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The Problem of the Classical Limit of Quantum Mechanics and the Role of Self-Induced Decoherence

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Our account of the problem of the classical limit of quantum mechanics involves two elements. The first one is self-induced decoherence, conceived as a process that depends on the own dynamics of a closed quantum system governed by a Hamiltonian with continuous spectrum; the study of decoherence is addressed by means of a formalism used to give meaning to the van Hove states with diagonal singularities. The second element is macroscopicity represented by the limit $\hbar \to 0$: when the macroscopic limit is applied to the Wigner transformation of the diagonal state resulting from decoherence, the description of the quantum system becomes equivalent to the description of an ensemble of classical trajectories on phase space weighted by their corresponding probabilities.

KEY WORDS: classical limit; Riemann–Lebesgue lemma; decoherence.

1. INTRODUCTION

The problem of the classical limit of quantum mechanics has been a point of debate since the birth of the theory. Although this problem is usually addressed in the context of measurement, it can be analyzed from a more general point of view, in terms of *how the classical world arises from an underlying quantum reality*, independently of whether there is a measurement involved or not. Of course, the problem of the classical limit relies on the assumption that, if quantum mechanics is correct, then its results must reproduce the results of classical mechanics in the appropriate limit.

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In the old days of the theory, Heisenberg and Bohr among others conceived the classical limit of quantum mechanics by analogy with the classical limit of special relativity: $\hbar \to 0$ in quantum mechanics should play the same role as $\beta \to 0$ in special relativity. This assumption was considered by Einstein as an oversimplification since, while relativity and classical mechanics have the same deterministic structure, quantum mechanics has a probabilistic structure. Nevertheless, since those days it has been usually claimed that classical mechanics can be recovered as a limiting case of quantum mechanics when $\hbar \to 0$. This assumption led to correct results when the classical limit was conceived in the following way:

$$\text{QM} \left\{ \begin{array}{c} \xleftarrow{\qquad\qquad\qquad\qquad} \\ \text{quantization} \\ \xrightarrow{\qquad\qquad\qquad\qquad} \\ \text{classical limit} \equiv \hbar \to 0 \end{array} \right\} \text{CM} \tag{1}$$

where QM and CM stand for quantum mechanics and classical mechanics, respectively. In this schema, the first step is to quantize a classical system, e.g., by means of the Weyl transformation, in order to obtain the corresponding quantum system (at present, quantization is also called "deformation"). Then, the original classical system is recovered by applying the inverse Weyl transformation, i.e., the Wigner transformation[4] to the quantum system previously obtained and by taking the limit $\hbar \to 0$. It is quite clear that this method is completely circular to the extent that it only recovers the classical system originally proposed.

When the theoretical structure of quantum mechanics finally lost its classical origin, the problem of the classical limit acquired a new formulation that became the traditional one:

$$\text{QM} \left\{ \begin{array}{c} \xrightarrow{\qquad\qquad\qquad\qquad} \\ \text{classical limit} \equiv \hbar \to 0 \end{array} \right\} \text{CM} \tag{2}$$

This means that, no matter how the original quantum system was described, a classical system should be obtained via the Wigner transformation when $\hbar \to 0$. However, this way of conceiving the problem of the classical limit leads, at least, to three problems:

1. In general, the Wigner state function $\rho(\phi)$ (where $\phi = (\mathbf{q}, \mathbf{p})$ is a point in phase space) is not non-negatively defined;[5] as a result, it cannot be interpreted as a probability distribution.

[4] Historically, Weyl proposed his transformation as a quantization method. Later and independently, Wigner proposed a transformation that mapped quantum states into classical density functions. Finally, Moyal proved that the Wigner transformation was equivalent to the inverse Weyl transformation.

[5] A function $\rho(\phi)$ is non-negative if and only if $\rho(\phi) \geq 0$, a.e..

2. Only Hamiltonians of degree ≤ 2 in p and q yield to Hamiltonian fluxes that maintain the deformation invariant (or covariant) (see Refs. 1 and 2). In fact, only in these cases $\rho_1\rho_2(t) = \rho_1(t)\rho_2(t)$ after performing the Wigner transformation.
3. In some cases, factors of the form $\hbar^{-1}$ may appear in the Wigner state function due to the features of the Wigner transformation (see Refs. 1 and 5). In these cases, the limit $\hbar \to 0$ of the Wigner function is singular.

In this paper we will follow a well known trend in contemporary physics, according to which the classical limit must not be conceived as a mere consequence of a limiting procedure, but as a result of a physical process. From this perspective, the explanation of the emergence of the classical world from the underlying quantum realm involves two steps: the first one consists in explaining the physical phenomenon of decoherence,[6] and the second one consists in taking the macroscopic limit $\hbar \to 0$. However, we will move away from the mainstream position with respect to the explanation of decoherence: the aim of this paper is to obtain the classical limit of quantum mechanics on the basis of the *self-induced approach*

[6] We are going to use in here the notion of self-induced decoherence (SID) considered as a property of systems with continuous spectrum. As is well known, this includes an enormous variety of quantum systems from field interactions, scattering processes, etc., which are in the core of quantum mechanics. Nevertheless, situations in which discrete systems may approach to the continuum may be considered. See Appendix C and Ref. 33.

Decoherence processes are usually considered as the effect of interactions between the system under study and its environment, the duality between Object+Environment. In fact, this is only one possibility. Decoherence has been considered at least for the following reasons:

(i) A quantum computer is subject of external interactions that can produce errors. Here the quantum computer is the Object and the surroundings the Environment. This would be a typical effect of decoherence in which the Object possesses pure discrete spectrum.

(iii) Decoherence could be used to derive macroscopic laws from quantum mechanics as proposed by Halliwell. In this case, the splitting between Object+Environment yields to the splitting between macroscopic variables and microscopic variables.(3)

(iii) Decoherence is a useful tool in the discussion of the correspondence principle between quantum and classical mechanics. And this is the aim and purpose of the paper under discussion. As seen in the present discussion, this tool makes sense for systems with continuous spectrum.

Finally, it is important to remark that there is a closed connection between the decoherence considered as the interaction between Environment and Object and SID. This is the objective of a forthcomming study.(4)

to decoherence, such as it was presented in Ref. 6 and discussed in depth in Refs. 7 and 8. In contrast to the traditional einselection approach (see Ref. 9),[7] from the self-induced perspective decoherence does not require the openness of the system and its interaction with the environment: a single closed system can decohere when it has continuous spectrum. This self-induced approach is based in the well-known phenomenon of destructive interference of the off-diagonal terms of the density matrix (see Refs. 5, 10, 11, 12, 15) We will show that, in this new scenario, the classical limit is described by the following diagram:

$$\text{QM} \left\{ \begin{array}{c} \xrightarrow{\quad\text{decoherence}\quad} \text{Boolean QM} \xrightarrow{\quad\text{macroscopicity} \equiv \hbar \to 0\quad} \\ \xrightarrow[\quad\text{classical limit}\quad]{} \end{array} \right\} \text{CSM} \tag{3}$$

Self-induced decoherence transforms quantum mechanics into a Boolean quantum mechanics where the interference terms that preclude classicality have vanished. Macroscopicity, expressed by the limit $\hbar \to 0$,[8] turns Boolean quantum mechanics into classical statistical mechanics (CSM) in phase space. According to this view, the classical limit of quantum mechanics is not classical mechanics but *classical statistical mechanics*, and it requires two physical conditions: *decoherence* and *macroscopicity*. In other words, in order to behave classically a quantum system must have decohered and must be macroscopic enough: each one of these conditions alone is necessary but not sufficient for its classical behavior. Furthermore, we will show how and under what conditions this explanation overcomes the three problems that arise from the traditional way of conceiving the classical limit.

This paper is organized as follows. In Sect. 2, we present the formalism for observables and states, necessary for developing our program. Section 3 is devoted to explain the self-induced approach to decoherence: decoherence in energy and in the remaining variables are considered. In Sect. 4, we study the operation known as Wigner transformation and its application to observables and states. In Sect. 5, we show how the classical limit leads to classical statistical mechanics when the macroscopic limit is applied to the Wigner transformation of the quantum state resulting

[7] Let us note that in the einselection approach the previous problems 1–3 are not even considered.

[8] It is quite clear that it is not possible to set the value of $\hbar$ equal to 0, since it is not a dimensionless parameter but an universal constant. This means that, strictly speaking, the macroscopic limit is $\hbar/S \to 0$, where S is the characteristic action of the system: this is a factual limit which represents realistic situations where $S \gg \hbar$.

from decoherence. In Sect. 6 we discuss the physical meaning of the results just obtained, arguing that classicality must be understood as an emergent property that objectively arises from an underlying quantum mechanical realm. Finally, in Sect. 7 we draw our conclusions. Three appendices complete the paper.

2. FORMALISM FOR OBSERVABLES AND STATES

The formalism for observables and states used in the present paper is inspired by the formalism introduced by Antoniou *et al.*[13,14] which, in turn, is based on the works of van Hove.[15] In several papers[16–22] we applied different versions of this formalism to the study of the properties of quantum systems with continuous spectrum. In particular, the formalism was used in paper[6] for explaining decoherence. In order to simplify the notation, here we will study a simplified model where all the observables have continuous spectrum (cases where all observables except H have discrete spectrum will be considered in the footnotes): this will allow us to improve the mathematical basis of our approach without a proliferation of indices that would not introduce conceptual advantages.

