Operator ordering for generally covariant systems

Rafael Ferraro[a,b*] and Daniel M. Sforza[b†]

[a]Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina

[b]Instituto de Astronomía y Física del Espacio, Casilla de Correo 67 - Sucursal 28, 1428 Buenos Aires, Argentina

The constraint operators belonging to a generally covariant system are found out within the framework of the BRST formalism. The result embraces quadratic Hamiltonian constraints whose potential can be factorized as a never null function times a gauge invariant function. The building of the inner product between physical states is analyzed for systems featuring either intrinsic or extrinsic time.

An essential aspect of a generally covariant system is the invariance of its action under reparametrizations; this means that the label that parametrizes the trajectories of the system is not the time but a physically irrelevant parameter. As a consequence, the system is constrained to remain on the hypersurface of the phase space where the Hamiltonian is null. In fact, since the "evolution" generated by the Hamiltonian can be regarded as a reparametrization of the classical trajectory, then the Hamiltonian behaves like a generator of a gauge transformation of the system; so the Hamiltonian is a first class constraint. Besides the Hamiltonian constraint $\mathcal{H}_o$ associated with the reparametrization invariance, the system can exhibit additional gauge invariance generated by first class constraints $\mathcal{H}_a$ linear and homogeneous in the momenta, telling that some canonical variables are not genuine degrees of freedom but mere spurious variables devoid of physical meaning. The *observables* are not sensitive to the values of these spurious variables, nor to the choice of the parametrization. The super-Hamiltonian and super-momenta constraints of General Relativity are an example of such a set of first class constraints.[1,2]

According to Dirac's method, the gauge invariance is preserved at the quantum level by including in the Hilbert space only those states that are annihilated by the constraint operators (*physical states*):

$$\hat{\mathcal{H}}_o\psi = 0, \qquad \hat{\mathcal{H}}_a\psi = 0, \tag{1}$$

and a linear operator must be inserted in the inner product to kill the integrations on the spurious degrees of freedom (gauge fixing) .[3]

In order that the prescription (1) be consistent with the algebra of first class constraints, a proper operator ordering should be found out such that the constraint algebra is realized at the quantum level in the following way:

$$[\hat{\mathcal{H}}_\alpha, \hat{\mathcal{H}}_\beta] = \hat{C}^\gamma_{\alpha\beta}(q^i, \hat{p}_j)\hat{\mathcal{H}}_\gamma \tag{2}$$

where α, β stand for both o and a.

Such an operator ordering can be found out by raising the *BRST generator* Ω to the status of a hermitian and nilpotent operator. The BRST generator is a fermionic magnitude defined in a phase space extended by the addition of pair of canonically conjugated variables $(\eta^\alpha, \mathcal{P}_\alpha)$ (*ghosts*) for each first class constraint. Ghosts have parity opposed to the one of the respective constraint. $\Omega(q, p, \eta, \mathcal{P})$ is defined by the conditions [3]

$$\{\Omega, \Omega\} = 0, \qquad \Omega = \eta^\alpha \mathcal{H}_\alpha + more \tag{3}$$

(the Poisson bracket is symmetric for fermionic quantities). In Eq. (3) "more" means terms of higher order in the ghosts. Ω is a conserved charge of the extended system, and generates a

*Electronic address: ferraro@iafe.uba.ar
†Electronic address: sforza@iafe.uba.ar

global (rigid) symmetry. A hermitian and nilpotent realization of Ω captures the structure of the algebra (2) in the equation

$$0 = [\hat{\Omega}, \hat{\Omega}] = 2\hat{\Omega}^2 \tag{4}$$

So, well ordered Dirac operator constraints can be identified from the form of $\hat{\Omega}$. Also Dirac physical states can be mapped into a cohomological class of physical states ($\hat{\Omega}\Psi = 0$) of the nilpotent operator $\hat{\Omega}$. The BRST extended system contains both classical and quantum behavior of the constrained system.

We will consider the following system:

$$\mathcal{H}_o = \frac{1}{2} g^{ij}(q) p_i p_j + v, \qquad \mathcal{H}_a = \xi_a^j(q) p_j, \tag{5}$$

where v is a gauge invariant potential (invariant under transformations generated by the supermomenta), and g^{ij} is an indefinite non-degenerate metric. The constraint algebra is

$$\{\mathcal{H}_o, \mathcal{H}_a\} = C_{oa}^b(q,p)\mathcal{H}_b = C_{oa}^{bj}(q) p_j \mathcal{H}_b, \tag{6}$$

$$\{\mathcal{H}_a, \mathcal{H}_b\} = C_{ab}^c(q)\mathcal{H}_c. \tag{7}$$

In this case the BRST generator is

$$\Omega = \eta^\alpha \mathcal{H}_\alpha + \frac{1}{2}\eta^\alpha \eta^\beta C_{\alpha\beta}^\gamma \mathcal{P}_\gamma, \tag{8}$$

