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Electronic Notes in Discrete Mathematics 44 (2013) 115–120

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

On minimal non [h,2,1] graphs

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Abstract

The class of graphs which admits a VPT representation in a host tree with maximum degree at most h is denoted by $[h, 2, 1]$. The classes $[h, 2, 1]$ are closed by taking induced subgraphs, therefore each one can be characterized by a family of minimal forbidden induced subgraphs. In this paper we associate the minimal forbidden induced subgraphs for $[h, 2, 1]$ which are VPT with (color) critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of minimal forbidden induced subgraphs which are VPT, split and have no dominated stable vertices. We conjecture that there are no other VPT minimal forbidden induced subgraphs. We also prove that the minimal forbidden induced subgraphs for $[h, 2, 1]$ that are VPT graphs belong to the class $[h + 1, 2, 1]$.

Keywords: Intersection graphs, representations on trees, forbidden subgraphs.

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http://dx.doi.org/10.1016/j.endm.2013.10.018

1 Introduction

A graph G is called a **VPT graph** if it is the vertex intersection graph of a family of paths in a tree. The class of graphs which admits a VPT representation in a host tree with maximum degree at most h is denoted by **[h,2,1]**. In [4], it is shown that the problem of recognizing VPT graphs is polynomial time solvable. In [1], it is proved that the problem of deciding whether a given VPT graph belongs to $[h, 2, 1]$ is NP-complete even when restricted to the class $VPT \cap split$ without dominated stable vertices. The classes $[h, 2, 1]$, $h \geq 2$, are closed by taking induced subgraphs, therefore each one can be characterized by a family of minimal forbidden induced subgraphs. Such a family is know only for $h = 2$ [6] and there are some partial results for $h = 3$ [3]. In this paper we associate the minimal forbidden induced subgraphs for $[h, 2, 1]$ which are VPT with (color) critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of minimal forbidden induced subgraphs which are VPT, split and have no dominated stable vertices. We conjecture that there are no other VPT minimal forbidden induced subgraphs. We also prove that the minimal forbidden induced subgraphs for $[h, 2, 1]$ that are VPT graphs belong to the class $[h + 1, 2, 1]$.

In the present work, all results that have no reference are original; many proofs are omitted for lack of space.

2 Preliminaries

In this paper all graphs are connected, finite and simple. Notation we use is that used by Bondy and Murty [2].

A **complete set** is a subset of vertices inducing a complete subgraph. A **clique** is a maximal complete set. The set of cliques of G is denoted by $\mathcal{C}(G)$. A **stable set** is a subset of pairwise non-adjacent vertices .

The graph G is **split** if $V(G)$ can be partitioned into a stable set S and a clique K. The pair $(\mathbf{S}, \mathbf{K})$ is the **split partition** of G and this partition is unique up to isomorphisms. The vertices in S are called **stable vertices**, and K is called the **central clique** of G. We say that a stable vertex $s \in S$ is **dominated** if there exists $s' \in S$ such that $N(s) \subseteq N(s')$. Notice that if G is split then $\mathcal{C}(G) = \{K, N[s] \text{ for } s \in S\}$.

The graph G is **k-colorable** if its vertices can be colored with at most k colors in such a way that no two adjacent vertices share the same color. The minimum k such that G is k-colorable is the **chromatic number** $\chi(G)$ of G. A vertex $v \in V(G)$ or an edge $e \in E(G)$ is a **critical element of G** if $\chi(G-v) < \chi(G)$ or $\chi(G-e) < \chi(G)$. A graph G with chromatic number h is **h-vertex critical** (resp. **h-edge critical**) if each of its vertices (resp. edges) is a critical element and it is **h-critical** if both hold.

Definition 2.1 [5] Let $C \in \mathcal{C}(G)$. The **branch graph** of G for the clique C noted $\mathbf{B(G/C)}$ is defined as follows: the vertices of $B(G/C)$ are the vertices of $V(G) - C$ adjacent to some vertex of C. Two vertices v and w are adjacent in $B(G/C)$ if and only if: (1) $vw \notin E(G)$; (2) there is a vertex of C adjacent to both v and w; (3) there are vertices v' and w' of C such that v' is adjacent to v and non-adjacent to w, and w' is adjacent to w and non-adjacent to v.