2.1. Quantum Operator Algebra

Let us consider a system with a complete set of commuting observables (CSCO) $\{H, O_1, \ldots, O_N\}$ where H has a continuous spectrum $0 \leq \omega < \infty$ and, for the sake of simplicity, the O_i, $i = 1, 2, \ldots, N$, have also continuous spectra.[9] We will assume that the observables $H, O_1, \ldots, O_N$ are Weyl observables, i.e., that they come from the Weyl transformation of classical observables. In order to simplify the notation we will use $\{H, O\}$ to denote the CSCO $\{H, O_1, \ldots, O_N\}$. The generalized eigenbasis of $\{H, O\}$ is $\{|\omega, o\rangle\}$, where ω and o satisfy:

$$H\,|\omega, o\rangle = \omega\,|\omega, o\rangle \quad \text{and} \qquad O\,|\omega, o\rangle = o\,|\omega, o\rangle. \tag{4}$$

Then, H and O can be expressed as:

$$H = \int_0^\infty \int_o \omega\,|\omega, o\rangle\langle\omega, o|\,d\omega\,do, \quad O = \int_0^\infty \int_o o\,|\omega, o\rangle\langle\omega, o|\,d\omega\,do. \tag{5}$$

[9] The continuous spectrum is relevant for the classical limit since, in the limit $\hbar \to 0$ (precisely, the high quantum number limit), many discrete spectra become continuous. Spectra with continuous and discrete parts are studied in Refs. 6 and 22.

In addition to H and O, there are additional observables that may or may not commute with H and O. Then, a generic observable A has the following form:

$$A = \int_0^\infty \int_0^\infty \int_o \int_{o'} \widetilde{A}(\omega, \omega', o, o')\ |\omega, o\rangle\langle\omega', o'|\ d\omega\, d\omega'\, do\, do', \tag{6}$$

where $\widetilde{A}(\omega, \omega', o, o')$ could be, in principle, a distributional kernel. However, we will not work with the set of all the possible observables of the system, but only with a subset of it. The condition that defines this subset is given by the choice of the kernel $\widetilde{A}(\omega, \omega', o, o')$, which it is usually taken to be:[13,14,18]

$$\widetilde{A}(\omega, \omega', o, o') = A(\omega, o, o')\ \delta(\omega - \omega') + A(\omega, \omega', o, o') \tag{7}$$

where $A(\omega, o, o')$ and $A(\omega, \omega', o, o')$ are sufficiently regular functions (see Ref. 19 for details). Then, we will work with observables whose generic form is:

$$\begin{aligned} A = &\int_0^\infty \int_o \int_{o'} A(\omega, o, o')\ |\omega, o, o'\rangle\ d\omega\, do\, do' \\ &+ \int_0^\infty \int_0^\infty \int_o \int_{o'} A(\omega, \omega', o, o')\ |\omega, \omega', o, o'\rangle\ d\omega\, d\omega'\, do\, do', \end{aligned} \tag{8}$$

where we have introduced the following notation

$$|\omega, o, o'\rangle = |\omega, o\rangle\langle\omega, o'| \qquad \text{and} \qquad |\omega, \omega', o, o'\rangle = |\omega, o\rangle\langle\omega', o'|.$$

With the condition $\langle\omega, o|\omega', o'\rangle = \delta(\omega - \omega')\,\delta(o - o')$, the set of the operators of the form (8) is an algebra $\mathcal{A}$, and the observables are the self-adjoint elements of $\mathcal{A}$ (See Refs. 18 and 19).[10]

The first term of (8) represents the observables that commute with those of the CSCO $\{H, O\}$, and it will be called the *singular component*

[10] Although we will work with a subset of all the possible observables of the system, the physical generality of the self-induced approach to decoherence relies on the fact that the coordinates of the observables not belonging to $\mathcal{A}$ in the generalized eigenbasis of $\{H, O\}$, being singular, cannot be measured in laboratory and, therefore, they must always be approximated by their averaged counterparts (for a full argument, see Ref. 7). On the other hand, the restriction from the generic $\widetilde{A}(\omega, \omega', o, o')$ to more specific kernels like those of (7) will take the role of the usual "coarse graining" or "trace" as we will see below (see below Eq. (31) and the end of Sect. 6).

A_S of A; the second term of (8) will be called the *regular component* A_R of A:

$$A_S := \int_o \int_{o'} \int_0^\infty A(\omega, o, o')\ |\omega, o, o')\ d\omega\, do\, do'$$

$$A_R := \int_o \int_{o'} \int_0^\infty \int_0^\infty A(\omega, \omega', o, o')\ |\omega, \omega', o, o')\ d\omega\, d\omega'\, do\, do'. \qquad (9)$$

The operators A_S and A_R form the singular and regular algebras $\mathcal{A}_S$ and $\mathcal{A}_R$ respectively: $A_S \in \mathcal{A}_S$ and $A_R \in \mathcal{A}_R$. The generalized eigenbases of $\mathcal{A}_S$ and $\mathcal{A}_R$ are $\{|\omega, o, o')\}$ and $\{|\omega, \omega', o, o')\}$ since they span the algebras $\mathcal{A}_S$ and $\mathcal{A}_R$ respectively. Note that these two algebras have a trivial intersection and that the algebra $\mathcal{A}$ is the direct sum of both: $\mathcal{A} = \mathcal{A}_S \oplus \mathcal{A}_R$.[19]

2.2. States as Linear Functionals

States are continuous linear functionals on the algebra $\mathcal{A}$ defined as above. Let $\mathcal{A}^*$ be the dual space of $\mathcal{A}$ (the vector space of all linear continuous functionals on $\mathcal{A}$). In our notation, the action of the functional $\rho \in \mathcal{A}^*$ onto the operator $A \in \mathcal{A}$ is denoted as $(\rho|A)$. With this notation we define $(\omega, o, o'|$ and $(\omega, \omega', o, o'|$ as:

$$(\omega, o, o'|A) = A(\omega, o, o'), \quad (\omega, \omega', o, o'|A) = A(\omega, \omega', o, o') \qquad (10)$$

for all $A \in \mathcal{A}$. It can be shown that $(\omega, o, o'|$ and $(\omega, \omega', o, o'|$ are in $\mathcal{A}^*$ for all values of ω, ω', o, o'.[19] In addition, if $\mathcal{A}_S^*$ is the dual of $\mathcal{A}_S$ and $\mathcal{A}_R^*$ is the dual of $\mathcal{A}_R$, it can be shown that $(\omega, o, o'| \in \mathcal{A}_S^*$ and $(\omega, \omega', o, o'| \in \mathcal{A}_R^*$,[18,19], and that the following relations hold:

$$(\omega, o, s|\omega', o', s') = \delta(\omega - \omega')\delta(o - o')\delta(s - s'),$$

$$(\omega, \sigma, o, s|\omega', \sigma', o', s') = \delta(\omega - \omega')\delta(\sigma - \sigma')\delta(o - o')\delta(s - s'),$$

$$(\omega, \sigma, |\omega', \sigma', o', s') = (\omega, \sigma, o, s|\omega', \sigma') = 0. \qquad (11)$$

An element of the dual $\mathcal{A}^*$ can be expressed as:

$$\rho = \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')\ (\omega, o, o'|\ d\omega\, do\, do'$$

$$+ \int_0^\infty \int_0^\infty \int_o \int_{o'} \rho(\omega, \omega', o, o')\ (\omega, \omega', o, o'|\ d\omega\, d\omega'\, do\, do', \quad (12)$$

which shows that $(\omega, o, o'|$ and $(\omega, \omega', o, o'|$ form a generalized basis of $\mathcal{A}^*$. Again, we will call the first term in (12) the *singular component* ρ_S of ρ, and the second term in (12) the *regular component* ρ_R of ρ, where $\rho_S \in \mathcal{A}_S^*$ and $\rho_R \in \mathcal{A}_R^*$:

$$\rho_S := \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')\ (\omega, o, o'|\ d\omega\, do\, do',$$

$$\rho_R := \int_0^\infty \int_0^\infty \int_o \int_{o'} \rho(\omega, \omega', o, o')\ (\omega, \omega', o, o'|\ d\omega\, d\omega'\, do\, do'. \quad (13)$$

The action of the functional ρ on the operator A is given by:

$$(\rho|A) = \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')\ A(\omega, o, o')\, d\omega\, do\, do'$$

$$+ \int_0^\infty \int_0^\infty \int_o \int_{o'} \rho(\omega, \omega', o, o')\ A(\omega, \omega', o, o')\, d\omega\, d\omega'\, do\, do' \quad (14)$$

It is interesting to remark that, although $A(\omega, o, o')$ and $A(\omega, \omega', o, o')$ must be regular, well behaved functions (polynomials and Schwartz functions, see Ref. 19), this is not the case of $\rho(\omega, o, o')$ and $\rho(\omega, \omega', o, o')$, which may be singular with the only condition that (14) be well defined. For instance, the functional $(\eta, s, s'|$ can be written in the form (12) with $\rho(\omega, o, o') = \delta(\eta - \omega)\, \delta(s - o)\, \delta(s' - o')$ and $\rho(\omega, \omega', o, o') = 0$.

