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A Painlevé II model in two-ion electrodiffusion with radiation boundary conditions

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ABSTRACT

Existence, uniqueness, and multiplicity properties are established via a variational formulation for a Painlevé II model subject to radiation boundary conditions in two-ion electrodiffusion. Numerical experiments using an adapted shooting method are also presented to support the theoretical results.

1. Introduction

A model equation of Painlevé II type was derived independently by Grafov and Chernenko [1] and Bass [2] in the context of charged ion transport. Thus, in [2], two-ion electrolytic phenomena in the presence of a planar boundary were investigated, and the classical Painlevé II equation was derived *ab initio* for the electric field via the Nernst–Planck system. Additional literature on the derivation of the Painlevé II equation in the context of ionic transport is cited in Volgin and Davydov [3]. Two-point Dirichlet and periodic boundary value problems (BVPs) for this Painlevé II equation and a non-integrable generalization in two-ion electrodiffusion were investigated successively in [4,5]. In particular, topological methods were applied to establish the existence of solutions under appropriate conditions on the physical parameters. Three-ion BVPs were also investigated in [5,6]. In [6], an integro-differential formulation was adopted, and boundedness properties were established via the method of upper and lower solutions (see, for example, De Coster and Habets [7]). The latter method was later applied to a generalized Painlevé II equation in [8], and existence results were obtained by a diagonal argument. A two-point Neumann BVP for the Painlevé II model of two-ion electrodiffusion was recently investigated by Amster et al. in [9,10]. The BVP is unconventional in that the model equation involves the yet-to-be-determined boundary values of the solution. A novel two-dimensional shooting method was used to establish existence properties, and a practical algorithm was presented for the numerical solution of the BVP.

In [11], Rogers et al. returned to the BVP originally posed and treated approximately by Bass in [2]; it determines the electric field distribution in a region $x > 0$ occupied by an electrolyte. An auto-Bäcklund transformation admitted by the Painlevé II equation was applied iteratively to construct exact representations for the electric field distribution for BVPs wherein the ratio of the fluxes of the positive and negative ions adopts one of an infinite sequence of values. These representations involve either Yablonski–Vorob'ev polynomials or classical Airy functions. The requirement that the electric field distributions and ion concentrations in these representations be non-singular imposes constraints on the physical parameters. These constraints, along with asymptotic properties, were investigated in detail by Bass et al. in [12].

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In recent work by Bracken et al. [13], novel flux quantization aspects associated with the iterative action of the Bäcklund transformations were investigated. Exact analytic expressions were obtained for the electric field and ionic concentrations in well-stirred reservoirs exterior to the junction boundaries. Radiation boundary conditions applied to two-point BVPs for the Painlevé II equation were derived in this connection.

To date, existence and uniqueness properties for such Robin-type BVPs for the Painlevé II equation do not seem to have been treated in the literature. In this paper, a variational approach is adopted in order to treat this class of two-ion BVPs.

2. The main results

Here, we consider a two-ion electrodiffusion BVP on $0 < x < 1$ in which the electric field $E(x)$ is given by the Painlevé II equation

$$\lambda^2 E''(x) = \frac{1}{2}\lambda^2 E(x)^3 + 2[c_{-\infty} + (c_{+\infty} - c_{-\infty})x]E(x) + A \tag{1}$$

subject to radiation-type boundary conditions (see [13]):

$$\lambda_0 E'(0) = E(0), \qquad \lambda_1 E'(1) = E(1), \tag{2}$$

where

$$\lambda_0 = \lambda/\sqrt{2c_{-\infty}}, \qquad \lambda_1 = \lambda/\sqrt{2c_{+\infty}}. \tag{3}$$

In the above, $\lambda > 0$ is a dimensionless constant, and the constants $c_{\pm\infty} > 0$ and A are regarded as given. Here,

$$A = -2(\alpha_+ - \alpha_-)(c_{+\infty} - c_{-\infty}) - 2j_0,$$

where

$$\alpha_\pm = \frac{D_\pm}{D_+ + D_-},$$

with $D_\pm$ being the diffusion constants corresponding to the two ionic species. It is noted that $\alpha_\pm > 0$ and $\alpha_+ + \alpha_- = 1$. The quantities $c_{+\infty}, c_{-\infty}$, and j_0 denote ionic concentrations and current density, respectively (see [13]). Replacing E by $-E$ if necessary, throughout the paper we shall assume, without loss of generality, that $A \geq 0$. By redefining φ and A, we can further assume, without loss of generality, that $\lambda = 1$, but we refrain from doing that.

The following results will be established.

Theorem 2.1. *Problem (1)–(2) has exactly one negative solution. Moreover, there are at most two positive solutions, and the set of all solutions is bounded in the C^2-norm.*

Theorem 2.2. (i) *If $c_{-\infty} \geq c_{+\infty}$, then (1)–(2) has a unique solution.*
(ii) *If $c_{-\infty} < c_{+\infty}$, then there exist positive constants $A_* < A^*$ such that the following hold.*
 (a) *If $A < A_*$, then (1)–(2) admits at least three classical solutions.*
 (b) *If $A > A^*$, then (1)–(2) has a unique solution.*

Section 3 is devoted to proving the preceding results by variational methods. It is worth recording that the boundary conditions (2) differ from the standard Robin-type conditions in the crucial fact that both λ_0 and λ_1 are positive. Thus, the associated functional has the form $J(E) := J_0(E) - \lambda_1 J_1(E)$, where J_0 is coercive and J_1 takes arbitrarily large positive values over a one-dimensional subspace of $H^1(0, 1)$. In this setting, the proof of the existence part of our first result will consist of showing that the functional is still coercive, and deducing from this fact the existence of a global minimum. In broad terms, we shall see that, if $\|E\|_{H^1}$ is large, then $J_1(E)$ is small enough when compared with $J_0(E)$. In fact, this is immediate when λ_1 is large. It is important to note that, from (3), it is seen that larger values of λ_1 imply that $J_0(E)$ tends to ∞ faster than $J_1(E)$ as $\|E\|_{H^1} \to \infty$. Theorem 2.1 is proved with this taken into consideration.

