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Classification of Integral Modular Categories of Frobenius–Perron Dimension pq^4 and p^2q^2

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Abstract. We classify integral modular categories of dimension pq^4 and p^2q^2, where p and q are distinct primes. We show that such categories are always group-theoretical, except for categories of dimension $4q^2$. In these cases there are well-known examples of non-group-theoretical categories, coming from centers of Tambara–Yamagami categories and quantum groups. We show that a non-group-theoretical integral modular category of dimension $4q^2$ is either equivalent to one of these well-known examples or is of dimension 36 and is twist-equivalent to fusion categories arising from a certain quantum group.

1 Introduction

Braided group-theoretical categories are well understood; they are equivalent to fusion subcategories of $\mathrm{Rep}(D^\omega(G))$, where G is a finite group and ω is a 3-cocycle on G [Na1, O]. Fusion subcategories of $\mathrm{Rep}(D^\omega(G))$ are determined by triples (K, H, B), where K, H are normal subgroups of G that centralize each other and B is a G-invariant ω-bicharacter on $K \times H$ [NNW, Theorem 5.11]. Triples (K, H, B) for which $HK = G$ and B satisfies a certain nondegeneracy condition determine the *modular* subcategories [NNW, Proposition 6.7]. Moreover, braided group-theoretical categories enjoy *property **F***: the braid group representations on endomorphism spaces have finite image [ERW]. Our approach to the classification of integral modular categories of a given dimension is to consider those that are group-theoretical as understood and then explicitly describe those that are not.

For distinct primes p, q, and r, any integral modular category of dimension p^n, pq, pqr, pq^2, or pq^3 is group-theoretical by [EGO, DGNO1, NR]. On the other hand, non-group-theoretical integral modular categories of dimension $4q^2$ were constructed in [GNN] and [NR]. Furthermore, there are non-group-theoretical integral

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modular categories of dimension p^2q^4 if p is odd and $p|(q+1)$, obtained as the Drinfeld centers of Jordan–Larson categories (see [JL, Theorem 1.1]).

If $\mathcal{C}$ is an integral, nondegenerate, braided fusion category, then the set of modular structures on $\mathcal{C}$ is in bijection with the set of isomorphism classes of invertible self-dual objects of $\mathcal{C}$. Thus, we view the problem of classifying integral modular categories as being equivalent to the problem of classifying integral nondegenerate braided fusion categories.

In this work, we classify integral modular categories of dimension pq^4 and p^2q^2. In particular, we prove the following theorem.

Theorem 1.1 *Let $\mathcal{C}$ be an integral modular category.*

(i) *If* $\mathrm{FPdim}(\mathcal{C}) = pq^4$, *then $\mathcal{C}$ is group-theoretical.*
(ii) *If* $\mathrm{FPdim}(\mathcal{C}) = p^2q^2$ *is odd, then $\mathcal{C}$ is group theoretical.*
(iii) *If $\mathcal{C}$ is a non-group-theoretical category of dimension* $4q^2$, *then either* $\mathcal{C} \cong \mathcal{E}(\zeta, \pm)$, *as braided fusion categories with ζ an elliptic quadratic form on $\mathbb{Z}_q \times \mathbb{Z}_q$, or $\mathcal{C}$ is twist-equivalent to* $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ *or* $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$.

Here, $\mathcal{E}(\zeta, \pm)$ are modular categories constructed in [GNN], $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ are the modular categories constructed from the quantum group $U_q(\mathfrak{sl}_3)$ where q^2 is a primitive 6th root of unity and $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ are the fusion categories defined in Subsection 4.1. The notion of twist-equivalence is defined in Subsection 4.1. The 36-dimensional categories $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ are new and will be investigated further in a future work.

We hasten to point out that a coarser classification of these categories has been obtained; in [ENO2] it is shown that any fusion category of dimension p^aq^b is *solvable*, that is, such categories can be obtained from the category Vec of finite dimensional vector spaces via a sequence of extensions and equivariantizations by groups of prime order. However, the question of whether a given category admits a $\mathbb{Z}_p$-extension or $\mathbb{Z}_p$-equivariantization is a somewhat subtle one (see [ENO3, G1]).

2 Some General Results

In this section we will recall some general results about modular categories. These will be used in the sections that follow.

The Frobenius–Perron dimension $\mathrm{FPdim}(X)$ of a simple object X in a fusion category $\mathcal{C}$ is defined to be the largest positive eigenvalue of the fusion matrix N_X with entries $N_{X,Y}^Z = \dim \mathrm{Hom}(Z, X \otimes Y)$ of X, that is, the matrix representing X in the left regular representation of the Grothendieck semiring $Gr(\mathcal{C})$ of $\mathcal{C}$. The Frobenius–Perron dimension $\mathrm{FPdim}(\mathcal{C})$ of the fusion category $\mathcal{C}$ is defined to be the sum of the squares of the Frobenius–Perron dimensions of (isomorphism classes of) simple objects. A fusion category $\mathcal{C}$ is called *integral* if $\mathrm{FPdim}(X) \in \mathbb{N}$ for all simple objects X. If $\mathcal{C}$ is an integral modular category, then $\mathrm{FPdim}(X)^2$ divides $\mathrm{FPdim}(\mathcal{C})$ for all simple objects $X \in \mathcal{C}$ [EG, Lemma 1.2] (see also [ENO1, Proposition 3.3]). Note that integral modular categories are *pseudounitary* [ENO1, Proposition 8.24]; that is, the Frobenius–Perron dimension coincides with the categorical dimension.

A fusion category is said to be *pointed* if all its simple objects are invertible. A fusion category whose Frobenius–Perron dimension is a prime number is necessarily pointed [ENO1, Corollary 8.30].

For a fusion category $\mathcal{C}$, the maximal pointed subcategory of $\mathcal{C}$ generated by invertible objects will be denoted by $\mathcal{C}_{\text{pt}}$. A complete set of representatives of non-isomorphic simple objects in $\mathcal{C}$ will be denoted $\text{Irr}(\mathcal{C})$. The full fusion subcategory generated by all simple subobjects of $X \otimes X^*$, where X runs through all simple objects of $\mathcal{C}$, is called the *adjoint category* of $\mathcal{C}$, and is denoted by $\mathcal{C}_{\text{ad}}$. If the (finite) sequence of categories $\mathcal{C} \supset \mathcal{C}_{\text{ad}} \supset (\mathcal{C}_{\text{ad}})_{\text{ad}} \supset \cdots$ converges to the trivial category Vec, then $\mathcal{C}$ is called *nilpotent*. Clearly, any pointed fusion category is nilpotent.