A hermitian and nilpotent realization of Ω is [4]

$$\hat{\Omega} = \hat{\Omega}^{linear} + \hat{\Omega}^{quad} \tag{9}$$

where

$$\hat{\Omega}^{linear} = f^{\frac{1}{2}} \left[\eta^a \xi_a^i \hat{p}_i + \frac{1}{2}\eta^a \eta^b C_{ab}^c \hat{\mathcal{P}}_c \right] f^{-\frac{1}{2}} \tag{10}$$

$$\begin{aligned} \hat{\Omega}^{quad} = \quad & \eta^o \left(\tfrac{1}{2} f^{-\frac{1}{2}} \hat{p}_i f g^{ij} \hat{p}_j f^{-\frac{1}{2}} + v \right) \\ & + \tfrac{1}{2} f^{-\frac{1}{2}} \hat{p}_i f \eta^o \eta^a C_{oa}^{bi} \hat{\mathcal{P}}_b f^{-\frac{1}{2}} \\ & + \tfrac{1}{2} f^{-\frac{1}{2}} \hat{\mathcal{P}}_a f \eta^o \eta^b C_{ob}^{aj} \hat{p}_j f^{-\frac{1}{2}}, \end{aligned} \tag{11}$$

and f solves the equation

$$C_{ab}^b = f^{-1}(f\xi_a^i)_{,i}. \tag{12}$$

(f is a volume in the gauge orbit of the supermomenta).

In order to read from $\hat{\Omega}$ the constraint operators fulfilling Eq. (2), $\hat{\Omega}$ must be rearranged in $\eta - \mathcal{P}$ order by repeatedly using the ghost (anti)-commutation relations. After this procedure is completed, the classical structure of Eq. (8) will be reproduced at the quantum level [3]

$$\hat{\Omega} = \eta^\alpha \hat{\mathcal{H}}_\alpha + \frac{1}{2}\eta^\alpha \eta^\beta \hat{C}_{\alpha\beta}^\gamma \hat{\mathcal{P}}_\gamma. \tag{13}$$

In our case the result is

$$\hat{\mathcal{H}}_o = f^{\frac{1}{2}} \left[\frac{1}{2} f^{-1} \hat{p}_i f g^{ij} \hat{p}_j + v + \frac{i}{2} C_{oa}^{aj} \hat{p}_j \right] f^{-\frac{1}{2}} \tag{14}$$

$$\hat{\mathcal{H}}_a = f^{\frac{1}{2}} \xi_a^i \hat{p}_i f^{-\frac{1}{2}} \tag{15}$$

$$\hat{C}_{oa}^b = \frac{1}{2}(f^{\frac{1}{2}} C_{oa}^{bj} \hat{p}_j f^{-\frac{1}{2}} + f^{-\frac{1}{2}} \hat{p}_j C_{oa}^{bj} f^{\frac{1}{2}}) \tag{16}$$

$$\hat{C}_{ab}^c = C_{ab}^c \tag{17}$$

Although it was supposed that the potential v is gauge invariant to render simpler the constraint algebra, the results can be generalized to potentials that can be factorized as a gauge invariant function v times a never null function $\vartheta(q)$. This can be accomplished by performing a unitary transformation leading to a different hermitian and nilpotent BRST generator

$$\hat{\Omega} \rightarrow e^{i\hat{G}} \ \hat{\Omega} \ e^{-i\hat{G}}. \tag{18}$$

By choosing

$$\hat{G} = \frac{1}{2}[\hat{\eta}^o \ ln \ \vartheta(q) \ \hat{\mathcal{P}}_o - \hat{\mathcal{P}}_o \ ln \ \vartheta(q) \ \hat{\eta}^o], \tag{19}$$

the operators (14-17) suffer the following changes:

$v \longrightarrow V = \vartheta \ v,$

$g^{ij} \longrightarrow G^{ij} = \vartheta \ g^{ij},$

$f \longrightarrow \vartheta^{-1} \ f,$

$C_{oa}^{bj} \longrightarrow \vartheta \ C_{oa}^{bj},$

and new structure functions appear:

$$C_{oa}^o = \xi_a^i (\ln \vartheta)_{,i} \ . \tag{20}$$

Thus the unitary transformation (18-19) amounts to the scaling of the classical super-Hamiltonian constraint. The scaling of a constraint does not modify the classical dynamics.