It is clear that if $C \in \mathcal{C}(G)$ and $v \in V(G) - C$ then $C \in \mathcal{C}(G - v)$, the following lemma says what happens with the branch graph when we remove such vertices.

Lemma 2.2 *Let $C \in \mathcal{C}(G)$ and let $v \in V(G) - C$: (i) If $v \notin V(B(G/C))$ then $B(G - v/C) = B(G/C)$; (ii) if $v \in V(B(G/C))$ then $B(G - v/C) = B(G/C) - v$.*

In [1] we proved the following theorem which shows that there is a relation between the VPT graphs that can be represented in a tree with maximum degree at most h and the chromatic number of their branch graphs.

Theorem 2.3 *[1] Let $G \in VPT$ and $h \geq 4$. The graph G belongs to $[h, 2, 1]-[h-1, 2, 1]$ if and only if $Max_{C\in\mathcal{C}(G)}(\chi(B(G/C))) = h$. The reciprocal implication is also true for $h = 3$.*

Definition 2.4 A clique K of G is called **principal** if $Max_{C\in\mathcal{C}(G)}(\chi(B(G/C))) = \chi(B(G/K))$.

3 Necessary conditions for minimal non [h,2,1] graphs

A **minimal non [h,2,1] graph** is a minimal forbidden induced subgraph for the class $[h, 2, 1]$, this means any graph G such that $G \notin [h, 2, 1]$ and $G - v \in [h, 2, 1]$ for every vertex $v \in V(G)$.

Theorem 3.1 *Let $G \in VPT$ and let $h \geq 3$. If G is a minimal non $[h, 2, 1]$ graph then $G \in [h + 1, 2, 1]$.*

Proof. Let $C \in \mathcal{C}(G)$ and let $v \notin C$. We know that $G - v \in [h, 2, 1]$ then, by Theorem 2.3, $\chi(B(G - v/C)) \leq h$. By Lemma 2.2, $\chi(B(G - v/C)) =$

$\chi(B(G/C) - v) \geq \chi(B(G/C)) - 1$. Thus, $\chi(B(G/C)) - 1 \leq h$ and hence $\chi(B(G/C)) \leq h + 1$. Then, by Theorem 2.3, $G \in [h + 1, 2, 1]$. □

Theorem 3.2 *Let K be a principal clique of a VPT minimal non $[h, 2, 1]$ graph G, with $h \geq 3$. Then: (i) $V(B(G/K)) = V(G) - K$; (ii) if $v \in V(G) - K$ then $|N(v) \cap K| > 1$; (iii) $B(G/K)$ is $(h+1)$-vertex critical.*

4 Building minimal non [h,2,1] graphs

The construction presented here is similar to that done in [1], and a generalization of that used in [3]. Given a graph H with $V(H) = \{v_1, \ldots, v_n\}$, let G_H be the graph with vertices: v_i, for each $1 \leq i \leq n$; v_{ij}, for each $1 \leq i < j \leq n$ such that $v_i v_j \in E(H)$; and $\tilde{v}_i$, for each $1 \leq i \leq n$ with $d_H(v_i) = 1$. The cliques of G_H are: $K = \{v_{ij}$, with $1 \leq i < j \leq n$ $\} \cup \{\tilde{v}_i$, for each $1 \leq i \leq n$ such that $d_H(v_i) = 1\}$, and $C_{v_i} = \{v_i\} \cup \{v_{ij}; v_j \in N_H(v_i)\} \cup \{\tilde{v}_i$, if $d_H(v_i) = 1\}$, for $1 \leq i \leq n$. (See an example in Figure 1).