The condition of positivity for a functional f means that, if $f \in \mathcal{A}^*$ and $A \in \mathcal{A}$, then $f(A^\dagger A) \geq 0$, where $A^\dagger$ is the adjoint of A. In our case we will require positivity to ρ_S, and this implies that:

$$\rho(\omega, o, o') \geq 0. \quad (15)$$

The condition of normalization for a functional f means that $f(I) = 1$. In our case, we will normalize only ρ_S:[11]

$$I = \int_0^\infty \int_o |\omega, o\rangle\langle\omega, o|\, d\omega\, do \Longrightarrow (\rho_S|I)$$
$$= \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')\, d\omega\, do\, do' = 1. \tag{16}$$

Note that $I \in \mathcal{A}$ since for I, $A(\omega, o, o') \equiv 1$ and $A(\omega, \omega', o, o') \equiv 0$.

Finally, note that $\mathcal{A}^* = \mathcal{A}_S^* \oplus \mathcal{A}_R^*$,[18] and that $\{(\omega, o, o'|\}$ is a generalized basis of $\mathcal{A}_S^*$ and $\{(\omega, \omega', o, o'|\}$ is a generalized basis for $\mathcal{A}_R^*$.

3. SELF-INDUCED DECOHERENCE

3.1. Decoherence in Energy

Let us now consider the time evolution of the system. Since ρ is a functional, its time evolution in the Schrödinger picture cannot be directly computed by means of the Liouville–von Neumann equation. Nevertheless, this equation also describes the time evolution of the observable A in the Heisenberg picture:

$$i\hbar\partial_t A = -[H, A] = \mathbb{L}A \Rightarrow A(t) = \exp(it\,\mathbb{L}/\hbar)\, A(0), \tag{17}$$

where H is the Hamiltonian that governs the time evolution, and $\mathbb{L}$ is the Liouville operator associated to the Hamiltonian $H : \mathbb{L}\, A = -[H, A]$. Once the time evolution of A has been computed, the time evolution of ρ can be obtained by means of the *duality formula*:

$$(\rho|\exp(-it\,\mathbb{L}/\hbar)\, A) = (\exp(it\,\mathbb{L}/\hbar)\rho|A). \tag{18}$$

This equation gives the time evolution of ρ, which satisfies the Liouville–von Neumann equation:

$$i\hbar\partial_t \rho = [H, \rho] = -\mathbb{L}\rho \Rightarrow \rho(t) = \exp(it\,\mathbb{L}/\hbar)\rho(0). \tag{19}$$

In order to follow this strategy in our case, we begin by applying the Liouville–von Neumann evolution equation to the generalized basis

[11] We require positivity and normalization for ρ_S since, as we will see, it is the only component of ρ that remains after decoherence. In addition, $(\rho_R|I) = 0$ for any $\rho_R \in \mathcal{A}_R^*$.

$\{|\omega, o, o') = |\omega, o\rangle\langle\omega, o'|,\ |\omega, \omega', o, o') = |\omega, o\rangle\langle\omega', o'|\}$. Since $H|\omega, o, o'\rangle = \omega\,|\omega, o, o'\rangle$, we have that:

$$\mathbb{L}|\omega, o, o') = -H|\omega, o\rangle\langle\omega, o'| + |\omega, o\rangle\langle\omega, o'|H = -(\omega - \omega)|\omega, o, o') = 0 \tag{20}$$

$$\mathbb{L}|\omega, \omega', o, o') = -H|\omega, o\rangle\langle\omega', o'| + |\omega, o\rangle\langle\omega', o'|H = -(\omega - \omega')|\omega, \omega', o, o'). \tag{21}$$

This means that the generalized basis $\{|\omega, o, o'),\ |\omega, \omega', o, o')\}$ is an eigenbasis of the operator $\mathbb{L}$. Moreover, $\mathbb{L}|\omega, o, o') = 0$ implies that not only the $|\omega, o, o')$, but also all the singular operators $A_S \in \mathcal{A}_S$ are time invariant ($e^{-it\mathbb{L}/\hbar}A_S = A_S$), since:

$$\begin{aligned}\mathbb{L}A_S &= \mathbb{L}\int_0^\infty \int_o\int_{o'} A(\omega, o, o')\,|\omega, o, o')\,d\omega\,do\,do' \\ &= \int_0^\infty \int_o\int_{o'} A(\omega, o, o')\,(\mathbb{L}\,|\omega, o, o'))\,d\omega\,do\,do' = 0.\end{aligned} \tag{22}$$

Therefore, for any $A \in \mathcal{A}$, $\mathbb{L}A = \mathbb{L}A_S + \mathbb{L}A_R = \mathbb{L}A_R$. Moreover, due to Eq. (21) $\mathbb{L}A_R \in \mathcal{A}_R$.

From Eq. (21), it can be obtained:

$$e^{-it\mathbb{L}/\hbar}\,|\omega, \omega', o, o') = e^{it(\omega-\omega')/\hbar}\,|\omega, \omega', o, o') \tag{23}$$

and, hence, for any $A_R \in \mathcal{A}_R$ we have:

$$e^{-it\mathbb{L}/\hbar}\,A_R = \int_o\int_{o'}\int_0^\infty\int_0^\infty A(\omega, \omega', o, o')\,e^{it(\omega-\omega')/\hbar}\,|\omega, \omega', o, o')\,d\omega\,d\omega'\,do\,do'. \tag{24}$$

Then, for any $A \in \mathcal{A}$, we obtain the following time evolution in the Heisenberg picture:

$$\begin{aligned}e^{-it\mathbb{L}/\hbar}\,A &= \int_0^\infty \int_o\int_{o'} A(\omega, o)|\omega, o, o')\,d\omega\,do\,do' \\ &\quad + \int_0^\infty\int_0^\infty \int_o\int_{o'} A(\omega, \omega', o, o')\,e^{it(\omega-\omega')/\hbar}\,|\omega, \omega', o, o')\,d\omega\,d\omega'\,do\,do'.\end{aligned} \tag{25}$$

A similar situation arises when we consider the time evolution of the states. Since $|\omega, o, o')$ is time invariant, the duality formula tells us that

the functionals $(\omega, o, o'|$ are time invariant; therefore, all the singular functionals $\rho_S \in \mathcal{A}_S^*$ are also time invariant.

Now we can compute $(\rho|A(t))$ which, by the duality formula (18), is equal to $(\rho(t)|A)$:

$$(\rho|A(t)) = (\rho(t)|A) = \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')\ A(\omega, o, o')\, d\omega\, do\, do'$$

$$+ \int_0^\infty \int_0^\infty \int_o \int_{o'} \rho(\omega, \omega', o, o')\, e^{i(\omega-\omega')t/\hbar} A(\omega', \omega, o', o)\, d\omega\, d\omega'\, do\, do'. \quad (26)$$

Considering that A is arbitrary, we obtain the final equation for the evolution of the functional ρ as:

$$\rho(t) = \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')(\omega, o, o'|\, d\omega\, do\, do'$$

$$+ \int_0^\infty \int_0^\infty \int_o \int_{o'} \rho(\omega, \omega', o, o') e^{i(\omega-\omega')t/\hbar} (\omega, \omega', o, o'|\, d\omega\, d\omega'\, do\, do' \quad (27)$$

where we will call the first term "invariant part" and the second term "fluctuating part" of $\rho(t)$.

If we now consider the states ρ such that the product

$$\rho(\omega, \omega', o, o')\, A(\omega, \omega', o, o')$$

is integrable, the Riemann–Lebesgue theorem[23] can be applied to Eq. (26) to conclude that:

$$\lim_{t\to\infty} (\rho(t)|A) = \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o') A(\omega, o, o')\, d\omega\, do\, do' = (\rho_*|A), \quad (28)$$

for any $A \in \mathcal{A}$, where the functional ρ_* is precisely the singular component ρ_S of ρ (see Eq. (13)):

$$\rho_* = \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')(\omega, o, o'|\, d\omega do\, do'. \quad (29)$$

The physical meaning of this process can be understood when we consider that the mean value of the observable A in the state ρ can be computed as $\langle A\rangle_\rho = (\rho|A)$. Therefore, Eq. (28) can be rewritten as:

$$\lim_{t\to\infty} \langle A\rangle_{\rho(t)} = \langle A\rangle_{\rho_*} \quad \text{for any} A \in \mathcal{A}. \quad (30)$$

Of course, this limit does not contradict the fact that the off-diagonal terms of a functional ρ representing the quantum state of a closed system never vanish through the *unitary* evolution described by the Liouville–von Neumann equation. What self-induced decoherence shows is that the mean value $\langle A\rangle_{\rho(t)}$ of any observable $A \in \mathcal{A}$ will evolve in such a way that, for $t \to \infty$, it can be computed *as if* the system were in a state ρ_* where the off-diagonal terms have vanished. Formally this is expressed by the fact that, although we strictly obtain the limit (30) (or (28)), the state $\rho(t)$ has only a *weak limit*:

$$w - \lim_{t\to\infty} \rho(t) = \rho_*. \tag{31}$$

This weak limit means that, even if $\rho(t)$ always follows a unitary evolution, the system decoheres *from an observational point of view*, that is, from the viewpoint given by the observable A, for any $A \in \mathcal{A}$.