Two objects X and Y in a braided fusion category $\mathcal{C}$ (with braiding c) are said to *centralize* each other if $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$. If $\mathcal{D}$ is a full (not necessarily fusion) subcategory of $\mathcal{C}$, then the *centralizer* of $\mathcal{D}$ in $\mathcal{C}$ is the full fusion subcategory

$$\mathcal{D}' := \left\{ X \in \mathcal{C} \mid c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y},\ \text{for all } Y \in \mathcal{D} \right\}.$$

If $\mathcal{C}' = \text{Vec}$ (the fusion category generated by the unit object), then $\mathcal{C}$ is said to be *nondegenerate*. If $\mathcal{C}' = \mathcal{C}$, then $\mathcal{C}$ is said to be *symmetric*. If $\mathcal{C}$ is nondegenerate and $\mathcal{D}$ is a full fusion subcategory of $\mathcal{C}$, then $(\mathcal{D}')' = \mathcal{D}$, and by [M2]

$$\text{FPdim}(\mathcal{D}) \cdot \text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C}).$$

Recall that a *modular category* is a nondegenerate braided fusion category equipped with a ribbon structure.

We record the following theorems for later use.

Theorem 2.1 ([GN, Corollary 6.8]) *If $\mathcal{C}$ is a pseudounitary modular category, then* $(\mathcal{C}_{\text{pt}})' = \mathcal{C}_{\text{ad}}$.

Theorem 2.2 ([DGNO1, Corollary 4.14]) *A modular category $\mathcal{C}$ is group-theoretical if and only if it is integral and there is a symmetric subcategory $\mathcal{L}$ such that $(\mathcal{L}')_{\text{ad}} \subset \mathcal{L}$.*

A *grading* of a fusion category $\mathcal{C}$ by a finite group G is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

of $\mathcal{C}$ into a direct sum of full abelian subcategories such that the tensor product $\otimes$ maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$ for all $g,\ h \in G$. The $\mathcal{C}_g$'s are called *components* of the G-grading of $\mathcal{C}$. A G-grading is said to be *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$. For a faithful grading, the dimensions of the components are all equal [ENO1, Proposition 8.20]. Every fusion category $\mathcal{C}$ is faithfully graded by its universal grading group $U(\mathcal{C})$ [GN], and this grading is called the *universal grading*. The trivial component of this grading is $\mathcal{C}_e = \mathcal{C}_{\text{ad}}$, where e is the identity element of $U(\mathcal{C})$. For a modular category $\mathcal{C}$, the universal grading group $U(\mathcal{C})$ is isomorphic to the group of isomorphism classes of invertible objects of $\mathcal{C}$ [GN, Theorem 6.2], in particular, $\text{FPdim}(\mathcal{C}_{\text{pt}}) = |U(\mathcal{C})|$.

Finally, we recall some standard algebraic relations involving the S-matrix $\widetilde{S}$, twists θ_i, and fusion constants $N_{i,j}^k$ of a pseudounitary modular category. The matrix $\widetilde{S}$ is symmetric and projectively unitary, with entries given by the *twist equation*

$$\widetilde{S}_{i,j} = \theta_i^{-1}\theta_j^{-1}\sum_k N_{i,j^*}^k \theta_k \mathrm{FPdim}(X_k).$$

The *Gauss sums* $p_\pm = \sum_k \theta_k^{\pm 1}\mathrm{FPdim}(X_k)^2$ satisfy $p_+p_- = \mathrm{FPdim}(\mathcal{C})$.

3 Dimension pq^4

Theorem 3.1 *Let p and q be distinct primes, and let $\mathcal{C}$ be an integral modular category of dimension pq^4. Then $\mathcal{C}$ is group-theoretical.*

Proof Since $\mathrm{FPdim}(X)^2$ must divide $\mathrm{FPdim}(\mathcal{C}) = pq^4$ for every simple object $X \in \mathcal{C}$, the possible dimensions of simple objects are 1, q, and q^2. Let a, b, and c denote the number of isomorphism classes of simple objects of dimension 1, q, and q^2, respectively. We must have $a + bq^2 + cq^4 = pq^4$, and so q^2 must divide $a = \mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$. Since the dimension of a fusion subcategory must divide $\mathrm{FPdim}(\mathcal{C})$, it follows that there are six possible values for $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}})$: q^2, q^3, q^4, pq^2, pq^3, and pq^4.

Case (i) $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) = q^3$. In this case, there are q^3 components in the grading of $\mathcal{C}$ by its universal grading group $U(\mathcal{C})$, and each component has dimension pq. For each $g \in U(\mathcal{C})$, let a_g, b_g, and c_g denote the number of isomorphism classes of simple objects of dimension 1, q, and q^2, respectively, contained in the component $\mathcal{C}_g$. We must have $a_g + b_g q^2 + c_g q^4 = pq$, and so q must divide a_g. Note that $a_g \neq 0$ for all $g \in U(\mathcal{C})$, since otherwise we would have $b_g q + c_g q^3 = p$, a contradiction. Thus, each component must contain at least q (non-isomorphic) invertible objects, and since there are q^3 components, it follows that there must be at least q^4 (non-isomorphic) invertible objects, a contradiction.

Cases (ii)–(v) $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) \in \{q^4, pq^2, pq^3, pq^4\}$. In each case, $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$ is a power of a prime, so $\mathcal{C}_{\mathrm{ad}}$ is nilpotent [GN]. Consequently, $\mathcal{C}$ is also nilpotent, and since it is integral and modular, it follows that it is group-theoretical [DGNO1].

Case (vi) $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) = q^2$. In this case, $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = pq^2$. This fact together with the possibilities for the dimensions of simple objects implies that $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ must be of dimension q^2, and so $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{C}_{\mathrm{ad}}$. Hence $\mathcal{C}_{\mathrm{pt}}$ is symmetric, since $\mathcal{C}_{\mathrm{ad}} = (\mathcal{C}_{\mathrm{pt}})'$.

