the self, which the action rediscovers provided it submits to certain external laws. Furthermore, the subject no longer occupies the center of the world, a center all the more limited because the child is unaware of this perspective; he places himself as an object among other objects and so becomes an integral part of the universe he has constructed by freeing himself of his personal perspective. Piaget [1999, p. 97]

To address the notion of velocity, first we need to address the notion of spatial relations. Take e.g., a look at a garden. Leave aside (project out) the moving

leaves and birds, and consider those elements that impress us as keeping a constant relation among themselves (stones, bushes, etc.). We seek for a universal instruction to move around the garden. We then identify the elements with labels, $i \in [1 \dots N]$ and summarise the instruction of "going from i to j" as x_{ij}. Any moving instruction can be given as a concatenation of moving instructions, this is the most essential condition of spatial relations. We use the symbol $\oplus$ to denote concatenation. x_{jj} denotes the instruction for remaining at the locus of j, or just "do nothing". We call x_{jj} the *neutral* element, $0 \equiv x_{jj}$. We then realise that $x_{ij} = x_{ij} \oplus x_{jj}$, and further

$$
\begin{aligned}
x_{ij} \oplus x_{jk} &= x_{ik} \\
x_{ij} \oplus x_{jj} &= x_{ij} \\
x_{jj} &\equiv 0 \\
x_{ij} &= \ominus x_{ji}
\end{aligned}
\tag{1}
$$

The last line in (1) express the perceived fact that the outcome of staying in one place in the end is the same as the concatenation of going from that place to any other and returning. Thus, *returning* becomes the inverse operation of *going*: $x_{ij} = \ominus x_{ji}$.

We need now to introduce velocities, and hence, we first need to introduce change, or its abstract form, time (that time is the abstract form of change was already known to Aristotle [Aristotle, 1994–2010]). Our observations may indicate/suggest that the organisation of the garden is not always the same, perhaps because we want to explain where a bird is feeding in the garden. We have then decided that there are things that, for our purposes, are permanent (do not change) such as trunks and stones as well as changing objects, e.g., the position of birds. "Going to bird 1" is not the same type of instruction as "going to tree 1". The birds cannot be located with "old instructions", the location instructions have to be updated as a function of other perceived changes. Each observer may have her/his own clock, so we write t_S to record the changes perceived by S[6], and understand that all the instructions in Eq. 1 were given for a determined time, namely:

$$
\begin{aligned}
x_{ij}(t_S) \oplus x_{jk}(t_S) &= x_{ik}(t_S) \\
x_{ij}(t_S) \oplus x_{jj}(t_S) &= x_{ij}(t_S) \\
x_{jj}(t_S) &\equiv 0 \\
x_{ij}(t_S) &= \ominus x_{ji}(t_S)
\end{aligned}
$$

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[6]Notice that clocks in general come from outside the phenomena in study, they are the remaining part of the universe from which our mind is isolating the observed. The perception that some events are recurrent forces us to avoid arbitrary decisions by assigning to the intervals between consecutive events the same duration. In turn, each interval can be divided using other series of events (more frequent) equally perceived as having the same essence, thus refining the division of time. The repetition and idealisation of the process gives us the time.

Since we have an intuition of regular processes, we may agree on the functioning of reference clocks and hence consider time-records as real numbers with the usual sum operation. We finally consider the relation between the rate of change for our (reference) clock and the rate of change of relative positions,

$$v_{ij}(t_S, \delta) \equiv \frac{x_{ij}(t_S + \delta) \ominus x_{ij}(t_S)}{(t_S + \delta) - t_S}$$

It follows that

$$v_{ij}(t_S, \delta) \oplus v_{jk}(t_S, \delta) = v_{ik}(t_S, \delta) \qquad (2)$$

This is, whatever the clock is, the composition law between velocities must be the same that the composition law of the space.

2.1 Descartes' mathematisation of space

The Cartesian view is always the view of an observer, the view that matches our intuitive construction, namely an extrinsic view. In Descartes' method, directions and distances are used instead of giving instructions to move around the garden based upon landmarks. Thus, the instruction that was "walk from i to j, x_{ij}" becomes "from i walk x steps in the direction $\hat{e}_{ij}$ to j", that we annotate $x_{ij} = \hat{e}_{ij}x$. If we now agree to consider only the path from a given reference position (the position of "ego"), all paths consist in concatenations of this kind of instructions. Our intuition tell us more, it tells us that there are only three independent directions, at least as much as we can perceive. Therefore, the space is three dimensional and the mathematical construct is Cartesian space, represented by $\mathbb{R}^3$, while $\oplus$ is the ordinary vector sum. Cartesian space not only inherits the rules developed for the concatenation of instructions, but adds new rules based on intuition, such as $x\hat{e} + x'\hat{e}' = x'\hat{e}' + x\hat{e}$ (addition is commutative), as well as the other rules of vector algebra. This idea is underlying the operational part of Newton's concepts of absolute space and true motion, seeking for a reference –somehow external to the process in study– from which changes in motion correspond to forces.