At this point it is interesting to add a few remarks:

(i) Here we are using the term decoherence in a broad sense. As a matter of fact, due to the nature of functions $A(\omega, o, o')$ and $A(\omega, \omega', o, o')$ used in the previous context (polynomials and Schwartz type functions see Ref. 19), decoherence is produced in the limit as $t \longmapsto \infty$ only. Note that for finite values of time, Eq. (27) implies that, if $\rho = \rho(0)$ is a regular state operator (for which is necessary and sufficient that $\rho(\omega, o, o') = \rho(\omega, \omega', o, o')$, see Refs. 16, 18 and 19), so is $\rho(t)$ *for any finite value of* t. Thus, in general, ρ will be a singular diagonal state (and hence decohered in this sense) only in the very limit $t \longmapsto \infty$ and this in a weak sense only. This property is sufficient for the discussion relevant to the classical limit. In order to introduce in the formalism systems with finite decoherence times, we need to restraint the algebra $\mathcal{A}$, by restraining the class of the allowed functions $A(\omega, o, o')$ and $A(\omega, \omega', o, o')$. For instance, if we add the condition that $A(\omega, \omega', o, o')$ would have an analytic continuation with respect to the variable $\omega - \omega'$ and this continuation has a pole at the point $z_0 = \omega_R - i\gamma$, with $\gamma > 0$ (resonance pole), the time of decoherence (as the time at which the second term in (26) is nearly equal to zero for the observable $A \in \mathcal{A}$) is estimated by $t_D = \hbar/\gamma$. Then, decoherence times of the order of $10^{-37} - 10^{-39}$ seconds are obtained.[21]

(ii) Unlike the usual decoherence a la Zurek[9] that transforms pure states into mixtures, this type of decoherence transform both pure and mixed states into states with diagonal singular a la van Hove.[13–15] See a characterization of these states, for instance, in Refs. 13, 14 and 18.

(iii) Observe that the limit in (31) is taking in a weak sense. Convergence is taking in the averages as in (28) and (30). This, together with

the selection of the algebra $\mathcal{A}$ of observables, suggest that this type of decoherence is produced by a kind of *generalized coarse graining*. For a thourogly discussion on this idea see Ref. 7.

(iv) We have mentioned that the second term in (26) and hence in (27) vanishes when $\rho(\omega, \omega', o, o')\, A(\omega, \omega', o, o')$ is an integrable function on the variable ω-ω', as a consequence of the Riemann–Lebesgue theorem. Nevertheless the most general form of the Riemann–Lebesgue theorem says that the Fourier transform of some singular measures vanishes as $t \longmapsto \pm\infty$.[24] This permits to enlarge the class of states that decohere.

3.2. Decoherence in the Remaining Variables

As we have seen, for $t \to \infty$ the system decoheres in energy since ρ_* turns out to be diagonal in ω. However, we would like to obtain a state diagonal in all the variables. To the extent that we have taken the limit $t \to \infty$, it is impossible that a new process diagonalizes the o-variables. As we will see, when a convenient basis is chosen, the diagonalization of ρ can be completed. This second stage necessarily depends on the initial condition ρ at $t = 0$, since ρ_* is a constant of motion.

Let us consider a unitary operator U that keeps the Hamiltonian invariant but changes the set of observables $\{O_1, O_2, \ldots, O_N\}$ into the set $\{P_1, P_2, \ldots, P_N\}$, where $\{H, P_1, P_2, \ldots, P_N\}$ is also a CSCO. The simplest form of U is given by:

$$U = \int_0^\infty \int_p \int_o U(\omega, p, o)\, |\omega, p\rangle\langle \omega, o|\, d\omega\, dp\, do. \tag{32}$$

It is probably worthy to note that, since the O_i and the P_i, $i = 1, 2, \ldots, N$, are unitarily equivalent, they have the same spectrum and therefore the variable p runs into the same domain than the variable o. The action of U on the ket $|\omega, o\rangle$ defines the action of U on any ket, since the kets $|\omega, o\rangle$ belong to a generalized basis. This action can be easily computed as:

$$|\omega, p\rangle := U\, |\omega, o\rangle = \int_p U(\omega, p, o)\, |\omega, o\rangle\, dp, \tag{33}$$

where obviously $P_i|\omega, p\rangle = p_i|\omega, p\rangle$ and this justifies the notation. The unitarity of U implies that $UU^{-1} = I$. From here, we obtain:

$$\int_o U(\omega, p, o)\, U^*(\omega, p', o)\, do = \delta(p - p'). \tag{34}$$

In the new representation, the operator A takes the form:

$$A = \int_0^\infty \int_p \int_{p'} A(\omega, p, p')\, |\omega, p\rangle\langle\omega, p'|\, d\omega\, dp\, dp'. \tag{35}$$

Introducing Eq. (33) into Eq. (35) gives:

$$A = \int_0^\infty \int_p \int_{p'} \int_o \int_{o'} U(\omega, p, o)\, A(\omega, p, p')\, U^*(\omega, p', o')\, |\omega, o\rangle\langle\omega, o'|\, d\omega\, dp\, dp'\, do\, do'. \tag{36}$$

Therefore, the coordinates of A in the old basis are:

$$A(\omega, o, o') = \int_p \int_{p'} U(\omega, p, o)\ A(\omega, p, p')\, U^*(\omega, p', o')\, dp\, dp'. \tag{37}$$

Since U is a unitary operator, Eq. (37) is invertible:

$$A(\omega, p, p') = \int_o \int_{o'} U(\omega, p', o')\ A(\omega, o, o')\, U^*(\omega, p, o)\, do\, do' \tag{38}$$

and, finally, by duality one finds that:

$$\rho(\omega, p, p') = \int_o \int_{o'} U(\omega, p', o')\, \rho(\omega, o, o')\, U^*(\omega, p, o)\, do\, do'. \tag{39}$$

If $\rho = \int_0^\infty \int_o \int_{o'} \rho(\omega, o, o')\, |\omega, o\rangle\langle\omega, o'|\, d\omega\, do\, do'$ is a state, it must be positively defined and self adjoint. This implies that $\rho(\omega, o, o') = \rho^*(\omega, o', o)$.[19] Then, we can choose $U(\omega, p, o)$ such that:

$$\rho(\omega, p, p') = \rho(\omega, p)\, \delta(p - p') \tag{40}$$

and this completes the diagonalization. In the new basis, Eqs. (28) and (29) of the previous subsection become:

$$\lim_{t\to\infty} (\rho(t)|A) = \int_p \int_0^\infty \rho(\omega, p) A(\omega, p)\, d\omega\, dp = (\rho_*|A) \quad \text{for any} A \in \mathcal{A}, \tag{41}$$

$$\rho_* = \int_p \int_0^\infty \rho(\omega, p)(\omega, p|\, d\omega\, dp, \tag{42}$$

where now ρ_* is completely diagonal in ω and p. The generalized basis given by $\{|\omega, p), |\omega, \omega', p, p')\}$ is the preferred basis (also called "pointer

basis"[12]), as presented in Ref. 6 and extensively discussed in Ref. 7. On the other hand, the decoherence time t_D can be computed[13] as in Ref. 22.

4. THE WIGNER TRANSFORMATION

4.1. Characterization of the Wigner Transformation

The Weyl transformation maps functions or generalized functions on phase space into operators.[1,2,25,26] Thus, the Wigner transformation maps operators into functions on the phase space.[25,26] If A is an operator, we will denote the function corresponding to A via the Wigner transformation by $symbA$ or $A(\phi)$, where $\phi = (\mathbf{q}, \mathbf{p}) = (q_1, q_2, \ldots, q_{N+1}, p_1, p_2, \ldots, p_{N+1})$ is a point in flat phase space. The function $A(\phi) = symbA$ is called *Wigner symbol* or *Wigner function* of the operator A. As the Weyl transformation is a one to one mapping, the image of the algebras $\mathcal{A}$, $\mathcal{A}_S$ and $\mathcal{A}_R$ by the Wigner transformation are non-commutative algebras of functions denoted by $\mathcal{L}$, $\mathcal{L}_S$ and $\mathcal{L}_R$, respectively, where $\mathcal{L} = \mathcal{L}_S \oplus \mathcal{L}_R$. Since the image of A by the Wigner transformation is the function $symbA$, it seems quite natural to use the notation $symb$ to denote the Wigner transformation itself, so that:

$$symb\colon \mathcal{A} \longmapsto \mathcal{L}, \qquad symb\colon \mathcal{A}_R \longmapsto \mathcal{L}_R, \qquad symb\colon \mathcal{A}_S \longmapsto \mathcal{L}_S. \quad (43)$$

Let us define the mapping $\mathcal{A}_R \longmapsto \mathcal{L}_R$ for regular observables as usual.[26] First, consider the phase space (in this case[14] $\mathbb{R}^{2(N+1)}$) and endow it with the symplectic form:

$$\omega_{ab} = \begin{pmatrix} 0 & I_{N+1} \\ -I_{N+1} & 0 \end{pmatrix}; \qquad \omega^{ab} = \begin{pmatrix} 0 & -I_{N+1} \\ I_{N+1} & 0 \end{pmatrix}. \quad (44)$$

Then, let $\widehat{f}$ be an operator such that $symb\widehat{f} = f(\phi)$. This transformation is defined by the usual Wigner recipe as:

$$symb\widehat{f} = f(\phi) := \int d^{2N+1}\psi \exp\left(\frac{i}{\hbar}\psi^a \omega_{ab}\phi^b\right) Tr\left(\widehat{T}(\psi)\widehat{f}\right) \quad (45)$$

[12] We prefer to use the term "preferred basis" instead of "pointer basis" since it arises in contexts other than measurements where the pointer of a measuring device is involved.