The beginning part of our second result states that, if $c_{-\infty} \geq c_{+\infty}$, then the solution is unique. In the contrary case, under a smallness assumption on the independent term of the equation, the functional satisfies the hypotheses of a linking-type theorem. This allows us to prove the existence of at least one extra local minimum and a saddle-type critical point.

In Section 4, we present some examples obtained using the shooting method. The purposes are to support our theoretical results with concrete numerical evidence, and to shed light on further properties of the solutions.

3. Proofs of the main results

In this section, we introduce a variational formulation for the BVP (1)–(2). For convenience, we set

$$\varphi(x) := c_{-\infty} + (c_{+\infty} - c_{-\infty})x.$$

Eq. (1) can be rewritten in the simpler form

$$E''(x) = \frac{1}{2}E(x)^3 + \frac{2\varphi(x)}{\lambda^2}E(x) + \frac{A}{\lambda^2}. \tag{4}$$

Note that $\varphi(x) > 0$ for all $x \in [0, 1]$.

Define the functional $J : H^1(0, 1) \to \mathbb{R}$ by

$$J(E) := \int_0^1 \left(\frac{\lambda^2}{2}E'(x)^2 + \frac{\lambda^2}{8}E(x)^4 + \varphi(x)E(x)^2 + AE(x)\right) dx + \lambda_0 c_{-\infty}E(0)^2 - \lambda_1 c_{+\infty}E(1)^2.$$

It is readily shown that $J \in C^1(H^1(0, 1), \mathbb{R})$, with

$$DJ(E)(\phi) = \int_0^1 \lambda^2 E'(x)\phi'(x) + \frac{\lambda^2}{2}E(x)^3\phi(x) + 2\varphi(x)E(x)\phi(x) + A\phi(x)dx \\ + 2[\lambda_0 c_{-\infty}E(0)\phi(0) - \lambda_1 c_{+\infty}E(1)\phi(1)].$$

Suppose that E is a critical point of J. We claim that it is a solution of our BVP.

First, by considering $\phi \in C_0^1(0, 1)$, we deduce that E is a weak solution of (1). As $H^1(0, 1) \hookrightarrow C([0, 1])$, it follows that E has a continuous second-order weak derivative, and hence it is classical. Now, taking arbitrary $\phi \in H^1(0, 1)$, and integrating by parts the equality $DJ(E)(\phi) = 0$, we have

$$\lambda^2[E'(1)\phi(1) - E'(0)\phi(0)] = 2[\lambda_1 c_{+\infty}E(1)\phi(1) - \lambda_0 c_{-\infty}E(0)\phi(0)].$$

By choosing ϕ such that $\phi(0) = 0 \neq \phi(1)$, we see that $\lambda^2 E'(1) = 2\lambda_1 c_{+\infty}E(1)$. In the same way, by choosing ϕ such that $\phi(1) = 0 \neq \phi(0)$, we deduce that $\lambda^2 E'(0) = 2\lambda_0 c_{-\infty}E(0)$. Hence, E satisfies (2), and is a solution of our BVP.

Proof of Theorem 2.1.

Existence

We claim that J achieves a global minimum in $H^1(0, 1)$, which is a critical point, and hence it is a solution of our BVP.

By standard results (see e.g. Mawhin and Willem [14]), J is weakly lower semi-continuous. Therefore, it suffices to prove that J is coercive in order to establish the claim. Suppose that this is false, namely, that there exists some sequence $\{E_n\}$ such that $\|E_n\|_{H^1} \to \infty$, but $\{J(E_n)\}$ is bounded above.

First, let us observe that, for given $E \in H^1(0, 1)$, we may fix an $x_0 \in [0, 1]$ such that $|E(x_0)| = \min_{x\in[0,1]} |E(x)|$, and write

$$E(1) = E(x_0) + \int_{x_0}^1 E'(x)dx.$$

Then

$$|E(1)| \le |E(x_0)| + \int_0^1 |E'(x)|\, dx \le \|E\|_{L^2} + \|E'\|_{L^2}. \tag{5}$$

It follows that, for arbitrary $\varepsilon > 0$,

$$E(1)^2 \le \left(1 + \frac{1}{\varepsilon}\right) \|E\|_{L^2}^2 + (1 + \varepsilon)\, \|E'\|_{L^2}^2.$$

If $\|E_n'\|_{L^2} \le M$ for some constant M and all n, then, from the assumption that $\|E_n\|_{H^1} \to \infty$, we see that $\|E_n\|_{L^2} \to \infty$, which implies that $\|E_n\|_{L^4} \to \infty$. Moreover, from (5), $|E_n(1)| \le \|E_n\|_{L^4} + M$, and thus

$$J(E_n) \ge \|E_n\|_{L^4}^2 \left(\frac{\lambda^2}{8}\|E_n\|_{L^4}^2 - \frac{A}{\|E_n\|_{L^4}} - \lambda_1 c_{+\infty}\frac{E_n(1)^2}{\|E_n\|_{L^4}^2}\right) \to +\infty,$$

a contradiction.

Hence, we may suppose that $\|E_n'\|_{L^2} \to \infty$, and observe that, for n large enough,

$$J(E_n) \ge \|E_n'\|_{L^2}^2 \left(\frac{\lambda^2}{4} + \frac{\lambda^2}{8}\frac{\|E_n\|_{L^4}^4}{\|E_n'\|_{L^2}^2}\right) - \lambda_1 c_{+\infty}E_n(1)^2.$$

If $\frac{\|E_n\|_{L^4}^4}{\|E_n'\|_{L^2}^2} \to \infty$, then, for any $\varepsilon > 0$, we have

$$J(E_n) \ge \|E_n'\|_{L^2}^2 \left[C(\varepsilon) + \frac{\|E_n\|_{L^4}^4}{\|E_n'\|_{L^2}^2}\left(\frac{\lambda^2}{8} - D(\varepsilon)\frac{\|E_n\|_{L^2}^2}{\|E_n\|_{L^4}^4}\right)\right],$$

where

$$C(\varepsilon) := \frac{\lambda^2}{4} - \lambda_1 c_{+\infty}(1+\varepsilon)$$

and

$$D(\varepsilon) := \lambda_1 c_{+\infty}\left(1+\frac{1}{\varepsilon}\right).$$

As $\|E_n\|_{L^2} \le \|E_n\|_{L^4} \to \infty$, we deduce again that $J(E_n) \to +\infty$, a contradiction.