2.2 Subjective and relational spaces

A central issue in Classical Mechanics has been that of formulating mechanics in a fully relational way, this is, not only suppressing the absolute space but suppressing a Cartesian view based upon a privileged reference system as well. Absolute space can be seen as the view of a privileged observer: God. Nothing is gained by changing the name of the privileged observer. Already with Leibniz the alternative idea of a relational space arose, i.e., a space free of the arbitrariness

of an extrinsic reference. How do we construct an intrinsic view (i.e., without external observers)?

Individual subjective spaces contain the arbitrariness of the choice of origin and the choice of references, but what is *real* presents "characters that are entirely independent of our opinions about them" [Peirce, 1955, p. 18]. In this sense, the Cartesian space is not *real*. In contrast, relational constructions as those we used to introduce this discussion, like x_{ij}, stand their chance of being real. Yet, the Cartesian view is not completely arbitrary because all arbitrary spaces that we can produce map into each other in a one-to-one form. Thus, the observations in one space only need to be translated into the observations in another space (characterised by different arbitrariness). We say that the descriptions are *intersubjective*. When the differences between subjective spaces correspond to arbitrary elections that influence the description in a systematic form, as it is the case of the choice of origin and the choice of directions of reference, the set of transformations relating the different descriptions must satisfy conditions of consistency that allow us to move in the set of arbitrary descriptions without contradictions. This is the core meaning of the No Arbitrariness Principle (NAP) [Solari and Natiello, 2018], in short: the set of transformations associated to arbitrary decisions must form a *group*. Actually, considering it in finer detail, there is a group associated with each class of equivalent arbitrary decisions, this is: a group for the election of reference point, a group for the election of reference directions, and so on.

We begin by considering one body alone in a 3-dimensional, universe. For such a body, relative space makes no sense at all. There is nothing else available to consider (such as e.g., relative positions), apart from the body. When we consider two bodies, only a one dimensional universe is conceivable. The distance between the two bodies is the only possibility for geometric change. When we consider at least three bodies, a distinct difference arises. Arbitrariness in the representation corresponds to the choice of different orientations and the location of one point in the system. If the Cartesian space for N bodies corresponds to R^{3N} and the group of transformations between arbitrary representations (after restricting the choice of directions to orthogonal directions) is $\mathbb{E}(3) = \mathbb{ISO}(3) = \mathbb{SO}(3) \ltimes \mathbb{R}^3$ (in words: $\mathbb{E}(3)$ equals the semi-direct product of its subgroups $\mathbb{SO}(3)$ and $\mathbb{R}^3$ - normal subgroup-, hence $\mathbb{SO}(3) \cong \mathbb{E}(3)/\mathbb{R}^3$). The two component groups correspond to a global orientation and the position of one point. The real space is what results of modding out the arbitrariness. This means: a point for $N = 1$, the positive line for $N = 2$. For $N \geq 3$ it acquires the characteristics we intuitively assign to the relational space by removing from the subjective space R^{3N} e.g., a global orientation and a distinguishable point.

2.3   Relatively inertial frames

The goal of this Section is to define a reference frame internal to a set of N bodies moving without (relative) interactions, i.e., N free bodies.

Lemma 2.1. *The law of motion for the relative distance x_{ik} among any pair of free bodies is*

$$\frac{dx_{ik}}{dt} = C_{ik},$$

for some constants C_{ik}.

Proof.   According to NAP, the law of motion for the relative position, x_{ik}, must be independent of the existence of other bodies since they cannot influence the motion (otherwise, the pair is not free or it is influenced by arbitrary decisions regarding the other bodies). Hence,

$$\frac{dx_{ik}}{dt} = f(x_{ik}) = f(x_{ij}) + f(x_{jk}) = f(x_{ij} + x_{jk}).$$

Therefore, f must be an affine transformation, $f(x_{ik}) = C_{ik} + Ax_{ik}$, with C_{ik}, A constants. We have the additional result that $C_{ik} = C_{ij} + C_{jk}$, a principle of addition of the velocities of free bodies. In addition, since the law must be the same for all times (there is no privileged time), C_{ik}, A must be constant. Finally, $A = 0$ since otherwise the bodies would not be independent of each other, being their evolution affected by their relative distance.□

Let $i, j \in (1, 2, 3)$. We have the relations x_{ij} (oriented distances) and we can consider a large number of different vectors, e.g. the set $\{x_{ij}, dx_{ij} = x_{ij}(t_S + \delta) - x_{ij}(t_S)\}$. In a three dimensional space those vectors are not completely arbitrary: some internal relations will become explicit. Three bodies define a plane, which along with the relative velocity vectors 1.3 to define a reference frame, except under singular (full coplanarity or zero velocity) circumstances. We are now in the position to consider *relatively inertial* systems.