The quantum system also remains unchanged provided that the Dirac physical states change according to

$$\varphi \to \varphi' = \vartheta^{-1/2}\ \varphi. \tag{21}$$

In order to define an inner product between Dirac physical states, the spurious variables associated with the super-momenta constraints should be frozen by the ordinary procedure of inserting the Dirac deltas of the gauge fixing functions and the corresponding Fadeev-Popov determinant. However the reparametrization invariance associated with the super-Hamiltonian constraint should be managed in a more specific way. If the potential is positive definite, then the system can be deparametrized as a relativistic particle, where the time is hidden in the configuration space (*intrinsic time*); time essentially is a canonical variable inside the light-cone of the metric g^{ij}. Therefore a variable behaving as *time* (i.e., monotonically increasing on every classical trajectory) should be also frozen together with the rest of the spurious variables.[1]

Things can be not so straightforward in other cases. As an example, let us consider the case where the potential is not definite positive but there exists a time-like vector $\vec{\xi}_o$ such that [5]

$$\mathcal{L}_{\vec{\xi}_o}(|\vec{\xi}_o|^{-2}\ V) = 1 \tag{22}$$

and

$$\mathcal{L}_{\vec{\xi}_o}(|\vec{\xi}_o|^{-2}\ \mathbf{G}) \approx 0 \tag{23}$$

(i.e. $\vec{\xi}_o$ is a Killing vector of the scaled metric $|\vec{\xi}_o|^{-2}$ $\mathbf{G}$ on the constraint surface). Then, one could factorize out the function $\vartheta = |\vec{\xi}_o|^2$ from the super-Hamiltonian to get a simpler although equivalent constraint:

$$\mathcal{L}_{\vec{\xi}_o} v = 1 \tag{24}$$

and

$$\mathcal{L}_{\vec{\xi}_o} \mathbf{g} \approx 0 \tag{25}$$

[1]This is not the case for General Relativity, where the potential is the spatial curvature ${}^3R$

$\sqrt{2}\ \vec{\xi}_o$ is a unitary Killing vector for the scaled metric $\mathbf{g}$. In a coordinate system where the parameter of $\vec{\xi}_o$ is the q^o coordinate, and the rest of the coordinate basis $-\{\partial/\partial q^\mu\}-$ is orthogonal to $\vec{\xi}_o$, the former geometrical properties become

$$\vec{\xi}_o = \frac{\partial}{\partial q^o}\ , \tag{26}$$

$$\frac{\partial v}{\partial q^o} = 1 \tag{27}$$

and

$$\frac{\partial g^{ij}}{\partial q^o} \approx 0\ . \tag{28}$$

Then the super-Hamiltonian looks

$$\mathcal{H}_o = -\frac{1}{2}p_o^2 + \frac{1}{2}g^{\mu\nu}(q^\lambda)p_\mu p_\nu + \mathcal{V}(q^\mu) + q^o \tag{29}$$

where $\mathcal{V}(q^\mu)$ is a gauge invariant potential. One could deparametrize this system by performing the following canonical transformation

$$q^o = p_t, \qquad\qquad p_o = -t \tag{30}$$

Thus the super-Hamiltonian becomes

$$\mathcal{H}_o = p_t + \frac{1}{2}g^{\mu\nu}p_\mu p_\nu + \mathcal{V}(q^\mu) - \frac{1}{2}t^2 \tag{31}$$

This is nothing but the Hamiltonian constraint of a trivially parametrized system. Coming back to the original variables, one realizes that the variable to be frozen in the inner product is not a coordinate but a momenta (*extrinsic time* [6]) [7]. This "gauge fixing" should be done in a non conventional way. However, the transformation (30) in the above examined example teach us that the required insertion in the inner product involves an integral operator performing a Fourier transform [5].

This work was supported by Universidad de Buenos Aires (Proy. TX 64) and Consejo Nacional de Investigaciones Científicas y Técnicas.

REFERENCES

1. K. V. Kuchař, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon, Oxford, 1981).

2. K. V. Kuchař, in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, edited by G. Kunstatter, D. Vincent and J. Williams, World Scientific, Singapore, 1992.
3. M. Henneaux and C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, Princeton, NJ, 1992.
4. R. Ferraro and D. M. Sforza, Phys. Rev. D **55** (1997) 4785.
5. R. Ferraro and D. M. Sforza, Phys. Rev. D **59** (1999) 107503.
6. J.W. York, Phys. Rev. Lett. **28** (1972) 1082.
7. S. C. Beluardi and R. Ferraro, Phys. Rev. D **52** (1995) 1963.