Finally, by taking a subsequence, we may suppose that $\frac{\|E_n\|_{L^4}^4}{\|E_n'\|_{L^2}^2} \le M$ for some constant M. Let $F_n := \frac{E_n}{\|E_n'\|_{L^2}}$; then $\|F_n\|_{L^2} \to 0$ and $\|F_n'\|_{L^2} = 1$. From the compact embedding $H^1(0,1) \hookrightarrow C([0,1])$, we may suppose that $F_n \to 0$ uniformly; in particular, $F_n(1) \to 0$. Now write

$$J(E_n) \ge \|E_n'\|_{L^2}^2\left(\frac{\lambda^2}{4} - \lambda_1 c_{+\infty} F_n(1)^2\right),$$

and the same contradiction $J(E_n) \to +\infty$ follows. This completes the proof of the coerciveness of J and, hence, proves the existence of a solution.

Boundedness of the solution set

We first establish several lemmas which are both of intrinsic interest and relevant to our purpose.

Lemma 3.1. *Let $b \in (0, +\infty]$, and let $E, \tilde{E} : [0, b) \to \mathbb{R}$ be classical solutions of (1) subject to the first boundary condition in (2). If there exists $x_0 \in [0, b)$ such that $E(x_0) > \tilde{E}(x_0)$, then $E(x) > \tilde{E}(x)$ and $E'(x) > \tilde{E}'(x)$ for all $x \in [0, b)$.*

Proof. Let $\phi := E - \tilde{E}$. Then

$$\lambda^2 \phi''(x) = \left[\frac{\lambda^2}{2}\mu(x) + 2\varphi(x)\right]\phi(x), \tag{6}$$

where $\mu(x) := E(x)^2 + E(x)\tilde{E}(x) + \tilde{E}(x)^2 \ge 0$. The "coefficient" of ϕ on the right-hand side of the equation (i.e. the expression in square brackets) is positive.

Let us first consider the special case $x_0 = 0$. This implies that $\phi(0) > 0$ and $\phi'(0) = \phi(0)/\lambda_0 > 0$. The conclusion follows from the fact that the right-hand side of (6) remains positive for all $x \in [0, b)$.

The case $x_0 > 0$ is an immediate corollary of the special case; the hypothesis implies that $E(0) > \tilde{E}(0)$, because the contrary case would have led to $E(x) \le \tilde{E}(x)$ for all $x \in [0, b)$ and, in particular, for $x = x_0$. $\square$

Although the hypotheses of Lemma 3.1 do not involve the right endpoint boundary condition of the solutions, in this section we will only apply the result to a set of solutions that satisfy both boundary conditions. The significance of the lemma is that solutions of the BVP do not intersect each other, and thus they can be linearly ordered.

We say that a solution $E(x)$ is positive (negative) if $E(x) > (<) 0$ for all $x \in (0, 1)$.

Lemma 3.2. *There are at most two positive solutions.*

Proof. Suppose on the contrary that there are three distinct positive solutions, $0 < E_1(x) < E_2(x) < E_3(x)$. Define $\phi_1(x) := E_2(x) - E_1(x)$ and $\phi_2(x) := E_3(x) - E_2(x)$. Then they satisfy, respectively, differential equations of the form

$$\phi_1''(x) = Q_1(x)\phi_1(x) \quad \text{and} \quad \phi_2''(x) = Q_2(x)\phi_2(x)$$

that are analogues of (6), where Q_1 and Q_2 are appropriate adaptations of the expression in square brackets on the right-hand side of (6). It is obvious that

$$Q_1(x) < Q_2(x), \quad \text{for all } x \in [0, 1].$$

Using the boundary condition at the initial point $x = 0$, we see that

$$\frac{\phi_1'(0)}{\phi_1(0)} = \frac{E_2'(0) - E_1'(0)}{E_2(0) - E_1(0)} = \frac{1}{\lambda_0} = \frac{E_3'(0) - E_2'(0)}{E_3(0) - E_2(0)} = \frac{\phi_2'(0)}{\phi_2(0)}.$$

From standard Sturm comparison theory, we conclude that

$$\frac{\phi_1'(x)}{\phi_1(x)} < \frac{\phi_2'(x)}{\phi_2(x)}, \quad \text{for } x \in (0, 1).$$

In particular,

$$\frac{\phi_1'(1)}{\phi_1(1)} < \frac{\phi_2'(1)}{\phi_2(1)}.$$

This contradicts the required boundary condition at $x = 1$, which implies that

$$\frac{\phi_1'(1)}{\phi_1(1)} = \frac{E_2'(1) - E_1'(1)}{E_2(1) - E_1(1)} = \frac{1}{\lambda_1} = \frac{E_3'(1) - E_2'(1)}{E_3(1) - E_2(1)} = \frac{\phi_2'(1)}{\phi_2(1)}.$$

Thus, there cannot be three positive solutions. □

We remark that exactly the same arguments as used in the above proof are also applicable to negative solutions. However, a stronger result actually holds for negative solutions.