We set

$$\hat{e}_{ij} = \begin{cases} \frac{x_{ij}}{|x_{ij}|} & if \ \frac{d}{dt_S}x_{ij} = 0 \\ \frac{v_{ij}}{|v_{ij}|} & if \ \frac{d}{dt_S}x_{ij} \equiv v_{ij} \neq 0 \end{cases} \tag{3}$$

where $v_{ij} = \lim_{\delta \to 0} v_{ij}(t_S, \delta)$. The use of a limit is not strictly necessary, since we will deal with constant v_{ij}. It is enough to ask for v_{ij} being independent of δ.

In terms of an arbitrary frame, the conditions $dx_{ij} = 0$ and $dv_{ij} = 0$ are ultimately a perception of the observer and as such they introduce subjectivity in the description. It is this decision made by the subject what creates the space that can be mathematically represented. Eq. 3 removes much of the subjectivity leaving

only the arbitrary origin of the subjective frame, which is not involved in the description of relative positions. Thus, the observer might be linearly accelerated but the description of relative motion will remain unchanged.

Definition 2.1. *Two bodies are* relatively inertial *if there exists a reference frame, a constant vector a, and a scalar b such that $\dfrac{db}{dt_S}$ is constant and*

$$
\begin{aligned}
\hat{e}_{ij} \times (x_{ij} \times \hat{e}_{ij}) &= a \\
\hat{e}_{ij} \cdot x_{ij} &= b \\
\frac{d}{dt_S} \hat{e}_{ij} &= 0
\end{aligned}
$$

where

$$
x_{ij} = a + b\hat{e}_{ij}
$$

Definition 2.2. $N \geq 3$ *bodies are relatively inertial if there exists a reference frame and a sequential order $1, \cdots, N$ such that body k is relatively inertial to body $k + 1$. We call this frame a* relatively inertial frame *for the N bodies.*

Corolary 2.1. *Free bodies are relatively inertial bodies.*

Proof. This is a consequence of Definition 1.4, Definition 2.2 and Lemma 2.1. □

Lemma 2.2. *Relatively inertial is an equivalence relation i.e., a relation $\sim$ such that for three bodies A, B, C it holds that $A{\sim}A$, $A{\sim}B \Rightarrow B{\sim}A$, $A{\sim}B$ and $B{\sim}C \Rightarrow A{\sim}C$.*

Proof. $A \sim A$ since whenever $x_{ii} = 0$ and for arbitrary $\hat{e}$ we have $a = 0$ and $b = 0$ in Definition 2.1. If a pair a, b exists such that $A \sim B$ it follows that $B \sim A$ with the associated pair $-a, b$. Since there is a common reference frame for the whole chain of relatively inertial bodies, the third relation follows from vector addition rules. We have

$$
\begin{aligned}
x_{AB} &= a + b\hat{e}_{AB} \\
x_{BC} &= a' + b'\hat{e}_{BC} \\
x_{AC} &= x_{AB} + x_{BC} \\
&= a + a' + b\hat{e}_{AB} + b'\hat{e}_{BC}
\end{aligned}
$$

Further, $b\hat{e}_{AB} + b'\hat{e}_{BC} = c + t_S \left(\frac{db}{dt}\hat{e}_{AB} + \frac{db'}{dt}\hat{e}_{BC} \right)$, where c is some constant vector and the constant quantity in parenthesis is either zero or some other constant nonzero vector. In the latter case, letting $\lambda = ||\frac{db}{dt}\hat{e}_{AB} + \frac{db'}{dt}\hat{e}_{BC}||$ and $\hat{e}_{AC} = \dfrac{1}{\lambda} \left(\dfrac{db}{dt_S}\hat{e}_{AB} + \dfrac{db'}{dt}\hat{e}_{BC} \right)$ we get $x_{AC} = (a + a' + c) + \lambda t_S \hat{e}_{AC}$. □

Lemma 2.3. *The relatively inertial frames referring to $N \geq 3$ relatively inertial bodies have as group of arbitrariness the translations (which may be time dependent) and the (time independent) rotations composed as a semi-direct product group.*