We claim that $\mathcal{C}_{\mathrm{pt}}$ is a Tannakian subcategory. This is true if q is odd [DGNO2, Corollary 2.50], so assume that $q = 2$ and suppose on the contrary that $\mathcal{C}_{\mathrm{pt}}$ is not Tannakian. Then $\mathcal{C}_{\mathrm{pt}}$ contains a symmetric subcategory $\mathcal{S}$ equivalent to the category of super vector spaces. Let $g \in \mathcal{S}$ be the unique nontrivial (fermionic) invertible object, and let $\mathcal{S}' \subseteq \mathcal{C}$ denote the Müger centralizer of $\mathcal{S}$. By [M1, Lemma 5.4], we have $g \otimes X \ncong X$ for any simple object $X \in \mathcal{S}'$.

On the other hand, we have $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{S}'$ and $\mathrm{FPdim}(\mathcal{S}') = 8p$. The possibilities for the dimensions of simple objects of $\mathcal{C}$ imply that the number of simple objects of dimension 2 in $\mathcal{S}'$ is necessarily odd. Therefore, the action by tensor multiplication of the group of invertible objects of $\mathcal{C}$ on the set of isomorphism classes of simple objects of FP-dimension 2 of $\mathcal{S}'$ must have a fixed point, which is a contradiction. Hence $\mathcal{C}_{\mathrm{pt}}$ is Tannakian, as claimed.

Therefore $\mathcal{C}_{pt} \cong \operatorname{Rep}(G)$ as symmetric tensor categories, where G is a group of order q^2. Let $\widehat{\mathcal{C}} := \mathcal{C}_G$ denote the corresponding de-equivariantization of $\mathcal{C}$. By the main result of [K, M3], $\widehat{\mathcal{C}}$ is a G-crossed braided fusion category (of dimension pq^2), and the equivariantization of $\widehat{\mathcal{C}}$ with respect to the associated G-action is equivalent to $\mathcal{C}$ as braided tensor categories (see [GNN, Theorem 2.12]).

Furthermore, since $\mathcal{C}$ is modular, the associated G-grading of $\widehat{\mathcal{C}}$ is faithful [GNN, Remark 2.13]. Thus, the trivial component $\widehat{\mathcal{C}}_e \supseteq \widehat{\mathcal{C}}_{ad}$ of this grading is of dimension p, and, in particular, it is pointed. Hence $\widehat{\mathcal{C}}$ is a nilpotent fusion category.

In view of [GN, Corollary 5.3], the square of the Frobenius–Perron dimension of a simple object of $\widehat{\mathcal{C}}$ must divide $\operatorname{FPdim}(\widehat{\mathcal{C}}_e) = p$. Since $\widehat{\mathcal{C}}$ is also integral, we see that $\widehat{\mathcal{C}}$ is itself pointed. It follows from [NNW, Theorem 7.2] that $\mathcal{C}$, being an equivariantization of a pointed fusion category, is group-theoretical. ■

Remark 3.2 In this remark, we show that two of the four cases addressed in Cases (ii)–(iv) of the proof above cannot occur. If $\operatorname{FPdim}(\mathcal{C}_{pt}) = q^4$, then $\operatorname{FPdim}((\mathcal{C}_{pt})') = p$, so $(\mathcal{C}_{pt})'$ must be pointed. Therefore, $(\mathcal{C}_{pt})'$ is contained in $\mathcal{C}_{pt}$, and this implies that p divides q^4, a contradiction. If $\operatorname{FPdim}(\mathcal{C}_{pt}) = pq^3$, then each component in the universal grading of $\mathcal{C}$ has FP-dimension q, and so it cannot accommodate a non-invertible object, a contradiction.

4 Dimension p^2q^2

In this section, we will make repeated use of the following result.

Theorem 4.1 ([ENO2, Theorem 1.6 and Proposition 4.5(iv)]) *If p and q are primes and $\mathcal{C} \neq \operatorname{Vec}$ is a fusion category of dimension p^aq^b, then $\mathcal{C}$ contains a non-trivial invertible object.*

Theorem 4.2 *Let $p < q$ be primes, and let $\mathcal{C}$ be an integral modular category of dimension p^2q^2. Then one of the following is true:*

(i) *$\mathcal{C}$ is group-theoretical;*

(ii) *$p = 2$, $q = 3$, and $\operatorname{FPdim}(\mathcal{C}_{pt}) = 3$;*

(iii) *$p \mid q - 1$ and $\operatorname{FPdim}(\mathcal{C}_{pt}) = p$.*

Proof Since $\operatorname{FPdim}(X)^2$ must divide $\operatorname{FPdim}(\mathcal{C}) = p^2q^2$ for every simple object $X \in \mathcal{C}$, the possible dimensions of the simple objects are 1, q, and p. Since $\mathcal{C}_{pt}$ is a fusion subcategory of $\mathcal{C}$, we know that $\operatorname{FPdim}(\mathcal{C}_{pt}) \mid p^2q^2$, and hence

$$\operatorname{FPdim}(\mathcal{C}_{pt}) \in \left\{1, p, q, p^2, q^2, pq, pq^2, p^2q, p^2q^2\right\}.$$

Applying Theorem 4.1, we conclude that $\operatorname{FPdim}(\mathcal{C}_{pt}) > 1$. The proof now proceeds by cases based on $\operatorname{FPdim}(\mathcal{C}_{pt})$. For each $g \in U(\mathcal{C})$, let a_g, b_g, and c_g denote the number of isomorphism classes of simple objects of dimension 1, p, and q in the component $\mathcal{C}_g$, respectively. Let e denote the identity element of $U(\mathcal{C})$.

Cases (i)–(v) $\operatorname{FPdim}(\mathcal{C}_{pt}) \in \{p^2q^2, pq^2, p^2q, p^2, q^2\}$. In each case, $\operatorname{FPdim}(\mathcal{C}_{ad})$ is a power of a prime, so $\mathcal{C}_{ad}$ is nilpotent [GN]. Consequently, $\mathcal{C}$ is also nilpotent, and since it is modular, it follows that it is group-theoretical [DGNO1, Corollary 6.2].

Case (vi) $\text{FPdim}(\mathcal{C}_{\text{pt}}) = pq$. In this case, $\text{FPdim}(\mathcal{C}_g) = pq$ for all $g \in U(\mathcal{C})$. Since $p < q$ we immediately conclude that $c_g = 0$ for all $g \in U(\mathcal{C})$ and thus $a_g \neq 0$ from the equation $pq = a_g + b_g p^2$. By Theorem 4.1, we know that $\mathcal{C}_{\text{ad}}$ contains a non-trivial invertible object. Since there are pq components, the number of invertible objects is at least $pq + 1$, a contradiction.