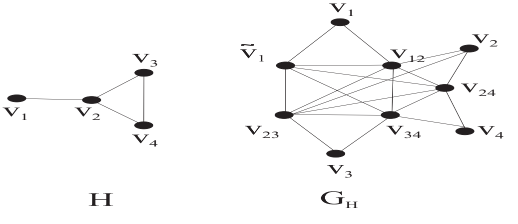

Fig. 1. A graph H and the graph G_H.

Notice that the vertices of G_H are partitioned in a stable set S of size $n = |V(H)|$ corresponding to the vertices v_i; and a central clique K of size $|E(H)| + |\{v \in V(H); d_H(v) = 1\}|$ corresponding to the remaining vertices. The usefulness of G_H relies on the properties described in the following lemma.

Lemma 4.1 *(i) G_H is a $VPT \cap$ split graph without dominated stable vertices; (ii) $B(G_H/K) = H$; (iii) $B(G_H/C_{v_i})$ is 1-colorable, for each $1 \leq i \leq n$; (iv) K is a principal clique of G_H.*

Theorem 4.2 *Let $h \geq 3$. The graph G_H is a minimal non $[h, 2, 1]$ graph if and only if H is $(h + 1)$-critical.*

Proof. $\Rightarrow$) Assume that G_H is a minimal non $[h, 2, 1]$ graph. By Lemma 4.1 and Theorem 3.2, $B(G_H/K) = H$ is $(h + 1)$-vertex critical. Let see that H is $(h + 1)$-edge critical. Let $e = v_i v_j \in E(H)$. Since H is $(h + 1)$-vertex critical, each of its vertices has degree at least 3. Thus, $G_{H-e} = G_H - v_{ij}$. Then, by Lemma 4.1, $B(G_H - v_{ij}/K - v_{ij}) = H - e$. Since G_H is a minimal non $[h, 2, 1]$

graph then $G_H - v_{ij} \in [h, 2, 1]$. Hence, $\chi(B(G_H - v_{ij}/K - v_{ij})) \leq h$, which implies that $\chi(H - e) \leq h$. Therefore, H is $(h+1)$-edge critical.

$\Leftarrow$) Let H be an $(h+1)$-critical graph with $V(H) = \{v_1, v_2, ..., v_n\}$. By Lemma 4.1, $Max_{C \in \mathcal{C}(G_H)}(\chi(B(G_H/C))) = \chi(B(G_H/K)) = \chi(H) = h+1$. Hence, by Theorem 2.3, $G_H \in [h+1, 2, 1] - [h, 2, 1]$. Let us see that $G_H - v \in [h, 2, 1]$, for all $v \in V(G_H)$. First, if $v = v_i \in V(H)$, using Lemma 2.2 and Lemma 4.1, $B(G_H - v_i/K) = B(G_H/K) - v_i = H - v_i$. Thus, since H is $(h+1)$-vertex critical, $\chi(B(G_H - v_i/K)) = h$. Hence, $G_H - v_i \in [h, 2, 1]$. Secondly, if $v = v_{ij}$ being $e = v_i v_j \in E(H)$, then $\chi(B(G_H - v_{ij}/K - v_{ij}))$ $= \chi(H - e) = h$, because H is $(h+1)$-edge critical. Hence, $G_H - v_{ij} \in [h, 2, 1]$. Since H has no degree 1 vertices, G_H has no more vertices. □

5 Split graphs without dominated stable vertices

In this Section we characterize minimal non $[h, 2, 1]$ graphs, with $h \geq 3$, which are VPT, split and have no dominated stable vertices.

Lemma 5.1 *Let $G = (S, K)$ be VPT, split and without dominated stable vertices. If G is a minimal non $[h, 2, 1]$ graph, with $h \geq 3$, then: (i) K is a principal clique of G; (ii) $V(B(G/K)) = S$; (iii) for all $k \in K$, $|N(k) \cap S| = 2$; (iv) $|E(B(G/K))| = |K|$; (v) $B(G/K)$ is $(h+1)$-critical.*

Proof. (iii) Since G has no dominated stable vertices and $G \in VPT$, $|N(k) \cap S| \leq 2$, for all $k \in K$.