Lemma 3.3. *There is at most one negative solution.*

Proof. Suppose on the contrary that there are two distinct negative solutions, $E_1(x) < E_2(x) < 0$, which satisfy the "linearized" equations

$$E_i''(x) = P_i(x)E_i(x), \quad i = 1, 2,$$

respectively, where

$$P_i(x) = \left[\frac{E_i(x)^2}{2} + \frac{2\varphi(x)}{\lambda^2} + \frac{A}{\lambda^2 E_i(x)}\right].$$

Since $E_i(x) < 0$, we see that P_i is a decreasing function of E_i, and so

$$P_1(x) > P_2(x).$$

Note that this is not true if E_i is not negative, and that is why positive solutions behave differently. At the initial boundary point,

$$\frac{E_1'(0)}{E_1(0)} = \frac{E_2'(0)}{E_2(0)} = \frac{1}{\lambda_0}.$$

Again, Sturm comparison theory yields

$$\frac{E_1'(1)}{E_1(1)} < \frac{E_2'(1)}{E_2(1)},$$

contradicting the second boundary condition. □

Lemma 3.4. *Assume that $A > 0$. Suppose that $E(x)$ is a solution that changes sign. Then it has a unique zero x_0, such that $E(x_0) = 0$. In $(0, x_0)$, $E(x) < 0$. In $(x_0, 1]$, $E(x) > 0$, $E'(x) > 0$, and $E''(x) > 0$.*

Proof. First, we show that $E(0) < 0$. Suppose the contrary, i.e. that $E(0) \geq 0$. Then $E'(0) \geq 0$, and, by (4), $E''(0) > 0$. Then usual comparison arguments can be used to show that $E(x)$ is convex and increasing. Thus, it remains positive for all $x \in (0, 1]$, contradicting the assumption.

Since $E(x)$ changes sign, it must have at least one zero. Let x_0 be the smallest of all the zeros. In $(0, x_0)$, $E(x) < 0$. At x_0, $E'(x_0) \geq 0$ and $E''(x_0) > 0$. Comparison arguments again show that $E(x)$ is strictly convex (i.e. that $E''(x) > 0$) and increasing for $x > x_0$, and so $E(x)$ cannot have another zero. □

Lemma 3.5. *Assume that $A > 0$, and let $E_{\min}$ be a global minimizer of J; that is, $J(E_{\min}) = \min_{E \in H^1(0,1)} J(E)$. Then $E_{\min}(x) < 0$ for all x.*

Proof. Suppose that the statement of the lemma is false. Then, from the previous lemma, there are only two possibilities.

1. If $E_{\min}(1) > 0$, then take $E(x) := -|E_{\min}(x)|$. It is seen that $E \in H^1(0, 1)$ and $J(E) < J(E_{\min})$, a contradiction.
2. If $E_{\min}(1) = 0$, then $E_{\min}(x) < 0$ for $x < 1$. As $E_{\min}'(1) = 0$ and $E_{\min}''(1) > 0$, a new contradiction is obtained. □

We are now in a position to prove the boundedness of the solution set. We first show that the set is bounded above. There are two cases.

In the first case, we assume that there is a positive solution. By Lemma 3.2, there are at most two positive solutions. Let $E_+(x)$ denote either the unique positive solution or the largest of the two positive solutions. Let $E(x)$ be any other solution. If $E(x)$ is positive, it must be the smallest of the two positive solutions, and so is bounded above by $E_+(x)$. If $E(x)$ is not positive, then there exists an $x_0 \in [0, 1]$ such that $E(x_0) \leq 0 < E_+(x_0)$. By Lemma 3.1, $E(x) < E_+(x)$ for all x. Therefore, $E_+(x)$ serves as an upper bound for the solution set.

In the complementary case, there is no positive solution. Suppose that the solution set is not bounded above. Then there exists a sequence of solutions $E_1(x) < E_2(x) < \cdots < E_n(x) < \cdots$, such that $E_n(1) \to +\infty$. Without loss of generality, we may assume that

$$E_n(1) > 0 \quad \text{and} \quad E_n(x_n) = 0 \quad \text{for some } x_n \in [0, 1). \tag{7}$$

We consider two subcases. First, suppose that $\lambda_1 \geq 1$. Since $E_n(x)$ is strictly convex in $(x_n, 1)$, $E_n'(x) < E_n'(1) = E_n(1)/\lambda_1$. Thus,

$$\begin{aligned} E_n(x_n) &= E_n(1) - \int_{x_n}^1 E_n'(x)\,dx \\ &> E_n(1) - (1-x_n)\frac{E_n(1)}{\lambda_1} \\ &= \left(1 - \frac{1-x_n}{\lambda_1}\right) E_n(1) \\ &> 0, \end{aligned} \tag{8}$$

contradicting (7). Now, suppose that $\lambda_1 < 1$, and define $\alpha = 1 - \lambda_1/2$. Observe that E_n is not necessarily convex in $[\alpha, 1]$; however, for n large, it is still true that $E_n'(x) < E_n'(1)$ for all $x < 1$. Indeed, otherwise (since $E_n'(0) = E_n(0)/\lambda_0 \leq 0$), the function E_n' achieves an absolute maximum at some value $x^* \in (0, 1)$, with $E_n'(x^*) \geq E_n'(1)$ and $E_n''(x^*) = 0$. The latter implies that $E_n(x^*) < 0$ is the (unique) root of the polynomial

$$P(z) := \frac{\lambda^2}{2} z^3 + 2\varphi(x^*)z + A,$$

and, consequently, $|E_n(x^*)|$ cannot be arbitrarily large. On the other hand,

$$\begin{aligned} \lambda^2 E_n'''(x^*) &= \left[\frac{3\lambda^2}{2} E_n(x^*)^2 + 2\varphi(x^*)\right] E_n'(x^*) + 2\varphi'(x^*)E_n(x^*) \\ &\geq \left[\frac{3\lambda^2}{2} E_n(x^*)^2 + 2\varphi(x^*)\right] E_n'(1) + 2\varphi'(x^*)E_n(x^*). \end{aligned}$$

As $E_n'(1) = E_n(1)/\lambda_1 \to +\infty$, it is seen that, when n is sufficiently large, $E_n'''(x^*) > 0$, and this contradicts the fact that a maximum of E_n' is achieved at x^*. Thus, the same argument as that used to obtain (8) can be used for any $x \in [\alpha, 1]$, instead of x_n, to obtain

$$\begin{aligned} E_n(x) &> \left(1 - \frac{1-x}{\lambda_1}\right) E_n(1) \\ &\geq \left(1 - \frac{1-\alpha}{\lambda_1}\right) E_n(1) \\ &= \frac{E_n(1)}{2} \\ &> 0. \end{aligned}$$

One consequence is that $x_n < \alpha$. When combined with (4), another consequence is that

$$E_n''(x) \geq \frac{E_n(x)^3}{2} \geq \frac{E_n(1)^3}{16} \quad \text{for } x \in [\alpha, 1].$$

Integrating this inequality over $[\alpha, 1]$ leads to

$$\begin{aligned} E_n'(\alpha) &\leq E_n'(1) - \frac{(1-\alpha)E_n(1)^3}{16} \\ &= \left[\frac{1}{\lambda_1} - \frac{(1-\alpha)E_n(1)^2}{16}\right] E_n(1). \end{aligned}$$

As n becomes very large, $E_n(1)$ becomes very large, and the number given by the expression inside the square brackets will eventually become negative. Thus $E_n'(\alpha) < 0$ for large n, contradicting the last conclusion of Lemma 3.4. This completes the proof of the upper-boundedness of the solution set.