Case (vii) $\text{FPdim}(\mathcal{C}_{\text{pt}}) = q$. In this case, $\text{FPdim}(\mathcal{C}_g) = p^2 q$ for all $g \in U(\mathcal{C})$. We can apply Theorem 4.1 to $\mathcal{C}_{\text{ad}}$ to deduce that $\mathcal{C}_{\text{pt}} \subset \mathcal{C}_{\text{ad}}$. Examining the dimension of $\mathcal{C}_{\text{ad}}$ we have $p^2 q = q + b_e p^2 + c_e q^2$. So q must divide b_e, and hence $b_e = 0$. Consequently, $c_e q = (p-1)(p+1)$. Since $p < q$, we must have $p = 2$ and $q = 3$.

Case (viii) $\text{FPdim}(\mathcal{C}_{\text{pt}}) = p$. As in case (vii) we immediately conclude that $\mathcal{C}_{\text{pt}} \subset \mathcal{C}_{\text{ad}}$, and $\text{FPdim}(\mathcal{C}_g) = pq^2$ for all $g \in U(\mathcal{C})$. Examining the dimension of $\mathcal{C}_{\text{ad}}$ we have $pq^2 = p + b_e p^2 + c_e q^2$. Therefore, $c_e = 0$ and $b_e = (q^2-1)/p$. Similar analysis of the nontrivial components reveals that $b_g = 0$ and $c_g = p$. We will identify the simple objects of $\mathcal{C}_{\text{pt}}$ with the elements of $U(\mathcal{C})$ and denote them by g^k, ordered such that $g^k \otimes g^\ell = g^{k+\ell}$ (exponents are computed modulo p). We will denote the objects of dimension p by Y_r and the objects of dimension q in the $\mathcal{C}_{g^k}$ component by X_i^k.

We will show that $p \mid q-1$. This is immediate in the case $p = 2$, so we will assume that $p \geq 3$. We first need to determine some of the fusion rules. Denote by $\text{Stab}_{U(\mathcal{C})}(Y_r)$ the stabilizer of the object Y_r under the tensor product action of $g^j \in \mathcal{C}_{\text{pt}}$. Computing the dimension of

$$Y_r \otimes Y_r^* = \bigoplus_{h \in \text{Stab}_{U(\mathcal{C})}(Y_r)} h \oplus \bigoplus_{s=1}^{\frac{q^2-1}{2}} N_{Y_r, Y_r^*}^{Y_s} Y_s$$

we see that p must divide $|\text{Stab}_{U(\mathcal{C})}(Y_r)|$ and hence $\text{Stab}_{U(\mathcal{C})}(Y_r) = U(\mathcal{C})$. An analogous argument shows that $\text{Stab}_{U(\mathcal{C})}(X_i^k)$ is trivial for all i and k. In particular, the action of $U(\mathcal{C})$ on $\mathcal{C}_{g^k}$ is fixed-point free, and so we may relabel such that $g \otimes X_i^k = X_{i+1}^k$ (with indices computed modulo p). These results about the stabilizers allow us to compute $N_{Y_r, X_i^k}^{X_j^k}$ as follows

$$\bigoplus_{j=1}^{p} N_{Y_r, X_i^k}^{X_j^k} X_j^k = Y_r \otimes X_i^k = (g^\ell \otimes Y_r) \otimes X_i^k = Y_r \otimes X_{i+\ell}^k = \bigoplus_{j=1}^{p} N_{Y_r, X_{i+\ell}^k}^{X_j^k} X_j^k.$$

Since this must hold for all ℓ we can conclude that

$$N_{Y_r, X_i^k}^{X_j^k} = N_{Y_r, X_h^k}^{X_j^k}$$

for all r, h, i, j, and k. A dimension count gives $N_{Y_r, X_i^k}^{X_j^k} = 1$.

Denote the S-matrix of $\mathcal{C}$ by $\widetilde{S}$ and the entries by $\widetilde{S}_{A,B}$ (normalized so that $\widetilde{S}_{\mathbf{1},\mathbf{1}} = 1$). Since $\text{FPdim}(Y_r) = p$ and $\text{FPdim}(X_i^k) = q$ are coprime, [ENO2, Lemma 7.1] implies that either

$$\widetilde{S}_{Y_r, X_i^k} = 0 \quad \text{or} \quad |\widetilde{S}_{Y_r, X_i^k}| = pq.$$

Since the columns of $\widetilde{S}$ have squared-length $(pq)^2$, we must have $\widetilde{S}_{Y_r, X_i^k} = 0$.

We compute $\widetilde{S}_{Y_r, X_i^k}$ another way using the fusion rules above and the twist equation to conclude that $0 = \sum_{j=1}^{p} \theta_{X_j^k}$. The vanishing of this sum allows us to compute the

Gauss sums as follows:

$$p_+ = \sum_{Z \in \mathrm{Irr}(\mathcal{C})} \theta_Z \mathrm{FPdim}(Z)^2 = p\left(1 + p \sum_{r=1}^{\frac{q^2-1}{2}} \theta_{Y_r}\right).$$

However, [Ng, Proposition 5.4] shows that $\mathcal{C}$ is anomaly free and in particular that $p_+ = pq$. From this it immediately follows that

$$\frac{q-1}{p} = \sum_{r=1}^{\frac{q^2-1}{p}} \theta_{Y_r}.$$

The right-hand side of this equation is an algebraic integer, and so we conclude that p must divide $q-1$. ■

Remark 4.3 In this remark, we show that two of the Cases (i)–(v) of the proof above cannot occur. If $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) = p^2$, then $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = q^2$, so applying Theorem 4.1 we see that there is a nontrivial invertible object in $\mathcal{C}_{\mathrm{ad}}$. Therefore, $a_e \geq 2$ and $a_e \mid q^2$. On the other hand, the invertible objects in $\mathcal{C}_{\mathrm{ad}}$ will form a fusion subcategory of $\mathcal{C}_{\mathrm{pt}}$, and so $a_e \mid p^2$, a contradiction. A similar argument shows that the case $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) = q^2$ cannot occur.