Proof. The instantaneous relative position of N bodies is invariant under $\mathbb{ISO}(3)$ as explained in Subsection 2.2. With relative positions, the arbitrariness of the origin of coordinates cancels out, even when it changes as a function of time. In contrast, a change in the arbitrary choice of orthogonal directions of reference as a function of time will make $\frac{d}{dt_S}\hat{e}_{ij} \neq 0$ in the definition 2.1, hence it will break the concept of relatively inertial set. $\square$ We observe that in the cited paragraphs of Thomson, the directions of reference in space correspond to the relative motion of free bodies. Relatively inertial frames correspond to the situation presented in Newton [1687], Collorary VI of the chapter Axioms. The introduction of relatively inertial sets of bodies deserves a detailed discussion. In subsection 2.2 we introduced the instantaneous space for the description of the relational problem of N bodies. Starting from subjective space, R^{3N}, we arrived to a relational space where the orbits of points by the action of the group $\mathbb{E}(3) = \mathbb{ISO}(3)$ were identified, this is: $R^{3N}/\mathbb{E}(3)$. The description of the evolution in time in the relational space is then represented by a function of time, R, into $R^{3N}/\mathbb{E}(3)$. Thus, each trajectory is given by a function $F(t) : R \mapsto \left(R^{3N}/\mathbb{E}(3)\right)$ and the set of functions will be called $\mathcal{F}$. Definition 2.1 uses relative positions, x_{ij}, which are invariant under changes of the origin of the (subjective) coordinates, hence, these mathematical objects are invariant under the action of $\mathbb{R}(3)$, the group of translations, and only rotations in $\mathbb{E}(3)$ may change them. From the original group of arbitrariness, $\mathbb{E}(3)$, we are left with the effective action of $\mathbb{E}(3)/\mathbb{R}(3) \sim \mathbb{SO}(3)$ (a result that can be intuited as well). Since we have to make such a choice for every time when considering a trajectory, the group of arbitrariness associated to the set $\{x_{ij}(t)\}$ consists of (continuous and twice differentiable) time-dependent rotations. Continuity and differentiability is requested because we have to deal with velocities in the definition. Because the translation of the subjective origin of coordinates does not intervene in the definition of the inertial set and inertial frame, we can allow any arbitrariness to such a collective translation. Note that the arbitrariness of the description encompasses $\mathcal{F}$ and time-dependent rotations, while the class of relatively inertial frames related to a relatively inertial set of bodies only contains time-independent rotations, because of Lemma 2.3.

Regarded from any arbitrary relatively inertial frame, the N relatively inertial bodies belonging to the relatively inertial system may display any possible type of trajectory $x_i(t)$ as indicated above. However, for any pair of bodies we have that $x_i(t) - x_j(t) = v_{ij}t + x_{ij}(0)$, with constant v_{ij}, when the reference frame is chosen with orientations as indicated in definition 2.1. Picking a reference frame that is

also relatively inertial to the N bodies, the standard description of the bodies from an inertial frame [ch. 1, Goldstein, 1980] is recovered. Indeed, the N bodies are then described as having coordinates $x_i(t) = x_i(0) + v_i t$. This view is equivalent to augmenting the set of bodies in one, adding an extra "body", representing the origin of coordinates. The transformations among inertial frames in the standard setting, the Galilean group, arise as a consequence of this choice (it corresponds to picking different extra bodies relatively inertial to the set of N, having different constant relative velocity).

Definition 2.3. Inertial frame. *Inertial frames are the relatively inertial frames associated to free bodies.*

Corolary 2.2. *Newton's laws hold only in inertial frames.*

It is possible for a set of bodies to be relatively inertial without being free, hence two separate sets of bodies can be each relatively inertial and yet no inertial frame may be available for all the bodies to pertain to a relatively inertial set.

Remark 2.1. *The "added value" of the concept of relatively inertial is twofold. On one side, the concept is intrinsic to a system and hence independent of the system being e.g., accelerated relative to some external reference. Moreover, it is fully compatible with classical, Newtonian mechanics, thus eliminating a metaphysical ingredient discussed for centuries, now without any reference to e.g., "distant stars". In an inertial system, any relative motion that deviates from constant velocity corresponds with forces/interactions, i.e., to bodies that are not free. However, the statement is not true if we replace "inertial system" by "relatively inertial system"*[7]*.*