First we will observe that if $k \in K$ then $K - k \in \mathcal{C}(G - k)$. If not there must exists $s \in S$ such that s is adjacent to all the vertices of $K - k$. But, by Theorem 3.1, $G \in [h+1, 2, 1]$ then, by Theorem 2.3 and item (i), $\chi(B(G/K)) = h + 1$ being $V(B(G/K)) = S$, thus $|S| \geq h + 1 \geq 4$. Hence, there exist $s_1, s_2, s_3 \in S$. Moreover, since G has no dominated stable vertices, $s_i k \in E(G)$, for $i = 1, 2, 3$. Then, $|N(k) \cap S| > 2$, which contradicts the fact that $G \in VPT$ without dominated stable vertices.

Now, suppose that there exists $\tilde{k} \in K$ such that $|N(\tilde{k}) \cap S| < 2$.

1. If $|N(\tilde{k}) \cap S| = 0$: It is clear that $B(G - \tilde{k}/K - \tilde{k}) = B(G/K)$. Then, $\chi(B(G - \tilde{k}/K - \tilde{k})) = \chi(B(G/K)) = h + 1$, which contradicts the fact that G is a minimal non $[h, 2, 1]$ graph.

2. If $|N(\tilde{k}) \cap S| = 1$: We will see that $B(G - \tilde{k}/K - \tilde{k}) = B(G/K)$. It is clear that $V(B(G - \tilde{k}/K - \tilde{k})) = V(B(G/K))$ and $E(B(G - \tilde{k}/K - \tilde{k})) \subseteq E(B(G/K))$. Let $uv \in E(B(G/K))$ such that $uv \notin E(B(G - \tilde{k}/K - \tilde{k}))$. Since $|N(\tilde{k}) \cap S)| = 1$ we can assume, without loss of generality, that $\{N(v) \cap K\} - \{N(u) \cap K\} = \{\tilde{k}\}$. Therefore, since, for all $k \in K$, $|N(k) \cap S| \leq 2$ we

have that $N_{B(G/K)}(v) = \{u\}$ then $d_{B(G/K)}(v) = 1$, which contradicts the fact that H is $(h+1)$-vertex critical. $\square$

Theorem 5.2 *Let G be VPT, split and without dominated stable vertices and let $h \geq 3$. G is a minimal non $[h, 2, 1]$ graph if and only if there exists an $(h+1)$-critical graph H such that $G \simeq G_H$.*

Proof. $\Leftarrow$) It follows directly applying Theorem 4.2.

$\Rightarrow$) By Theorem 3.1, $G \in [h+1, 2, 1]$. Let $H = B(G/K)$. By Lemma 5.1, H is an $(h+1)$-critical graph. Let us see that $G \simeq G_H$. Let $G_H = (S', K')$. By Lemma 4.1, $B(G_H/K') = H$ then $B(G_H/K') = B(G/K)$. Also, by Lemma 5.1, $S' = S$, $|K'| = |K|$ and, for all $k \in K$, $|N(k) \cap S| = 2$. Since G has no dominated stable vertices, if $N(k) \cap S = \{v_i, v_j\}$ then $v_iv_j \in E(H)$. Moreover, if k, $\tilde{k} \in K$, with $k \neq \tilde{k}$ then $N(k) \cap S \neq N(\tilde{k}) \cap S$. Hence, we can define a function that assigns to each vertex $k \in K$ an edge $v_iv_j \in E(H)$, that is, an element of K'. Note that in G_H the vertex $v_iv_j \in K'$ is adjacent exactly to v_i and v_j. Hence, the function f can be extended to a new function $\tilde{f}$ from $K\dot{\cup}S$ to $K'\dot{\cup}S'$, being the identity function from S to S'. Moreover, $\tilde{f}$ is an isomorphism between G and G_H. $\square$

Conjecture 5.3 *If G is VPT and minimal non $[h, 2, 1]$ then G is split without dominated stable vertices.*

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