Next, we recall a general fact about modular categories. Let $\mathcal{C}$ be a modular category and suppose that it contains a Tannakian subcategory $\mathcal{E}$. Let G be a finite group such that $\mathcal{E} \cong \mathrm{Rep}(G)$, as symmetric categories. The de-equivariantization $\mathcal{C}_G$ is a braided G-crossed fusion category of Frobenius–Perron dimension $\mathrm{FPdim}(\mathcal{C})/|G|$ (see [K, M3]).

Since $\mathcal{C}$ is modular, the associated G-grading of $\mathcal{C}_G$ is faithful, and the trivial component $\mathcal{C}_G^0$ is a modular category of Frobenius–Perron dimension $\mathrm{FPdim}(\mathcal{C})/|G|^2$. Furthermore, as a consequence of [DMNO, Corollary 3.30] we have an equivalence of braided fusion categories

$$\mathcal{C} \boxtimes (\mathcal{C}_G^0)^{\mathrm{rev}} \cong \mathcal{Z}(\mathcal{C}_G).$$

Notice that this implies that $\mathcal{C}$ is group-theoretical if and only if $\mathcal{C}_G$ is group-theoretical [Na2, Proposition 3.1].

Proposition 4.4 *Let $p < q$ be prime numbers. Let $\mathcal{C}$ be an integral modular category of dimension p^2q^2, and let $G \cong U(\mathcal{C})$ be the group of invertible objects of $\mathcal{C}$. Suppose that $\mathcal{C}$ is not group-theoretical. Then there exists a G-crossed braided fusion category $\widehat{\mathcal{C}}$ such that the equivariantization of $\widehat{\mathcal{C}}$ with respect to the associated G-action is equivalent to $\mathcal{C}$ as braided fusion categories. The corresponding G-grading on $\widehat{\mathcal{C}}$ is faithful; the trivial component $\widehat{\mathcal{C}}_e$ is a modular category of Frobenius–Perron dimension* $\mathrm{FPdim}(\mathcal{C})/|G|^2$, *and there is an equivalence of braided fusion categories*

$$\mathcal{C} \boxtimes (\widehat{\mathcal{C}}_e)^{\mathrm{rev}} \cong \mathcal{Z}(\widehat{\mathcal{C}}).$$

Moreover, $\widehat{\mathcal{C}}$ is not group-theoretical.

Proof In view of the preceding comments, it will be enough to show that the category $\mathcal{E} = \mathcal{C}_{\mathrm{pt}}$ is a Tannakian fusion subcategory.

By Theorem 4.2, we may assume that $|G| = p$, or $\mathrm{FPdim}(\mathcal{C}) = 36$ and $|G| = 3$. Let $\mathcal{D} \subseteq \mathcal{C}$ be a nontrivial fusion subcategory. Since the order of G is a prime number, it follows from Theorem 4.1 that $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{D}$, so $\mathcal{D}_{\mathrm{pt}} = \mathcal{C}_{\mathrm{pt}} = \mathcal{E}$. In particular, $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{C}_{\mathrm{ad}} = \mathcal{C}'_{\mathrm{pt}}$ and therefore $\mathcal{C}_{\mathrm{pt}}$ is symmetric. If the order of G is odd, this implies that $\mathcal{E}$ is Tannakian.

So we may assume that $|G| = 2$. Suppose on the contrary that $\mathcal{E}$ is not Tannakian. Then $\mathcal{E}$ is equivalent, as a symmetric category, to the category sVec of finite-dimensional super vector spaces. Therefore $\mathcal{E}'$ is a slightly degenerate fusion category of Frobenius–Perron dimension $2q^2$. Let $g \in \mathcal{E}$ be the unique nontrivial invertible object. By [M1, Lemma 5.4], we have $g \otimes X \ncong X$, for any simple object $X \in \mathcal{E}'$.

The possible Frobenius–Perron dimensions of simple objects of $\mathcal{C}$ in this case are 1, 2, and q. This leads to the equation $\mathrm{FPdim}(\mathcal{E}') = 2q^2 = 2 + 4a + q^2 b$, where a, b are non-negative integers. If $b \neq 0$, then $\mathcal{E}'$ contains a Tannakian subcategory $\mathcal{B}$, by [ENO2, Proposition 7.4]. Hence, in this case, $\mathcal{E}$ is Tannakian, since $\mathcal{E} \subseteq \mathcal{B}$.

Otherwise, if $b = 0$, every non-invertible simple object X of $\mathcal{E}'$ is of Frobenius–Perron dimension 2, and therefore the stabilizer $\mathrm{Stab}_G(X)$ of any such object under the action of the group G by tensor multiplication is not trivial, as follows from the relation

$$X \otimes X^* \cong \bigoplus_{g \in \mathrm{Stab}_G(X)} g \oplus \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C}) \setminus \mathrm{Irr}(\mathcal{C}_{\mathrm{pt}})} N_{X,X^*}^{Y} Y.$$

Then we see that the action of G on the set of isomorphism classes of simple objects of Frobenius–Perron dimension 2 of $\mathcal{E}'$ must be trivial, which is a contradiction. Hence $\mathcal{E}$ is Tannakian, as claimed. ■

Remark 4.5 Keep the notation in Proposition 4.4. Suppose that $\mathcal{C}$ is not group-theoretical and $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) = p < q$. Then $\mathrm{FPdim}(\widehat{\mathcal{C}}) = pq^2$ and $\widehat{\mathcal{C}}_e$ is a modular category of Frobenius–Perron dimension q^2, and hence it is pointed. Since $\widehat{\mathcal{C}}_{\mathrm{ad}} \subseteq \widehat{\mathcal{C}}_e$, $\widehat{\mathcal{C}}$ is a nilpotent integral fusion category. Then $\mathrm{FPdim}(X) = 1$ or q, for all simple object X of $\widehat{\mathcal{C}}$ [GN, Corollary 5.3]. Moreover, since $\widehat{\mathcal{C}}$ is not pointed, it is of type $(1, q^2; q, p-1)$ (that is, having q^2 non-isomorphic simple objects of dimension 1 and $p - 1$ non-isomorphic simple objects of dimension q.)