Remark 2.2. *The relatively inertial class is larger that the inertial class associated to the inertial frames 2.3. It contains any kind of global time-dependent displacement of the frame and the objects under study and, in particular, accelerated systems as those considered in Einstein's equivalence principle (stating the complete physical equivalence of a homogeneous gravitational field and a corresponding acceleration of the reference system [Einstein, 1907]). Regarded in this way, the equivalence principle is not an independent principle but a particular case of the more general NAP and the presently derived concept of relatively inertial.*

Remark 2.3. *The concept of force in Classical Mechanics has been gradually "naturalised" by habit (in the sense that forces became treated as observables, rather than as meta-observables –ideated–). Consequently, an inertial-mass has*

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[7]In [Solari and Natiello, 2018] it is assumed explicitly that the reference corresponds to a free body.

followed, relating force and the actually observable acceleration. However, a detailed construction of mechanics [Solari and Natiello, 2018] shows that the concept of inertial mass rests completely on the concept of gravitational mass. Therefore, it makes no sense to distinguish them. Hence, the "weak equivalence principle" [Dicke, 1981] (stating that inertial mass equals gravitational mass) is a social construction rather than a basic ingredient of mechanics.

3   Discussion and conclusions

It is worth to discuss the implications of Definition 1.3. In practice, we always deal with approximate notions, be it of free bodies, relatively inertial bodies or inertial frames. The possibility of interacting or not with other objects/bodies is always considered from a limited perspective and it is subject to revision whenever necessary. While perfection is probably unattainable outside the mathematical universe, the pair of concepts *approximation* and *tolerance* are quantifiable and admit the possibility of refinement and improvement.

We have shown that there are two different conceptions of relational space, one that preserves the intuition of the subjective space and constitutes a secular version of absolute space resting upon hypothetical fixed stars, and a second one, almost forgotten, that rests upon the concept of inertial frames, free bodies and internal relations. Such distinction parallels the notion of Absolute space (Definition 1.1) in Newton and that of absolute motion (Definition 1.2) which is the operative concept in Newton's *Principia*. The first one can be called metaphysical as well as it is not true that for any practical use we identify inertial systems by referring them to the fixed stars or any other equivalent primary reference as Neumann's Alpha Body or the distribution of masses of a –necessarily assumed– finite universe. The second version is of practical use but does not support the simple intuition as crystallised in the Cartesian representation of space, but rather requires a philosophical intuition Husserl [1983] or equivalently, to acknowledge the process of abduction in general and the production of reality by the building intelligence of the child.

Leibniz' relationism is the result of a discipline of mind based on the "Principle Sufficient Reason" and the "Principle of Identity of Indiscernibles" which relate deeply with the "No Arbitrariness Principle" and the process of modding out arbitrariness (subjective views) applied in the present construction. In contrast, providing an arbitrary reference for the space, such as fixed stars, violates NAP, and as such accepts Leibniz' conclusions regarding the space but disdains Leibniz' main rational principles.

In terms of classical mechanics we have shown that it is possible to introduce inertial frames and sets of inertial bodies without introducing an Absolute Space

(or universal references of no actual use) complementing the work in [Solari and Natiello, 2018]. We have also shown that the new approach includes a larger variety of reference systems.

In the view of the present work, Newton's definition 1.1 of absolute space plays only a metaphysical role. There is no way of working out from absolute space the meaning of true motion (definition 1.2) and there is no way to put absolute space to test, as Newton explains in the Scholium. We suggest that its need must be considered in psychological terms as it relates to initial perceptions of the child in the construction of the notion of space. Absolute space can be safely replaced with a secular relational space such as the one produced by taking the "fixed stars" as alleged reference, since we are only changing the useless metaphysics. Why should free bodies be in uniform motion in the frame of reference given by the fixed stars? Yet, the metaphysical discussion has entertained most philosophers [Barbour, 1982].

In contrast, the notion of true motion (definition 1.2), closely linked to the principle of inertia, can be inferred from observations and the abducted result has stood firmly all falsification attempts. It stands as a true belief, one that "shapes our actions" [Peirce, 1994, CP 5.371], on its own right. In this work we have shown how inertial frames follow logically from it. Since inertial frames correspond to the setting where Newton's axioms hold true, hence all of Newton's mechanics rests on the notion of true motion.

In achieving our result we rested, much as Leibniz did, on a sanitising concept, in our case the "No Arbitrariness Principle" (NAP) and its mathematical formulation. While NAP operates before Newton's laws of mechanics, a residual action is left. Since there is not a unique inertial system, there should be a group of operations transforming statements in one inertial system into statements in another one, while the laws of mechanics remain the same. Thus, this relativity principle is not an independent construct somehow unique to physics but only a consequence of NAP, having to account for a residual arbitrariness.

4   Acknowledgements

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