[13] Of course, in the case where $t_D \to \infty$, the system does not decohere.

[14] The fact that the dimension of the phase space is $2(N+1)$, where $N+1$ is the number of observables of the CSCO, amounts to the integrability of the classical system resulting from the Wigner transformation. Non-integrable cases will be considered elsewhere.

where ψ^a and ϕ^b denote the ath and the bth components of the points ψ and ϕ on phase space, respectively. Here:

$$\widehat{T}(\psi) = \exp\left(\frac{i}{\hbar}\psi^a \omega_{ab} \phi^b\right), \tag{46}$$

where:

$$\widehat{\phi} = \left(\widehat{q}_1, \ldots, \widehat{q}_{N+1}, -i\hbar\frac{\partial}{\partial q_1}, \ldots, -i\hbar\frac{\partial}{\partial q_{N+1}}\right) \tag{47}$$

and $\widehat{q}_i$, $i = 1, 2, \ldots, N+1$, is the i-th component of the position operator on the Hilbert space $L^2(\mathbb{R}^{N+1})$.

The non-commutative product that corresponds to the product of operators in $\mathcal{L}$ (or $\mathcal{L}_S$ and $\mathcal{L}_R$) is called *star product*, and it is given by:

$$symb(\widehat{f}\widehat{g}) = symb\,\widehat{f} * symb\widehat{g} = (f * g)(\phi) \tag{48}$$

where $f(\phi)$ and $g(\phi)$ are the Wigner symbols of the operators $\widehat{f}$ and $\widehat{g}$, respectively. It can be proven (Ref. 25, Eq. (2.59); for more general expansions, see Ref. 27) that:

$$\begin{aligned}(f * g)(\phi) &= f(\phi)\exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b\right)\\ g(\phi) &= g(\phi)\exp\left(-\frac{i\hbar}{2}\overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b\right) f(\phi).\end{aligned} \tag{49}$$

The *Moyal bracket* is the Wigner symbol corresponding to the commutator in $\mathcal{L}$:

$$\{f, g\}_{mb} = \frac{1}{i\hbar}(f * g - g * f) = symb\left(\frac{1}{i\hbar}[f, g]\right). \tag{50}$$

Then, if we expand the last two equations in power series of $\hbar$, we obtain:[(1)]

$$(f * g)(\phi) = f(\phi)g(\phi) + \sum_{r=1} \hbar^r P^r(f(\phi)g(\phi)), \tag{51}$$

$$\{f, g\}_{mb} = \{f, g\}_{pb} + \sum_{r=1} \hbar^{2r} P^{2r+1}(f(\phi)g(\phi)), \tag{52}$$

where the P^r are the coefficients obtained by means of (49) and pb means Poisson bracket. This suggests that, in the limit $\hbar \to 0$, the star product should become the ordinary product and the Moyal bracket should become the Poisson bracket. In fact, this is the case in many circumstances although not in all of them, because in some cases the coefficients P^r may contain factors of the form $\hbar^{-1}$, making the limit $\hbar \to 0$ singular (see Refs. 1 and 5). In those cases, the problem 3 mentioned in the Introduction arises. From (49) we see that factors $\hbar^{-1}$ can only come from the symbols $f(\phi)$ or $g(\phi)$; then, if these functions do not depend on $\hbar^{-1}$, the limit $\hbar \to 0$ is regular and can be considered as the proper macroscopic limit.

Finally, let us observe that if $\widehat{f}$ commute with $\widehat{g}$, Eqs. (49) and (51) become:

$$(f * g)(\phi) = f(\phi) \cos\left(-\frac{i\hbar}{2} \overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b\right) g(\phi) \tag{53}$$

and, hence, in the simple cases with no factors of the form $\hbar^{-1}$ we obtain:

$$(f * g)(\phi) = f(\phi) g(\phi) + 0(\hbar^2). \tag{54}$$

4.2. The Wigner Transformation of Observables and States

In the previous subsection we have considered the Wigner transformation for regular observables, as it is usually defined. But the Wigner transformation has not been defined when singular distributions are involved; therefore, the transformation must be defined in this case.

Let us go back to the algebra $\mathcal{A}_S$ of the observables that commute with the CSCO $\{H, P_1, \ldots, P_N\}$, also denoted by $\{H, P\}$ for simplicity. An element of this algebra is given by:

$$A_S = \int_0^\infty \int_p A(\omega, p) \, |\omega, p) \, d\omega \, dp, \tag{55}$$

where $A(\omega, p)$ is a regular function on its variables. The functional calculus gives:

$$A_S = A(H, P), \tag{56}$$

where $P := (P_1, P_2, \ldots, P_N)$. Since we are assuming that the observables H and P_i are Weyl observables, they have Wigner functions $H(\phi)$

and $P_i(\phi)$, $i = 1, 2, \ldots, N$, respectively, not depending on $\hbar^{-1}$. As a consequence, if $\widehat{f}$ and $\widehat{g}$ are any powers of H or P_i, (54) holds. Therefore, since we have assumed[19] that $A(\omega, p)$ is differentiable on its variables, the Taylor theorem gives:

$$symbA_S = A_S(\phi) = A(H(\phi), P(\phi)) + 0(\hbar^2). \tag{57}$$

This means that the problem 3 mentioned in the Introduction does not arise in the singular algebra $\mathcal{A}_S$ when we work with Weyl operators. As a consequence, the Wigner symbol of any observable $A \in \mathcal{A}_S$, when $\hbar \to 0$, is $A(H(\phi), P(\phi))$.

Now let us study the Wigner transformation for states of $\mathcal{A}^*$. There are two cases that we have to consider. The first case is given by the states in $\mathcal{A}^*$ that can be written as regular density operators. These states are characterized by $\rho(\omega, \omega, p, p') = \rho(\omega, p, p')$ (see Refs. 13, 14, 16, 18 and 19). Regular density operators have well defined Wigner functions:[26] we only must add a $(2\pi\hbar)^{-(N+1)}$ factor to (45) in order to obtain the usual normalization for the Wigner function. The second case includes any other possibility, i.e., density operators for which either $\rho(\omega, \omega, p, p') \neq \rho(\omega, p, p')$ or $\rho(\omega, \omega, p, p')$ or $\rho(\omega, p, p')$ are defined only in a distributional sense and not as regular functions. The question arises about whether the operators of this second type do or do not have a well defined Wigner function. The answer is given by the duality formula (18). If $\rho \in \mathcal{A}^*$ and $A \in \mathcal{A}$, we can define $symb\rho \equiv \rho(\phi)$ in such a way that it satisfies:

$$\begin{aligned}(symb\rho|symbA) := (\rho|A) = &\int_0^\infty \int_p \int_{p'} A(\omega, p, p')\, \rho(\omega, p, p')\, d\omega\, dp\, dp' \\ &+ \int_0^\infty \int_0^\infty \int_p \int_{p'} A(\omega, \omega', p, p')\, \rho(\omega, \omega', p, p') \\ &\times d\omega\, d\omega'\, dp\, dp'. \end{aligned} \tag{58}$$

As the integrals in (58) are well defined, for any $\rho \in \mathcal{A}^*$, $symb\rho$ is also well defined and belongs to the dual space $\mathcal{L}^*$ of the algebra $\mathcal{L}$. Let us recall the decomposition $\mathcal{L}^* = \mathcal{L}_S^* \oplus \mathcal{L}_R^*$ and the fact that the operation $symb$ is a bijection: $symb\, \mathcal{A}_S^* \longmapsto \mathcal{L}_S^*$. Definition (58) will allow us to obtain $symb\rho_S$ in the next section.

5. THE CLASSICAL LIMIT

We have seen that, as the result of decoherence, the regular part ρ_R of ρ vanishes and only the singular part $\rho_S = \rho_*$ remains (see (42)):

$$\rho_* = \rho_S = \int_0^\infty \int_p \rho(\omega, p)(\omega, p|\, d\omega\, dp. \tag{59}$$

Then, the problem is to find the classical distribution $\rho_c(\phi)$ resulting from applying the macroscopic limit $\hbar \to 0$ to the Wigner transformation of $\rho_* = \rho_S$:

$$\rho_c(\phi) = \lim_{\hbar\to 0} symb\rho_S = \lim_{\hbar\to 0} \int_0^\infty \int_p \rho(\omega, p)\, symb(\omega, p|\, d\omega\, dp$$

$$= \int_0^\infty \int_p \rho(\omega, p) \lim_{\hbar\to 0} [symb(\omega, p|]\, d\omega\, dp. \tag{60}$$

The first step consists in obtaining the limit of $symb|\omega, p)$ for $\hbar \to 0$. From (55) and (57) we know that:

$$\lim_{\hbar\to 0} [symbA_S] = A(H(\phi), P(\phi)) = \lim_{\hbar\to 0} \int_0^\infty \int_p A(\omega, p)\, symb|\omega, p)\, d\omega\, dp$$

$$= \int_0^\infty \int_p A(\omega, p) [\lim_{\hbar\to 0} symb|\omega, p)] d\omega\, dp. \tag{61}$$

The function $A(H(\phi), P(\phi))$ can also be written as:

$$A(H(\phi), P(\phi)) = \int_0^\infty \int_p A(\omega, p)\ \delta(H(\phi) - \omega)\, \delta(P(\phi) - p)\, dp\, d\omega. \tag{62}$$

If we compare (61) and (62), we can conclude that:[15]

$$\lim_{\hbar\to 0} symb\,|\omega, p) = \delta(H(\phi) - \omega)\, \delta(P(\phi) - p). \tag{63}$$

The second step begins by remembering that, from (58) and (11), $symb(\omega, p|$ must satisfy:[16]

$$(symb(\omega, p|\,|symb\,|\omega', p')) = (\omega, p|\omega', p') = \delta(\omega - \omega')\, \delta(p - p'). \tag{64}$$

[15] In the case of discrete spectrum we would have $A(\omega, p) = \delta(\omega - \omega')\delta^N_{pp'}$ instead of $A(\omega, p) = \delta(\omega - \omega')\, \delta(p - p')$. Then, we would obtain $symb|\omega', p') = \delta(H(\phi) - \omega')\delta^N_{P(\phi)p'}$, where $\delta^N_{P(\phi)p'}$ is a N Kronecker δ.