Theorem 4.6 *Let $2 < p < q$ be prime numbers, and let $\mathcal{C}$ be an integral modular category of dimension p^2q^2. Then $\mathcal{C}$ is group-theoretical.*

Proof By Theorem 4.2, we may assume that $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) = p$ and $p \mid q - 1$. Keep the notation in Proposition 4.4. The category $\widehat{\mathcal{C}}$ has Frobenius–Perron dimension pq^2. Observe that $\widehat{\mathcal{C}}$ must be group-theoretical. Otherwise, by [JL, Theorem 1.1] we should have $p \mid q + 1$, leading to the contradiction $p = 2$. Therefore Proposition 4.4 implies that $\mathcal{C}$ is group-theoretical, as claimed. ■

Let $A = \mathbb{Z}_q \times \mathbb{Z}_q$, with $2 \neq q$ prime, let ζ be an elliptic quadratic form on A, and let $\tau = \pm 1/q$. Then the associated Tambara–Yamagami fusion categories $\mathcal{TY}(A, \zeta, \tau)$

are inequivalent and not group-theoretical. By [JL, Theorem 1.1], these are the only non-group-theoretical fusion categories of dimension $2q^2$.

Examples of non-group-theoretical modular categories $\mathcal{C}$ of Frobenius–Perron dimension $4q^2$ such that $\text{FPdim}(\mathcal{C}_{\text{pt}}) = 2$ were constructed in [GNN, Subsection 5.3]; these examples consist of two equivalence classes, denoted $\mathcal{E}(\zeta, \pm)$, according to the sign choice of $\tau = \pm 1/q$. By construction, there is an embedding of braided fusion categories $\mathcal{E}(\zeta, \pm) \subseteq \mathcal{Z}(\mathcal{TY}(A, \zeta, \tau))$.

Theorem 4.7 *Let $q \neq 2$ be a prime number, and let $\mathcal{C}$ be an integral modular category such that* $\text{FPdim}(\mathcal{C}) = 4q^2$ *and* $\text{FPdim}(\mathcal{C}_{\text{pt}}) = 2$. *Then either $\mathcal{C}$ is group-theoretical or $\mathcal{C} \cong \mathcal{E}(\zeta, \pm)$ as braided fusion categories.*

Proof Keep the notation in Proposition 4.4 and suppose that $\mathcal{C}$ is not group-theoretical. Then $\text{FPdim}(\widehat{\mathcal{C}}) = 2q^2$ and $\widehat{\mathcal{C}}$ is not group-theoretical. Hence, by [JL, Theorem 1.1], $\widehat{\mathcal{C}} \cong \mathcal{TY}(A, \zeta, \tau)$ as fusion categories, where ζ is an elliptic quadratic form on $A = \mathbb{Z}_q \times \mathbb{Z}_q$, and $\tau = \pm 1/q$. In view of Proposition 4.4, we have an equivalence of braided fusion categories

$$\mathcal{C} \boxtimes (\widehat{\mathcal{C}}_e)^{\text{rev}} \cong \mathcal{Z}(\mathcal{TY}(A, \zeta, \tau)), \tag{4.8}$$

where $\widehat{\mathcal{C}}_e$ is a pointed modular category of Frobenius–Perron dimension q^2.

The center of $\mathcal{TY}(A, \zeta, \tau)$ is described in [GNN, Section 4]. The group of invertible objects of $\mathcal{Z}(\mathcal{TY}(A, \zeta, \tau))$ is of order $2q^2$. In particular, $\mathcal{Z}(\mathcal{TY}(A, \zeta, \tau))$ contains a unique pointed fusion subcategory $\mathcal{B}$ of dimension q^2, which is nondegenerate. We note that, since the Müger centralizer $\mathcal{E}(\zeta, \pm)'$ inside of $\mathcal{Z}(\mathcal{TY}(A, \zeta, \tau))$ is of dimension q^2, whence pointed, this implies that $\mathcal{E}(\zeta, \pm) = \mathcal{B}'$.

We must have $\mathcal{B} = (\widehat{\mathcal{C}}_e)^{\text{rev}}$. Hence, by (4.8), $\mathcal{C} \cong \mathcal{B}' = \mathcal{E}(\zeta, \pm)$, finishing the proof. ■

Using Theorem 4.6 and Theorem 4.7, we can now strengthen Theorem 4.2.

Theorem 4.9 *Let $p < q$ be primes, and let $\mathcal{C}$ be an integral modular category of dimension p^2q^2. Then one of the following is true:*

(i) *$\mathcal{C}$ is group-theoretical;*

(ii) *$p = 2$, $q = 3$, and* $\text{FPdim}(\mathcal{C}_{\text{pt}}) = 3$;

(iii) *$p = 2$,* $\text{FPdim}(\mathcal{C}_{\text{pt}}) = 2$, *and $\mathcal{C} \cong \mathcal{E}(\zeta, \pm)$, as braided fusion categories, for some elliptic quadratic form ζ on $\mathbb{Z}_q \times \mathbb{Z}_q$.*

Remark 4.10 In view of [JL] there are three equivalence classes of non-group-theoretical integral fusion categories of Frobenius–Perron dimension 36. The argument in the proof of Theorem 4.9 implies that a non-group-theoretical integral modular category that satisfies Theorem 4.9(ii) is equivalent to a fusion subcategory of the center of one of these.

In Subsection 4.1, we investigate further the non-group-theoretical categories that satisfy Theorem 4.9(ii).

4.1 Modular Categories of Dimension 36

We begin by classifying the possible fusion rules corresponding to non-group-theoretical modular categories satisfying the conditions of Theorem 4.9(ii).

Proposition 4.11 *Let* $\mathcal{C}$ *be a non-group-theoretical integral modular category of dimension* 36 *with* $\mathrm{Irr}(\mathcal{C}_{\mathrm{pt}}) = \{\mathbf{1}, g, g^2\}$.

(i) *Then* $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{C}_2$ *as a* $\mathbb{Z}_3$*-graded fusion category with respective isomorphism classes of simple objects*

$$\{\mathbf{1}, g, g^2, Y\} \cup \{X, gX, g^2X\} \cup \{X^*, gX^*, g^2X^*\},$$

where $\mathrm{FPdim}(g^i) = 1$, $\mathrm{FPdim}(g^iX) = 2$ *and* $\mathrm{FPdim}(Y) = 3$.

(ii) *Up to relabeling* $g \leftrightarrow g^{-1}$*, the fusion rules are determined by*

$$g \otimes Y \cong Y, \qquad Y^{\otimes 2} \cong \mathbf{1} \oplus g \oplus g^2 \oplus 2Y$$
$$g^i \otimes X \cong g^iX, \qquad Y \otimes X \cong X \oplus gX \oplus g^2X \tag{4.12}$$
$$X \otimes X^* \cong \mathbf{1} \oplus Y$$

and either

(a) $X^{\otimes 2} \cong X^* \oplus gX^*$, *or*
(b) $X^{\otimes 2} \cong g^2X^* \oplus gX^*$.