The proof of the lower-boundedness of the solution set proceeds in a similar manner, and is actually simpler. Indeed, from Lemma 3.5, we know that the global minimizer $E_{\min}$ is strictly negative, and from Lemma 3.3 there are no other negative solutions. Using now Lemma 3.1, we conclude that $E_{\min}$ serves as a lower bound of all solutions. This completes the proof of the lower-boundedness of the solution set.

Once we know that $|E(x)|$ is uniformly bounded, by (4), $|E''(x)|$ is also uniformly bounded. Hence, $E(x)$ is uniformly bounded in the C^2-norm.

Proof of Theorem 2.2. Let Φ denote the unique solution of the initial value problem

$$-\lambda^2\Phi''(x) + 2\varphi(x)\Phi(x) = 0$$
$$\Phi(0) = \lambda_0 \qquad \Phi'(0) = 1,$$

and define the functional

$$M(E) := \int_0^1 \left(\frac{\lambda^2}{2}E'(x)^2 + \varphi(x)E(x)^2\right) dx.$$

Lemma 3.6. *The following conditions are equivalent.*

(A0) $c_{-\infty} < c_{+\infty}$.
(A1) $\lambda_1\Phi'(1) < \Phi(1)$.
(A2) $M(E) < \lambda_1 c_{+\infty}E(1)^2 - \lambda_0 c_{-\infty}E(0)^2$ *for some* $E \in H^1(0,1)$.

Proof. Let us first prove the equivalence of (A1) and (A2).

If (A1) holds, then it suffices to take $E = \Phi$ to verify (A2). Conversely, let us consider the functional I defined by $I(E) := M(E) + \lambda_0 c_{-\infty}E(0)^2$. It is clear that I is coercive and that $I(E) > 0$ for $E \neq 0$. It is easy to verify that its restriction over the set $\{E \in H^1(0,1) : E(1) = 1\}$ achieves a minimum at some function E_1.

Then

$$\int_0^1 [\lambda^2 E_1'(x)\xi'(x) + 2\varphi(x)E_1(x)\xi(x)]dx + 2\lambda_0 c_{-\infty}E_1(0)\xi(0) = \mu\xi(1) \tag{9}$$

for all $\xi \in H^1(0,1)$, where μ is a Lagrange multiplier. It follows that

$$-\lambda^2 E_1''(x) + 2\varphi(x)E_1(x) = 0,$$
$$\lambda_0 E_1'(0) = E_1(0), \qquad \lambda^2 E_1'(1) = \mu,$$

and (A2) implies that $\mu = 2I(E_1) < 2\lambda_1 c_{+\infty}$. Thus, (A1) follows from the fact that $\Phi(x) = E_1(x)/E_1(0)$.

Next, we show that (A0) is equivalent to (A1). Suppose that (A0) holds. Then $\lambda_0 > \lambda_1$ and $\varphi(x) < c_{+\infty}$ for $x \in [0,1)$. Let $v(x) := e^{x/\lambda_1}$. A simple computation shows that

$$\int_0^1 \Phi''(x)v(x)dx < \int_0^1 \Phi(x)v''(x)dx,$$

and hence

$$\Phi'(1)v(1) - \frac{v(0)}{\lambda_0} < \Phi(1)\frac{v(1)}{\lambda_1} - \frac{v(0)}{\lambda_1}.$$

We conclude that $\lambda_1\Phi'(1) < \Phi(1)$, so (A1) holds. If, on the contrary, we assume that $c_{-\infty} \geq c_{+\infty}$, the previous computations still hold, with all the inequalities reversed, and we deduce that $\lambda_1\Phi'(1) \geq \Phi(1)$. □

In order to prove (i), assume that $c_{-\infty} \geq c_{+\infty}$, and suppose that the BVP has two different solutions E and $\tilde{E}$. As before, $\phi := E - \tilde{E}$ satisfies (6). Multiplying (6) by ϕ and integrating yields

$$2c_{+\infty}\lambda_1\phi(1)^2 = 2c_{-\infty}\lambda_0\phi(0)^2 + \lambda^2\|\phi'\|_{L^2}^2 + \int_0^1 \left[\frac{\lambda^2}{2}\mu(x) + 2\varphi(x)\right]\phi(x)^2\,dx. \tag{10}$$

From this, we deduce that (A2) holds, a contradiction. Thus, the first statement of Theorem 2.2 is proved.

For the proof of (ii)(a), we shall make use of a linking theorem by Rabinowitz [15]. Let us recall the following definitions for a Banach space $\mathbb{B}$ and $I \in C^1(\mathbb{B}, \mathbb{R})$.

1. A sequence $\{u_n\} \subset \mathbb{B}$ is called a Palais–Smale sequence if $|I(u_n)| \leq c$ for some constant c and $DI(u_n) \to 0$, and
2. I is said to satisfy condition (PS) if any Palais–Smale sequence has a convergent subsequence in $\mathbb{B}$.