[16] This is a definition of $symb(\omega, p|$ as a distribution. The aim of the following discussion is to show that this distribution is proportional to $\delta(H(\phi) - \omega)\delta(P(\phi) - p)$. This discussion is necessary but has the same level of rigor as the Dirac delta itself.

Since the r.h.s. of the last equation does not depend on $\hbar$, the limit for $\hbar \to 0$ results:

$$\lim_{\hbar\to 0} (symb(\omega, p| \,|symb\,|\omega', p')) = (\lim_{\hbar\to 0} [symb(\omega, p|] \,|\, \lim_{\hbar\to 0} [symb\,|\omega', p')])$$

$$= \delta(\omega - \omega')\,\delta(p - p'). \tag{65}$$

We will use $\rho_{S\omega p}(\phi)$ to denote the limit for $\hbar \to 0$ of $symb(\omega, p|$:

$$\rho_{S\omega p}(\phi) := \lim_{\hbar\to 0} symb(\omega, p|. \tag{66}$$

Therefore, replacing (63) and (66) into (65) we obtain:

$$(\rho_{S\omega p}(\phi) \,|\, \delta(H(\phi) - \omega')\,\delta(P(\phi) - p')) = \delta(\omega - \omega')\,\delta(p - p'). \tag{67}$$

The final step consists in obtaining $\rho_{S\omega p}(\phi)$. Since we have assumed that the number of operators in the CSCO $\{H, P_1, \ldots, P_N\}$ coincides with the number of degrees of freedom of the system under consideration and, as a consequence, the phase space has dimension $2(N + 1)$ (see footnote 11), then there exists a canonical transformation that carries the position-momentum variables $\phi = (\mathbf{q}, \mathbf{p})$ into the variables

$$\psi = (\tau(\phi), \alpha_1(\phi), \ldots, \alpha_N(\phi), H(\phi), P_1(\phi), \ldots, P_N(\phi)),$$

where $H(\phi) = symbH$, $P_i(\phi) = symbP_i$, $i = 1, 2, \ldots, N$, $\tau(\phi)$ is the conjugate variable of $H(\phi)$, and the $\alpha_i(\phi)$ are the conjugate variables of the $P_i(\phi)$. On the other hand, from Subsect. 3.1 we know that each $A_S \in \mathcal{A}_S$ and each functionals of the form $\rho_S \in \mathcal{A}_S^*$ are time invariant. Since the function *symb* does not introduce time variables, all the $A_S(\phi) = symbA_S \in \mathcal{L}_S$ and all the $\rho_S(\phi) = symb\rho_S \in \mathcal{L}_S^*$ are also time invariant, and the same holds for their limits for $\hbar \to 0$. Thus, if $A(\phi) \in \mathcal{L}_S$, $\rho(\phi) \in \mathcal{L}_S^*$, $A(\phi) = f(H(\phi), P(\phi))$, and $\rho(\phi) = g(H(\phi), P(\phi))$, then $(\rho(\phi)|A(\phi))$ can be expressed as:

$$(\rho(\phi)|A(\phi)) = \int_{\phi^{2(N+1)}} f(H(\phi), P(\phi))\; g(H(\phi), P(\phi))\; d\phi^{2(N+1)}$$

$$= \int_\tau d\tau \int_\alpha d\alpha \int_H \int_P f(H, P)\; g(H, P)\, dH\, dP. \tag{68}$$

We can call:

$$C(H, P) = \int_{\mathcal{M}(H,P)} d\tau \, d\alpha, \tag{69}$$

where $C(H, P)$ is the volume of the region $\mathcal{M}(H, P)$ of the configuration manifold defined by the conditions $H = \text{const}$ and $P = \text{const}$. Then, (68) becomes:

$$(\rho(\phi)|A(\phi)) = C(H, P) \int_H \int_P f(H, P)\ g(H, P)\, dH\, dP. \tag{70}$$

In the simplest case of bounded integrable systems described by action-angle variables, $C(H, P)$ is a time-constant equal to $(2\pi)^{N+1}$.[6] Anyway, $C(H, P)$ is always a constant that we will ignore from now on in order to simplify notation. If we now apply the result expressed by (70) to (67), we obtain:

$$\int_H \int_P \rho_{S\omega p}(H, P)\, \delta(H - \omega')\, \delta(P - p')\, dH\, dP = \delta(\omega - \omega')\, \delta(p - p'). \tag{71}$$

This means that:[17]

$$\rho_{S\omega p}(H, P) = \delta(H - \omega)\, \delta(P - p). \tag{72}$$

If we now go back to the variables ϕ:

$$\rho_{S\omega p}(\phi) = \lim_{\hbar \to 0} symb(\omega, p| = \delta(H(\phi) - \omega)\, \delta(P(\phi) - p). \tag{73}$$

Finally, we can obtain the classical distribution ρ_c by replacing the just obtained result (73) into (60):

[17] In the discrete case, (71) reads:

$$\int_H \int_P \rho_{S\omega p}(\phi)\, \delta(H - \omega')\, \delta^N_{Pp'}\, dH\, dP^N = \delta(\omega - \omega')\, \delta^N_{pp'}.$$

Then:

$$\rho_{S\omega' p'}(\phi) = \delta(H(\phi) - \omega')\, \delta^N(P(\phi) - p').$$

$$\rho_c(\phi) = \int_0^\infty \int_p \rho(\omega, p)\, \rho_{S\omega p}(\phi)\, d\omega\, dp$$

$$= \int_0^\infty \int_p \rho(\omega, p)\, \delta(H(\phi) - \omega)\, \delta(P(\phi) - p)\, d\omega\, dp. \qquad (74)$$

As we can see, $\rho_c(\phi)$ is a *constant of motion*, as it was expected due to the results obtained in Subsect. 3.1. Equation (74) has a clear *physical meaning*: $\rho_c(\phi)$ is a sum of densities infinitely strongly peaked on the classical trajectories defined by the constants of motion $H(\phi) = \omega$ and $P(\phi) = p$ and averaged by the density function $\rho(\omega, p)$ which is properly normalized according to (16). As a consequence, $\rho_c(\phi)$ can be conceived as sum of classical trajectories weighted by their corresponding probabilities. This leads to the expected result: classical motion takes place along a classical trajectory, and the probability of each possible trajectory is given by the initial condition ρ at $t = 0$.

It is also interesting to consider the case where the initial condition ρ at $t = 0$ is such that the factor $\rho(\omega, \omega', p, p')\, A(\omega, \omega', p, p')$ of (26) is strongly peaked around $\omega - \omega' = E$. In this case, the evolution factor $e^{i(\omega-\omega')t/\hbar}$ can be approximated by $e^{iEt/\hbar}$. This shows two facts. First, there is an interplay between the characteristic energy E and the decoherence time: the decoherence time becomes shorter as the energy is higher. Second, the limit $E \to \infty$ plays the same mathematical role as $t \to \infty$, and represents the well known "high energy limit":[5] for high energies many systems behave in an almost classical way, e.g., high energy orbits of atoms can be approximated by classical trajectories.[18]

In this section we have found the classical limit by applying the macroscopic limit to the Wigner transformation of the state ρ_* resulting from decoherence. Nevertheless, it is also possible to translate the quantum evolution equation into the classical language via the Wigner transformation in order to obtain the complete process of decoherence in classical terms. This strategy leads to the same result as the one obtained in the present section (see Appendix A).

Finally, let us recall the three problems of the traditional way of conceiving the classical limit as they were presented in sec. 1 and let us consider how and under what conditions they can be overcome from the present approach:

[18] It is worth noting, however, that while t is a perfectly well defined variable, E is just a characteristic energy that may be not well defined in some cases. For this reason, the mathematically precise strategy is to find the limit for $t \to \infty$ as we have done in the present work.