Proof First note that $\mathcal{C}$ is faithfully $\mathbb{Z}_3$-graded, so that each graded component has dimension 12, and simple objects can only have dimension $1, 2$ or 3. Solving the Diophantine equations provided by the dimension formulas (observing that $\mathcal{C}$ is not pointed), we see that $\mathcal{C}_{\mathrm{pt}} \subset \mathcal{C}_0 = \mathcal{C}_{\mathrm{ad}}$, which gives us the dimensions and objects described in (i).

The fusion rules given in (4.12) are determined by the dimensions and the symmetry rules for the fusion matrices. The remaining fusion rules will be determined from $X^{\otimes 2}$. Clearly $X^{\otimes 2} \in \mathcal{C}_2$, so

$$X^{\otimes 2} \cong a_0X^* \oplus a_1gX^* \oplus a_2g^2X^*,$$

where $\sum_i a_i = 2$. We claim that no $a_i = 2$. For suppose $X^{\otimes 2} \cong 2g^iX^*$ for some $0 \le i \le 2$. Then $(X^*)^{\otimes 2} \cong 2g^{-i}X$, and so

$$(X \otimes X^*)^{\otimes 2} \cong 4(g^i \otimes g^{-i}) \otimes X \otimes X^* \cong 4(\mathbf{1} \oplus Y).$$

On the other hand, $X \otimes X^* \cong \mathbf{1} \oplus Y$, so

$$(X \otimes X^*)^{\otimes 2} \cong (\mathbf{1} \oplus Y)^{\otimes 2} \cong 2\mathbf{1} \oplus g \oplus g^2 \oplus 4Y,$$

a contradiction. Therefore $X^{\otimes 2}$ is multiplicity free. This leaves three possibilities: (a) $a_0 = 1$ and $a_1 = 1$, or (b) $a_0 = 1$ and $a_2 = 1$, or (c) $a_0 = 0$ and $a_1 = a_2 = 1$. The first two are equivalent under the labelling change $g \leftrightarrow g^2$ proving (ii). ∎

Remark 4.13 The non-group-theoretical integral modular categories $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ have fusion rules as in Proposition 4.11(ii)(a). The category $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ is obtained from the quantum group $U_q(\mathfrak{sl}_3)$ with q^2 a primitive 6th root of unity. The data of this category and a proof of non-group-theoreticity may be found in [NR, Example 4.14].

Next, we classify, up to equivalence of fusion categories, modular categories realizing the fusion rules described in Proposition 4.11. To this end, we will need the notion of twist-equivalence, defined next. Let G be a finite group, and let e denote its identity element. Given a G-graded fusion category $\mathcal{C} := (\mathcal{C}, \otimes, \alpha)$ and a 3-cocycle $\eta \in Z^3(G, \mathbb{C}^*)$, the natural isomorphism

$$\alpha^{\eta}_{X_\sigma, X_\tau, X_\rho} = \eta(\sigma, \tau, \rho)\alpha_{X_\sigma, X_\tau, X_\rho}, \quad (X_\sigma \in \mathcal{C}_\sigma, X_\tau \in \mathcal{C}_\tau, X_\rho \in \mathcal{C}_\rho, \sigma, \tau, \rho \in G),$$

defines a new fusion category $\mathcal{C}^\eta := (\mathcal{C}, \otimes, \alpha^\eta)$. The fusion categories $\mathcal{C}$ and $\mathcal{C}^\eta$ are equivalent as G-graded fusion categories if and only if the cohomology class of η is zero; see [ENO3, Theorem 8.9]. We shall say that two G-graded fusion categories $\mathcal{C}$ and $\mathcal{D}$ are *twist-equivalent* if there is a $\eta \in Z^3(G, \mathbb{C}^*)$ such that $\mathcal{C}^\alpha$ is G-graded equivalent to $\mathcal{D}$ (compare with [KW]).

If $(\mathcal{C}, c)$ is a G-graded strict braided fusion category, then each $g \in (\mathcal{C}_e)_{\text{pt}}$ defines a $\mathcal{C}_e$-bimodule equivalence $L_g \colon \mathcal{C}_\sigma \to \mathcal{C}_\sigma, X \mapsto g \otimes X$ with natural isomorphism $c_{g,V} \otimes \text{id}_X \colon L_g(V \otimes X) \to V \otimes L_g(X)$, for every $\sigma \in G$.

Let $\mathcal{C} = \mathcal{C}(\mathfrak{sl}_3, q, 6)$ and consider the normalized symmetric 2-cocycle $\chi \colon \mathbb{Z}_3 \times \mathbb{Z}_3 \to \pi$ defined by $\chi(1,1) = \chi(1,2) = g^2, \chi(2,2) = 1$, where $\pi = \text{Irr}(\mathcal{C}_{\text{pt}}) = U(\mathcal{C}) = \{\mathbf{1}, g, g^2\}$. We define a new tensor product $\overline{\otimes} \colon \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ as

$$\overline{\otimes}|_{\mathcal{C}_i \boxtimes \mathcal{C}_j} = L_{\chi(i,j)} \circ \otimes.$$

Since $\text{H}^4(\mathbb{Z}_3, \mathbb{C}^*) = 0$, it follows by [ENO3, Theorem 8.8] that we can find isomorphisms

$$\omega_{i,j,k} \colon \chi(i+j, k) \otimes \chi(i, j) \longrightarrow \chi(i, j+k) \otimes \chi(j, k)$$

such that the natural isomorphisms

$$\widehat{\alpha}^{\omega}_{X_i, X_j, X_k} = (\text{id}_{\chi(i,j+k)} \otimes c_{\chi(j,k), X_i} \otimes \text{id}_{X_k}) \circ (\omega_{i,j,k} \otimes \text{id}_{X_i \otimes X_j \otimes X_k}),$$

define an associator with respect to $\overline{\otimes}$ and we get a new $\mathbb{Z}_3$-graded fusion category

$$\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6) := (\mathcal{C}, \overline{\otimes}, \widehat{\alpha}^\omega).$$

Remark 4.14 The notation $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ is ambiguous, because we are not specifying ω. However, $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ is unique up to twist-equivalence.