Theorem 3.7. *(Rabinowitz [15].) Let $\mathbb{B}$ be a Banach space, and let $J \in C^1(\mathbb{B}, \mathbb{R})$ satisfy (PS). Furthermore, assume that $\mathbb{B} = \mathbb{B}_1 \oplus \mathbb{B}_2$, with* $\dim(\mathbb{B}_1) < \infty$.

$$\max_{u\in\mathbb{B}_1:\|u\|=r} J(x) < \inf_{u\in\mathbb{B}_2} J(u) := \rho \tag{11}$$

for some $r > 0$. Then J has at least one critical point E_0 with $J(E_0) \geq \rho$.

In our case, we make use of $\Phi \in H^1(0,1) := \mathbb{B}$ as defined above, and take

$$\mathbb{B}_1 = span\{\Phi\},$$
$$\mathbb{B}_2 := \{E \in H^1(0,1) : E(1) = 0\}.$$

A simple computation shows that

$$\inf_{E\in\mathbb{B}_2} J(E) \geq -\int_0^1 \frac{A^2}{4\varphi(x)}\,dx.$$

On the other hand, if $\varepsilon > 0$, then

$$J(\varepsilon\Phi) = k(\varepsilon) + \varepsilon A \int_0^1 \Phi(x)\,dx,$$

where

$$k(\varepsilon) := \varepsilon^2(M(\Phi) + \lambda_0 c_{-\infty}\Phi(0)^2 - \lambda_1 c_{+\infty}\Phi(1)^2) + \frac{\lambda^2\varepsilon^4}{8}\int_0^1 \Phi(x)^4\,dx.$$

Fix ε small enough such that $k(\varepsilon) < 0$; thus, if $|A|$ is sufficiently small, then (11) holds with $r := \varepsilon\|\Phi\|$. Moreover, we know from Theorem 2.1 that J achieves a global minimum at some $E_{\min} \in H^1(0,1)$, and hence the linking theorem will provide a solution E_0 such that $J(E_0) \geq \rho > J(E_{\min})$, which implies that $E_0 \neq E_{\min}$. When $A = 0$, it is clear that $\rho = 0$ and that $E_0 \equiv 0$ is a critical point; thus, $\pm E_{\min}$ are two nontrivial solutions. When $A > 0$, we know from Lemma 3.5 that $E_{\min}$ is strictly negative. From the properties of the functional, we deduce that the infimum

$$\min_{E:E(1)\geq 0} J(E)$$

is achieved at some $E^* \neq E_{\min}$. As $\Phi(1) > 0$, it follows that $J(E^*) < \rho$; in particular, $E^* \notin \mathbb{B}_2$, and hence $E^*(1) > 0$. We conclude that E^* is a critical point and that $E^* \neq E_0, E_{\min}$.

To conclude the proof, let us verify that J satisfies the (PS) condition.

Let $E_n \in H^1(0,1)$ such that $|J(E_n)| \leq c$ for some constant c and $DJ(E_n) \to 0$. If $\|E_n\|_{H^1} \not\to \infty$, then, by taking a subsequence, we may assume that $E_n \to E$ weakly in H^1 and uniformly. As $DJ(E_n)(E) \to 0$, we deduce that

$$\int_0^1 \lambda^2 E'(x)^2 + \frac{\lambda^2}{2}E(x)^4 + 2\varphi(x)E(x)^2 + AE(x)dx + 2[\lambda_0 c_{-\infty}E(0)^2 - \lambda_1 c_{+\infty}E(1)^2] = 0.$$

Using now the fact that $DJ(E_n)(E_n) \to 0$, it is seen that $\int_0^1 E_n'(x)^2\,dx \to \int_0^1 E'(x)^2\,dx$, and thus $E_n \to E$ strongly.

Next, assume that $\|E_n\|_{H^1} \to \infty$, and let $V_n := E_n/\|E_n\|_{H^1}$.

As $DJ(E_n)(V_n) \to 0$, from the identity

$$\frac{1}{2}DJ(E_n)(V_n) = \frac{J(E_n)}{\|E_n\|_{H^1}} + \frac{\lambda^2}{8}\int_0^1 \frac{E_n(x)^4}{\|E_n\|_{H^1}}\,dx - \frac{A}{2}\int_0^1 \frac{E_n(x)}{\|E_n\|_{H^1}}\,dx,$$

the fact that $J(E_n)$ is bounded, and that $\left|\int_0^1 E_n(x)\,dx\right| \leq \|E_n\|_{L^4}$, we deduce that $\|E_n\|_{L^4}^4/\|E_n\|_{H^1} \to 0$, and thus also $\|E_n\|_{L^2}^4/\|E_n\|_{H^1} \to 0$.

Next, for arbitrary ϕ and $r \neq 0$, compute

$$DJ(rV_n)(\phi) = DJ(E_n)\left(\frac{r\phi}{\|E_n\|_{H^1}}\right) + A\int_0^1 \phi(x)\left(1 - \frac{r}{\|E_n\|_{H^1}}\right)dx + \frac{\lambda^2}{2}\int_0^1 \left(\frac{r^3E_n(x)^3}{\|E_n\|_{H^1}^3} - \frac{rE_n(x)^3}{\|E_n\|_{H^1}}\right)\phi(x)\,dx.$$

Hence,

$$\left|DJ(rV_n)(\phi) - A\int_0^1 \phi(x)dx\right| \leq c_n\|\phi\|_{H^1}$$

for some $c_n \to 0$; that is, $DJ(rV_n) \to A$.

By taking a subsequence, we may suppose that V_n converges weakly in $H^1(0,1)$ and uniformly to some V. In particular, it is readily seen that $DJ(rV_n)(\phi) \to DJ(rV)(\phi)$ for all ϕ, so

$$DJ(rV)(\phi) = A\int_0^1 \phi(x)\,dx.$$

Let us consider now the functional given by

$$\tilde{J}(E) := J(E) - A\int_0^1 E(x)\,dx.$$

Then $D\tilde{J}(rV_n) \to 0$, and it is easy to verify that $\tilde{J}(rV_n)$ is bounded. Thus, $\{rV_n\}$ is a bounded Palais–Smale sequence for $\tilde{J}$, and, as before, we deduce that it has a convergent subsequence. Hence, we may assume that $V_n \to V$ strongly and, in particular, that $\|V\|_{H^1} = 1$. Moreover, rV is a critical point of $\tilde{J}$; in other words, rV is a solution of (1)–(2) with $A = 0$. As r is arbitrary, it follows that $V = 0$. This contradiction completes the proof.