1. Although in general the Wigner function $\rho(\phi)$ of a state ρ is not non-negatively defined, we can guarantee the non-negativeness of $\rho_c(\phi)$ (74). In fact, $\rho(\omega, p)$ is non-negatively defined due to its origin, since it represents the diagonal components of the original state ρ (it can be also proved that, in general, if $\rho = \rho(\omega) = \rho(\omega, \omega)$, then the limit for $\hbar \to 0$ of $\rho(\phi)$ is non-negatively defined a.e.; see Appendix B).
2. Although only Hamiltonians of degree ≤ 2 in p and q (quadratic Hamiltonians) yield to Hamiltonian fluxes that maintain the deformation invariant, this is not a problem from the present perspective since, after decoherence, only the singular algebra remains, and in this algebra both states and observables are constants of motion.
3. Although in some cases factors of the form $\hbar^{-1}$ may appear in the Wigner state function making the limit $\hbar \to 0$ singular, we have shown that this possibility is blocked when the observables of the CSCO are Weyl observables. The requirement of working with a CSCO consisting of Weyl observables is very weak since it does not impose artificial constraints on the state ρ. Furthermore, to apply the Weyl transformation to classical observables is the usual strategy in practice for obtaining the corresponding quantum observables.

6. THE PHYSICAL MEANING OF THE CLASSICAL LIMIT

As we have seen, the classical limit of quantum mechanics involves two elements:

1. *Decoherence*: According to the self-induced approach, decoherence is a physical process that depends on the own dynamics of a closed quantum system governed by a Hamiltonian with continuous spectrum. As the result of decoherence, in the infinite time limit the mean value of any relevant observable can be computed as if the system were in the diagonal state ρ_*. In other words, decoherence transforms standard (non-Boolean) quantum mechanics into a Boolean quantum mechanics restricted to states that are diagonal in the basis defined by the CSCO $\{H, P\}$.
2. *Macroscopicity*: For $\hbar \to 0$, the Wigner transformation of the diagonal state ρ_* turns out to be $\rho_c(\phi)$, and it is resolved into

an ensemble of classical trajectories on phase space weighted by their corresponding probabilities. This means that, in the macroscopic limit, the Wigner transformation maps the Boolean description resulting from decoherence onto classical statistical mechanics.

This shows that, strictly speaking, the classical limit of quantum mechanics is not classical mechanics but classical statistical mechanics. This point deserves some further remarks.

In the classical distribution $\rho_c(\phi)$ resulting from the classical limit, the ensemble of trajectories is weighted by the non-negative function $\rho(\omega, p)$: it is precisely the fact that $\rho(\omega, p)$ is non-negatively defined what permits it to be interpreted as a probability function. Nevertheless, the formal agreement between $\rho_c(\phi)$ and a density distribution in standard classical statistical mechanics does not mean that both have the same physical meaning. In fact, in classical statistical mechanics probabilities are conceived as a sort of measure of our ignorance about the real deterministic classical trajectory. On the contrary, since the $\rho(\omega, p)$ are the diagonal components of the original quantum state ρ, they represent quantum probabilities which, as many no-go theorems show, are irreducible. Of course, this fact does not mean that a particular classical trajectory cannot be chosen. Let us suppose that we prepare the system at $t = 0$ in an initial condition ρ such that its singular part ρ_S is an eigenstate $(\omega, p|$. In this case, as a consequence of decoherence and macroscopicity, we will obtain the particular trajectory defined by the constants of motion $H(\phi) = \omega$ and $P(\phi) = p$ with certainty. This shows that, although $\rho(\omega, p)$ in the classical distribution $\rho_c(\phi)$ represents quantum irreducible probabilities, a particular classical trajectory can always be selected by means of the proper preparation of the quantum initial conditions.

Finally, it is worth stressing the emergent nature of classicality as explained by the present approach. As we have seen, the off-diagonal terms of the quantum state $\rho(t)$ never vanish through the unitary quantum evolution. Strictly speaking, what self-induced approach shows is that, in the infinite time limit, for any $A \in \mathcal{A}$, $\langle A \rangle_{\rho(t)}$ can be computed *as if* the system were in the diagonal state ρ_*. In fact, the limit for $t \to \infty$ of $\langle A \rangle_{\rho(t)}$ could also be computed in the Heisenberg picture as the limit for $t \to \infty$ of $\langle A(t) \rangle_\rho$; in this case we would obtain a diagonal operator A_*. This fact clearly shows that the fundamental magnitude in the explanation of decoherence is $\langle A \rangle_{\rho(t)} = \langle A(t) \rangle_\rho$ and not ρ_* nor A_*. In other words, decoherence should be conceived as a coarse-grained process that describes the evolution of the state $\rho(t)$ from the observational viewpoint

given by the observable A.[19] As a consequence, classicality is an emergent property that arises in a coarse-grained level of description.[20] The classical limit shows that, from the point of view given by A, as the result of decoherence and macroscopicity the quantum system behaves *as if* it were a classical statistical system. This means that our measurements of the mean value of any relevant observable A on the quantum system will give the same results as those we would obtain on a classical statistical system described as an ensemble of classical trajectories weighted by their corresponding probabilities. The distinction between the fundamental and the coarse-grained levels of description permits us to understand how the Boolean and deterministic classical world objectively arises from an underlying non-Boolean and indeterministic quantum level.

7. CONCLUSION

Einstein was right when he considered the idea that the limit $\hbar \to 0$ is the right classical limit as an oversimplification. On the basis of the assumption that the problem of the classical limit of quantum mechanics amounts to the question of how the classical world arises from an underlying quantum reality, our account of the problem involves two elements. The first one is self-induced decoherence, conceived as a process that depends on the own dynamics of a closed quantum system governed by a Hamiltonian with continuous spectrum; the study of decoherence was addressed by means of formal tools derived from the van Hove formalism. The second element is macroscopicity represented by the limit $\hbar \to 0$; we have shown that, when the macroscopic limit is applied to the Wigner transformation of the diagonal state resulting from decoherence, the description of the quantum system becomes equivalent to the description of an ensemble of classical trajectories on phase space weighted by their corresponding probabilities. Furthermore, this approach to the classical limit explains under what conditions the problems arising from traditional approach can be avoided. Finally, when these formal results are considered in the light of a generalized concept of coarse-graining, decoherence turns out to be a coarse-grained process that, in the infinite time limit, leads

[19] $\langle A\rangle_{\rho(t)} = (\rho|A)$ can be thought as a generalized "projection" of the state ρ. In fact, we can define a projector Π belonging to $\mathcal{A}\otimes\mathcal{A}^*$ as $\Pi = A\rho_A$, where ρ_A satisfies $(\rho_A|A) = 1$ (this condition guarantees that $\Pi^2 = \Pi$). In this case, $\rho_{\text{rel}} = (\rho|A)\rho_A$, where ρ_{rel} is the projected part of ρ relevant for decoherence. Since coarse-graining amounts to a projection (see Ref. 28), $\langle A\rangle_{\rho(t)}$ can be conceived as a coarse-grained magnitude.

[20] An interesting discussion about emergence and reductionistic relations between the various levels of the quantum mechanical descriptions can be found in Refs. 29 and 30.

to classicality when the system is macroscopic enough. Since there is no subjective element involved in this process, from our approach classicality is a property that objectively emerges from the underlying quantum world.

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APPENDIX A: THE QUANTUM EVOLUTION IN CLASSICAL TERMS

The quantum evolution of the system can be translated into the classical language via the Wigner transformation. The phase space analogue of the Liouville–von Neumann equation for $\hbar \to 0$, Eq. (52), is:

$$\partial_t \rho(t) = \{H, \rho\}_{mb}. \tag{A.1}$$

Let us now compute $\rho(\phi, t) = symb\rho(t)$, which is equal, up to the order $\hbar^2$ (hence, it is in fact the limit when $\hbar \to 0$), to:

$$\rho(\phi, t) = \rho\left(H(\phi), P(\phi)\right) + \int_0^\infty \int_0^\infty \int_p \int_{p'} \rho(\omega, \omega', p, p') \times e^{i(\omega-\omega')t/\hbar}\, symb(\omega, \omega', p, p' |\, d\omega\, d\omega'\, dp\, dp'. \tag{A.2}$$

Here we have used the Liouville equation for the regular part, and we have kept the singular part unchanged since we know that it is time invariant. As in the quantum case, we can call the first term the "invariant part" and the second term the "fluctuating part" of $\rho(\phi, t)$.[16,21] On this basis we can also compute:

$$\begin{aligned}(\rho(\phi, t)|A(\phi)) = & \int_H \int_P \rho(H(\phi), P(\phi))\, A(H(\phi), P(\phi))\, dH\, dP \\ & + \int_0^\infty \int_0^\infty \int_p \int_{p'} \rho(\omega, \omega', p, p')\, e^{i(\omega-\omega')t/\hbar} \\ & \times A(\omega, \omega', p, p')\, d\omega\, d\omega'\, dp\, dp',\end{aligned} \tag{A.3}$$

where, again, we call the first term "invariant part" and the second term "fluctuating part". Now, if the product $\rho(\omega, \omega', p, p')A(\omega', \omega, p', p)$ is integrable, we can use the Riemann–Lebesgue theorem to conclude that:

$$\lim_{t\to\infty}(\rho(\phi, t)|A(\phi)) = \int_H \int_P \rho(H(\phi), P(\phi))\ A(H(\phi), P(\phi))\, dH\, dP$$

$$= (\rho_*(\phi)|A(\phi)) \tag{A.4}$$

for any $A(\phi) \in \mathcal{L}$ and for any $\rho(\phi) \in \mathcal{L}^*$ with the right properties. Therefore:

$$\rho_*(\phi) = \int_H \int_P \rho(H(\phi), P(\phi))\, dH\, dP$$

$$= \int_0^\infty \int_p \rho(\omega, p)\, \delta(H(\phi) - \omega)\, \delta(P(\phi) - p)\, d\omega\, dp. \tag{A.5}$$

This last equation is precisely the Wigner transformation of ρ_* for $\hbar \to 0$, as obtained in Eq.(74). In $\rho_*(\phi)$ the non-diagonal terms have disappeared and only the diagonal (singular) terms remain. Thus, we have found the weak limit:

$$w - \lim_{t\to\infty} \rho(\phi, t) = \rho_*(\phi), \tag{A.6}$$

which express the result of decoherence.