Theorem 4.15 *Let $\mathcal{A}$ be a fusion category.*

(i) *If $\mathcal{A}$ has fusion rules given by Proposition* 4.11(ii)(a)*, then $\mathcal{A}$ is twist-equivalent to $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ for some choice of q.*

(ii) *If $\mathcal{A}$ is braided and has fusion rules given by Proposition* 4.11(ii)(b)*, then $\mathcal{A}$ is twist-equivalent to $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ for some choice of q.*

(iii) *Any fusion category twist-equivalent to $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ or $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ is non-group-theoretical.*

Proof (i) This follows from the classification results in [KW, Theorem A_ℓ].

(ii) Since $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6) = \mathcal{C}(\mathfrak{sl}_3, q, 6)$ as abelian categories, their simple objects are the same. However, since the tensor product is different, the duals of simple objects can be different, so we shall use the following notation for the simple objects of $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$:

$$\mathbf{1},\ g,\ g^2,\ Y,\ gX,\ g^2X,\ \overline{X},\ g\overline{X},\ g^2\overline{X},$$

where $\overline{X} = X^*$ with respect to the original tensor product of $\mathcal{C}(\mathfrak{sl}_3, q, 6)$.

Next, we investigate the fusion rules of $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$. First note that $X^* \in \mathcal{C}_2$ and $X\overline{\otimes}\overline{X} = g^2 \oplus Y$, so $X^* = g\overline{X}$ and $X\overline{\otimes}X^* = \mathbf{1} \oplus Y$. Since χ is normalized, the only important fusion rule that changes is

$$X\overline{\otimes}X = g^2 \otimes (\overline{X} \oplus g\overline{X}) = g^2\overline{X} \oplus \overline{X} = gX^* \oplus g^2X^*.$$

Note that the fusion rules of $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ are the same as Proposition 4.11(ii)(b).

If $\mathcal{A}$ is a braided fusion category with fusion rules given by Proposition 4.11(ii)(b), then, using the 2-cocycle χ^{-1}, we can construct a fusion category $\mathcal{D}$ with the same fusion rules of $\mathcal{C}(\mathfrak{sl}_3, q, 6)$. By (i), there exists a q such that $\mathcal{D}$ is twist-equivalent to $\mathcal{C}(\mathfrak{sl}_3, q, 6)$, and, again using the 2-cocycle χ on $\mathcal{C}(\mathfrak{sl}_3, q, 6)$, we get a fusion category twist-equivalent to $\mathcal{A}$.

(iii) Let G be a finite group, and let e denote its identity element. In [G2, Theorem 1.2], it was proved that a G-graded fusion category $\mathcal{A}$ is group-theoretical if and only if there is a pointed $\mathcal{A}_e$-module category $\mathcal{M}$ such that $\mathcal{A}_\sigma \boxtimes_{\mathcal{A}_e} \mathcal{M} \cong \mathcal{M}$ as $\mathcal{A}_e$-module categories for all $\sigma \in G$. If $\mathcal{A}$ is twist-equivalent to $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ or $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$, then we have $\mathcal{A}_e = \mathcal{C}(\mathfrak{sl}_3, q, 6)_e$ as fusion categories and $\mathcal{A}_\sigma = \mathcal{C}(\mathfrak{sl}_3, q, 6)_\sigma$ as $\mathcal{A}_e$-bimodule category. Since $\mathcal{C}(\mathfrak{sl}_3, q, 6)$ is non-group-theoretical [NR, Example 4.14], $\mathcal{A}$ is also non-group-theoretical. ■

$$S := \begin{pmatrix} 1 & 1 & 1 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 3 & 2q^2 & 2q^{-2} & 2q^2 & 2q^{-2} & 2q^2 & 2q^{-2} \\ 1 & 1 & 1 & 3 & 2q^{-2} & 2q^2 & 2q^{-2} & 2q^2 & 2q^{-2} & 2q^2 \\ 3 & 3 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2q^2 & 2q^{-2} & 0 & 2q^{-1} & 2q & 2q & 2q^{-1} & -2 & -2 \\ 2 & 2q^{-2} & 2q^2 & 0 & 2q & 2q^{-1} & 2q^{-1} & 2q & -2 & -2 \\ 2 & 2q^2 & 2q^{-2} & 0 & 2q & 2q^{-1} & -2 & -2 & 2q^{-1} & 2q \\ 2 & 2q^{-2} & 2q^2 & 0 & 2q^{-1} & 2q & -2 & -2 & 2q & 2q^{-1} \\ 2 & 2q^2 & 2q^{-2} & 0 & -2 & -2 & 2q^{-1} & 2q & 2q & 2q^{-1} \\ 2 & 2q^{-2} & 2q^2 & 0 & -2 & -2 & 2q & 2q^{-1} & 2q^{-1} & 2q \end{pmatrix}$$

$$T := \mathrm{Diag}(1, 1, 1, -1, q^2, q^2, 1, 1, q^{-2}, q^{-2})$$

Figure 1: $q = e^{\pi i/3}$.

4.2 Conclusions

In this section, we have classified integral modular categories of dimension p^2q^2 up to equivalence of braided fusion categories with the exception of non-group-theoretical $\mathbb{Z}_3$-graded 36-dimensional modular categories, which are classified only up to equivalence of fusion categories.

It can be shown that the category $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ is not of the form Rep(H) for a Hopf algebra H using the same technique as in [GHR, Theorem 5.27]. More generally, a non-group-theoretical fusion category $\mathcal{C}$ with fusion rules as in Proposition 4.11 cannot be equivalent to a category of the form Rep H, H a Hopf algebra. If $\mathcal{C} \cong \text{Rep}(H)$ for some Hopf algebra H, then since $\mathcal{C}$ admits a faithful $\mathbb{Z}_3$-grading, we would have a central exact sequence $k^{\mathbb{Z}_3} \to H \to \overline{H}$, where $\dim \overline{H} = 12$ and $\text{Rep}\, \overline{H} \cong \mathcal{C}_0$. The classification of semisimple Hopf algebras of dimension 12 implies that $\overline{H}$ is a group algebra. Hence $k^{\mathbb{Z}_3} \to H \to \overline{H}$ is an abelian exact sequence, and therefore H is group-theoretical, a contradiction.

The existence of a modular structure on the category $\overline{\mathcal{C}}(\mathfrak{sl}_3, q, 6)$ will be discussed in a future work, but for the interested reader we provide the modular data for $q = e^{\pi i/3}$ in Figure 1.

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