Finally, we proceed with case (ii)(b). In order to emphasize the dependence on A, for fixed $A > 0$, the absolute minimizer $E_{\min}$ of the functional shall be denoted E_A. From Lemma 3.5, we know that E_A is strictly negative.

Claim. $E_A(x) \to -\infty$ uniformly as $A \to +\infty$.

Indeed, let $x_0 \in [0, 1]$ be the point where the absolute maximum of E_A is achieved. Then $x_0 < 1$, since E_A decreases in a neighborhood of 1.

If $x_0 > 0$, then $E_A''(x_0) \le 0$. From (1), it follows that $E_A(x_0) \le r(x_0, A) \to -\infty$ uniformly in x_0 as $A \to +\infty$, where $r(x, A)$ denotes the unique root of the polynomial

$$P_{x,A}(z) := \frac{\lambda^2}{2} z^3 + 2\varphi(x)z + A.$$

Finally, suppose that $x_0 = 0$, and that $E_A(0) \ge -M$ for some constant M independent of A. Take A large enough in order to have $r(x, A) \le -(1 + 1/\lambda_0)M$ for all x. Then $E_A''(0) > 0$. Assume that $E_A''(x) \ge 0$ for $x \in [0, \delta]$ with δ maximum. Then $E_A(\delta) - E_A(0) = \delta E_A'(\xi) > \delta E_A'(0) \ge -M/\lambda_0$. Thus $E_A(\delta) \ge -(1 + 1/\lambda_0)M$, and we deduce that $\delta = 1$. In particular, $E_A'(x) > E_A'(0)$ for all x, and hence $E_A(x) \ge -M(1 + 1/\lambda_0)$. This implies that $E_A''(x) \to +\infty$ uniformly, and this contradicts the fact that $E_A(1) < 0$.

Now, we are able to prove uniqueness for A large. Suppose that $E \ne E_A$ is a solution, and write $E = E_A + V$. As solutions do not intersect and E_A is the only negative solution, it follows that $V(x) > 0$ for all x. Moreover, V satisfies the radiation boundary conditions, and

$$\lambda^2 V''(x) = \frac{\lambda^2}{2}\left(3E_A(x)^2 + 3E_A(x)V(x) + V(x)^2\right)V(x) + 2\varphi(x)V(x).$$

Next, observe that

$$3E_A(x)^2 + 3E_A(x)V(x) + V(x)^2 \ge \frac{3}{4}E_A^2(x),$$

so we conclude that $V''(x) > c^2 V(x)$ for some constant c satisfying $c \to +\infty$ as $A \to +\infty$. Now, consider $W(x) := e^{cx}$. Then

$$V'(1)W(1) - V'(0)W(0) > V(1)W'(1) - V(0)W'(0);$$

that is,

$$\frac{V(1)}{\lambda_1}e^c - \frac{V(0)}{\lambda_0} > (V(1)e^c - V(0))c.$$

Take $c > 1/\lambda_0, 1/\lambda_1$. Then

$$\frac{V(1)}{V(0)}e^c < \frac{c - 1/\lambda_0}{c - 1/\lambda_1}.$$

Furthermore, observe that V is convex and that $V'(0) > 0$. Thus, $V(1) > V(0)$, and we conclude that c (and hence A) cannot be too large.

Remark. As the functional J satisfies the (PS) condition and $J(0) = 0$, it might be natural to ask if, under appropriate conditions on the parameters, J satisfies the standard mountain pass geometry; that is,

$$\inf_{\|E\|=\rho} J(E) > 0 \quad \text{and} \quad J(\tilde{E}) \le 0 \quad \text{for some } \tilde{E} \text{ such that } \|\tilde{E}\| > \rho \tag{12}$$

for some $\rho > 0$. It is worth noting that, if $A \ne 0$, then $\inf_{\|E\|=\rho} J(E) < 0$ when $\rho > 0$ is small. Indeed, it suffices to write

$$J(rE) = r^2 J(E) + (r^4 - r^2)\frac{\lambda^2}{8}\int_0^1 E(x)^4\,dx + (r - r^2)A\int_0^1 E(x)\,dx,$$

and to fix E such that $A\int_0^1 E(x)\,dx < 0$, so $J(rE) < 0$ for small values of r.

When $A = 0$, (12) cannot be satisfied for any value of ρ. This follows from the fact that, if $\inf_{\|E\|=\rho} J(E) > 0$ and $\|\tilde{E}\| > \rho$, then, setting $r := \|\tilde{E}\|/\rho > 1$ and $E := \tilde{E}/r$, we obtain

$$J(\tilde{E}) = r^2 J(E) + (r^4 - r^2)\frac{\lambda^2}{8}\int_0^1 E(x)^4\,dx > 0.$$

However, in this particular case, $E = 0$ is a critical point, and J achieves a global minimum, so a sufficient condition for the existence of a pair $\pm E$ of nontrivial solutions (since J is even) is that $J(E) \le 0$ for some $E \ne 0$. This is obviously the case when (A2) holds.

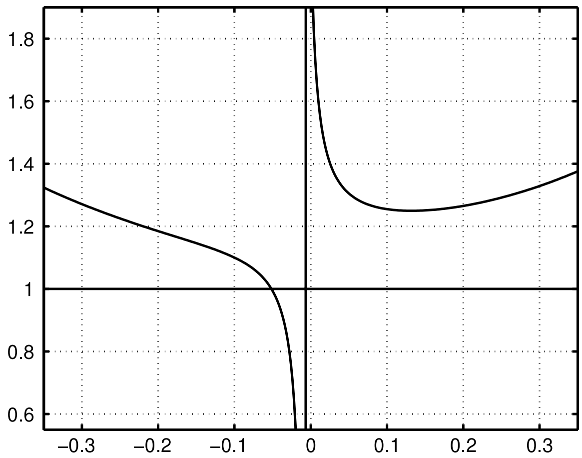

Fig. 1. Graph of $T(\gamma)$ $\{c_{-\infty}=1,\ c_{+\infty}=0.5,\ A=0.05\}$.