APPENDIX B: POSITIVITY OF THE WIGNER FUNCTION OF $\rho(\omega)$

Here we will prove that the Wigner function of a state represented by a density operator of the form $\rho(\omega) = \rho(\omega, \omega)$ is positively defined a.e. in the limit $\hbar \to 0$. This proof is a reformulation of an argument due to Narcovich.[31,32].

Let us call $\rho_\hbar(q, p)$ the Wigner function for the density operator ρ in terms of $\hbar$. We will call:

$$\lim_{\hbar\to 0} \rho_\hbar(q, p) = G(q, p). \tag{B.1}$$

We will prove that $G(q, p)$ is non-negative (a.e.):

$$G(q, p) \geq 0. \tag{B.2}$$

Let us call:

$$a = \begin{pmatrix} q \\ p \end{pmatrix}, \qquad z = \begin{pmatrix} q' \\ p' \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{B.3}$$

and

$$\sigma(a, z) = aJz = (q, p) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = qp' - pq'. \tag{B.4}$$

Now we consider the inverse symplectic Fourier transformation of $\rho_\hbar(q, p) = \rho_\hbar(\phi)$ given by:

$$f_\hbar(z) = \int \rho_\hbar(\phi)\, e^{i\sigma(\phi, z)}\, dz, \tag{B.5}$$

where $dz = d\mathbf{q}\, d\mathbf{p}$. This transformation (i.e., the function $_\hbar(z)$) is of $\hbar$-positive type (see Refs. Refs. 31 and 32), which means that:

$$\sum_{j,k=1}^{m} f_\hbar(a_j - a_k)\, e^{i(\hbar/2)\sigma(a_k, a_j)}\, \lambda_j^* \lambda_k \geq 0, \tag{B.6}$$

where:

1. $a_1, a_2, \ldots, a_m$ are arbitrary points of the phase space:

$$a_k = \begin{pmatrix} q_k \\ p_k \end{pmatrix}. \tag{B.7}$$

2. $\lambda_1, \lambda_2, \ldots, \lambda_m$ are arbitrary numbers.

3 $m = 1, 2, \ldots$ is an arbitrary finite positive integer.

Let us now consider:

$$g(z) = \lim_{\hbar \to 0} f_\hbar(z) \tag{B.8}$$

and see whether $g(z)$ is positive in the sense of Bochner. If we make the limit $\hbar \to 0$ in Eq. (B.5), we obtain:

$$\sum_{j,k=1}^{m} g(a_j - a_k)\, \lambda_j^* \lambda_k \geq 0, \tag{B.9}$$

where the notation has been defined in 1, 2, 3 right above. Thus, the function $g(z)$ is positive in the sense of Bochner. Therefore, the Fourier transform $\varphi(q, p)$ is also positive (a.e.):

$$\varphi(q, p) = \int g(z)\, e^{-iza}\, dz \geq 0. \tag{B.10}$$

This property is inherited by the symplectic Fourier transform of $g(z)$, that we have called $G(q, p)$:

$$G(q, p) = \varphi(p, -q). \tag{B.11}$$

Then, from Eq. (B.8) we have:

$$\begin{aligned} \lim_{\hbar \mapsto 0} \rho_\hbar(q, p) &= \int \left[\lim_{\hbar \mapsto 0} f_\hbar(z) \right] e^{-i\sigma(a,z)}\, dz \\ &= \int g(z)\, e^{-i\sigma(a,z)}\, dz = G(q, p) \geq 0 \qquad (a.e) \end{aligned} \tag{B.12}$$

where $\rho_\hbar(q, p)$ is an arbitrary regular Wigner function, q.e.d. Nevertheless, it is worth stressing that the non-negativeness obtained through decoherence is stronger than this result: $\rho_c(\phi)$ is non-negatively defined on the whole phase space, and not only a.e.

APPENDIX C: CHARACTERISTIC TIMES AND TRANSITIONS FROM THE DISCRETE TO THE CONTINUOUS

We add here a brief appendix in which we discuss the problem of characteristic times and the discrete to continuous transitions. Thermodynamic limit, normalization in a box followed by a limiting process, etc. have been used very often to describe systems with continuous spectrum in which these systems are taking as a limit of a sequence of systems with discrete spectrum. All these are typical situations of transitions discrete-continuum. In addition, this problem has been studied in the context of self-induced decoherence (SID), a concept used in the present paper. The results sketched here are extensively discussed in Refs. 9, 8, 33, 34, 35 and 21.

C.1. Characteristic Times

In order to compare the behavior of models with discrete spectrum with those showing a continuous spectrum, one should make an analysis on characteristic times. In particular, we compare the characteristic times that arose in the self-induced decoherence and the most traditional point of view associated to the environment induced decoherence (EID).

In the SID one uses a closed system,[6–8,21] named U, that can be split into the system under study S plus the environment E, so that $U = S \cup E$. This decomposition, although sometimes rather artificial, is necessary if we want to compare SID with EID.[4] Conventionally, the whole space U is often called the "Universe". There are two kinds of characteristic times, those in connection with the system S, considered as an unstable state which tends toward a state of equilibrium, and those related with the universe U. Thus, we have:

(i) Two characteristic times related with S:

- Decoherence *dynamical* time τ_{DS} of the system S in the pointer basis defined by the *predictability sieve* of EID. This decoherence receives the name of *dynamical decoherence*.
- Relaxation time τ_{RS}. The state $\rho_S(t)$ of S goes to the equilibrium state given by ρ_{*S}, which trivially decoheres in the eigenbasis for ρ_{*S}. Then, the eigenbasis for ρ_{*S} is the pointer basis. This decoherence is called *statistical decoherence*.

These decoherence times are related through a very simple equation as for instance:

$$\tau_{DS} = \tau_{RS} \left(\frac{L_{\mathrm{DB}}}{L_0} \right)^2,$$

where L_{DB} is the de Broglie length and L_0 a characteristic macroscopic length.[9]

(ii) Two times related with the Universe U.

- Relaxation time of the universe U, which is also the decoherence time of the static decoherence of SID.[21] This is the time for the state $\rho_U(t)$ to arrive to the equilibrium state ρ_{*U}, which is also the decoherence time because ρ_{*U} obviously decoheres in the eigenbasis of ρ_{*U}.
- The recurrence or revival time of U, that we denote as τ_{RU}.

C.2. Discrete Versus Continuous Spectrum

Fortunately, we have an interesting discussion on the transition discrete continuous on a simple model (A harmonic oscillator on a bath of harmonic oscillators with a linear coupling, this can be considered as a *discrete Friedrichs model*) in Ref. 33. The Hamiltonian of the universe U has discrete spectrum with eigenvalues $\{\alpha_\nu\}$. It is proven in Ref. 33 that

$$\tau_{RU} = \frac{2\pi\hbar}{\min(\alpha_{\nu+1} - \alpha_\nu)}.$$

Now, if the number of particles goes to infinite (with some conditions such that the spectrum goes to a dense set in the real line), this discrete model approaches a similar model with continuous spectrum for times fulfilling the condition (see Ref. 33 also Ref. 34):

$$t \ll \tau_{RU}.$$

With respect to the behavior of characteristic times for systems having discrete and continuous spectrum, we can summarize the situation as follows:

(i) For systems with a discrete spectrum:

- Systems with discrete spectrum never decohere as $\tau_{RU} \neq \infty$. Poincaré recurrence theorem is then valid and we recover the initial conditions after the revival time τ_{RU}.
- The subsystem S dynamically decoheres for all practical purposes if $t > \tau_{DS}$.
- Decoherence for discrete spectrum systems is produced for some limit as previously explained.[33] In this case, if $t \ll \tau_{RU}$ the system with a discrete spectrum is almost indistinguishable from another with continuous spectrum. A similar situation in the context of EID is discussed in Ref. 36.
- In the example discussed in Ref. 33, we see S arrives to a state of equilibrium and then U as a whole does arrive to equilibrium. Thus, both statistcally decohere.

(ii) For system with a continuous spectrum:

- The revival time $\tau_{RU} \longmapsto \infty$ and U decoheres.[4]

- If the system is described by a Hamiltonian whose resolvent has no poles that can be looked as resonance poles, there is no decoherence in proper sense as $\tau_{DU} \longmapsto \infty$. Only systems with finite decoherence time decohere. In the case that U reaches the equilibrium, also $S \subset U$ reaches equilibrium and therefore S statistically decoheres.

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