4. Numerical experiments

An alternative approach to study the boundary value problem (1)–(2) is to employ a shooting method. It is possible to derive rigorous theoretical results using this alternative approach, but our main concern here is with numerical evidence. Thus, we study the initial value problem that consists of the differential equation (1) (or equivalently (4)) subject to the initial conditions

$$E(0)=\gamma, \qquad E'(0)=\frac{\gamma}{\lambda_0}, \tag{13}$$

where γ is a real parameter. Corresponding to each $\gamma \in \mathbb{R}$ is a solution $E(x;\gamma)$. It satisfies the first boundary condition in (2), but not necessarily the second one. Due to the superlinear term $E(x)^3/2$ in the equation, it is possible that the solution may blow up to ∞ or down to $-\infty$, at some point before reaching $x=1$. As Lemma 3.1 shows, for γ within some bounded interval (γ_1,γ_2), $E(x;\gamma)$ can be extended to $x=1$. We define the function $T:(\gamma_1,\gamma_2)\to\mathbb{R}$ by

$$T(\gamma)=\frac{E'(1;\gamma)}{E(1;\gamma)}.$$

If we can find a γ such that $T(\gamma)=1/\lambda_1$, then the corresponding $E(x;\gamma)$ will be a solution of the BVP.

MATLAB is adopted for our numerical experiments. For each computation, specific numerical values are chosen for the set of constants:

$$\lambda,\ c_{-\infty},\ c_{+\infty},\ A.$$

For convenience, we set $\lambda=1$. Then, we choose a set of values for γ. For each γ, we solve the initial value problem (1)–(13) using the built-in MATLAB function `ode45`. The value of $T(\gamma)$ is then computed. The figures provided are plots of the function $T(\gamma)$ for different choices of the constants involved.

The graph of each $T(\gamma)$ consists of two components, a decreasing curve on the left and a U-shaped curve on the right, with a common asymptote, represented by the vertical solid lines in Figs. 1 and 2. The asymptote occurs at $\gamma=\gamma_0$, determined by requiring that $E(1;\gamma_0)=0$. The horizontal solid line in each figure is drawn at $T(\gamma)=1/\lambda_1$. Its intersections with the graph of $T(\gamma)$ give the solutions of (1)–(2).

Fig. 1 illustrates the situation when $c_{-\infty}>c_{+\infty}$. The right-hand component of the graph of $T(\gamma)$ lies above the horizontal line, while the left-hand component intersects the latter at one point, thus confirming part (i) of Theorem 2.1.

Fig. 2 illustrates the situation when $c_{-\infty}<c_{+\infty}$. The right-hand component of the graph of $T(\gamma)$ cuts the horizontal line at two points, while the left-hand component cuts the latter at one more point, giving a total of three solutions, thus confirming part (ii)(a) of Theorem 2.1.

Fig. 3 extends the experiment of Fig. 2 by varying the constant A. The dotted, solid, and dashed curves correspond to the three choices $A=0.02$, 0.05, and 0.1, respectively. It is observed empirically that, as A is increased, the right-hand component of the graph is raised, while the left-hand component is lowered. In any case, the left-hand component always yields one solution. On the other hand, for large A, the right-hand component can be raised clear of the horizontal line. This confirms part (ii)(b) of Theorem 2.1.

Remarks/Conjectures. In all the previous examples, the left-hand component contains the (unique) negative solution. When $c_{-\infty}<c_{+\infty}$, the right-hand component yields two positive solutions. The existence of a sign-changing solution seems to be quite exceptional, since the vertical asymptote lies very close to $\gamma=0$. In any case, it seems that there are always at most three solutions. It also seems reasonable to conjecture that $\min_{\gamma\ge 0}T(\gamma)$ always exists and increases with A (it is already known that it is greater than $1/\lambda_1$ if A is large).

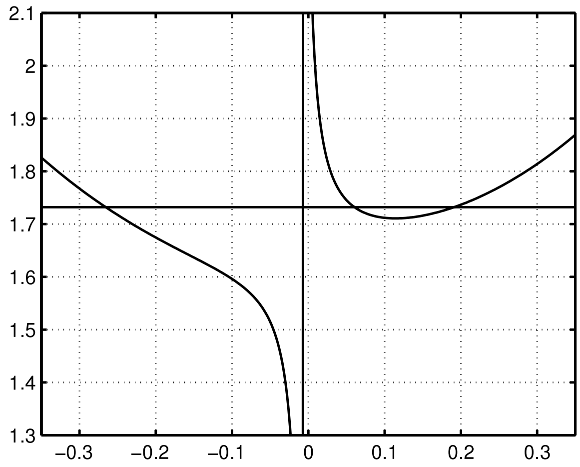

Fig. 2. Graph of $T(\gamma)$ $\{c_{-\infty} = 1,\ c_{+\infty} = 1.5,\ A = 0.05\}$.

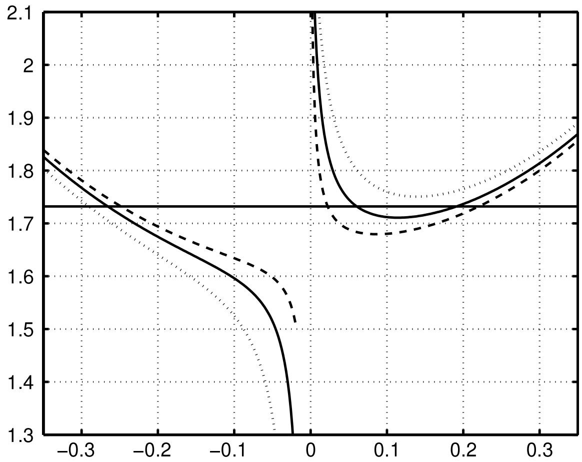

Fig. 3. $\{c_{-\infty} = 1,\ c_{+\infty} = 1.5,\ A = 0.02,\ 0.05,\ 0.1\}$.

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